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GEOMETRIC ERGODICITY AND R-POSITIVITY FOR GENERAL MARKOV CHAINS

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We show that for positive recurrent Markov chains on a general state space, a geometric rate of convergence to the stationary distribution π in a "small" region ensures the existence of a uniform rate $\rho < 1$ such that for π-a.a. x, $\|P^n(x, \cdot) - \pi(\cdot)\| = O(\rho^n)$. In particular, if there is a point α in the space with $\pi(\alpha) > 0$, the result holds if $|P^n(\alpha, \alpha) - \pi(\alpha)| = O(\rho_\alpha^n)$ for some $\rho_\alpha < 1$. This extends and strengthens the known results on a countable state space. Our results are put in the more general R-theoretic context, and the methods we use enable us to establish the existence of limits for sequences $\{R^n P^n(x, A)\}$, as well as exhibiting the solidarity of a geometric rate of convergence for such sequences. We conclude by applying our results to random walk on a half-line.

1. Introduction. In [5] and [19], Kendall and Vere-Jones proved that for $\{X_n\}$ an irreducible Markov chain on the integers, with n-step transition probabilities $P^n(i, j)$, the existence of a state i, a rate $\rho_i < 1$ and a limit $\pi(i)$ such that

$$|P^n(i, j) - \pi(i)| = O(\rho_i{}^n)$$

imply the existence of a uniform rate $\rho < 1$ and limits $\pi(j)$ such that for any pair (k, j),

$$|P^n(k, j) - \pi(j)| = O(\rho^n) .$$

In the transient case, where $\pi(j) \equiv 0$, the rate ρ is exact for all (k, j); but in the positive recurrent case, with $\pi(j) > 0$ for all j, ρ can only be chosen as a (uniform) upper bound on the rates of convergence.

In [15], one of us has extended the transient version of this result to Markov chains with values in general space. The purpose of this paper is to provide an analogue, for general chains, of the somewhat deeper positive recurrent result.

We consider a Markov chain $\{X_n\}$ on a state space $(S, \mathscr{F})$, with transition probabilities

$$P^n(x, A) = \Pr(X_n \in A \mid X_0 = x) , \qquad x \in S,\ A \in \mathscr{F} ;$$

we assume that for each $x \in S$, $P^n(x, \cdot)$ is a measure on $\mathscr{F}$, and for each $A \in \mathscr{F}$, $P^n(\cdot, A)$ is a measurable function on S. We will assume throughout that $\{X_n\}$ is φ-irreducible for some probability measure φ, and that $\{X_n\}$ is aperiodic (cf. Orey [10] for definitions). The most interesting results we shall prove are for the case where $\mathscr{F}$ is countably generated, and there is a finite invariant measure

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π for $\{X_n\}$. We then know from [4] that for π-a.a. x,

$$\|P^n(x, \cdot) - \pi(\cdot)\| \to 0 , \quad n \to \infty , \tag{1.1}$$

where $\|\cdot\|$ denotes total variation. Also, every set A with $\pi(A) > 0$ contains a subset C such that $\pi(C) > 0$ and for some $k \geqq 1$

$$p_\pi^k(x, y) \geqq \delta_C > 0 , \qquad x, y \in C , \tag{1.2}$$

where $p_\pi^n(x, y)$ is the density of $P^n(x, \cdot)$ with respect to π. Suppose C is such a set, and let $\pi_C = \pi I_C/\pi(C)$ be the probability measure given by restricting π to C. We shall prove

THEOREM 1. *Suppose $\mathscr{F}$ is countably generated and π is a finite invariant measure for $\{X_n\}$. Let C be a set satisfying* (1.2), *and assume that, when X_0 has the initial distribution π_C on C, convergence to the* (*single*) *limit $\pi(C)$ is geometrically fast*, i.e., *for some $\rho_C < 1$,*

$$|\textstyle\int_C \pi_C(dy)P^n(y, C) - \pi(C)| = O(\rho_C{}^n) .$$

Then there exists $\rho < 1$ such that, for π-a.a. x,

$$\|P^n(x, \cdot) - \pi(\cdot)\| = O(\rho^n) .$$

This result improves the pointwise result of Vere-Jones [19], even when S is countable; for it implies, for geometrically ergodic chains on the integers, the existence of a common rate $\rho < 1$ and constants M_i *independent of j* such that

$$\sup_j |P^n(i, j) - \pi(j)| \leqq M_i \rho^n .$$

Our result is in a sense best possible, in that in general the constants M_i cannot be taken independent of i; in fact it can be shown [3] that if there is any completely uniform rate $\{a_n\}$ (not even necessarily geometric) over all i, j such that

$$|P^n(i, j) - \pi(j)| \leqq a_n \to 0 , \quad n \to \infty ,$$

then there exists $\rho < 1$ such that

$$\sup_i \|P^n(i, \cdot) - \pi(\cdot)\| = O(\rho^n) ,$$

so the Markov chain is strongly ergodic; geometrically ergodic chains which are not strongly ergodic are well known, examples being provided by random walk on a half-line (see our Section 6, or [7], for conditions for geometric ergodicity; and [18] for the fact that random walks on a half-line are not strongly ergodic). We are grateful to Dean Isaacson for pointing out to us this optimality aspect of Theorem 1.

As with the result given in [19], an analogue of Theorem 1 holds in the wider context of the R-theory of Markov chains; it is in this context that we shall present it. Our methods will hinge crucially on the splitting technique introduced in [8]; in the next section we shall review both the R-theory from [15], and those elements of the splitting method which we shall use. The splitting technique also enables us to fill some of the gaps in the theory of R-positive chains, and in Section 3 we shall exhibit R-positivity analogues of (1.1) which are a considerable improvement on those of [15]. In Section 4 we turn to questions of geometric ergodicity for R-positive chains, and prove our main results, of which Theorem 1 is a special case. In Section 5 we give more concrete versions of our results when the state space is topological, and, in Section 6, we give an application of our results to random walk on a half-line, using results of [7].

2. Preliminaries: R-theory and the splitting technique. We denote the generating function of $P^n(x, A)$ by

$$G_z(x, A) = \sum_{n=1}^{\infty} P^n(x, A)z^n .$$

We define, for any initial probability measure λ on $\mathscr{F}$, the measure $\mathbb{P}_\lambda$ on the canonical version of $\{X_n\}$ (cf. [9]) and for any set $A \in \mathscr{F}$, we define $\tau_A = \inf(n \geqq 1 : X_n \in A)$. We shall use the taboo probabilities

$${}_AP^n(x, B) = \mathbb{P}_x(X_n \in B, \tau_A \geqq n) ,$$

with generating function

$${}_AG_z(x, B) = \sum_{n=1}^{\infty} {}_AP^n(x, B)z^n .$$

For any nonnegative function g and measure μ we write

$$\mu g = \int_S g(y)\mu(dy) ,$$

and we write 1_A for the indicator function of a set A. For any nonnegative function g we write the transition kernel

$$I_g(x, E) = g(x)1_E(x) ;$$

if $g = 1_A$ then we write $I_{1_A} = I_A$. Inequalities on measures are taken setwise on $\mathscr{F}$, and on functions pointwise. For $r > 0$, a σ-finite measure μ on $\mathscr{F}$ is called r-subinvariant if $\mu \geqq r\mu P$; and a nonnegative function g is called r-subinvariant if $g \geqq rPg$ φ-a.e. Some basic results of [15] and [16] can be summarised in the following way.

