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THE EQUIVALENCE OF ABSORBING AND REFLECTING BARRIER PROBLEMS FOR STOCHASTICALLY MONOTONE MARKOV PROCESSES[1]

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The equivalence between absorbing and reflecting barrier problems for random walks is shown to hold for stochastically monotone Markov processes. For Markov chains in continuous time this relation is expressed directly in terms of the Q-matrices of the chains. Some examples are given.

1. Introduction. Let $p(t, x, dy)$, $0 \leqq t < \infty$ or $t = 0, 1, 2, \cdots$ be a Markov transition probability on the state space $[0, \infty]$ and denote by $(P^x, X(t), 0 \leqq t < \infty)$ (or $t = 0, 1, 2, \cdots$) the Markov process associated with p, i.e., satisfying $P^x\{X(t) \in A\} = \int_A p(t, x, dy)$ for Borel sets $A \subset [0, \infty]$, $0 \leqq x \leqq \infty$, $0 \leqq t < \infty$. For most applications $P^x\{X(t) < \infty\} = 1$ $(0 \leqq x < \infty)$, and then the process is said to be honest.

In what follows it will be notationally convenient to denote the paths of Markov processes generically by $X(t)$, $0 \leqq t < \infty$ (or $t = 0, 1, 2, \cdots$) and to distinguish different processes by attaching subscripts to the probability measures governing their evolution. For purposes of terminology it will be convenient to refer to the process P^x (or P_1^x or P_2^x).

The central problem of this paper is to find conditions under which, given a process P_1^x, one can find a process P_2^x such that for all $0 \leqq x, y, t < \infty$

$$P_1^x\{X(t) \geqq y\} = P_2^y\{X(t) \leqq x\} . \tag{1}$$

Putting $y = 0$ in (1) shows that P_2^x (if it exists) necessarily has an absorbing barrier at 0. (The methods of this paper show that one may easily formulate and prove analogous results for the converse problem: given P_2^x with an absorbing barrier at 0 find P_1^x so that (1) holds. For the sake of brevity details concerning this converse problem are omitted.)

The relation (1) when P_1^x is Brownian motion on $[0, \infty)$ with reflection at 0 and P_2^x Brownian motion with absorption at 0 has been systematically exploited by P. Lévy (1948, pages 210 ff.). The same relation for random walks was noticed by D. Lindley (1952) in his work on queueing theory and has subsequently been applied in this context by others. To be more precise let $x_1, x_2, \cdots$ be independent and identically distributed, and let P_1^x denote the probability

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governing the random walk with initial value x, increments x_t, and reflection at 0; so under P_1^x the random variables $X(t)$ may be defined recursively by $X(0) = x$, $X(t) = \max(0, X(t-1) + x_t)$, $t = 1, 2, \cdots$. Let P_2^x denote the probability governing the random walk with initial value x, increments $-x_t$, and absorption at 0, i.e., in terms of $s_t = \sum_{n \leqq t} x_n$ ($s_0 = 0$), under P_2^x the random variables $X(t)$ may be defined by

$$\begin{aligned} X(t) &= x - s_t \qquad \text{if} \quad x - s_n > 0 \quad \text{for all} \quad n \leqq t \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Then (1) follows easily from the fact that $(x_1, \cdots, x_t)$ have the same joint distribution as $(x_t, \cdots, x_1)$.

A classical application of (1) for random walks is as follows. Let $\tau_0 = \inf\{t : X(t) = 0\}$, where by convention $\inf \varnothing = +\infty$. Then under mild conditions which are satisfied by the random walk example of the preceding paragraph, for all $x \geqq 0$

$$\lim_{t\to\infty} P_1^x\{X(t) \geqq y\} = P_2^y\{\tau_0 < \infty\}, \tag{2}$$

and, under additional conditions on the distribution of the increments x_t of the random walk, a result of Cramér on the probability of ruin for a risk process provides an estimate for the right-hand side of (2) for large values of y (cf. Feller, 1966, page 393). More recently the author (1976) has described Monte Carlo techniques which permit one to estimate the right-hand side of (1) or (2) by simulation with considerably more accuracy than direct simulation of the left-hand side. These applications motivate the present study.

