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A MARTINGALE APPROACH TO THE POISSON CONVERGENCE OF SIMPLE POINT PROCESSES[1]

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The paper concerns the Doob-Meyer increasing processes of simple point processes on the positive half line. It is shown that the weak convergence of such point processes to a simple Poisson process is implied by the pointwise weak convergence of their increasing processes, provided that the increasing processes satisfy a mild regularity condition. Conditions under which the regularity is satisfied are investigated. One condition is that the increasing process is that of the point process with its generated σ-fields. The Poisson convergence theorem is applied to superpositions of point processes.

1. Introduction, definitions and notation. The cumulative number of points of a point process on the positive half-line is a positive local submartingale. Such a process therefore has a Doob-Meyer increasing process associated with it and this has been the subject of much work recently. Attention has focused on showing that for simple point processes the increasing process determines the distribution of the point process (e.g., [10], [11]), on obtaining representations for martingales using it (e.g., [2], [3], [5], [7], [10]) and in its applications to statistics (Aalen (1976)). In this paper it is shown that weak convergence of simple point processes to a simple Poisson process is implied by the weak convergence of their increasing processes to a continuous (deterministic) function at each point of the positive half-line (Theorem 1, Section 2). Section 3 presents some information about the structure of increasing processes of simple point processes. This information is needed to apply the convergence result to superpositions of point processes (Section 4).

Define $\underline{X}$ to be the space of right continuous, increasing, step functions, $x: \mathbb{R}^+ \to \mathbb{N}$, with jumps of size one and $x(0) = 0$. Let $\underline{\mathscr{B}}$ be the least σ-algebra making coordinate projections from X measurable. Let $(\Omega, \mathscr{A}, P)$ be a complete probability space. A set, $\{\mathscr{F}(t)\}_{t\geq0}$, of sub-$\sigma$-fields of $\mathscr{A}$ will always be increasing, right continuous, with all null sets in $\mathscr{F}(0)$. Define a *point process* $\mathbf{N}$ to be a measurable mapping N from $(\Omega, \mathscr{A}, P)$ to $(X, \mathscr{B})$, and an associated set of sub-σ-fields, $\{\mathscr{F}(t)\}_{t\geq0}$ of $\mathscr{A}$ such that $\{N(t)\}_{t\geq0}$ ($N(t)$ is the rv whose value at $\omega(\in\Omega)$ is $N(\omega)$'s value at $t(\in\mathbb{R}^+)$) is adapted to $\{\mathscr{F}(t)\}_{t\geq0}$. We will often write $\mathbf{N} = \{N(t), \mathscr{F}(t)\}_{t\geq0}$.

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The *Doob–Meyer process*, $A = \{A(t)\}_{t \geqq 0}$ associated with a point process, $\mathbf{N}$, is the natural increasing stochastic process such that $\{N(t) - A(t)\}_{t \geqq 0}$ is a local martingale with respect to $\{\mathscr{F}(t)\}_{t \geqq 0}$. We shall call A the *compensator* of $\mathbf{N}$ (following Kabanov, Liptser and Shiraev (1974)). If $N: \Omega \to X$ is a fixed measurable mapping and A, A' are compensators of $\{N(t), \mathscr{F}(t)\}_{t \geqq 0}$, $\{N(t), \mathscr{F}'(t)\}_{t \geqq 0}$ (respectively), then in general A and A' will differ. This is one reason why the definition of a point process includes the associated σ-fields.

If, for $i \geqq 1$, τ_i is the time of the ith jump of N, then τ_i is a stopping time. We shall frequently use the martingale facts relating to stopping times which allow us to conclude, for instance, that $\{A(t \wedge \tau_i)\}_{t \geqq 0}$ is the compensator of $\{N(t \wedge \tau_i)\}_{t \geqq 0}$ (these facts may be found in, e.g., Meyer (1966), Chapters VI and VII).

We shall be interested in approximations to A. For this purpose, if $r < s \in \mathbb{R}^+$, we define a *partition*, Q, of $(r, s]$ to be a set of disjoint half-open intervals whose union is $(r, s]$. An *R-sequence of partitions*, $\{Q_n\}_{n \geqq 1}$, will denote a sequence of partitions such that

(1) if $n \geqq m$ then Q_n is a refinement of Q_m;
(2) $\max_{(t,t'] \in Q_n} (t' - t) \to 0$.

If $\mathbf{N}$ is a point process and Q a partition of $(r, s]$ ($r < s \in \mathbb{R}^+$), then $\{N(t') - N(t)\}_{(t,t'] \in Q}$ is a discrete submartingale difference sequence, and its increasing process evaluated at s will be denoted $a(Q)$. We will say that $a(Q)$ is a *discrete approximation to* $A(s) - A(r)$ and we have

$$a(Q) = \textstyle\sum_{(t,t'] \in Q} E(N(t') - N(t) \mid \mathscr{F}(t)) \, .$$

We shall say that A is *calculable* if, for any s ($\geqq 0$) and any R-sequence of partitions, $\{Q_n\}_{n \geqq 1}$, of $(0, s]$, we have

$$a(Q_n) \to_P A(s) \, .$$

We shall call A *locally calculable* (*L-calculable*) if A^i, the compensator of N stopped at τ_i, is calculable, for each i. Conditions under which this holds are given in Section 3.

The notation used above is a prototype for all the notation in this paper; for example, if $\tilde{\mathbf{N}}_n$ is a point process, its compensator will be $\tilde{A}_n$ and, for $r < s \in \mathbb{R}^+$ and Q_m a partition of $(r, s]$, then $\tilde{a}_n(Q_m)$ will be the discrete approximation to $\tilde{A}_n(s) - \tilde{A}_n(r)$ using Q_m.

Since knowledge of the compensator of a point process (with respect to the generated σ-fields) determines the distribution of the process (e.g., [10], [11]), we may define a *Poisson process* by specifying its compensator. Hence, a *Poisson process* will mean a point process which has compensator (with respect to the σ-fields generated by the process), $m: \mathbb{R}^+ \to \mathbb{R}^+$, which is (deterministic and) continuous. Note that m will then be its mean function, and that this definition includes all Poisson processes in the usual sense, which do not have multiple points.