THEOREM R1. (i) *There exists a real number $R \geqq 1$ and a partition $\{A(j)\}$ of S such that R is the radius of convergence of* φ-a.a. *the series $G_z(x, A)$ for all $A \in \mathscr{F}$ with $\varphi(A) > 0$ and $A \subset A(j)$ for some j.*

(ii) *Either $G_R(x, A) \equiv \infty$ for all x and all A with $\varphi(A) > 0$ (the R-recurrent case) or $G_R(x, A(j)) < \infty$ for* φ-a.a. *x and all j (the R-transient case).*

(iii) *In both cases, there exists at least one R-subinvariant measure $\mu \gg \varphi$. If $\mathscr{F}_\mu = \{A \in \mathscr{F} : 0 < \mu(A) < \infty\}$, then any set in $\mathscr{F}_\mu$ has the "correct" convergence properties of* (i) *and* (ii).

(iv) *In the R-recurrent case, there is an R-subinvariant measure π and R-subinvariant function f, satisfying $\pi = \pi RP$ and $f = RPf$* π-a.e.; *π is unique up to constant multiples, and f is unique up to constant multiples and definition on π-null* *sets. For any set A with $\varphi(A) > 0$,*

$$\pi(\cdot) = \int_A \pi(dy)\, {}_AG_R(y, \cdot)\,, \tag{2.1}$$

$$f(x) = \int_A {}_AG_R(x, dy) f(y)\,, \quad \pi\text{-a.e.} \tag{2.2}$$

Part (i) of this theorem gives the general version of geometric ergodicity for transient chains: for if, for some $\rho_0 < 1$, some x_0 and some A_0 with $\varphi(A_0) > 0$,

$$P^n(x_0, A_0) = O(\rho_0{}^n)\,,$$

then it follows that for φ-a.a. x, and all "suitable" A,

$$P^n(x, A) = O(R^{-n})\,.$$

Part (iii) of the theorem then identifies in an exact way how to choose "suitable" A in this result: in [11] and [12] results are given (which we mention in Section 5) which show that under some conditions, one can identify $\mathscr{F}_\mu$ further.

THEOREM R2. (i) *In the R-recurrent case, there is a partition $\{B(j)\}$ of S such that for every $A \subset B(j)$, there is a π-null set N_A and for $x \notin N_A$,*

$$R^n P^n(x, A) \to f(x)\pi(A)/\pi f\,, \tag{2.3}$$

where π, f are as in (2.1) *and* (2.2). *Either $\pi f < \infty$ and $\{X_n\}$ is called R-positive, or $\pi f = \infty$ and $\{X_n\}$ is called R-null.*

(ii) *If A is any set with $\pi(A) < \infty$ and* $\inf_{x \in A} f(x) > 0$, *then* (2.3) *holds for A.*

One of the major purposes of the next section is to show that, in fact, (2.3) holds assuming only $A \in \mathscr{F}_\pi$, and to investigate the class of initial distributions for which the analogue of (2.3) is true.

We now turn to the splitting technique of [8]. We suppose that $\{X_n\}$ satisfies the following

MINORIZATION CONDITION. For some $k \geq 1$, some measurable nonnegative function $h < 1$, with $\varphi h > 0$, and some probability measure ν on $\mathscr{F}$,

$$P^k(x, A) \geq h(x)\nu(A)\,, \qquad x \in S,\ A \in \mathscr{F}\,. \tag{2.4}$$

We shall use k, h and ν exclusively for the quantities in (2.4). The minorization condition is far from restrictive; when $\mathscr{F}$ is countably generated, then (see Section 5) we can always choose ν as φ restricted to a C-set for φ (defined as in (1.2)), and h as $\alpha 1_C$ for some $\alpha > 0$.

We now use (2.4) to "split" the chain $\{X_n\}$; a more detailed description is given in [8], but the following will suffice for our purposes. Suppose first that $k = 1$ in (2.4). Write, for all $x \in S$, $A \in \mathscr{F}$,

$$x_0 = (x, 0)\,, \qquad x_1 = (x, 1)\,;$$
$$A_0 = A \times \{0\}\,, \qquad A_1 = A \times \{1\}\,, \qquad A^* = A \times \{0, 1\}\,.$$

Let $\mathscr{F}^*$ be the σ-algebra generated on S^* by the sets $\{A_i;\ A \in \mathscr{F},\ i = 0, 1\}$; we identify subsets A of S with corresponding subsets A^* of S^*.

Any function g on S is given the value $g(x_i) = g(x)$, $i = 0, 1$, on S^*; but any measure λ on $\mathscr{F}$ is *split* onto $\mathscr{F}^*$ by setting

$$\lambda(A_0) = \lambda I_{1-h}(A) , \qquad \lambda(A_1) = \lambda I_h(A) . \tag{2.5}$$

We can now define the split chain $\{X_n^*\}$ corresponding to $\{X_n\}$. We first define a transition probability $\bar{P}$ from S^* to $\mathscr{F}$ by setting, for $x \in S$, $A \in \mathscr{F}$,

$$\bar{P}(x_0, A) = [1 - h(x)]^{-1}[P(x, A) - h(x)\nu(A)] ; \tag{2.6a}$$

$$\bar{P}(x_1, A) = \nu(A) ; \tag{2.6b}$$

let $\{X_n^*\}$ be a chain on $(S^*, \mathscr{F}^*)$ whose transition probabilities $P^*(x_i, \cdot)$ are the splittings of the measures $\bar{P}(x_i, \cdot)$ from $\mathscr{F}$ onto $\mathscr{F}^*$.

If λ is any initial measure on $\mathscr{F}^*$, we write $\mathbb{P}_\lambda^*$ for the probability measure of $\{X_n^*\}$ given X_0^* has distribution λ. In [8] it is shown that the marginal distribution of $\{X_n^*\}$ on S is the same as that of $\{X_n\}$, but $\{X_n^*\}$ has the advantage that, from (2.6b), the set S_1 is an *atom*, i.e., transitions from every point in S_1 are identical. This enables us to use renewal theory arguments not available for $\{X_n\}$, and these provide the main tool in the sequel. If $k > 1$ in (2.4), then we split the k-step chain $\{X_{nk}\}$, exactly as in (2.6) with P^k in place of P. Hence to use (2.4) for general k, we need to verify that the R-theory of $\{X_n\}$ is inherited in a suitable way by both $\{X_{nk}\}$ and $\{X_{nk}^*\}$; and this we now do.

