The Annals of Probability
1978, Vol. 6, No. 4, 585–614

STOCHASTIC AND MULTIPLE WIENER INTEGRALS FOR GAUSSIAN PROCESSES[1]

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Multiple Wiener integrals and stochastic integrals are defined for Gaussian processes, extending the related notions for the Wiener process. It is shown that every L_2-functional of a Gaussian process admits an adapted stochastic integral representation and an orthogonal series expansion in terms of multiple Wiener integrals. Also some results of Wiener's theory of nonlinear noise are generalized to noises other than white.

0. Introduction. Let us first fix our basic notation and terminology. We will consider throughout a Gaussian process $X = (X_t, t \in T)$ defined on a probability space $(\Omega, \mathscr{B}, P)$, with zero mean (for simplicity) and covariance function $R(t, s)$. T will be an interval of the real line, even though more general index sets could clearly be used. There are two important Hilbert spaces associated to a Gaussian process. The nonlinear space of X, $L_2(X) = L_2(\Omega, \mathscr{B}(X), P)$, consists of all $\mathscr{B}(X)$-measurable random variables with finite second moment which are called (nonlinear) L_2-functionals of X; $\mathscr{B}(X)$ is the σ-field generated by the process X. The linear space of X, $H(X)$, is the closed subspace of $L_2(X)$ spanned by X_t, $t \in T$, and its elements are called linear L_2-functionals of X.

The first useful notion in the study of the nonlinear space of a Wiener process is the multiple Wiener integral. This notion was first introduced by Wiener (1938), who termed it "polynomial chaos," and was redefined in a somewhat deeper way by Itô (1951). Itô showed that his multiple integrals of different degree are mutually orthogonal and also presented their connection with the celebrated Fourier–Hermite expansion of L_2-functionals of Cameron and Martin (1947). In his work on nonlinear problems Wiener (1958) reinterpreted the multiple Wiener integrals for a Wiener process in an extremely simple and intuitive way and made some interesting applications. Finally Kakutani (1961), Neveu (1968) and Kallianpur (1970) studied the connection between the nonlinear space of a Gaussian process and the tensor products of its linear space, which sheds new light and gives more insight on the structure of the nonlinear space.

The first objective of this work is to define multiple Wiener integrals for

Received September 24, 1976; revised June 7, 1977.

[1] This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-75-2796.

AMS 1970 *subject classifications.* Primary 60G15, 60H05, 60G10.

Key words and phrases. Gaussian processes, multiple Wiener integrals, nonlinear noise, stochastic integrals, iterated integrals, adapted and future increments independent integrands.

general Gaussian processes and to use them in extending Wiener's theory of nonlinear noise. The groundwork is in Section 1 where the Hilbert spaces of appropriate integrands for the multiple Wiener integrals are introduced and studied. The multiple Wiener integrals are then defined in Section 2. Section 5 includes the extension of some basic results of Wiener's nonlinear noise theory from noises generated by the Wiener process to noises generated similarly by processes with stationary Gaussian increments (Theorems 5.1 and 5.2), as well as a simple but interesting result on processes with stationary increments which we could not find in the literature (Lemma 5.5).

The second useful notion in the study of the nonlinear space of a Wiener process is the stochastic integral. The stochastic integral was first introduced by Itô (1944) for the Wiener process. Every L_2-functional of a Wiener process has a representation as a stochastic integral, where the integrand is adapted to the Wiener process. That Wiener process is a Gaussian martingale suggests possible extensions of Itô's integral to martingales and to Gaussian processes. The stochastic integral for martingales was successfully defined by Meyer (1962) and thoroughly studied by Kunita and Watanabe (1967). The idea involved remains the same. But in order to extend Itô's integral to general Gaussian processes one should take a rather different approach using the tensor product structure of nonlinear Gaussian spaces.