If $\{\mathbf{N}_n\}_{n \geqq 1}$ is a sequence of point processes, then its paths are all in $D[0, \infty)$.

Skorohod J_1 weak convergence to a Poisson process, $\mathbf{N}$, in this case reduces to convergence in distribution of $(N_n(t_1), \cdots, N_n(t_m))$ to the appropriate Poisson random vector, for all $m(\in \mathbb{N})$ and $0 \leqq t_1 < \cdots < t_m < \infty$ (Straf (1972) and Lindvall (1973)). This will be denoted $N_n \to_d N$ and $\to_d$ will also serve for convergence in distribution of random variables and vectors. We note that this is the same as viewing $\{N_n\}_{n \geqq 1}$ as random measures and requiring weak convergence (with respect to the vague topology) or "finite dimensional distributions" convergence (Kallenberg (1973)).

For notational convenience (e.g., when taking complicated conditional expectations), if X, Y are random variables and $A \in \mathscr{A}$, then $X - Y: A$ will be the rv $(X - Y)I(A)$. The notation $\|X\|_0$ will denote the norm in probability of X (i.e., $E|X|/(1 + |X|)$). If $\{X_n\}_{n \geqq 1}$ and $\{Y_n\}_{n \geqq 1}$ are sequences of random variables then $X_n \sim_P Y_n$ will denote $\|X_n - Y_n\|_0 \to 0$.

A process, $\mathbf{N} = \{N(t), \mathscr{F}(t)\}_{t \geqq 0}$, will be called a *counting process*, if it satisfies all the conditions of a point process, except that its jump sizes, $\{S_i\}_{i \geqq 1}$, may be positive integer random variables with $ES_i < \infty$. A counting process is clearly a positive local submartingale, and its *compensator* is defined as for that of a point process. All concepts for point processes (e.g., calculability of the compensator) are extended to counting processes. A sequence of counting processes, $\{\mathbf{N}_n\}_{n \geqq 1}$, will be called an *asymptotic point process sequence*, if, for each $s \in \mathbb{R}^+$,

$$(1.1) \qquad E(N_n(s)\colon\ N_n \text{ has a jump of size } > 1 \text{ on } (0, s]) \to 0\,.$$

We shall write (1.1) as

$$E(N_n(s)\colon\ N_n \textit{ not simple} \text{ on } (0, s]) \to 0\,.$$

2. The Poisson convergence theorem. In this section the main theorem is stated and proved.

THEOREM 1. *Let $\{\mathbf{N}_n\}_{n \geqq 1}$ be a sequence of point processes with compensators $\{A_n\}_{n \geqq 1}$. Suppose $\mathbf{N}$ is a Poisson process with (continuous) mean function, $m: \mathbb{R}^+ \to \mathbb{R}^+$, and that either of the following conditions hold.*

(a) *Each A_n ($n \geqq 1$) is L-calculable and, for all $t \in \mathbb{R}^+$,*

$$A_n(t) \to_d m(t)\,.$$

(b) *For all $t \in \mathbb{R}^+$, and for any R-sequence of partitions, $\{Q_n\}_{n \geqq 1}$ of $(0, t]$, $EN_n(t) < \infty$ and*

$$a_n(Q_n) \to_d m(t)\,.$$

Then

$$N_n \to_d N\,.$$

Theorem 1 is analogous to Theorem 5.2 of Kallenberg (1976). The main differences are that Kallenberg's theorem concerns a.s. diffuse conditional intensity measures (which involve conditioning on two sides instead of one), it presupposes L_1 convergence of the conditional intensity measures and it applies to point processes on general spaces with their generated σ-fields. Because of the order properties of the half-line, compensators are more natural. For instance, they determine the distribution of the point process (e.g., [10], [11]) while Kallenberg (1976) shows that conditional measures do not.

The idea of the proof of Theorem 1 is to apply suitably the following lemma, which is implied by Proposition 43 of Freedman (1974), and is also a straightforward consequence of Theorem 2 of Brown and Eagleson (1971).

LEMMA 1. *Suppose* $s \in \mathbb{R}^+$ *and* $\{Q_n\}_{n\geq 1}$ *is an R-sequence of partitions of* $(0, s]$. *Let* $\{D_n(t')\}_{(t,t']\in Q_n}$ *be a double array of events and* $\{\mathscr{F}_n(t)\}_{(t,t']\in Q_n}$ *of σ-fields such that* $D_n(t') \in \mathscr{F}_n(t')$ *for all* t' $((t, t'] \in Q_n,\ n = 1, 2, \cdots)$. *If*

$$\max_{(t,t']\in Q_n} P(D_n(t') \mid \mathscr{F}_n(t)) \to_P 0 \tag{2.1}$$

and

$$\sum_{(t,t']\in Q_n} P(D_n(t') \mid \mathscr{F}_n(t)) \to_P \lambda \tag{2.2}$$

then $\sum_{(t,t']\in Q_n} I(D_n(t'))$ *converges in distribution to the Poisson law with parameter* λ.

The proof now proceeds via two technical lemmas.

LEMMA 2. *To prove Theorem* 1 *under condition* (a), *we may* (*and will hereafter*) *assume that* $EN_n(t) < \infty$ $(t > 0)$ *and that each increasing process*, A_n, *is calculable*.

PROOF. To prove Theorem 1, it is required to show that, for all fixed $0 < t_1 < \cdots < t_m < \infty$, $(N_n(t_1), \cdots, N_n(t_m)) \to_d (N(t_1), \cdots, N(t_m))$. Of course, to show this, we shall only use the conditions on $A_n(t)$ and its approximands, for $t \leqq t_m$. For each n, choose a jump time, σ_n, of N_n such that $N_n(\sigma_n) = k_n \in R^+$ and $P(\sigma_n \leqq t_m) \to 0$, $n \to \infty$. The process $N_n'(t) = N_n(t \wedge \sigma_n)$ has compensator $A_n'(t) = A_n(t \wedge \sigma_n)$, so that $A_n'(t) \to_d m(t)$, for $t \leqq t_m$. Moreover, $EN_n'(t) \leqq EN_n(\sigma_n) < \infty$, for all $t > 0$. For an R-sequence, $\{Q_j\}$, of partitions of $(0, t]$, we have $a_n'(Q_j) \to_P A_n'(s)$ $(j \to \infty)$, by the definition of L-calculability. Hence A_n' is calculable, for each n. Finally, note $(N_n'(t_1), \cdots, N_n'(t_m)) \sim_P (N_n(t_1), \cdots, N_n(t_m))$.