LEMMA 1. (i) *If $\{X_n\}$ is φ-irreducible, then so are $\{X_{nk}\}$ and $\{X_{nk}^*\}$.*
(ii) *The R-properties of $\{X_n\}$ are inherited as R^k-properties by $\{X_{nk}^*\}$.*

PROOF. (i) The argument used in Lemma 1.5 of [8] can be imitated to prove $\{X_{nk}\}$ is φ-irreducible; that $\{X_{nk}^*\}$ is then φ-irreducible follows directly from the nature of the splitting (2.5), and (2.11) of [8].

(ii) Suppose $k = 1$. From (i) and Theorem R1, there exist convergence parameters R and R^* for $\{X_n\}$ and $\{X_n^*\}$, and clearly $R \leqq R^*$; we have to prove that if for some $x \in S^*$ and $A \in \mathscr{F}^*$ with $\varphi(A) > 0$

$$\textstyle\sum_n r^n P^{*n}(x, A) < \infty , \tag{2.7}$$

then $\{X_n\}$ is r-transient. But the solidarity properties of r-transience imply that for some point $x \in S$, and some $B \in \mathscr{F}$ with $\varphi(B) > 0$ both $\sum_n r^n P^{*n}(x_0, B) < \infty$ and $\sum_n r^n P^{*n}(x_1, B) < \infty$; and so $\sum_n r^n P^n(x, B) = [1 - h(x)] \sum_n r^n P^{*n}(x_0, B) + h(x) \sum_n r^n P^{*n}(x_1, B) < \infty$. We next establish the equivalence of R-positivity for $\{X_n\}$ and for $\{X_n^*\}$. This follows because of the uniqueness of the R-invariant measure and function in the R-recurrent case: it is simple to check that if π is R-invariant for $\{X_n\}$, then π split onto $\mathscr{F}^*$ is R-invariant for $\{X_n^*\}$; whilst if f^* is R-invariant for $\{X_n^*\}$, f defined by $f(x) = [1 - h(x)]f^*(x_0) + h(x)f^*(x_1)$ is R-invariant for $\{X_n\}$. Hence, in particular, $\pi f = \pi^* f^*$, and so the two are finite or not together.

Now let $k > 1$. We need to prove only that $\{X_{nk}\}$ inherits the R^k-properties corresponding to the R-properties of $\{X_n\}$; and again it is clear that the only thing that needs to be proved is that, if $\sum_n r^{nk}P^{nk}(x, A) < \infty$ for some x, A with $\varphi(A) > 0$, then in fact $\sum_n r^n P^n(x, C) < \infty$ for some C with $\varphi(C) > 0$. By using the same argument as in the proof of Lemma 1.5(i) of [8], we can find $C \in \mathscr{F}$ with $\varphi(C) > 0$, $\gamma > 0$ and m_0 an integer such that

$$\inf\{r^{m_0k-i}P^{m_0k-i}(y, A); i = 0, \cdots, k-1, y \in C\} = \gamma \tag{2.8}$$

Then we have

$$\begin{aligned} \infty &> k \textstyle\sum_{n=m_0}^{\infty} r^{nk}P^{nk}(x, A) \\ &= \textstyle\sum_{n=0}^{\infty} \sum_{i=0}^{k-1} r^{(n+m_0)k}P^{(n+m_0)k}(x, A) \\ &\geqq \textstyle\sum_{n=0}^{\infty} \sum_{i=0}^{k-1} \int_C r^{nk+i}P^{nk+i}(x, dy)r^{m_0k-i}P^{m_0k-i}(y, A) \\ &\geqq \gamma \textstyle\sum_{n=0}^{\infty} r^n P^n(x, C) . \end{aligned}$$

3. Existence of R-limits. Throughout this and the next section, we will always assume that $\{X_n\}$ satisfies the minorization condition, and $\{X_n\}$ is R-recurrent. As we have seen, this implies that by a splitting procedure, we can always introduce an atom into the space, at least for $\{X_{nk}^*\}$. If α is such an atom for $\{X_n\}$, we write $P(\alpha, \cdot) \equiv P(x, \cdot)$ and $f(\alpha) \equiv f(x)$, $x \in \alpha$, where f is the unique R-invariant function of Theorem R1; and we let

$$N_\alpha' = \{y : f(y) \neq \textstyle\sum_{n=1}^{\infty} R^n {}_\alpha P^n(y, \alpha) f(\alpha)\} ,$$

and put

$$N_\alpha = N_\alpha' \cup \{y : G_{\frac{1}{2}}(y, N_\alpha') > 0\} .$$

From Theorem R1(iv), $\pi(N_\alpha) = 0$. If λ is any measure on $\mathscr{F}$ and g is any nonnegative measurable function on S, we put

$$\Gamma_{\lambda g}(n) = \int_S \lambda(dx) \int_S R^n {}_\alpha P^n(x, dy) g(y) ;$$

if $\lambda = \varepsilon_x$, the point mass at x, we put $\Gamma_{\varepsilon_x g}(n) = \Gamma_{xg}(n)$, and if $g = 1_E$ we put $\Gamma_{\lambda 1_E}(n) = \Gamma_{\lambda E}(n)$. We write $\Gamma_{\alpha g} \equiv \Gamma_{xg}$, $x \in \alpha$. If g is any nonnegative measurable function, and μ is any signed measure, we put

$$\|\mu\|_g = \sup_\eta \{|\mu\eta| ; 0 \leqq \eta \leqq g\} ;$$

if E is any set in $\mathscr{F}$, we write

$$\|\mu\|_E = \|\mu\|_{1_E} .$$

We write

$$u(n) = R^n P^n(\alpha, \alpha) ;$$

we can then use the first-entrance-last-exit decomposition (3.14) of [8] to write, for any nonnegative measurable g, and measure λ

$$R^n \lambda P^n g = \Gamma_{\lambda g}(n) + \Gamma_{\lambda\alpha} * u * \Gamma_{\alpha g}(n) , \qquad n \geqq 1 , \tag{3.1}$$

where $*$ denotes convolution of sequences. In particular, we have

$$R^n \lambda P^n(\alpha) = \Gamma_{\lambda\alpha} * u(n) , \qquad n \geqq 1 , \tag{3.2}$$

the usual first entrance decomposition.