The second objective of this work is to define a stochastic integral for general Gaussian processes, and this is done in Section 3. The general properties of the stochastic integral are stated in Theorem 3.2 and some specific stochastic integrals are calculated (Theorems 3.5 and 3.7). The differential rule of the stochastic integral will be developed elsewhere. The stochastic integral is defined for general integrands, not necessarily adapted. In Section 4 it is shown that each L_2-functional of a general Gaussian process has a representation as a stochastic integral where the integrand is adapted to the Gaussian process (Theorem 4.2). The stochastic integral of integrands independent of future increments of the Gaussian process is also considered (Theorem 4.3), but the L_2-functionals which have stochastic integral representations with such integrands have not been characterized yet.

It should be noted that the two representations of L_2-functionals of a (general) Gaussian process presented here (the first as a series of multiple Wiener integrals and the second as a stochastic integral) open the way to the study of nonlinear devices with (general) Gaussian inputs.

Notation. Integrals over T^p are denoted by the integral sign with no subscript. 1_E denotes the characteristic function of the set E. $\otimes$ denotes tensor product and $\tilde{\otimes}$ symmetric tensor product.

The two types of multiple Wiener integrals (MWI) are denoted by I_p and J_p (I and J for $p = 1$), and the two types of stochastic integrals by $\mathscr{I}$ and $\mathscr{J}$; attention is focused on I_p and $\mathscr{I}$. There is also a tensor product integral denoted by $I_\otimes$.

$\mathscr{L}$ and $\mathscr{S}$, with superscripts indicating dimensionality and subscripts indicating the MWI considered, denote classes of Lebesgue integrable functions ($\mathscr{L}$) and of step or Riemann integrable functions ($\mathscr{S}$). These classes of functions are dense in the domains $\Lambda_2(\otimes^p R)$ and $\lambda_2(\otimes^p R)$ of the MWI's I_p and J_p respectively. The domain of the stochastic integral $\mathscr{I}$ is denoted by $\Lambda^*_{2;L_2(X)}(R)$, with additional superscripts to indicate adapted (ad) or future increments independent (fii).

Finally Φ and Ψ denote certain maps used in the definition of the stochastic integral.

1. The Hilbert spaces $\Lambda_2(R)$ and $\lambda_2(R)$. In this section X need not be Gaussian but merely a second order process with mean zero and covariance R. It is shown in Loève (1955, page 472) that the following two integrals

$$I(f) = \mathscr{R} \int f(t)\, dX_t \qquad \text{and} \qquad J(f) = \mathscr{R} \int f(t) X_t\, dt$$

can be defined as the mean square limits of the corresponding sequences of approximating Riemann sums if and only if the following double Riemann integrals exist,

$$\mathscr{R} \iint f(t)f(s)\, d^2R(t, s) \qquad \text{and} \qquad \mathscr{R} \iint f(t)f(s)R(t, s)\, dt\, ds\,,$$

and then $I(f)$ and $J(f)$ are random variables with means zero and variances the corresponding double Riemann integrals.

1.1. *The Hilbert spaces $\Lambda_2(R)$ and $\lambda_2(R)$.* Consider the set $\mathscr{S}_I$ of all step functions on T, $f(t) = \sum_1^N f_n 1_{(a_n, b_n]}(t)$, $(a_n, b_n] \subset T$, and define

$$\int f(t)\, dX_t = \sum_1^N f_n (X_{b_n} - X_{a_n})\,.$$

$\mathscr{S}_I$ is clearly a linear space and for all $f, g \in \mathscr{S}_I$, we have

$$\mathscr{E} \int f(t)\, dX_t = 0$$

$$\mathscr{E}\left(\int f(t)\, dX_t \cdot \int g(t)\, dX_t\right) = \iint f(t)g(s)\, d^2R(t, s)$$

where the double integral is defined in the obvious way. Two step functions f, g will be considered identical if