LEMMA 3. *Suppose the conditions of Theorem* 1 *hold. If* $s \in \mathbb{R}^+$, *there exists an R-sequence of partitions*, $\{Q_n\}_{n\geq 1}$, *of* $(0, s]$ *satisfying the following equations. If* B_n $(n = 1, 2, \cdots)$ *is the event* $[\max_{(t,t']\in Q_n} N_n(t') - N_n(t) \leqq 1]$, *then*

$$P(B_n) \to 1\ . \tag{2.3}$$

$$\max_{(t,t']\in Q_n} E(N_n(t') - N_n(t) \mid \mathscr{F}_n(t)) \to_P 0\ . \tag{2.4}$$

$$a_n(Q_n) \to_P m(s)\ . \tag{2.5}$$

$$\sum_{(t,t']\in Q_n} E(N_n(t') - N_n(t) : B_n^c) \mid \tag{2.6}$$

(*provided that* $EN_n(s) < \infty$).

REMARK. Lemma 3 contains the technical information which will be needed to apply Lemma 1. Although the proof of Lemma 3 appears complicated, the idea is simple. Firstly, partitions are produced to satisfy (2.3). Secondly, these partitions are refined so that (2.4) and (2.5) hold. The refinement uses the fact that m is continuous and that, under condition (a) or (b) of Theorem 1, discrete approximations to the compensator A_n can, for large n, be made close in probability to m.

PROOF. Produce a sequence $\{Q_n\}_{n\geq 1}$ of partitions which satisfy (2.3) and (2.6). It is possible to do this because of our definition of point processes; for any fixed $n\,(\geqq 1)$ and an R-sequence, $\{R_m\}_{m\geq 1}$, of partitions of $(0, s]$, $C_m = [\max_{(t,t']\in R_m} N_n(t') - N_n(t) \leqq 1]$ increases to $[N_n$ simple on $(0, s]]$. Hence $P(C_m) \to 1$ and, because $EN_n(s) < \infty$, dominated convergence gives $E(N_n(s)\colon C_m{}^c) \to 0$, $m \to \infty$. Note that any refined sequence also satisfies (2.3) and (2.6).

Suppose condition (a) of Theorem 1 is satisfied. Fix $j \in \mathbb{N}$. Find a partition, $R(j)$, of $(0, s]$ such that

$$\max_{(t,t']\in R(j)} |m(t') - m(t)| < 2^{-j-2}. \tag{2.7}$$

Suppose $R(j)$ has $L(\in \mathbb{N})$ elements in it. We may choose $n(j)$ large enough so that for $n \geqq n(j)$ and $t(\in \mathbb{R}^+)$, any end point of one of the intervals of $R(j)$,

$$P(|A_n(t) - m(t)| > 2^{-j-2}) < (L2^{j+2})^{-1}. \tag{2.8}$$

Since the compensators A_n $(n = 1, 2, \cdots)$ are all calculable, we may choose a partition $S_n(j)$ $(n = n(j), \cdots)$ including all the points of Q_n and of $R(j)$ such that,

$$P(|a_n(S_n(j)) - A_n(s)| > 2^{-j-2}) < (L2^{j+2})^{-1} \tag{2.9}$$

and such that if $(t, t'] \in R(j)$ and $R(n, t)$ is the partition of $(t, t']$ formed by the points of $S_n(j)$ inside $(t, t']$ then

$$P(|A_n(t') - A_n(t) - a_n(R(n, t))| > 2^{-j-2}) < (L2^{j+2})^{-1}. \tag{2.10}$$

Combining (2.7), (2.8) and (2.10) we see that

$$P(|a_n(R(n, t))| > 2^{-j}) < (L2^{j})^{-1}. \tag{2.11}$$

Together (2.8) and (2.9) give

$$P(|a_n(S_n(j)) - m(s)| > 2^{-j-1}) < (L2^{j+1})^{-1} < 2^{-j-1}. \tag{2.12}$$

Now let j vary and for $n(j) \leqq n < n(j + 1)$ redefine Q_n to be $S_n(j)$ (for $1 \leqq n < n(1)$, Q_n remains the same). This definition and (2.11) gives for $n(j) \leqq n < n(j + 1)$

$$\begin{aligned} P(\max_{(t,t']\in Q_n} E(N(t') - N(t) \mid \mathscr{F}(t)) > 2^{-j}) \\ \leqq P \quad (\text{for some } \ (t, t'] \ \text{ in } \ R(j), |a_n(R(n, t))| > 2^{-j}) \\ < 2^{-j}. \end{aligned}$$

Hence (2.4) is satisfied. Likewise, using (2.12), we see (2.5) is also satisfied.

Now suppose condition (b) of Theorem 1 is satisfied. Start again with the R-sequence of partitions of $(0, s]$, $\{Q_n\}_{n\geq 1}$, which satisfy (2.3) and (2.6). Again fix $j \in \mathbb{N}$ and follow the procedure of (2.7). Define partitions $S_n(j)$ $(n = 1, 2, \cdots)$ to include Q_n and $R(j)$ so that there exists $n(j)$ for which $n \geqq n(j)$ implies

$$P(|a_n(S_n(j)) - m(s)| > 2^{-j}) < 2^{-j}$$

and, letting $R(n, t)$ and L be as before,

$$P(|m(t') - m(t) - a_n(R(n, t))| > 2^{-j-2}) < (L2^{j+1})^{-1} .$$

Redefining $\{Q_n\}_{n\geqq 1}$ as previously, the same reasoning shows that (2.4) and (2.5) are also satisfied under the hypothesis of condition (b) of Theorem 1.

