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BESSEL FUNCTIONS AND THE INFINITE DIVISIBILITY OF THE STUDENT *t*-DISTRIBUTION[1]

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Using the representation theorem and inversion formula for Stieltjes transforms, we give a simple proof of the infinite divisibility of the student t-distribution for all degrees of freedom by showing that $x^{-\frac{1}{2}}K_\nu(x^{\frac{1}{2}})/K_{\nu+1}(x^{\frac{1}{2}})$ is completely monotonic for $\nu \geqq -1$. Our approach proves the stronger and new result, that $x^{-\frac{1}{2}}K_\nu(x^{\frac{1}{2}})/K_{\nu+1}(x^{\frac{1}{2}})$ is a completely monotonic function of x for all real ν. We also derive a new integral representation.

1. Introduction. A function $f(x)$ is completely monotonic if and only if $(-1)^n(d^nf(x)/dx^n) \geqq 0$ for $x \in (0, \infty)$. It is clear that the Laplace transform of an L_1 function is completely monotonic. We follow Watson's [8] notations for the Bessel functions.

Ismail and Kelker [6] proved that the infinite divisibility of the student t-distribution of $2\nu + 2$ degrees of freedom is equivalent to the complete monotonicity of $f_\nu(x)$,

$$f_\nu(x) \equiv x^{-\frac{1}{2}}K_\nu(x^{\frac{1}{2}})/K_{\nu+1}(x^{\frac{1}{2}}) \,. \tag{1.1}$$

They obtained partial results and conjectured that $f_\nu(x)$ is completely monotonic for $\nu \geqq -1$. Grosswald [3] established this conjecture by proving that

$$\mathscr{L}\left[\frac{2t^{-1}}{\pi^2}\Big/\{J_{\nu+1}^2(t^{\frac{1}{2}}) + Y_{\nu+1}^2(t^{\frac{1}{2}})\}\right] = \mathscr{L}^{-1}[f_\nu(t)] \,, \tag{1.2}$$

$\mathscr{L}$, $\mathscr{L}^{-1}$ being the Laplace transform and its inverse respectively. The observation

$$\int_0^\infty e^{-xu} \int_0^\infty e^{-ut}f(t)\,dt = \int_0^\infty \frac{f(t)}{x+t}\,dt$$

means that a two fold Laplace transform is a Stieltjes transform and that Grosswald's result (1.2) is nothing but

$$x^{-\frac{1}{2}}K_\nu(x^{\frac{1}{2}})/K_{\nu+1}(x^{\frac{1}{2}}) = \frac{2}{\pi^2}\int_0^\infty \frac{t^{-1}}{x+t}\{J_{\nu+1}^2(t^{\frac{1}{2}}) + Y_{\nu+1}^2(t^{\frac{1}{2}})\}^{-1}\,dt \,, \quad x > 0 \,, \quad \nu \geqq -1 \,, \tag{1.3}$$

which obviously implies that both $f_\nu(x)$ and its inverse Laplace transforms are completely monotonic functions of x for $\nu \geqq -1$. This note contains a very

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simple proof of (1.3) based on the representation and inversion theory of the Stieltjes transform. As a matter of fact our proof shows that (1.3) holds for complex x, that is

$$(1.4)\qquad z^{-\frac{1}{2}}K_\nu(z^{\frac{1}{2}})/K_{\nu+1}(z^{\frac{1}{2}}) = \frac{2}{\pi^2}\int_0^\infty \frac{t^{-1}}{z+t}\{J^2_{\nu+1}(t^{\frac{1}{2}}) + Y^2_{\nu+1}(t^{\frac{1}{2}})\}^{-1}\,dt\,,$$
$$|\arg z| < \pi\,,\quad \nu \geqq -1\,.$$

Furthermore we shall establish the following natural companion to (1.4)

$$(1.5)\qquad z^{-\frac{1}{2}}\frac{K_{\nu+1}(z^{\frac{1}{2}})}{K_\nu(z^{\frac{1}{2}})} = \frac{2\nu}{z} + \frac{2}{\pi^2}\int_0^\infty \frac{t^{-1}}{z+t}\{J_\nu{}^2(t^{\frac{1}{2}}) + Y_\nu{}^2(t^{\frac{1}{2}})\}^{-1}\,dt\,,$$
$$|\arg z| < \pi\,,\quad \nu \geqq 0\,.$$

The relationship (1.5) is a natural companion to (1.4) because (1.5), owing to the fact $K_\nu(z) = K_{-\nu}(z)$, is really (1.4) for $\nu < -1$. It is plain that (1.4) and (1.5) establish the complete monotonicity of $f_\nu(x)$ for all real ν.