Before proceeding with the general problem of determining P_2^x from P_1^x so that (1) holds, it is of interest to consider another example suggested by Lindley (1959). Now let P_1^x denote the random walk with initial value x, increments x_t, and reflection at 0 and $b > 0$. Hence under P_1^x, $X(0) = x$ and $X(t) = \min(b, \max(0, X(t-1) + x_t))$, $t = 1, 2, \cdots$. Under P_2^x let $X(t)$ be a random walk with initial value x, increments $-x_t$, and absorption at 0 and $b+$. More precisely let $T = \inf\{t : x - s_t \notin (0, b]\}$, and under P_2^x let $X(0) = x$, $X(t) = \max(0, x - s_{\min(T,t)})$, $t = 1, 2, \cdots$. Then (1) holds for all $0 \leqq x, y \leqq b$, $t = 0, 1, 2, \cdots$. This result, which is slightly more subtle than for a single barrier, follows easily from Theorem 1 in Section 2.

2. Stochastically monotone Markov processes. Theorem 1 below gives simple necessary and sufficient conditions on $P_1^x\{X(t) \geqq y\}$ in order that there exist a process P_2^x satisfying (1). In many applications P_1^x is given in terms of its infinitesimal characteristics. Theorems 2 and 3 are restricted to the case of a denumerable state space and are concerned with the problem of determining P_2^x directly from the infinitesimal characteristics of P_1^x.

A Markov process P^x is called stochastically monotone by Daley (1968) if for some $h > 0$ ($h \geqq 1$ for discrete time)

$$P^x\{X(t) \geqq y\} \tag{3}$$

is nondecreasing in x for each $0 \leqq y < \infty$, $0 \leqq t \leqq h$. It will be convenient to refer to the process P^x as right continuous if the probability (3) is right continuous in x for all $0 \leqq y < \infty$ and $0 \leqq t \leqq h$. (Note that this concept of right continuity has nothing to do with the sample path behavior of the process.)

THEOREM 1. *Given a process P_1^x which is honest or for which ∞ is an absorbing state, in order that there exist a process P_2^x satisfying* (1) *it is necessary and sufficient that P_1^x be stochastically monotone and right continuous. Under the sufficient conditions the process P_2^y may be assumed to satisfy* (1) *for all $0 \leqq y \leqq \infty$.*

PROOF. The necessity of the conditions is obvious, since the right-hand side of (1) is nondecreasing and right continuous in x for all y, t. For this part of the theorem the additional hypotheses on P_1^x are unnecessary.

To prove the sufficiency, for each fixed $0 \leqq t \leqq h$, $0 \leqq y \leqq \infty$ define a set function $p_2(t, y, A)$ by

$$p_2(t, y, [0, x]) = P_1^x\{X(t) \geqq y\}, \qquad 0 < x < \infty, \tag{5}$$

and

$$p_2(t, y, \{\infty\}) = 1 - \lim_{x\to\infty} p_2(t, y, [0, x]). \tag{6}$$

By assumption for each $0 \leqq t \leqq h$, $0 \leqq y < \infty$, p_2 is a nondecreasing right continuous function of x and hence extends to a measure on the Borel subsets of $[0, \infty)$. By (6) it is actually a probability measure on the Borel subsets of $[0, \infty]$. As the limit of the decreasing sequence $P_1^x\{X(t) \geqq n\}$ as $n \uparrow \infty$, $P_1^x\{X(t) = \infty\}$ is nondecreasing and right continuous in x, so (5) and (6) also define $p_2(t, \infty, \cdot)$ as a probability measure.

To show that this family of probabilities are actually Markov transition probabilities on $[0, h]$ and hence define a Markov process, it suffices to prove that they satisfy the Chapman-Kolmogorov equations on $[0, h]$, i.e., for $0 \leqq y \leqq \infty$, $0 \leqq t, s, t + s \leqq h$, and Borel sets $A \subset [0, \infty]$

$$p_2(t + s, y, A) = \int_{[0,\infty]} p_2(t, y, dz) p_2(s, z, A). \tag{7}$$

Since the left and right-hand sides of (7) are measures, it suffices to give the proof for sets of the form $A = [0, x]$ for $0 \leqq x < \infty$ and $A = \{\infty\}$. The following calculation is justified by (5), the Chapman-Kolmogorov equations for P_1^x, the assumption that P_1^x is honest or has an absorbing state at ∞, and Fubini's theorem.