LEMMA 4. *Suppose the conditions of Theorem* 1 *hold. If* $s \in \mathbb{R}^+$, *then* $N_n(s) \to_d$ *Poisson law with parameter* $m(s)$.

PROOF. Let $\{Q_n\}_{n\geqq 1}$ be an R-sequence of partitions satisfying the requirements of Lemma 3. For $(t, t'] \in Q_n$ $(n = 1, 2, \cdots)$, let $D_n(t') = [N_n(t') - N_n(t) = 1]$ and recall that B_n, as in Lemma 3, is the event that all intervals of Q_n have $\leqq$ one jump in them. Then from (2.3),

$$\sum_{(t,t']\in Q_n} I(D_n(t')) = \sum_{(t,t']\in Q_n} N_n(t') - N_n(t) \quad \text{on} \quad B_n \sim_P N_n(s) . \tag{2.13}$$

Hence $\sum_{(t,t']\in Q_n} I(D_n(t'))$ has a limit distribution iff $N_n(s)$ does and, in this case, the two coincide. Now

$$\begin{aligned} &\|a_n(Q_n) - \textstyle\sum_{(t,t']\in Q_n} P(D_n(t') \mid \mathscr{F}_n(t))\| \\ &\qquad \leqq \textstyle\sum_{(t,t']\in Q_n} E(N_n(t') - N_n(t) : B_n^c) . \end{aligned} \tag{2.14}$$

Together (2.5), (2.6) and (2.14) produce

$$\sum_{(t,t']\in Q_n} P(D_n(t') \mid \mathscr{F}_n(t)) \to_P m(s) . \tag{2.15}$$

Also from (2.4)

$$\begin{aligned} &\max_{(t,t']\in Q_n} P(D_n(t') \mid \mathscr{F}_n(t)) \\ &\qquad \leqq \max_{(t,t']\in Q_n} E(N(t') - N(t) \mid \mathscr{F}_n(t)) \to_P 0 . \end{aligned} \tag{2.16}$$

Using (2.13), (2.15) and (2.16) in Lemma 1 concludes the proof.

PROOF OF THEOREM 1. We must show, for each $k(\in \mathbb{N})$, that

$$\begin{aligned} &0 \leqq s_1 < s_2 < \cdots < s_k < \infty \\ &\qquad \Rightarrow (N_n(s_1), \cdots, N_n(s_k)) \to_d (N(s_1), \cdots, N(s_k)) . \end{aligned} \tag{2.17}$$

For $k = 1$ Lemma 4 proves (2.17). Suppose (2.17) is true for some $k > 1$. We prove (2.17) in general, by induction. It suffices to show that for $0 \leqq s_1 < \cdots < s_{k+1}$ and arbitrary $m_1 \leqq \cdots \leqq m_{k+1}$ $(\geqq 0)$

$$\begin{aligned} &P(N_n(s_1) = m_1, \cdots, N_n(s_{k+1}) = m_{k+1}) \\ &\qquad \to P(N(s_1) = m_1, \cdots, N(s_{k+1}) = m_{k+1}) . \end{aligned} \tag{2.18}$$

Let F_n be the event $[N_n(s_1) = m_1, \cdots, N_n(s_k) = m_k]$. Since $P(F_n)$ converges to a number greater than zero, $P(F_n) > 0$ for all n sufficiently large. Without loss of generality we assume $P(F_n) > 0$ for all n. To prove (2.18) is equivalent to proving

$$(2.19)\qquad P(N_n(s_{k+1}) - N_n(s_k) = m_{k+1} - m_k \mid F_n) \\ \to P(N(s_{k+1}) - N(s_k) = m_{k+1} - m_k \mid F)$$

in view of the induction assumption (here F is the event $[N(s_1) = m_1, \cdots, N(s_k) = m_k]$). To prove this we shall use the following lemma.

LEMMA 5. *Let* $\mathbf{N}$ *be a point process with compensator* A. *Let* $F \in \mathscr{F}(s)$, $F \subseteq [N(s) = m]$ $(m \in \mathbb{N}, s > 0)$ *and* $P(F) > 0$. *Define the probability space* $(F, \mathscr{F} \cap F, P_F)$ *where* $\mathscr{F} \cap F = \{A \cap F: A \in \mathscr{F}\}$ *and* P_F *is the probability on* $\mathscr{F}$ *conditional on* F. *Define* $\mathbf{NF}$ *to be the point process on* F *where, for* $t \geqq 0$, $NF(t)$ *is the restriction of* $N(s+t) - N(s)$ *to* F *and* $\mathscr{F}F(t) = \{A \cap F: A \in \mathscr{F}(s+t)\}$. *If the compensator of* $\mathbf{NF}$ *is denoted* AF, *then, for* $t \geqq 0$, $AF(t)$ *is the restriction of* $A(s+t) - A(s)$ *to* F. *Moreover, if* A *is calculable, then so is* AF. *Finally, if* Q *is a partition of* $(0, t]$ $(t \geqq 0)$, *then* $aF(Q)$ *is the restriction of* $a(Q')$ *to* F, *where* $Q' = \{(s+z, s+z']: (z, z'] \in Q\}$.

PROOF. Straightforward, e.g., AF is natural if, for each jump time τ_i $(i > m)$ of $\mathbf{N}$ and for any positive bounded right-continuous martingale $\{Y(t)\}_{t \ge 0}$ (w.r.t. $\{\mathscr{F}F(t)\}_{t \ge 0}$), we have

$$(2.20)\qquad E_F(\textstyle\int_0^{\tau_i - s} Y(t)\, dAF(t)) = E_F(\int_0^{\tau_i - s} Y(t-)\, dAF(t))\,,$$

where E_F is expectation with respect to P_F (Meyer (1966), Theorem 19, VII). But the left-hand side of (2.20) equals $(PF)^{-1}E(\int_s^{\tau_i} Z(t)\, dA(t))$, where $Z(t)$ $(t \geqq s)$ is defined to be $Y(t)$ on F and zero outside F.

A similar computation applies to the right-hand side of (2.20). However $\{Z(t)\}_{t \ge s}$ is a martingale (w.r.t. $\{\mathscr{F}(t)\}_{t \ge s}$). Hence the fact that A is natural gives the validity of (2.20) and thus the naturalness of AF.