THEOREM 2. *Suppose that $\{X_n\}$ is R-positive, and normalize π, f so that $\pi f = 1$. Let g be any nonnegative function with $\pi g < \infty$. Then there is a set N_g with $\pi(N_g) = 0$ such that, for all $x \notin N_g$,*

$$\|R^n P^n(x, \cdot) - f(x)\pi(\cdot)\|_g \to 0\,, \quad n \to \infty\,. \tag{3.3}$$

PROOF. (i) We first assume that there is an atom $\alpha \subset S$. Let η be any measurable function with $0 \leqq \eta \leqq g$. From (3.1), letting $1 = 1_S$ denote the function identically unity on S,

$$\begin{aligned} |R^n \varepsilon_x P^n \eta - f(x)(\pi\eta)| &\leqq \Gamma_{xg}(n) + |\Gamma_{x\alpha} * u * \Gamma_{\alpha\eta}(n) - f(x)\pi(\alpha)\Gamma_{\alpha\eta} * 1(n)| \\ &\qquad + |f(x)\pi(\alpha)\Gamma_{\alpha\eta} * 1(n) - f(x)(\pi\eta)| \\ &\leqq \Gamma_{xg}(n) + \Gamma_{\alpha g} * |\Gamma_{x\alpha} * u(n) - f(x)\pi(\alpha)| \\ &\qquad + f(x)\pi(\alpha) \textstyle\sum_{j=n+1}^{\infty} \Gamma_{\alpha g}(j)\,, \end{aligned} \tag{3.4}$$

since from (2.1), $\pi\eta = \pi(\alpha) \sum_{j=1}^{\infty} \Gamma_{\alpha\eta}(j)$. Also from (2.1),

$$\infty > \pi g = \pi(\alpha) \textstyle\sum_{j=1}^{\infty} \Gamma_{\alpha g}(j)\,, \tag{3.5}$$

so that the last term in (3.4) tends to zero for all x, as $n \to \infty$. From (3.2) and Theorem R2(i), we know that for x outside a null set N_1, $|\Gamma_{x\alpha} * u(n) - f(x)\pi(\alpha)| \to 0$ as $n \to \infty$; so from (3.5) and a standard convolution argument, the second term in (3.4) goes to zero for $x \notin N_1$.

To handle the first term, we note that for any fixed $m < n$,

$$\Gamma_{\alpha g}(n) \geqq \textstyle\int_{\alpha^c} \Gamma_{\alpha dy}(m)\Gamma_{yg}(n - m)\,,$$

so from (3.5) again,

$$\infty > \textstyle\sum_{n=m+1}^{\infty} \Gamma_{\alpha g}(n) \geqq \int_{\alpha^c} \Gamma_{\alpha dy}(m) \sum_{n=1}^{\infty} \Gamma_{yg}(n)\,. \tag{3.6}$$

Since, from (2.1), $\pi(\cdot) = \pi(\alpha) \sum_{m=1}^{\infty} \Gamma_{\alpha\cdot}(m)$, (3.6) shows that there is some π-null set N_g' such that $\sum_{n=1}^{\infty} \Gamma_{yg}(n) < \infty$ for $y \notin N_g'$, and so for such y, the first term in (3.4) converges to zero also. Hence (3.3) is proved for $y \notin N_1 \cup N_g' = N_g$.

(ii) Now assume only that the minorization condition holds. The result proved in (i) holds for $\{X_{nk}^*\}$, and so if g is $\mathscr{F}$-measurable, it holds also for $\{X_{nk}\}$; so we have (3.3) as $n = mk \to \infty$. But if we choose $g_j = R^j P^j g$, the R-invariance of π gives $\pi g_j = \pi g < \infty$, and so for $j = 1, \cdots, k$,

$$\|R^{mk+j} P^{mk+j}(x, \cdot) - f(x)\pi(\cdot)\|_g \leqq \|R^{mk} P^{mk}(x, \cdot) - f(x)\pi(\cdot)\|_{g_j} \to 0\,,$$

which gives (3.3). □

By choosing $E \in \mathscr{F}_\pi = \{A : 0 < \pi(A) < \infty\}$, we get the following

COROLLARY. *Suppose $\{X_n\}$ is R-positive with $\pi f = 1$. Then for each $E \in \mathscr{F}_\pi$, there is a π-null set N_E such that for $x \notin N_E$,*

$$\|R^n P^n(x, \cdot) - f(x)\pi(\cdot)\|_E \to 0\,, \quad n \to \infty\,. \tag{3.7}$$

As a consequence of this corollary, we can see that the condition (3.10) in [17] is in fact always true. Hence Theorem 2 of [17] can be amended to give a completely general analogue of the countable space quasistationarity results of [13].

In the R-null case, by using (3.1) and the same methods as in Theorem 2, we can prove the following, a detailed proof of which we omit.

THEOREM 3. *Suppose $\{X_n\}$ is R-null, and let g be any nonnegative measurable function with $\pi g < \infty$. For x outside a π-null set N_g*

$$R^n P^n g(x) \to 0 , \quad n \to \infty .$$

In particular, if $\pi(E) < \infty$, for x outside a π-null set N_E,

$$R^n P^n(x, E) \to 0 .$$

In [15], where Theorem R2 was proved by reduction to the 1-recurrent case we could only show convergence for those sets in $\mathscr{F}_\pi$ which were also in

$$\mathscr{F}_f = \{A \in \mathscr{F}: \inf_{x \in A} f(x) \geqq \delta_A \text{ for some } \delta_A > 0, \text{ and } \pi(A) > 0\} ,$$

as noted in Theorem R2(ii).

We now investigate the limiting behaviour of sequences $R^n \lambda P^n g$ for arbitrary initial distributions λ, and we shall see that for such sequences, $\mathscr{F}_f$ plays a very natural role.

THEOREM 4. *Suppose $\{X_n\}$ is R-positive, with $\pi f = 1$. There is a single null set N such that, if λ is any measure with $\lambda(N) = 0$ and $\lambda f < \infty$, and g is any function with $f(x) \geqq cg(x) \geqq 0$ for some $0 < c < \infty$ (except perhaps for $x \in N$), then we have*

$$\|R^n \lambda P^n(\cdot) - (\lambda f)\pi(\cdot)\|_g \to 0 , \quad n \to \infty . \tag{3.8}$$

In particular, if $E \in \mathscr{F}_f$, for such a λ

$$\|R^n \lambda P^n(\cdot) - (\lambda f)\pi(\cdot)\|_E \to 0 , \quad n \to \infty . \tag{3.9}$$

PROOF. (i) Assume that there is an atom α, and let $N = N_\alpha$; assume $\lambda(N) = 0$. As in (3.4), we have for $0 \leqq \eta \leqq g$,

$$\begin{aligned} |R^n \lambda P^n \eta - (\lambda f)(\pi \eta)| \leqq \Gamma_{\lambda g}(n) + \Gamma_{\alpha g} * |\Gamma_{\lambda\alpha} * u(n) - (\lambda f)\pi(\alpha)| \\ + (\lambda f)\pi(\alpha) \textstyle\sum_{j=n+1}^{\infty} \Gamma_{\alpha g}(j) . \end{aligned} \tag{3.10}$$

Since $\infty > \pi f \geqq c\pi g$, from (3.5) the final term in (3.10) goes to zero as $n \to \infty$. To handle the first term, this time we note that

$$\infty > \lambda f = \textstyle\sum_{n=1}^{\infty} \Gamma_{\lambda\alpha}(n) f(\alpha) \tag{3.11}$$

since $\lambda(N_\alpha) = 0$. Now for any m, as in (3.6), again since $\lambda(N_\alpha) = 0$,

$$\begin{aligned} \textstyle\sum_{n=m+1}^{\infty} \Gamma_{\lambda\alpha}(n) &= \textstyle\int_{\alpha^c} \Gamma_{\lambda dy}(m) \sum_{n=1}^{\infty} \Gamma_{y\alpha}(n) \\ &= \textstyle\int_{\alpha^c} \Gamma_{\lambda dy}(m) f(y) [f(\alpha)]^{-1} \\ &\geqq c\Gamma_{\lambda g}(m) [f(\alpha)]^{-1} - \Gamma_{\lambda\alpha}(m) . \end{aligned} \tag{3.12}$$