2. Proofs of (1.4) **and** (1.5). Our proof of (1.3) is based on the following lemmas.

LEMMA 2.1. *A function $F(z)$ has the representation*

$$(2.1)\qquad F(z) = \int_0^\infty \frac{d\mu(t)}{z+t}\,,\qquad \int_0^\infty |d\mu| < \infty$$

if and only if

(i) *$F(z)$ is analytic for $|\arg z| < \pi$,*
(ii) *$F(z) = o(1)$ as $|z| \to \infty$ and $F(z) = o(|z|^{-1})$ as $|z| \to 0$,*

uniformly in every sector $|\arg z| \leqq \pi - \varepsilon$, $\varepsilon > 0$.

PROOF. See Hirschman and Widder ([4], pages 235 and 238).

LEMMA 2.2. *If $F(z)$ has the representation* (2.1) *and $\mu(t)$ is normalized by $\mu(0) = 0$ and $\mu(t) = \frac{1}{2}\{\mu(t+) + \mu(t-)\}$ then*

$$(2.2)\qquad \mu(t_2) - \mu(t_1) = \frac{1}{2\pi i}\lim_{\eta\to 0}\int_{t_1}^{t_2}\{F(-t-i\eta) - F(-t+i\eta)\}\,dt\,.$$

PROOF. See Stone [7].

We now prove (1.4) and (1.5).

PROOF OF (1.4). Condition (i) of Lemma 2.1 is satisfied since $K_\mu(z^{\frac{1}{2}})$ has no zeros in $|\arg z| < \pi$, Watson ([8], page 511) and $K_{-\mu}(z) = K_\mu(z)$, (here $F(z) = f_\nu(z)$). As $|z| \to \infty$, from Erdélyi ([1], page 23) $K_\mu(z) \sim (\pi/2z)^{\frac{1}{2}}e^{-z}$, $|\arg z| < 3\pi/2$. Hence $f_\nu(z) = o(1)$ as $|z| \to \infty$ for $|\arg z| \leqq \pi - \varepsilon$. As $|z| \to 0$, $\lim_{z\to 0} z^\mu K_\mu(z)$ exists and is not zero, for $\mu > 0$, as can be seen from (13) page 5 and (37) page 9 in Erdélyi et al. [1]. Furthermore $\lim_{z\to 0}(K_0(z)/\log z)$ exists and is not zero. All these limits are uniform for $|\arg z| \leqq \pi/2 - \varepsilon/2$. Therefore $zf_\nu(z) = o(1)$, $\nu \geqq 0$, as $|z| \to 0$ and is uniform for $|\arg z| \leqq \pi - \varepsilon$. Consider next the range $-1 \leqq \nu < 0$, and use $K_{-\nu}(x) = K_\nu(x)$ to obtain $zf_\nu(z) = o(1)$ in a similar way. Thus Lemma 2.1 implies that $f_\nu(z)$ is a Stieltjes transform. We now use Lemma 2.2 to invert it. Clearly for $t > \eta > 0$, $-t + i\eta = (t - i\eta)e^{i\pi}$ while $-t - i\eta = (t + i\eta)e^{-i\pi}$. This implies

$$K_\mu(-t + i\eta)^{\frac{1}{2}} = K_\mu(e^{i\pi/2}(t - i\eta)^{\frac{1}{2}}) = \frac{-i\pi}{2} e^{-i\mu\pi/2} H_\mu^{(2)}(t - i\eta)^{\frac{1}{2}} , \tag{2.3}$$