$$\begin{aligned} p_2(t + s, y, [0, x]) &= P_1^x\{X(t + s) \geqq y\} \\ &= \int_{[0,\infty)} P_1^x\{X(t) \in dz\} P_1^z\{X(s) \geqq y\} + P_1^x\{X(t) = \infty\} P_1^\infty\{X(s) \geqq y\} \\ &= \int_{[0,\infty)} P_1^x\{X(t) \in dz\} \int_{[0,z]} p_2(s, y, du) + P_1^x\{X(t) = \infty\} \\ &= \int_{[0,\infty)} p_2(s, y, du) P_1^x\{u \leqq X(t) < \infty\} + P_1^x\{X(t) = \infty\} \\ &= \int_{[0,\infty]} p_2(s, y, du) P_1^x\{X(t) \geqq u\} \\ &= \int_{[0,\infty]} p_2(s, y, dz) p_2(t, z, [0, x]). \end{aligned} \tag{8}$$

Also by (6), (8) and the monotone convergence theorem

$$\begin{aligned} p_2(t+s, y, \{\infty\}) &= 1 - \lim_{x\to\infty} \int_{[0,\infty]} p_2(s, y, dz) p_2(t, z, [0, x]) \\ &= \int_{[0,\infty]} p_2(s, y, dz)(1 - \lim_{x\to\infty} p_2(t, z, [0, x])) \\ &= \int_{[0,\infty]} p_2(s, y, dz) p_2(t, z, \{\infty\}) . \end{aligned}$$

Hence p_2 satisfies the Chapman–Kolmogorov equations and defines a process P_2^x satisfying (1) on $[0, h]$.

To extend this process to $[0, \infty)$ observe that except for the first equality the calculation given in (8) is valid for arbitrary $0 \leqq t, s \leqq h$. By the monotone convergence theorem applied to the final expression in (8) one sees that $P_1^x\{X(t+s) \geqq y\}$ is nondecreasing and right continuous in x for $t + s \leqq 2h$. Hence (7) holds on $[0, 2h]$ and by repetitions of the argument on $[0, \infty)$.

REMARKS. The condition of stochastic monotonicity is the crucial one in Theorem 1. The continuity condition will typically be satisfied, as it is for example if $\int_{[0,\infty)} p_1(t, x, dy) f(y)$ is a continuous function of x on $[0, \infty)$ for bounded continuous f.

Theorems 2 and 3 are restricted to the case of a countable state space. In many applications a standard substochastic transition matrix $p(t, j, k)$, $0 \leqq t < \infty$, $j, k = 0, 1, \cdots$ is given in terms of its Q-matrix (and appropriate boundary conditions), $Q = (q(j, k))$ defined by

$$q(j, k) = \lim_{t\to 0} t^{-1}\{p(t, j, k) - \delta(j, k)\} , \tag{9}$$

where $\delta(j, k) = 1$ or 0 according as $j = k$ or $j \neq k$. The matrix Q defined by (9) necessarily satisfies

$$\infty > q(j, k) \geqq 0 \qquad j \neq k \tag{10}$$

and

$$-q(j, j) \geqq \sum_{k\neq j} q(j, k) , \tag{11}$$

and an arbitrary matrix having these properties is called a Q-matrix (Chung, 1967, page 251). It is called conservative if equality holds in (11). In what follows all entries $q(j, j)$ are assumed finite.

THEOREM 2. *Let Q_1 be a conservative Q-matrix and define*

$$q_2(k, j) = \sum_{\nu\geqq k} (q_1(j, \nu) - q_1(j-1, \nu)) \qquad k, j = 0, 1, \cdots \tag{12}$$

$(q_1(-1, k) \equiv 0)$. *Then $Q_2 = (q_2(k, j))$ is a Q-matrix if and only if*

$$q_2(k, j) \geqq 0 \qquad j \neq k , \tag{13}$$

which is conservative if and only if

$$\lim_{j\to\infty} q_1(j, k) = 0 \qquad k = 0, 1, 2, \cdots . \tag{14}$$

THEOREM 3. *Let Q_1 be a conservative Q-matrix and P_1^j the minimal Q_1 process. Assume that* (13) *and* (14) *hold, so that* (*by Theorem* 2) *Q_2 defined by* (12) *is also a* *conservative Q-matrix. Then there exists a process P_2^k with Q-matrix Q_2, which satisfies*

$$P_1^j\{X(t) \geqq k\} = P_2^k\{X(t) \leqq j\}, \qquad j, k = 0, 1, 2, \cdots, +\infty. \tag{15}$$

If the minimal Q_i process is honest for $i = 1$ or 2, then P_2^k is the minimal Q_2-process. If neither minimal process is honest, then P_2^k is an honest process which satisfies the forward Kolmogorov equations.