From (3.12) and (3.11), the first term of (3.10) goes to zero as $n \to \infty$. From (3.11), we can also put

$$\begin{aligned} &|\Gamma_{\lambda\alpha} * u(n) - (\lambda f)\pi(\alpha)| \\ &\leqq |\Gamma_{\lambda\alpha} * u(n) - f(\alpha)\pi(\alpha)\Gamma_{\lambda\alpha} * 1(n)| + |f(\alpha)\pi(\alpha)\Gamma_{\lambda\alpha} * 1(n) - (\lambda f)\pi(\alpha)| \\ &\leqq \Gamma_{\lambda\alpha} * |u(n) - f(\alpha)\pi(\alpha)| + f(\alpha)\pi(\alpha) \textstyle\sum_{j=n+1}^{\infty} \Gamma_{\lambda\alpha}(n) . \end{aligned} \tag{3.13}$$

Since $u(n) \to f(\alpha)\pi(\alpha)$, (3.11) implies that both terms in (3.13) go to zero. Hence the second term in (3.10) goes to zero from (3.5), and (3.8) is proved.

(ii) Suppose now the minorization condition holds. From (i), we have (3.7) for $\{X_{nk}^*\}$ and hence for $\{X_{nk}\}$ when λ is a measure on $\mathscr{F}$ and g is a function on S. To deduce the result for $\{X_n\}$, it suffices again to consider the functions $g_j = R^j P^j g$; for if $g \leqq cf$, then $R^j P^j g \leqq cR^j P^j f = f$ by R-invariance, except perhaps on N_α. □

A version of this theorem can be proved for R-null chains: we give without further details

THEOREM 5. *Suppose that $\{X_n\}$ is R-null. There is a π-null set N such that, for all λ with $\lambda(N) = 0$ and $\lambda f < \infty$, and all g with $f \geqq cg \geqq 0$ except on N, some $0 < c < \infty$, and with $\pi g < \infty$,*

$$R^n \lambda P^n g \to 0\,, \quad n \to \infty\,.$$

In particular, for all such λ and for $E \in \mathscr{F}_f \cap \mathscr{F}_\pi$,

$$R^n \lambda P^n(E) \to 0\,, \quad n \to \infty\,.$$

Finally, we remark that the splitting technique can be used to provide a simple proof of the existence of the R-invariant measure π: this complements the indirect construction in [15], or the direct construction in [9]. The details are similar to those for $R = 1$ in Section 4 of [8], and we will not pursue them here.

4. Geometric ergodicity for R-positive chains. In this section we come to the generalization of the solidarity results for rates of convergence with which we began the introduction. We shall again assume in this section that we deal with chains which satisfy the minorization condition; we shall also assume that $\{X_n\}$ is R-positive, with π and f normalized so that $\pi f = 1$. We need two preliminary results:

KENDALL'S LEMMA. *Suppose $\{u(n)\}$ is an aperiodic renewal sequence with corresponding first renewal probabilities $\{b(n)\}$,* i.e., *$u(0) = 1$ and*

$$u(n) = u * b(n)\,, \qquad n = 1, 2, \cdots\,.$$

Let $u_\infty = \lim_{n\to\infty} u(n)$. The following conditions are equivalent:

(i) *$|u(n) - u_\infty| = O(\rho_0^n)$, some $\rho_0 < 1$;*

(ii) *$\sum_{n=1}^\infty b(n) r_1^n < \infty$, some $r_1 > 1$;*

(iii) *$(1 - z)U(z) = (1 - z) \sum_{n=0}^\infty z^n u(n)$ can be extended as an analytic function with no zeros except a simple zero at $z = 1$, in a region $\{|z| < r_2\}$, some $r_2 > 1$.*

PROOF. See [5].

LEMMA 2. *If $\{a_n, n \geqq 1\}$ is a sequence with $a_n \geqq 0$, then for any $r > 1$, $\sum_{n=1}^\infty a_n r^n < \infty$ if and only if $\sum_{n=1}^\infty \sum_{m=n}^\infty a_m r^n < \infty$.*

PROOF. Use Fubini's theorem.

Because they are of independent interest, we give first two geometric ergodicity theorems for the case when an atom exists, and then extend these results to the general case using the splitting technique.

THEOREM 6. *Suppose S has an atom α, and for some $\rho_\alpha < 1$,*

$$|R^nP^n(\alpha, \alpha) - f(\alpha)\pi(\alpha)| = O(\rho_\alpha{}^n)\,, \quad n \to \infty\,. \tag{4.1}$$

Then there exists $\rho < 1$ and a single π-null set N such that, if g is any function with $f \geqq cg \geqq 0$ for some $0 < c < \infty$ (except perhaps on N), we have for all $x \notin N$,

$$\|R^nP^n(x, \cdot) - f(x)\pi(\cdot)\|_g = O(\rho^n)\,, \quad n \to \infty\,. \tag{4.2}$$

In particular, for $x \notin N$ and $E \in \mathscr{F}_f$,

$$\|R^nP^n(x, \cdot) - f(x)\pi(\cdot)\|_E = O(\rho^n)\,, \quad n \to \infty\,. \tag{4.3}$$

PROOF. As in [19], if we take $u(n) = R^nP^n(\alpha, \alpha)$, then from (3.2), the corresponding sequence $b(n) = \Gamma_{\alpha\alpha}(n)$. Our assumption (4.1) is then, from Kendall's lemma, equivalent to assuming that for some $r > 1$ (which we shall take such that $r < \rho_\alpha{}^{-1}$),

$$\textstyle\sum_{n=1}^\infty r^n\Gamma_{\alpha\alpha}(n) < \infty\,. \tag{4.4}$$

We shall show that for this r, and g as in the theorem,

$$\textstyle\sum_{n=1}^\infty r^n\|R^nP^n(x, \cdot) - f(x)\pi(\cdot)\|_g \tag{4.5}$$

is finite; thus (4.2) will hold with $\rho = r^{-1}$. As in (3.4) and (3.13), provided $x \notin N_\alpha$, we can bound (4.5) above by

$$\begin{aligned}&\textstyle\sum_{n=1}^\infty \Gamma_{xg}(n)r^n + [\sum_{n=1}^\infty \Gamma_{\alpha g}(n)r^n][\sum_{n=1}^\infty \Gamma_{x\alpha}(n)r^n \sum_{m=1}^\infty r^m|u(m) - f(\alpha)\pi(\alpha)|\\ &\qquad + f(\alpha)\pi(\alpha)\sum_{n=1}^\infty r^n \sum_{m=n+1}^\infty \Gamma_{x\alpha}(m)] + f(x)\pi(\alpha)\sum_{n=1}^\infty r^n \sum_{m=n+1}^\infty \Gamma_{\alpha g}(m)\,.\end{aligned}$$

Since we have (4.1) and Lemma 2, we thus need to show

$$\textstyle\sum_{n=1}^\infty \Gamma_{\alpha g}(n)r^n < \infty\,; \tag{4.6}$$

$$\textstyle\sum_{n=1}^\infty \Gamma_{x\alpha}(n)r^n < \infty\,; \tag{4.7}$$

$$\textstyle\sum_{n=1}^\infty \Gamma_{xg}(n)r^n < \infty\,; \tag{4.8}$$

and the latter two need hold only outside some (fixed) π-null set.