Returning to the proof of Theorem 1, the notation of Lemma 5 will be directly transferred to the point process $\mathbf{N}_n$. Suppose condition (a) of Theorem 1 holds. Lemma 5 informs us that each $A_n F_n$ $(n \in \mathbb{N})$ is calculable. In addition, for any $t \in \mathbb{R}^+$ and $\varepsilon > 0$,

$$P_{F_n}(|A_n F_n(t) - (m(t+s) - m(s))| > \varepsilon) \\ \leqq \left(P\left(|A_n(t+s) - m(t+s)| > \frac{\varepsilon}{2}\right) + P\left(|A_n(s) - m(s)| > \frac{\varepsilon}{2}\right)\right)\Big/ P(F_n)$$

and the latter $\to 0$. Hence $A_n F_n(t) \to m(t+s) - m(s)$, for all $t \in \mathbb{R}^+$, and thus condition (a) of Theorem 1 holds for $\{A_n F_n\}_{n \ge 1}$ and $m(\cdot + s) - m(s)$. Similarly, if condition (b) of Theorem 1 holds, then for any R-sequence of partitions, $\{Q_n\}_{n \ge 1}$, of $(0, t]$ $(t \in \mathbb{R}^+)$, $a_n(Q_n) \to m(t+s) - m(s)$. Applying Lemma 4 to the processes $\{\mathbf{N}_n \mathbf{F}_n\}_{n \ge 1}$ gives equation (2.21), and the proof of Theorem 1 is complete.

REMARK. Notice that the proof of Theorem 1 applies to arbitrary counting processes, except for the first paragraph of the proof of Lemma 3. This argument is easily seen to be valid if we assume that $\{\mathbf{N}_n\}_{n\geq 1}$ is an asymptotic sequence of point processes, and hence so is Theorem 1.

3. Compensators which are L-calculable. To satisfy Theorem 1 condition (a) all the compensators must be L-calculable. This section is devoted to demonstrating processes which have L-calculable compensators. Murali-Rao (1969) showed that if the compensator is continuous, then it is also L-calculable. However, Dellacherie and Doleans-Dade (1970) gave a counterexample to the conjecture that a compensator is always L-calculable. We need the following two technical lemmas, the first of which is easy to check.

LEMMA 6. *If $\mathscr{G}$ and $\mathscr{H}$ are sub σ-fields of $\mathscr{A}$, B is an arbitrary event, $C \in \mathscr{G}$ and*

$$\mathscr{G} \cap C = \mathscr{H} \cap C \tag{3.1}$$

then

$$\begin{aligned} P(B \cap C \mid \mathscr{G}) &= P(B \cap C \mid \mathscr{H})/P(C \mid \mathscr{H}) \qquad && \text{on} \quad C \\ &= 0 && \text{on} \quad C^c \end{aligned} \tag{3.2}$$

where $P(C \mid \mathscr{H})$ is a version for which $P(C \mid \mathscr{H}) \neq 0$ on C.

If $\mathbf{N}$ is a point process, recall that $\{\tau_i\}_{i\geq 1}$ is the sequence of jump times of N. Write T_i for the random vector $(\tau_1, \cdots, \tau_i)$ $(i = 1, 2, \cdots)$.

LEMMA 7. *Suppose that $\{N(t), \mathscr{F}(t)\}$ is a point process such that $\mathscr{F}(t) = \sigma(N(z), z \leq t)$, for each $t \geq 0$. For all i $(= 1, 2, \cdots)$ and $t' > t \geq 0$ we have*

$$\begin{aligned} P(\tau_i < t < \tau_{i+1} \leq t' \mid \mathscr{F}(t)) &= P(t < \tau_{i+1} \leq t' \mid T_i)/P(t < \tau_{i+1} \mid T_i) \\ &\qquad \text{on} \quad [\tau_i < t < \tau_{i+1}] \\ &= 0 \qquad \text{outside} \quad [\tau_i < t < \tau_{i+1}] \end{aligned}$$

where $P(t < \tau_{i+1} \mid T_i)$ is a version which is nonzero on $[\tau_i < t < \tau_{i+1}]$.

PROOF. Let $\mathscr{G}$ be $\mathscr{F}(t)$, $\mathscr{H}$ be $\sigma(T_i)$, $B = [\tau_{i+1} \leq t']$ and $C = [\tau_i < t < \tau_{i+1}]$. It is easily seen that $C \in \mathscr{G}$ and that (3.1) holds (by checking it on generating sets of $\mathscr{G}$ and $\mathscr{H}$). The conclusion of the lemma is obtained by applying Lemma 6 and noting that $[\tau_i < t] \in \sigma(T_i)$.

Much of the following proposition appears in the literature already. However it seems that the elementary proof given here is new and that the L-calculability of compensators of point processes (w.r.t. their generated σ-fields) has not appeared elsewhere. Papangelou (1972) gives a.s. calculability for a stationary point process N, with $EN(t) < \infty$, $t \geq 0$. Kallenberg (1976) has an analogous result for conditional intensity measures of point processes in general spaces. The representation (3.3)—(3.6) of a compensator is well known (e.g., [10], [11]).

PROPOSITION 1. *Let $\{N(t), \mathscr{F}(t)\}_{t\geq 0}$ be a point process such that $\mathscr{F}(t) = \sigma(N(z), z \leq t)$ for each $t \geq 0$. Then its compensator A is L-calculable. Moreover, if $s \in \mathbb{R}^+$,* *a version of* $A(s)$ *has the representation*

$$A(s) = \tilde{A}_0(s) + \cdots + \tilde{A}_{N(s)}(s) \tag{3.3}$$

where, for $i = 1, 2, \cdots, \tilde{A}_i(s)$ *satisfies*

$$\tilde{A}_i(s) = \int_{(\tau_i, s\wedge\tau_{i+1}]} (1 - F_i(u-, \cdot))^{-1} dF_i(u, \cdot) . \tag{3.4}$$

In (3.4), *the mapping* $F_i\colon [0, \infty) \times \Omega \to [0, 1]$ *is a regular conditional distribution function for* τ_{i+1} *given* $\tau_1, \cdots, \tau_i$ *such that*

$$F_i(u-, \omega) < 1 \qquad \text{if} \quad \tau_{i+1}(\omega) \geqq u . \tag{3.5}$$

The random variable $\tilde{A}_0(s)$ *is defined to be zero on* $[F(\tau_1) = 1]$ *and otherwise*

$$\tilde{A}_0(s) = \int_{(0, s\wedge\tau_1]} (1 - F(u-))^{-1} dF(u) . \tag{3.6}$$

In (3.6), $F\colon [0, \infty) \to [0, 1]$, *is the distribution function of* τ_1.