PROOF OF THEOREM 2. First note that since $q_1(j, j) < \infty$ by assumption it follows from (11) that $q_2(k, j)$ defined in (12) is in fact a well-defined, finite number. The condition (13) is obviously necessary in order that Q_2 be a Q-matrix. To see that it is also sufficient, by (11) it is enough to verify that $\sum_{j\geqq 0} q_2(k, j) \leqq 0$. By (12)

$$\sum_{j=0}^{J} q_2(k, j) = \sum_{j=0}^{J} \sum_{\nu\geqq k} (q_1(j, \nu) - q_1(j-1, \nu)) = \sum_{\nu\geqq k} q_1(J, \nu) \tag{16}$$

which is nonpositive for all $J \geqq k$, since Q_1 is a Q-matrix. By (16) Q_2 is conservative if and only if $\lim_{J\to\infty} \sum_{\nu\geqq k} q_1(J, \nu) = 0$. But since Q_1 is conservative,

$$\sum_{\nu\geqq k} q_1(J, \nu) = -\sum_{\nu<k} q_1(J, \nu) \to 0 \tag{17}$$

as $J \to \infty$ if and only if (14) holds.

PROOF OF THEOREM 3. Let N be a positive integer and define the $(N+1) \times (N+1)$ matrix

$$Q_{1N} = (q_{1N}(j, k)) = \begin{pmatrix} q_1(0, 0), & \cdots, & q_1(0, N-1), & \sum_{\nu\geqq N} q_1(0, \nu) \\ \vdots & & & \\ q_1(N, 0), & \cdots, & q_1(N, N-1), & \sum_{\nu\geqq N} q_1(N, \nu) \end{pmatrix}$$

and the $(N+2) \times (N+2)$ matrix

$$Q_{2N} = (q_{2N}(k, j)) = \begin{pmatrix} q_2(0, 0), & \cdots, & q_2(0, N), & \sum_{\nu\geqq N+1} q_2(0, \nu) \\ \vdots & & & \\ q_2(N, 0), & \cdots, & q_2(N, N), & \sum_{\nu\geqq N+1} q_2(N, \nu) \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is easily verified that Q_{2N} is defined in terms of Q_{1N} by the relation (12) (with $q_{1N}(N+1, k) \equiv 0$), that both Q_{1N} and Q_{2N} are conservative Q-matrices, and for the Markov chain determined by Q_{2N} the state $N+1$ is absorbing. Let $p_{1N}(t, j, k)$ denote the standard transition probability determined by the finite Q-matrix Q_{1N} and define

$$\begin{aligned} p_{2N}(t, k, j) &= \sum_{\nu=k}^{N} (p_{1N}(t, j, \nu) - p_{1N}(t, j-1, \nu)) && 0 \leqq j, k \leqq N, \\ p_{2N}(t, N+1, j) &= \delta(N+1, j) && 0 \leqq j \leqq N+1, \\ p_{2N}(t, k, N+1) &= 1 - \sum_{j=0}^{N} p_{2N}(t, k, j) && 0 \leqq k \leqq N+1. \end{aligned} \tag{18}$$

It is shown below that p_{2N} defined by (18) satisfies the forward Kolmogorov equations

$$\frac{\partial}{\partial t} p_{2N}(t, k, j) = \sum_{\nu=0}^{N+1} p_{2N}(t, k, \nu) q_{2N}(\nu, j) \tag{19}$$

for $0 \leqq j, k \leqq N+1$ and $t > 0$, together with the initial conditions

(20) $$p_{2N}(0, k, j) = \delta(k, j) .$$

To prove (19) for $0 \leqq j, k \leqq N$, observe that from (18), the relation (12) between the matrices Q_{1N} and Q_{2N}, and the fact that p_{1N} satisfies the backward Kolmogorov equations, follows