To see (4.6), note that for any fixed m, from (4.4)

$$\infty > \textstyle\sum_{n=1}^\infty \Gamma_{\alpha\alpha}(n)r^n \geqq \int_{\alpha^c} \sum_{n=1}^\infty \Gamma_{\alpha dy}(n)r^n\Gamma_{y\alpha}(m)r^m\,; \tag{4.9}$$

multiplying by r^{-m} and summing over m gives

$$\begin{aligned}\infty > (r-1)^{-1}\textstyle\sum_{n=1}^\infty \Gamma_{\alpha\alpha}(n)r^n &\geqq \textstyle\int_{\alpha^c} \sum_{n=1}^\infty \Gamma_{\alpha dy}(n)r^nf(y)/f(\alpha)\\ &\geqq c\textstyle\sum_{n=1}^\infty \Gamma_{\alpha g}(n)r^n/f(\alpha) - \sum_{n=1}^\infty \Gamma_{\alpha\alpha}(n)r^n\,,\end{aligned} \tag{4.10}$$

since $\Gamma_{\alpha N_\alpha}(n) = 0$ for all n. A similar operation on the inequality

$$\infty > \textstyle\sum_{n=1}^\infty \Gamma_{\alpha\alpha}(n)r^n \geqq \int_{\alpha^c} \Gamma_{\alpha dy}(m)r^m \sum_{n=1}^\infty \Gamma_{y\alpha}(n)r^n$$

gives us, from (4.4),

$$(4.11)\qquad \infty > (r-1)^{-1} \sum_{n=1}^{\infty} \Gamma_{\alpha\alpha}(n) r^n \geqq \int_{\alpha^c} \sum_{m=1}^{\infty} \Gamma_{\alpha dy}(m) \sum_{n=1}^{\infty} \Gamma_{y\alpha}(n) r^n = \int_{\alpha^c} \pi(dy) \sum_{n=1}^{\infty} \Gamma_{y\alpha}(n) r^n ,$$

so for all x outside a π-null set N_1, say, (4.7) holds. But now for $x \notin N = N_1 \cup N_\alpha$, we can imitate (4.10) starting from x rather than α, to find that (4.8) holds, and the theorem is proved. □

The value ρ occurring in (4.3) may not be the best possible for given x, E: Vere-Jones ([19], page 26) gives a chain on three states in which the rates of convergence differ from state to state whilst Teugels [14] constructs an $(n+1)$-state chain which has n different "decay parameters."

We now prove a version of Theorem 6 for arbitrary initial distribution λ. Since the step (4.11) is unavailable to us, however, we need to assume (4.12) below; this ensures that the analogue of (4.7) is true.

THEOREM 7. *Suppose S has an atom such that* (4.1) *holds. If λ is such that* $\lambda(N_\alpha) = 0$, $\lambda f < \infty$, *and for some* $\rho_\lambda < 1$

$$(4.12)\qquad |R^n \lambda P^n(\alpha) - (\lambda f)\pi(\alpha)| = O(\rho_\lambda{}^n) , \quad n \to \infty ,$$

then there exists $\beta_\lambda < 1$ such that for any g with $f \geqq cg \geqq 0$, some $0 < c < \infty$ *(except perhaps on N_α),*

$$(4.13)\qquad \|R^n \lambda P^n(\cdot) - (\lambda f)\pi(\cdot)\|_g = O(\beta_\lambda{}^n), \quad n \to \infty .$$

PROOF. From the proof of Theorem 6, we need only show that

$$(4.14)\qquad \sum_{n=1}^{\infty} \Gamma_{\lambda\alpha}(n) r_0{}^n < \infty ,$$

for some $r_0 > 1$; we can then take $\beta_\lambda = \max(r^{-1}, r_0{}^{-1})$, where r is as in the previous proof. But for $|z| < 1$, from (3.2)

$$(4.15)\qquad (1-z) \sum_{n=1}^{\infty} \lambda R^n P^n(\alpha) z^n = [\sum_{n=1}^{\infty} \Gamma_{\lambda\alpha}(n) z^n](1-z)U(z) ;$$

(4.12) ensures that the left-hand side of (4.15) can be extended analytically in a region $\{|z| < r_1\}$ for some $r_1 > 1$, and Kendall's lemma, together with (4.1), then implies that $r_2 > 1$ exists such that $\sum_n \Gamma_{\lambda\alpha}(n) z^n$ extends analytically to $\{|z| < r_2\}$. Since $\Gamma_{\lambda\alpha}(n) \geqq 0$, this gives (4.14) for any $r_0 < r_2$. □

We now move to the general situation, employing the minorization condition to extend the results above in exactly the same way the theorems of Section 3 were proved.

THEOREM 8. *Suppose that $\{X_n\}$ satisfies the minorization condition, and that for some* $\rho_1 < 1$,

$$(4.16)\qquad |R^{nk} \nu P^{nk} h - (\nu f)(\pi h)| = O(\rho_1{}^n) , \quad n \to \infty .$$

Then there exists $\rho < 1$ and a single π-null set N such that, if g is any function with $f \geqq cg \geqq 0$ *(except perhaps on N),*

(i) *for all* $x \notin N$, *we have*

$$\|R^n P^n(x, \cdot) - f(x)\pi(\cdot)\|_g = O(\rho^n), \quad n \to \infty ; \tag{4.17}$$

(ii) *for any* λ *with* $\lambda(N) = 0$, $\lambda f < \infty$ *and such that for some* $\rho_\lambda < \rho$,

$$|R^{nk}\lambda P^{nk}h - (\lambda f)(\pi h)| = O(\rho_\lambda{}^n), \quad n \to \infty , \tag{4.18}$$

we have

$$\|R^n \lambda P^n(\cdot) - (\lambda f)\pi(\cdot)\|_g = O(\rho^n), \quad n \to \infty . \tag{4.19}$$

PROOF. The results follow immediately for $\{X_{nk}^*\}$ on noticing that (4.16) and (4.18) are respectively (4.1) and (4.12) for the atom $S_1 \subset S^*$. They thus hold also for $\{X_{nk}\}$ when g is an $\mathscr{F}$-measurable function. To extend them to $\{X_n\}$ it suffices, as in Section 3, to consider the functions $g_j = R^j P^j g$, $j = 1, \cdots, k-1$. □