PROOF. Define, for $i \geqq 0$,

$$\tilde{N}_i(s) = N(s \wedge \tau_{i+1}) - N(s \wedge \tau_i) , \qquad (\tau_0 = 0) .$$

This has compensator

$$\tilde{A}_i(s) = A(s \wedge \tau_{i+1}) - A(s \wedge \tau_i) .$$

It is easily seen that (3.3) holds with $\tilde{A}_i(s)$ ($i = 0, 1, \cdots$) thus defined. Fix $i \geqq 1$. We will now show that, if $\{Q_n\}_{n\geq 1}$ is an R-sequence of partitions of $(0, s]$, then $\tilde{a}_i(Q_n)$ converges in probability to the right-hand side of (3.4).

Note that, for $t, t' \in \mathbb{R}^+$,

$$\tilde{N}_i(t') - \tilde{N}_i(t) = I[t < \tau_{i+1} \leqq t'] \tag{3.7}$$

and

$$\begin{aligned}\|\textstyle\sum_{(t,t']\in Q_n} E(I[t \leqq \tau_i < \tau_{i+1} \leqq t'] \mid \mathscr{F}(t))\|_1 \\ = \textstyle\sum_{(t,t']\in Q_n} P[t \leqq \tau_i < \tau_{i+1} \leqq t'] .\end{aligned} \tag{3.8}$$

The right-hand side of (3.8) converges to zero because it is dominated by the probability that there are two or more jumps of the process in one of the subintervals of Q_n. The limit of this is the probability that the path of $\mathbf{N}$ does not lie in X. Hence, using (3.7),

$$\tilde{a}_i(Q_n) \sim_P \textstyle\sum_{(t,t']\in Q_n} P(\tau_i < t < \tau_{i+1} \leqq t' \mid \mathscr{F}(t)) . \tag{3.9}$$

Now we use the special form of the σ-fields of $\mathbf{N}$. That form allows us to apply Lemma 7 to evaluate the terms of the right-hand side of (3.9). Use the version of $P(\tau_{i+1} > t \mid T_i)$ in this lemma to construct F_i, a regular conditional distribution function of τ_{i+1} given T_i satisfying (3.5). We then have

$$\tilde{a}_i(Q_n) \sim_P \textstyle\sum_{(t,t']\in Q_n} X_i(t, t') , \tag{3.10}$$

where $X_i(t, t')$ is the rv whose value is $(F_i(t', \omega) - F_i(t, \omega))/(1 - F_i(t, \omega))$ for $\omega \in [\tau_i < t < \tau_{i+1}]$ and 0 for $\omega \notin [\tau_i < t < \tau_{i+1}]$. Fix $\omega_0 \in [\tau_i < s]$, and recognize the sum on the right-hand side of (3.10) as, for each n, the integral of a step function w.r.t. the measure on the half-line generated by $F_i(\cdot, \omega_0)$. Apply dominated convergence to these step functions, for each $\omega_0 \in [\tau_i < s]$, and obtain

$$\tilde{a}_i(Q_n) \to_P \int_{(\tau_i, s\wedge \tau_{i+1}]} (1 - F_i(u-, \cdot))^{-1} dF_i(u, \cdot) . \tag{3.11}$$

Here, of course, the integral is interpreted as zero outside $[\tau_i < s]$. Similar considerations yield the analogue for $i = 0$.

Since $\tilde{\mathbf{N}}_i$ $(i \geqq 0)$ is bounded by 1 and is a point process, it is a submartingale of class D. It can be decomposed into a martingale and a potential of class D (Meyer (1966), Theorem 11, VI). By applying Murali-Rao's (1969) proof of the Meyer decomposition theorem we see that $\tilde{a}_i(Q_n) \to \tilde{A}_i(s)$ in the weak L_1 topology. Hence $\{\tilde{a}_i(Q_n)\}_{n\geq 1}$ is uniformly integrable. This and (3.11) imply that $\tilde{a}_i(Q_n) \to$ in the weak L_1 topology to the right-hand side of (3.11). The a.s. uniqueness of weak L_1 limits then gives (3.4), (3.6) and the calculability of $\tilde{A}_i$ $(i \geqq 0)$.

Let $M \geqq 1$ and N^M the process, N, stopped at τ_M. Then, from above,

$$\begin{aligned} a^M(Q_n) &= \textstyle\sum_{i=0}^{M-1} \tilde{a}_i(Q_n) \\ &\to_P \textstyle\sum_{i=0}^{M-1} \tilde{A}_i(s) \\ &= A^M(s) . \end{aligned}$$

An advantage of the proof of Proposition 1 is that the same technique can be used, in conjunction with the proposition, to show that the compensators of a wider class of point processes are calculable.

COROLLARY 1. *Suppose that* $\{N_i(t), \mathscr{F}_i(t)\}_{t\geq 0}$, $i = 1, 2, \cdots, r$, *is a sequence of independent point processes and that* $\mathscr{F}_i(t) = \sigma(N_i(z), z \leqq t)$ *for each* $t \in \mathbb{R}^+$ *and* $i = 1, 2, \cdots, r$. *Further suppose that* $\{N(t), \mathscr{F}(t)\}_{t\geq 0}$ *is a point process where*

$$N(t) = N_1(t) + \cdots + N_r(t) ,$$
$$\mathscr{F}(t) = \sigma(\mathscr{F}_1(t), \cdots, \mathscr{F}_r(t)) .$$

Then the compensator, A, *of* $\mathbf{N}$ *is L-calculable. For each* s,

$$A(s) = A_1(s) + \cdots + A_r(s) \quad \text{a.s.} , \tag{3.12}$$

where A_i *is the compensator of* $\mathbf{N}_i$ $(i = 1, 2, \cdots, r)$.