$$\begin{aligned}\frac{\partial}{\partial t} p_{2N}(t, k, j) &= \sum_{\nu=k}^{N} \left(\frac{\partial}{\partial t} p_{1N}(t, j, \nu) - \frac{\partial}{\partial t} p_{1N}(t, j-1, \nu)\right) \\ &= \sum_{\nu=k}^{N} \sum_{i=0}^{N} (q_{1N}(j, i) - q_{1N}(j-1, i)) p_{1N}(t, i, \nu) \\ &= \sum_{i=0}^{N} (q_{1N}(j, i) - q_{1N}(j-1, i)) \sum_{\nu=0}^{i} p_{2N}(t, k, \nu) \\ &= \sum_{\nu=0}^{N} p_{2N}(t, k, \nu) \sum_{i=\nu}^{N} (q_{1N}(j, i) - q_{1N}(j-1, i)) \\ &= \sum_{\nu=0}^{N+1} p_{2N}(t, k, \nu) q_{2N}(\nu, j) .\end{aligned}$$

The case $j = N+1$, $k \leqq N$ follows from the preceding case by subtraction, and the case $k = N+1$ is trivial. The initial condition (20) follows easily from (18) and the same condition for p_{1N}. Hence by known results (Doob, 1954, page 240) $p_{2N}(t, k, j)$ is the Markov transition probability determined by Q_{2N}, and by (18) for $0 \leqq j \leqq N$, $0 \leqq k \leqq N+1$

(21) $$\sum_{\nu=0}^{j} p_{2N}(t, k, \nu) = \sum_{\nu=k}^{N} p_{1N}(t, j, \nu) .$$

Let P_{iN}^{j} denote the process with initial state j and Q-matrix Q_{iN} ($i = 1, 2$; $N = 1, 2, \cdots$). Then (21) is equivalent to

(22) $$P_{1N}^{j}\{X(t) \geqq k\} = P_{2N}^{k}\{X(t) \leqq j\} \qquad 0 \leqq j \leqq N,\ 0 \leqq k \leqq N+1 .$$

The rest of the argument involves some sample path analysis, and it may be assumed (with no loss of generality) that the processes P_{iN}^{j} ($i = 1, 2$), $P_1^{\,j}$, and $P_2^{\,j}$ when the latter represents the minimal Q_2 process are strong Markov and have right continuous sample paths. Let $\tau_N = \inf\{t\colon X(t) \geqq N\}$, $\tau_\infty = \lim_{N\to\infty} \tau_N$. The sample paths of the processes P_{1N}^{j} and $P_1^{\,j}$ agree for all $t < \tau_N$ and hence

(23) $$P_{1N}^{j}\{X(t) \geqq k\} = P_1^{\,j}\{\tau_N > t, X(t) \geqq k\} + P_{1N}^{j}\{\tau_N \leqq t, X(t) \geqq k\} .$$

Also

(24) $$P_{1N}^{j}\{\tau_N \leqq t, X(t) \geqq k\} \leqq P_1^{\,j}\{\tau_N \leqq t\} .$$

Assume that the minimal Q_1 process is honest, i.e., that

(25) $$P_1^{\,j}\{\tau_{\infty'} \leqq t\} = 0 .$$

It follows from (23), (24) and (25) that

(26) $$\lim_{N\to\infty} P_{1N}^{j}\{X(t) \geqq k\} = P_1^{\,j}\{\tau_\infty > t, X(t) \geqq k\} = P_1^{\,j}\{X(t) \geqq k\} .$$

Let $P_2^{\,k}$ denote the minimal Q_2 process. Since $N+1$ is absorbing for the P_{2N}^{k} process, for $N+1 > j$

$$P_{2N}^{k}\{X(t) \leqq j\} = P_2^{\,k}\{\tau_{N+1} > t, X(t) \leqq j\} ,$$

and letting $N \to \infty$ yields

(27) $\lim_{N\to\infty} P_{2N}^k\{X(t) \leqq j\} = P_2^k\{\tau_\infty > t, X(t) \leqq j\} = P_2^k\{X(t) \leqq j\}$.

Hence from (22), (26) and (27) follows

(28) $$P_1^j\{X(t) \geqq k\} = P_2^k\{X(t) \leqq j\} \qquad j, k = 0, 1, 2, \cdots.$$

That (28) continues to hold when $j = \infty$ or $k = \infty$ follows at once from (25) and the fact that P_1^j and P_2^k are minimal (hence ∞ is absorbing).