$$\iint (f(t) - g(t))(f(s) - g(s))\, d^2R(t, s) = 0\,.$$

If we define for $f, g \in \mathscr{S}_I$,

$$\langle f, g \rangle = \iint f(t)g(s)\, d^2R(t, s)$$

then $(\mathscr{S}_I, \langle \cdot, \cdot \rangle)$ is an inner product space. Indeed $\langle f, g\rangle$ has the ordinary bilinear and symmetric properties, $\langle f, f\rangle = \mathscr{E}(\int f\, dX)^2 \geqq 0$, and $\langle f, f \rangle = 0$ only when f is the zero element of $\mathscr{S}_I$ according to the convention introduced above. Now let $\Lambda_2(R)$ be the completion of $\mathscr{S}_I$, so that it is a Hilbert space with inner product denoted again by $\langle \cdot, \cdot \rangle$. A typical element in $\Lambda_2(R)$ is a Cauchy sequence of step functions. However, we will find it convenient to treat elements in $\Lambda_2(R)$ as "formal" functions in $t \in T$ and to write $\iint f(t)g(s)\, d^2 R(t, s)$ for the inner product $\langle f, g \rangle$ (see Theorem 1.1 for a partial justification).

Notice that for $f \in \mathscr{S}_I$ the integral $\int f(t)\, dX_t$ depends on X only through its increments. Thus we may suppose without loss of generality that there is a point $t_0 \in T$ such that $X_{t_0} = 0$ a.s. Under this assumption we can establish an isomorphism between $H(X)$ and $\Lambda_2(R)$ as follows. The map

$$\mathscr{S}_I \to H(X) : f \mapsto \int f\, dX$$

preserves inner products and hence it can be extended to an isomorphism on $\Lambda_2(R)$ to a closed subspace of $H(X)$. But the set $X_t = \int 1_t(u)\, dX_u$, $t \in T$, where $1_t = 1_{(t_0,t]}$ for $t \geqq t_0$ and $= -1_{(t,t_0]}$ for $t < t_0$, generates $H(X)$ and $1_t \in \mathscr{S}_I$. It follows that the isomorphism is onto $H(X)$, i.e., $\Lambda_2(R) \cong H(X)$. We denote this isomorphism by I and we define the integral of $f \in \Lambda_2(R)$ with respect to X (which we write as $\int f(t)\, dX_t$ following our convention to view elements of $\Lambda_2(R)$ as formal functions) by

$$\int f(t)\, dX_t = I(f)\,.$$

The properties of this integral follow from those of I and are the analogues of the properties of the integral when X has orthogonal increments (see, e.g., Doob (1953)). The integral is defined for "functions" in $\Lambda_2(R)$ and thus it is of interest to identify usual functions in $\Lambda_2(R)$ besides the step functions. Two such classes of functions are identified in the following.

Under the additional assumption that $R(t, s)$ is of bounded variation on every bounded domain of $T \times T$, Cramér (1951) defined $\Lambda_2(R)$ as the completion (with respect to the same inner product) of the set $\mathscr{S}_I^*$ of all functions f whose double Riemann integral $\mathscr{R} \int\int f(t)f(s)\, d^2R(t, s)$ exists. However, Cramér's definition is not appropriate for the general case (where R is not necessarily of bounded variation on bounded domains) since then 1_t may not be in $\Lambda_2(R)$ and thus $\Lambda_2(R)$ may not be isomorphic to $H(X)$. It can be shown that when R is of bounded variation on bounded domains then the two definitions of $\Lambda_2(R)$ coincide.