REMARKS. 1. The structure of A_i is given by Proposition 1. Hence Proposition 1 and Corollary 1 give us a good idea of the structure of A.

2. Since the paths of an increasing process are defined to be right continuous, there is a set of probability zero, outside of which any version of the compensator will satisfy (3.12) for each $s \in \mathbb{R}^+$. Likewise in equations (3.3)—(3.6) of Proposition 1.

3. If $r = 1$, then Corollary 1 asserts the calculability part of Proposition 1.

4. Suppose $\mathscr{F}(t) = \sigma(\mathscr{F}_1(t), \cdots, \mathscr{F}_r(t), \mathscr{I}(t))$, where $\{\mathscr{I}(t)\}_{t\geq 0}$ is a set of sub-σ-fields of $\mathscr{A}$ which is independent of $\mathbf{N}$. Corollary 1 is still true, since the argument is unchanged.

5. It is an assumption (from the definition of a point process) that the σ-fields $\{\mathscr{F}(t)\}_{t\geq 0}$ in Corollary 1 (and, also, Proposition 1 and Theorem 1) are right continuous. Take $(\Omega, \mathscr{A})$ to be a product of r copies of X with the product σ-field. An argument, similar to that of Lemma 1 of Kabanov, Liptser and Shiraev (1974), shows that in this case the $\{\mathscr{F}(t)\}_{t\geq 0}$ are right continuous. Hence we may always apply Corollary 1 by changing to the canonical space.

PROOF. Let $\{Q_n\}$ be an R-sequence of partitions of $(0, s]$.
For $M = 1, 2, \cdots$ we have

$$a^M(Q_n) = \textstyle\sum_{i=1}^r a_{iM}(Q_n) \tag{3.13}$$

where N^M is N stopped at τ_M, the Mth jump of N and N_{iM} is N_i stopped at τ_M. Fix $i \in \{1, \cdots, r\}$. Let σ_n be τ_M, if $\tau_M \geq s$, and otherwise the first endpoint of Q_n which is $\geq \tau_M$. Define X_n to be $\sum_{(t,t']\in Q_n} E(N_i(t' \wedge \tau_M{}^i \wedge \sigma_n) - N_i(t \wedge \tau_M{}^i \wedge \sigma_n \mid \mathscr{F}(t))$, where $\tau_M{}^i$ is the Mth jump of N_i. Now

$$\|Y_n - a_{iM}(Q_n)\|_1 = E(N_i(\sigma_n \wedge \tau_M{}^i) - N_i(\tau_M)) \to 0\,, \tag{3.14}$$

as $n \to \infty$, since $|N_i(\sigma_n \wedge \tau_M{}^i) - N_i(\tau_M)| \leq M$ and $N_i(\sigma_n \wedge \tau_M{}^i) \to N_i(\tau_M)$. However

$$Y_n = \sum E(N_i(t' \wedge \tau_M{}^i) - N_i(t \wedge \tau_M{}^i) \mid \mathscr{F}_i(t))\,,$$

where the summation is over $(t, t']$ in Q_n for which $t' \leq \sigma_n$. From the proof of the previous proposition it can now be seen that

$$Y_n \to_P A_i(s \wedge \tau_M{}^i \wedge \tau_M) = A_i(s \wedge \tau_M)\,. \tag{3.15}$$

From (3.13), (3.14) and (3.15) we see that

$$a^M(Q_n) \to_P \textstyle\sum_{i=1}^r A_i(s \wedge \tau_m)\,.$$

Hence, from the argument at the end of the last proof, the proof of the corollary is complete.

REMARK. Corollary 1 is true if $\mathbf{N}$ is a counting process and $\{\mathbf{N}_i\}_1^r$ are point processes, as the same proof applies.

4. Application. Throughout this section we will consider a triangular array of counting processes, $\mathbf{N}_{ni} = \{N_{ni}(t), \mathscr{F}_{ni}(t)\}_{t\geq 0}$, $n \in \mathbb{N}$ and $i = 1, \cdots, k_n$. It will be assumed that $\{\mathbf{N}_{ni}\}_{i=1}^{k_n}$ is an independent sequence, and that $\mathscr{F}_{ni}(t) = \sigma(N_{ni}(z), z \leq t)$, $(t \geq 0, n \in \mathbb{N})$. The point process, $\mathbf{N}$, will be a Poisson process with continuous mean function, $m\colon \mathbb{R}^+ \to \mathbb{R}^+$. Define, for each n and $t \geq 0$,

$$N_n(t) = \textstyle\sum_{i=1}^{k_n} N_{ni}(t)\,; \tag{4.1}$$

$$\mathscr{F}_n(t) = \sigma(\mathscr{F}_{ni}(t), \cdots, \mathscr{F}_{nk_n}(t))\,. \tag{4.2}$$

The combination of Theorem 1 and Corollary 1 immediately produces

COROLLARY 2. *Suppose* $\mathbf{N}_n = \{N_n(t), \mathscr{F}_n(t)\}_{t \geq 0}$ *is a sequence of point processes. If, for each* $s \in \mathbb{R}^+$,

$$\sum_{i=1}^{k_n} A_{ni}(s) \to_d m(s)$$

then

$$N_n = \sum_{i=1}^{k_n} N_{ni} \to_d N .$$

REMARK. The remarks after the proofs of Theorem 1 and Corollary 1 show that Corollary 2 is true if each $\mathbf{N}_{ni}$ ($n \in \mathbb{N}$, $i = 1, \cdots, k_n$) is a point process and $\{\mathbf{N}_n\}_{n \geq 1}$ is asymptotically a point process sequence.