This proves the theorem in the case that the minimal Q_1 process is honest. For the dishonest case define truncated matrices Q_{1N} and Q_{2N} as above but with the difference that N is absorbing under P_{1N}^j and reflecting under P_{2N}^k. It is easy to show as above that (22) continues to hold and hence $P_{1N}^j\{X(t) \geqq k\}$ is nondecreasing in $j \leqq N$. The argument leading to (27) applied now to P_{1N}^j and P_1^j yields

$$\lim_{N\to\infty} P_{1N}^j\{X(t) \geqq k\} = P_1^j\{\tau_\infty > t, X(t) \geqq k\} + P_1^j\{\tau_\infty \leqq t\} = P_1^j\{X(t) \geqq k\}$$

and hence $P_1^j\{X(t) \geqq k\}$ is nondecreasing in j. It follows from Theorem 1 that there exists a process P_2^k satisfying (15). (This process is not, in general, the minimal Q_2 process.) Since $p_1(t, j, k) = P_1^j\{X(t) = k\}$ satisfies both the forward and backward Kolmogorov equations (cf. Chung, 1967, page 253), it is easy to modify the proof of (19) to show that

$$\begin{aligned} p_2(t, k, j) = P_2^k\{X(t) = j\} &= \sum_{\nu \geqq k} (p_1(t, j, \nu) - p_1(t, j-1, \nu)) \\ &= \sum_{\nu=0}^{k-1} (p_1(t, j-1, \nu) - p_1(t, j, \nu)) \end{aligned}$$

satisfies both the backward and forward Kolmogorov equations with Q-matrix Q_2. If the minimal Q_2 process is honest, there exists only one such process, which thus must be P_2^k. It remains to show that if the minimal Q_2 process and minimal Q_1 process are both dishonest, then P_2^k is honest (and hence not minimal).

By (15) $P_2^k\{X(t) \leqq j\} = P_1^j\{X(t) \geqq k\} \geqq P_1^j\{X(t) = \infty\}$ and hence for all $k = 1, 2, \cdots, +\infty$

(29) $$P_2^k\{X(t) < \infty\} \geqq \lim_{j\to\infty} P_1^j\{\tau_\infty \leqq t\}.$$

Thus to show that P_2^k is honest it suffices to prove that for all $t > 0$

(30) $$P_1^j\{\tau_\infty > t\} \to 0 \qquad j \to \infty.$$

The proof of (30) is complicated slightly by the possibility that under P_1^j there exists a finite number of states from which explosion cannot occur. For purposes of proving (30) these states may be lumped together into a single absorbing state, which by relabeling the state space if necessary may be taken to be the state 0. The proof of (30) in the (more usual) case that ∞ is accessible from 0 under P_1^j is similar but easier and hence omitted. Assume then that 0 is absorbing under P_1^j and let $\tau^* = \inf\{t: X(t) = 0\}$. Then

$$\begin{aligned} P_1^j\{\tau_\infty > t\} &\leqq P_1^j\{\tau_\infty \wedge \tau^* > t\} + P_1^j\{\tau_\infty > \tau^*\} \\ &\leqq t^{-1}E_1^j(\tau_\infty \wedge \tau^*) + P_1^j\{\tau_\infty > \tau^*\}, \end{aligned}$$

and since by stochastic monotonicity the probability in (30) is nonincreasing in j, it suffices to prove

$$\lim_{j\to\infty} P_1^j\{\tau_\infty > \tau^*\} = 0 \tag{31}$$

and

$$\liminf_{j\to\infty} E_1^j(\tau_\infty \wedge \tau^*) = 0\,. \tag{32}$$

By stochastic monotonicity $P_1^j\{\tau_\infty > \tau^*\} = 1 - \lim_{t\to\infty} P_1^j\{X(t) = \infty\}$ is nonincreasing in j and hence converges to a limit $p \geqq 0$ as $j \to \infty$. Thus for each $N > j > 0$

$$\begin{aligned} P_1^j\{\tau_\infty > \tau^*\} &= P_1^j\{\tau_N > \tau^*\} + \textstyle\sum_{k=N}^\infty P_1^j\{\tau_N < \tau^*, X(\tau_N) = k\}P_1^k\{\tau_\infty > \tau^*\} \\ &\geqq P_1^j\{\tau_N > \tau^*\} + pP_1^j\{\tau_N < \tau^*\}\,. \end{aligned}$$