It is of some interest to identify $\sum_n \Gamma_{\lambda\alpha}(n)r^n$ in the case where $\alpha = S_1$ for the split chain $\{X_{nk}^*\}$. Suppose $k = 1$ in the minorization condition, and let $\tau = \tau_{S_1}$ for $\{X_n{}^*\}$. Then for any initial measure λ on $\mathscr{F}^*$, and $r \geqq 0$, we have

$$\sum_{n=1}^\infty r^n \mathbb{P}_\lambda{}^*(\tau = n) = \lambda \sum_{n=0}^\infty r^{n+1}(P^* I_{S_0})^n P^*(S_1) = \xi(S_1) ,$$

say, where ξ is the minimal solution of

$$\xi = r\lambda P^* + r\xi I_{S_1} P^* .$$

As in the proof of Lemma 5.12 of [8], we have

$$\xi = \lambda \bar{P} \sum_{n=0}^\infty r^n (P - h \otimes \nu)^n ,$$

where $h \otimes \nu(x, A) = h(x)\nu(A)$; and so

$$\xi(S_1) = \lambda \bar{P} \sum_{n=0}^\infty r^n (P - h \otimes \nu)^n h .$$

Bearing in mind that we need (4.18) only to establish the analogue of (4.14), we have the following criteria for the assumptions of our geometric ergodicity theorems to hold:

THEOREM 9. (i) *The following two statements are equivalent*:

(a) *for some* $r_0 > 1$,

$$|R^{nk}\nu P^{nk}h - (\nu f)(\pi h)| = O(r_0{}^{-n}), \quad n \to \infty ;$$

(b) *for some* $r_1 > R$,

$$\nu \sum_{n=0}^\infty r_1{}^n (P^k - h \otimes \nu)^n h < \infty .$$

(ii) *There is a null set* N *such that for any measure* λ *with* $\lambda(N) = 0$, *the following two statements are equivalent provided either one of the two equivalent statements of* (i) *holds*:

(a) *for some* $r_0 > 1$,

$$|R^{nk}\lambda P^{nk}h - (\lambda f)(\pi h)| = O(r_0{}^{-n}), \quad n \to \infty ;$$

(b) *for some* $r_1 > R$

$$\lambda P^k \sum_{n=0}^\infty r_1{}^n (P^k - h \otimes \nu)^n h < \infty .$$

PROOF. We need only note that for the split chain $\{X^*_{nk}\}$, $x \in S_1 = \alpha$ implies $\varepsilon_x P^k = \nu$; the remainder follows from our previous results. □

5. Initial distributions, admissibility and topological spaces. In this section we give some miscellanea which make more concrete the notions in Theorem 8, which is the main result of this paper. Suppose now that $\mathscr{F}$ is countably generated, and let φ be any measure such that $\varphi(A) > 0$ implies $\sum_{n=1}^{\infty} P^n(x, A) > 0$ for each $x \in S$; i.e., any irreducibility measure. From [10], any set A with $\varphi(A) > 0$ contains a *C-set for* φ; that is, if $p_\varphi{}^n(x, y)$ denotes the density of $P^n(x, \cdot)$ with respect to φ, there is a set $C \subset A$ with $\varphi(C) > 0$, such that, for some $m > 0$,

$$p_\varphi{}^m(x, y) \geqq \delta_C > 0\,, \qquad x, y \in C\,. \tag{5.1}$$

Hence the minorization condition is satisfied with ν as $\varphi I_C/\varphi(C) = \varphi_C$, $h = [\delta_C \varphi(C)]1_C$, and $k = m$.

Hence we have immediately that (4.16) holds if, for some irreducibility measure φ, and some C-set for φ, satisfying (5.1)

$$|R^{mn}\varphi_C P^{mn}(C) - (\varphi_C f)\pi(C)| = O(\rho_\varphi{}^n)\,, \quad n \to \infty\,, \tag{5.2}$$

where $\rho_\varphi < 1$. Theorem 1 is a special case of this, since π is an irreducibilty measure for $\{X_n\}$; but so is any φ with $\pi \gg \varphi$, so that (5.2) holds for a much wider class of initial distributions than restrictions of π to C-sets for π. Identification of C-sets which are not atomic may not be trivial, although in cases where φ is known the densities $p_\varphi{}^n(x, y)$ will probably also be known. However, the fact that *every* A with $\varphi(A) > 0$ contains a C-set enables us to assert that, (again putting $\varphi_B = \varphi I_B/\varphi(B)$), (4.1) will hold provided that for some A with $\varphi(A) > 0$, and every $B \subset A$ with $\varphi(B) > 0$, there is $\rho_B < 1$ with

$$|R^n \varphi_B P^n(B) - (\varphi_B f)\pi(B)| = O(\rho_B{}^n)\,, \quad n \to \infty\,. \tag{5.3}$$

This formulation enables us to extend our results to the case where $\mathscr{F}$ may not be countably generated.

THEOREM 10. *Suppose $\mathscr{F}$ is not countably generated, and that* (5.3) *holds. There is a single convergence rate $\rho < 1$ such that, for any $E \in \mathscr{F}_f$,*

$$|R^n P^n(x, B) - f(x)\pi(B)| = O(\rho^n)$$

for any $B \subset E$ provided x is outside a π-null set N_B (which may depend on B).

PROOF. If this is not true, we can find a sequence of sets $E_j \in \mathscr{F}_f$ with sets $A_j \subset E_j$, such that for x in a set of positive π-measure, for infinitely many n

$$|R^n P^n(x, A_j) - f(x)\pi(A_j)| \geqq \rho_j{}^n\,, \quad \rho_j \uparrow 1\,.$$

Now let $\mathscr{F}_0$ be an admissible σ-field containing A for which (5.3) holds, and containing the sets E_j, A_j, $j = 1, \cdots$ (cf. [10]); since Theorem 8 holds for $\{X_n\}$ on $(S, \mathscr{F}_0)$, we have a contradiction. □

Now let us turn to the conclusions of the theorems. The functions g which occur there are of most use in enabling us to deduce results for $\{X_n\}$ from $\{X_{nk}\}$; the more interesting results are the total variation convergence results on $\mathscr{F}_f$ which follow from them. In the case $R = 1$, we have $f \equiv 1$ and so S itself is in $\mathscr{F}_f$; hence we have the usual total variation convergence at a geometrically fast rate provided such a rate exists for convergence on a C-set.

When $R > 1$, we will not usually have $S \in \mathscr{F}_f$; and the identification of elements of $\mathscr{F}_f$ is then of particular interest if S admits a topology. In another connection (now rendered somewhat redundant because of Theorem 2), these sets are identified in [11] and [12] for a large class of chains: the proof of the following is in these papers.

PROPOSITION 1. (i) *Suppose $Pg(x)$ is a bounded continuous function of x for every bounded measurable g. Then every relatively compact set E with $\varphi(E) > 0$ is in $\mathscr{F}_f$.*

(ii) *Suppose $Pg(x)$ is a bounded continuous function of x for every bounded continuous g, and that $\mathscr{F}$ is the Borel σ-field on S. If φ is regular, and the support of φ* (supp φ) *satisfies*

(a) $\varphi((\text{supp }\varphi)^c) = 0$;

(b) supp φ *is of second category in the relativised topology;*

then every relatively compact set E with $\varphi(E) > 0$ is in $\mathscr{F}_f$.