and

$$K_\mu(-t - i\eta)^{\frac{1}{2}} = K_\mu(e^{-i\pi/2}(t + i\eta)^{\frac{1}{2}}) = \frac{i\pi}{2} e^{i\mu\pi/2} H_\mu^{(1)}(t + i\eta)^{\frac{1}{2}} , \tag{2.4}$$

by (16) page 6 of Eredélyi et al. [1]. Therefore by (2.2) $\mu(t)$ is absolutely continuous and

$$\mu'(t) = \frac{t^{-\frac{1}{2}}}{2\pi i} \left\{ \frac{H_\nu^{(1)}(t^{\frac{1}{2}})H_{\nu+1}^{(2)}(t^{\frac{1}{2}}) - H_\nu^{(2)}(t^{\frac{1}{2}})H_{\nu+1}^{(1)}(t^{\frac{1}{2}})}{H_{\nu+1}^{(1)}(t^{\frac{1}{2}})H_{\nu+1}^{(2)}(t^{\frac{1}{2}})} \right\} . \tag{2.5}$$

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are related to the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, by Eredélyi et al. ([1], page 4).

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \tag{2.6}$$

and

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) . \tag{2.7}$$

New (2.5), (2.6), and (2.7) yield

$$\begin{aligned} \mu'(t^2) &= \frac{t^{-1}}{\pi} \{J_{\nu+1}(t)Y_\nu(t) - Y_{\nu+1}(t)J_\nu(t)\}\{J_{\nu+1}^2(t) + Y_{\nu+1}^2(t)\}^{-1} \\ &= \frac{2t^{-2}}{\pi^2} \{J_{\nu+1}^2(t) + Y_{\nu+1}^2(t)\}^{-1} , \end{aligned}$$

by (35) page 80 of [1]. This completes the proof.

Formula (1.5) can be proved by showing that $z^{-\frac{1}{2}}(K_{\nu+1}(z^{\frac{1}{2}})/K_\nu(z^{\frac{1}{2}})) - 2\nu/z$ is a Stieltjes transform, then invert it. The proof is similar to that of (1.4) and is omitted.

3. **Remarks.** Using special properties of the Bessel polynomials Grosswald [2] established the complete monotonicity of $f_\nu(x)$, for $\nu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots$, by expressing its inverse Stieltjes transform in terms of its poles. In [3] he compared (1.2) with his earlier result in [2] and obtained

$$J_{n+\frac{1}{2}}^2(x) + Y_{n+\frac{1}{2}}^2(x) = \frac{2}{\pi} x^{-(2n+1)} \prod_{j=1}^n (x^2 + \alpha_j^2) , \qquad n = 0, 1, 2, \cdots , \tag{3.1}$$

where $\alpha_1, \cdots, \alpha_n$ are the zeros of $K_{n+\frac{1}{2}}(z)$. We now show that (3.1) is an immediate consequence of (2.3), (2.4), (2.6), and (2.7). Clearly

$$J_{n+\frac{1}{2}}^2(x) + Y_{n+\frac{1}{2}}^2(x) = H_{n+\frac{1}{2}}^{(1)}(x)H_{n+\frac{1}{2}}^{(2)}(x) = \frac{4}{\pi^2} K_{n+\frac{1}{2}}(e^{i\pi/2}x)K_{n+\frac{1}{2}}(e^{-i\pi/2}x) . \tag{3.2}$$

The function $x^{\frac{1}{2}}e^{-x}K_{n+\frac{1}{2}}(x)$ is a polynomial of degree n in x^{-1}, see (40) page 10 of [1]. Therefore

$$K_{n+\frac{1}{2}}(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (x)^{-n-\frac{1}{2}}e^{-x} \prod_{j=1}^{n} (x - \alpha_j) \,. \tag{3.3}$$

It is clear now that (3.2) and (3.3) imply (3.1).

Added in proof. In a forthcoming paper entitled *Special functions, Stieltjes transform and infinite divisibility*, Ismail and Kelker used the methods of the present paper, i.e., Stieltjes transform and special function methods, to establish the infinite divisibility of an F distribution for any degree of freedom, including fractional degrees of freedom. This solves a problem of F. W. Steutel. Further results on monotonicity of quotients of Bessel functions have also been obtained in the abovementioned paper of Ismail and Kelker, and in [5].

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