Letting $N \to \infty$ yields $0 \geqq pP_1^j\{\tau_\infty < \tau^*\}$, and since by assumption $P_1^j\{\tau_\infty < \tau^*\} > 0$, it follows that $p = 0$ and (31) holds. Similarly by the lemma given below $E_1^j(\tau_\infty \wedge \tau^*) < \infty$ for all j and

$$\begin{aligned} E_1^j(\tau_\infty \wedge \tau^*) &= E_1^j(\tau_N \wedge \tau^*) + \textstyle\sum_{k\geqq N}^\infty P_1^j\{\tau_N < \tau^*, X(\tau_N) = k\}E_1^k(\tau_\infty \wedge \tau^*) \\ &\geqq E_1^j(\tau_N \wedge \tau^*) + \inf_{k\geqq N} E_1^k(\tau_\infty \wedge \tau^*)P_1^j\{\tau_N < \tau^*\}\,, \end{aligned}$$

so letting $N \to \infty$ yields (32).

LEMMA. *If for some $t_0 < \infty$, $P_1^1\{\tau_\infty \leqq t_0\} = \gamma > 0$, then $E_1^j(\tau_\infty \wedge \tau^*) < \infty$ for all j.*

PROOF. It obviously suffices to prove

$$\sup_{1\leqq j<\infty} P_1^j\{\tau_\infty \wedge \tau^* > nt_0\} \leqq (1-\gamma)^n \qquad n = 1, 2, \cdots\,. \tag{33}$$

By stochastic monotonicity $P_1^j\{\tau_\infty > t\} \leqq P_1^1\{\tau_\infty > t\}$ and hence (33) is true for $n = 1$ by hypothesis. Since

$$P_1^j\{\tau_\infty \wedge \tau^* > nt_0\} \leqq \textstyle\sum_{k=1}^\infty p_1(t_0, j, k)P_1^k\{\tau_\infty \wedge \tau^* > (n-1)t_0\}\,,$$

(33) follows for $n = 2, 3, \cdots$ by induction.

3. Examples and remarks. (a) For discrete time honest Markov processes Theorem 1 implies that to prove (1) for all $t = 1, 2, \cdots$ it suffices that it be satisfied for $t = 1$. Thus for the two barrier random walk example at the end of Section 1, if F denotes the left continuous distribution function of the x_t and $G(x) = 1 - F(-x)$ is the right continuous distribution function of $-x_t$, then

$$\begin{aligned} P_1^x\{X(1) \geqq y\} &= 1 - F(y-x)\,, \qquad 0 \leqq x\,,\ \ y \leqq b \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad y > b \end{aligned}$$

and

$$\begin{aligned} P_2^y\{X(1) \leqq x\} &= G(x-y)\,, \qquad 0 \leqq x\,,\ \ y \leqq b \\ &= 0 \qquad\qquad\qquad\qquad\quad y > b\,, \end{aligned}$$

so by Theorem 1 equation (1) holds for all $t = 1, 2, \cdots$.

(b) For the Markov chain imbedded in the $G/M/s$ queue at the instants immediately preceding new arrivals, the transition probability matrix is of the form (cf. Kendall, 1953)

$$P_1 = \left(\begin{array}{c|cccccc} & 0 & 0 & - & - & - & - \\ A_{s\times s} & - & - & - & - & - & - \\ & f_0 & 0 & - & - & - & - \\ \hline & f_1 & f_0 & 0 & - & - & - \\ B_{\infty\times s} & f_2 & f_1 & f_0 & 0 & - & - \\ & - & - & - & - & - & - \end{array}\right),$$

where $f_k \geqq 0$, $\sum f_k = 1$. By direct computation

$$P_2 = \left(\begin{array}{cccccc|cccccc} 1 & 0 & - & - & - & 0 & 0 & 0 & - & - & - & - \\ & & \tilde{A}_{(s-1)\times s} & & & & & & \tilde{C}_{(s-1)\times\infty} & & & \\ \hline & & & & & & f_0 & f_1 & f_2 & - & - & - \\ & & \tilde{B}_{\infty\times s} & & & & 0 & f_0 & f_1 & - & - & - \\ & & & & & & 0 & 0 & f_0 & - & - & - \\ & & & & & & - & - & - & - & - & - \end{array}\right).$$