In [11] and [12], these relatively compact sets are shown to have "correct" properties in the R-transient case also. From Proposition 1, Theorem R1, and the results of this paper, we can thus state the following general result.

THEOREM 11. *Suppose $\mathscr{F}$ is countably generated, and let φ be some irreducibility measure for $\{X_n\}$. Suppose that P satisfies either of the continuity assumptions of Proposition 1. If there is a C-set C for φ such that, for some $\rho_\varphi < 1$ and constants $\pi(x, C)$, $x \in C$,*

$$|\varphi_C P^n(C) - \varphi_C \pi(C)| = O(\rho_\varphi{}^n)\,, \quad n \to \infty\,,$$

then there is a φ-null set N, constants $\pi(x, A)$ for all $x \in S$ and $A \in \mathscr{F}$ (where $\pi(x, A) \equiv 0$ in the transient case, and $\pi(x, A) = \pi(A)$ in the positive recurrent case) and $\rho < 1$, such that, for all $x \notin N$,

$$\|P^n(x, \cdot) - \pi(x, \cdot)\|_E = O(\rho^n)\,, \quad n \to \infty\,,$$

whenever E is a relatively compact set with $\varphi(E) > 0$.

6. Random walk on a half-line. In this section we apply our results to random walk on $[0, \infty)$. We consider a sequence $Y_1, Y_2, \cdots$ of independent and identically distributed random variables, and write

$$W_n = (W_{n-1} + Y_n)^+\,, \qquad n = 1, 2, \cdots;$$

this is well defined once the initial distribution of W_0 is given. We suppose that both $\Pr(Y_i > 0) > 0$ and $\Pr(Y_i < 0) > 0$; the latter assumption ensures that

$\{W_n\}$ is φ_0-irreducible for $\varphi_0(\{0\}) = 1$, $\varphi_0((0, \infty)) = 0$. We will assume that $\mathbb{E}(Y_j)$ exists in the extended manner described in [7].

It is simple (cf. [6]) to verify that $\{W_n\}$ satisfies the continuity condition in Proposition 1 (ii), and we can then apply Theorem 11 to $\{W_n\}$. However, because of some monotonicity properties enjoyed by $\{W_n\}$, we are able to eliminate the null sets which occur for general chains, and achieve a more satisfying solidarity result.

THEOREM 12. *Let P be the transition law of $\{W_n\}$.*

(i) *A necessary and sufficient condition for the existence of $\rho_0 < 1$ such that*

$$P^n(0, 0) = O(\rho_0{}^n), \quad n \to \infty \tag{6.1}$$

is the existence of $0 < D < \infty$ and $\eta > 0$ such that

$$\mathbb{E}(Y_i) > 0, \qquad \Pr(Y_i < -x) \leqq De^{-\eta x}, \quad x \geqq 0. \tag{6.2}$$

If (6.2) holds, then there exists $\rho < 1$ such that for all x and all intervals $[0, a]$,

$$P^n(x, [0, a]) = O(\rho^n), \quad n \to \infty. \tag{6.3}$$

(ii) *A necessary and sufficient condition for the existence of $\rho_0 < 1$ and $\pi(0) > 0$ such that*

$$|P^n(0, 0) - \pi(0)| = O(\rho_0{}^n), \quad n \to \infty \tag{6.4}$$

is the existence of $0 < D < \infty$, $\eta > 0$ such that

$$\mathbb{E}(Y_i) < 0, \qquad \Pr(Y_i > x) \leqq De^{-\eta x}, \quad x \geqq 0. \tag{6.5}$$

If (6.5) holds then there is probability measure π on $[0, \infty)$ and $\rho < 1$ such that for all x,

$$\|P^n(x, \cdot) - \pi(\cdot)\| = O(\rho^n), \quad n \to \infty. \tag{6.6}$$

PROOF. The equivalence of (6.1) and (6.2), and (6.4) and (6.5), is proved by Miller in [7]. He then uses the results of Vere–Jones to deduce analogues of (6.3) and (6.6) under the additional assumption that Y_i takes on only integer values. From Theorem 11, we know that (6.3) holds for φ-almost all x, for any irreducibility measure φ. If we look at the proof of Theorem 1 of [15], we see that the null set on which (6.3) may fail is in fact precisely the set on which

$$\textstyle\sum_{n=1}^{\infty} P^n(x, 0) r^n = \infty \tag{6.7}$$

where r is some value for which

$$\textstyle\sum_{n=1}^{\infty} P^n(0, 0) r^n < \infty. \tag{6.8}$$

Since $\{W_n\}$ is stochastically monotone, $P^n(x, 0) \leqq P^n(0, 0)$ for all n and all $x \in [0, \infty)$ (see [1]), so (6.8) implies that the set on which (6.7) holds is void. This gives (i).

Now from Theorem 6 we see that (6.4) implies (6.6) except on a null set, since

$f \equiv 1$. To show that the null set is void, it suffices from the proof of Theorem 6 to show the emptiness of the sets

$$N_0 = \{y : \sum_{n=1}^{\infty} {}_0P^n(y, 0) \neq 1\}$$

and

$$N_1 = \{y : \sum_{n=1}^{\infty} {}_0P^n(y, 0)r^n = \infty\},$$

where r is such that

$$\sum_{n=1}^{\infty} {}_0P^n(0, 0)r^n < \infty .$$

Although we do not know the structure of π in detail, we do know that π is not concentrated on a bounded set, since $\Pr(Y_1 > 0) > 0$ (and in fact when Y_1 is not a lattice variable, $\pi(U) > 0$ for every open set U (see [2], page 147)). Now let us look at $\tau_0 = \inf(n \geqq 1 : W_n = 0)$, and write

$$S_n = \sum_{i=1}^{n} Y_i + S_0$$

(where $S_0 = W_0$ has a prescribed initial distribution), for the random walk underlying $\{W_n\}$. Then $\tau_0 = \inf(n \geqq 1 : S_n \leqq 0)$. We put $\tau_w = \inf(n \geqq 1 : S_n \leqq w)$, $w \geqq 0$. Then by translation invariance of $\{S_n\}$,

$$\begin{aligned} \Pr(\tau_0 \geqq n \mid S_0 = x) &= \Pr(\tau_w \geqq n \mid S_0 = x + w) \\ &\leqq \Pr(\tau_0 \geqq n \mid S_0 = x + w), \qquad w \geqq 0 . \end{aligned}$$

Hence firstly we see that if $y \in N_0$, then $y + w \in N_0$ for all $w \geqq 0$; since $\pi(N_0) = 0$, this cannot happen and so N_0 is void. Secondly, we have from Lemma 2 that

$$N_1 = \{y : \sum_{n=1}^{\infty} r^n [\sum_{m=n}^{\infty} {}_0P^m(y, 0)] = \infty\},$$

i.e., since N_0 is void

$$N_1 = \{y : \sum_{n=1}^{\infty} r^n \Pr(\tau_0 \geqq n \mid S_0 = y) = \infty\};$$

so if $y \in N_1$ then $y + w \in N_1$ for all $w \geqq 0$, and so N_1 is also void. □

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