Classically it was just assumed that each $\mathbf{N}_{ni}$ was a counting process. However, a uniform asymptotic negligibility condition was imposed—(4.4) in the following corollary—and a condition to ensure that with large probability each process contributes only one jump (cf. (4.5)). Define, for each $t \in \mathbb{R}^+$,

$$F_{ni}(t) = P(N_{ni}(t) \geqq 1) \tag{4.3}$$

so that F_{ni} is the distribution of the time to the first jump of N_{ni}. We can obtain the classical conditions (cf. Grigelionis (1963)) from Corollary 2, quite easily.

COROLLARY 3 (Grigelionis and Franken). *Suppose, for each* $s \in \mathbb{R}^+$,

$$\max_{1 \leqq i \leqq k_n} F_{ni}(s) \to 0 , \tag{4.4}$$

$$\sum_1^{k_n} P(N_{ni}(s) \geqq 2) \to 0 \tag{4.5}$$

and

$$\sum_1^{k_n} F_{ni}(s) \to m(s) . \tag{4.6}$$

Then

$$N_n \to_d N .$$

PROOF. Define for $n \in \mathbb{N}$, $i = 1, \cdots, k_n$, $s \geqq 0$

$$N'_{ni}(s) = N_{ni}(s) \wedge 1 ,$$

so that N'_{ni} is a point process. For the rest of the proof the range of summation of all sums will be $\{1, \cdots, k_n\}$ and they will be sums over i, unless otherwise stated. Define the counting processes,

$$N_n' = \sum N'_{ni} ,$$

so that from (4.5),

$$P(N_n'(s) \neq N_n(s)) \to 0 .$$

It is therefore clear that N_n' and N_n have the same limit distribution, if any.

We have

$$P(N_n' \text{ not simple on } (0, s]) \leqq \sum_{i \neq j} \int_0^s F_{nj}(u) - F_{nj}(u-)\, dF_{ni}(u)$$
$$\leqq \sum_i \int_0^s \sum_j F_{nj}(u) - \sum_j F_{nj}(u-)\, dF_{ni}(u) .$$

Since $\sum F_{nj}$ is bounded, monotonic on $(0, s]$ and m is bounded, continuous on $(0, s]$, the convergence of $\sum F_{ni}$ to m is uniform on $(0, s]$. Hence $P(N_n'$ not simple on $(0, s]) \to 0$. By looking at the sets $[N_n' - N'_{ni}$ simple on $(0, s]]$, it can be seen that $E(N_n'(s)\colon N_n'$ not simple on $(0, s]) \leqq$ right-hand side of the last inequality $+ \sum F_{ni}(s) \times P(N_n'$ not simple on $(0, s])$. Hence $E(N_n'(s)\colon N_n'$ not simple on $(0, s])$ also $\to 0$.

Now, note from Proposition 1,

$$\sum A'_{ni}(s) = \sum \int_{(0,s\wedge\sigma_{ni}]} (1 - F_{ni}(u-))^{-1} dF_{ni}(u)$$

where A'_{ni} is the compensator of N'_{ni} and σ_{ni} is the time of the jump of N'_{ni}. Suppose F is an arbitrary distribution function, with $F(0) = 0$, and $t \in \mathbb{R}^+$. Since

$$\begin{aligned}|\textstyle\int_{(0,t]} (1 - F(u-))^{-1} dF(u) - F(t)| &= \textstyle\int_{(0,t]} F(u-)/(1 - F(u-))\, dF(u) \\ &\leqq F^2(t)/(1 - F(t))\end{aligned}$$

and, from (4.4) and (4.6),

$$\begin{aligned}\sum F^2_{ni}(s \wedge \sigma_{ni})/(1 - F_{ni}(s \wedge \sigma_{ni}) &\leqq \max_{1\leqq i\leqq k_n} \frac{F_{ni}(s)}{1 - F_{ni}(s)} \sum F_{ni}(s) \\ &\to 0\,,\end{aligned}$$

we have

$$\sum A'_{ni}(s) \sim_P \sum F_{ni}(s \wedge \sigma_{ni})\,.$$

But

$$\begin{aligned}|\sum F_{ni}(s \wedge \sigma_{ni}) - \sum F_{ni}(s)| &\leqq N_n'(s) \max_{1\leqq i\leqq k_n} F_{ni}(s) \\ &\to 0\,,\end{aligned}$$

on $[N_n'(s) \leqq l]$, for any $l \in \mathbb{N}$. Indeed,

$$\begin{aligned}P(N_n'(s) \geqq l) &\leqq (l!)^{-1} \textstyle\sum_{j_1\neq\cdots\neq j_l} l!\, F_{nj_1}(s) \times \cdots \times F_{nj_l}(s) \\ &\leqq (l!)^{-1}(\textstyle\sum F_{ni}(s))^l\,,\end{aligned}$$

which is small for large l, uniformly in n, by (4.6).

Hence, combining the last three statements and (4.6),

$$\begin{aligned}\sum A'_{ni}(s) &\sim_P \sum F_{ni}(s) \\ &\to m(s)\,.\end{aligned}$$

The remark after Corollary 2 yields Corollary 3.

Finally we note that Corollary 2 applies to more situations than that where $\{\mathbf{N}_{ni}\}$ is a uniformly asymptotically negligible array. Suppose $\{\mathbf{N}_n\}_{n\geq 1}$ is asymptotically a sequence of point processes. Let $\mathbf{N}_{n1}$ be a Poisson process with mean function m for each n, and suppose $\sum_{i=2}^{k_n} A_{ni}(s) \to_d m'(s)$, for each $s \in \mathbb{R}^+$. Then $\sum_{i=1}^{k_n} A_{ni}(s) \to_d m(s) + m'(s)$ (so Corollary 2 applies) but $\{\mathbf{N}_n\}_{n\geq 1}$ is not uniformly asymptotically negligible. However if the situation is analogous to normal convergence of random variables, then, to have Poisson convergence of $\{\mathbf{N}_n\}$, N_n would decompose into $\sum_{i=1}^{j_n} N_{ni}$ and $\sum_{i=j_n+1}^{k_n} N_{ni}$. The processes in the first sum would be individually close to Poisson and those in the second would be infinitesimal (cf. Zolotarev (1967)). A discussion of this is not possible here as Theorem 1 has no necessary condition for Poisson process convergence.

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