Let $\tau_i = \inf\{n : X_n \leqq i\}$. It is easy to see from the queueing context that under P_2 all states communicate except 0 and hence passage to the limit $t \to \infty$ in (1) yields (2) for $x, y = 0, 1, 2, \cdots$. Moreover, from the form of P_2 one may show by standard arguments that

$$\begin{aligned} P_2^{\,k}\{\tau_0 < \infty\} &= 1 \qquad \text{for all} \quad k \quad \text{if} \quad \textstyle\sum kf_k \leqq 1 \\ &= \lambda^{k-s+1} P_2^{\,s-1}\{\tau_0 < \infty\}\,, \qquad k \geqq s-1 \quad \text{if} \quad \textstyle\sum kf_k > 1\,, \end{aligned}$$

where $0 < \lambda < 1$ is the unique solution of $\sum \lambda^k f_k = \lambda$. This limiting distribution for P_1 was calculated by Kendall (1953). The present derivation has the advantage that it "explains" the geometric tail of the distribution.

(c) For the Markov chain imbedded in the $M/G/1$ queue at the instants immediately following departures, the transition probability matrix is of the form

$$P_1 = \begin{pmatrix} g_0 & g_1 & g_2 & - & - & - \\ g_0 & g_1 & g_2 & - & - & - \\ 0 & g_0 & g_1 & - & - & - \\ 0 & 0 & g_1 & - & - & - \\ - & - & - & - & - & - \end{pmatrix}$$

from which by simple algebra

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & - & - & - \\ \sum_1^\infty g_k & 0 & g_0 & 0 & - & - & - \\ \sum_2^\infty g_k & 0 & g_1 & g_0 & - & - & - \\ - & - & - & - & - & - & - \end{pmatrix}.$$

It is interesting to compare these matrices with those of the preceding example for the special case $s = 1$. Again properties of the P_1 chain can be deduced from those of the P_2 chain, and vice versa.

(d) Let

$$Q_1 = \begin{pmatrix} -\lambda_0, & \lambda_0, & 0, & 0, & - & - & - \\ \mu_1, & -(\lambda_1 + \mu_1), & \lambda_1, & 0, & - & - & - \\ 0, & \mu_2, & -(\lambda_2 + \mu_2), & \lambda_2, & - & - & - \\ - & - & - & - & - & - & - \end{pmatrix}$$

where $\lambda_j, \mu_j > 0$ for all j, so that P_1^j is a birth and death process. Then Q_2 defined by (12) is

$$Q_2 = \begin{pmatrix} 0, & 0, & 0, & 0, & - & - & - \\ \lambda_0, & -(\lambda_0 + \mu_1), & \mu_1, & 0, & - & - & - \\ 0, & \lambda_1, & -(\lambda_1 + \mu_2), & \mu_2, & - & - & - \\ - & - & - & - & - & - & - \end{pmatrix},$$

so that P_2^j is also a birth and death process. According to Reuter (1957), the minimal Q_1 (Q_2) process is honest if and only if

$$\sum_1^\infty \left(\frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_0}\right) = \infty \tag{34}$$

$$\left(\sum_1^\infty \left(\frac{1}{\mu_n} + \frac{\lambda_{n-1}}{\mu_n \mu_{n-1}} + \cdots + \frac{\lambda_{n-1} \cdots \lambda_1}{\mu_n \cdots \mu_1}\right) = \infty\right);$$

and if both series converge, there exist infinitely many Q_2 processes which satisfy the forward Kolmogorov equations but only one of these is honest. This is the process P_2^j of Theorem 3. It is interesting to inquire in the more general stochastically monotone context whether the conditions that P_2^j be honest and satisfy the forward Kolmogorov equations determine P_2^j uniquely. It would also be interesting to know whether a stochastically monotone Markov Q-matrix can give rise to processes having essentially more complicated boundary behavior than that possible for birth and death processes and the extent to which boundary behavior more complicated that absorption and reflection is compatible with stochastic monotonicity of the Markov chain.

For birth and death processes it is easy to see that (2) follows from (1) and an easy exercise to derive the limiting behavior of P_1^j from the absorption properties of P_2^j and vice versa.

(e) Formal calculations suggest that if P_1^x is a linear diffusion with generator $D_1 f = af'' + bf'$, then for P_2^x to satisfy (1) it should be a diffusion with generator $D_2 f = af'' + (a' - b)f'$. The formulation and proof of an analogue to Theorem 3 for a general (real-valued) Markov process is an open problem.

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