Suppose that R is of bounded variation on every finite domain of $T \times T$. Then it determines uniquely, in the usual way, a σ-finite signed measure on the Borel subsets of $T \times T$, denoted again by R. Let $\mathscr{L}_I$ be the set of all measurable functions f on T such that the following Lebesgue integrals

$$\int\int |f(t)f(s)|\, d^2|R|(t, s) < \infty\,, \qquad \int\int |f(t)|1_{(a,b]}(s)\, d^2|R|(t, s) < \infty$$

are finite for all $(a, b] \subset T$, where $|R|$ is the total variation measure of R. We say that the function f in $\mathscr{L}_I$ represents an element in $\Lambda_2(R)$ if there is an $f' \in \Lambda_2(R)$ such that for all $g \in \mathscr{S}_I$,

$$\langle f', g\rangle = \int\int f(t)g(s)\, d^2R(t, s)\,.$$

Notice that if such an f' exists it is unique since $\mathscr{S}_I$ is dense in $\Lambda_2(R)$. We will then denote f' by f and we will write $f \in \Lambda_2(R)$. With this convention we have the following

THEOREM 1.1. *Let $R(t, s)$ be of bounded variation on every finite domain of $T \times T$. Then $\mathscr{L}_I$ is a dense subset of $\Lambda_2(R)$. Also if $f_1, f_2 \in \mathscr{L}_I$ and $\iint |f_1(t)f_2(s)|\, d^2|R|(t, s) < \infty$ then*

$$\langle f_1, f_2 \rangle = \iint f_1(t) f_2(s)\, d^2R(t, s) .$$

PROOF. Let E be a bounded Borel subset of T. Then $1_E \in \mathscr{L}_I$ and we will prove that $1_E \in \Lambda_2(R)$, i.e., there is an $f \in \Lambda_2(R)$ such that for all $g \in \mathscr{S}_I$,

$$\langle f, g \rangle = \iint 1_E(t) g(s)\, d^2R(t, s) .$$

Let I be a finite interval containing E (so that $|R|(I \times I) < \infty$). We can always find $I_n \subset I$, $n = 1, 2, \cdots$, with each I_n a finite union of half open intervals such that

$$|R|((I_n \Delta E) \times I) \to 0 \qquad \text{as} \quad n \to \infty .$$

Since

$$|R(I_n \times I_m) - R(E \times E)| \leqq |R|((I_n \Delta E) \times E) + |R|((I_m \Delta E) \times E) \to 0 ,$$

it follows that $\langle 1_{I_n}, 1_{I_m} \rangle \to R(E \times E)$ and thus $\{1_{I_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Lambda_2(R)$. Define $f \in \Lambda_2(R)$ by $f = \lim 1_{I_n}$. Similarly for all $g \in \mathscr{S}_I$ we have $\langle 1_{I_n}, g \rangle \to \iint 1_E(t) g(s)\, d^2R(t, s)$ and thus

$$\langle f, g \rangle = \iint 1_E(t) g(s)\, d^2R(t, s) .$$

It follows from the remark preceding the theorem that f is thus uniquely determined by E, independently of the choice of approximating sequence $\{I_n\}$ and of the interval I containing E.

Thus we have shown that $1_E \in \Lambda_2(R)$ for each bounded Borel subset E of T. The rest of the proof is standard (using approximation by simple functions and bounded convergence) and is thus omitted. □

Consider now the set $\mathscr{S}_J$ of all functions f on T such that the Riemann integral $\mathscr{R}\iint f(t)f(s)R(t, s)\, dt\, ds$ exists and is finite. $\mathscr{S}_J$ is a linear space. Two functions f and g in $\mathscr{S}_J$ will be considered identical if

$$\mathscr{R}\iint (f(t) - g(t))(f(s) - g(s))R(t, s)\, dt\, ds = 0 .$$

For $f, g \in \mathscr{S}_J$ we define $\int f(t)X_t\, dt = \mathscr{R}\int f(t)X_t\, dt$ and then we have

$$\mathscr{E}\left(\int f(t)X_t\, dt \cdot \int g(t)X_t\, dt\right) = \mathscr{R}\iint f(t)g(s)R(t, s)\, dt\, ds .$$

Define for $f, g \in \mathscr{S}_J$,

$$\langle f, g \rangle = \mathscr{R}\iint f(t)g(s)R(t, s)\, dt\, ds .$$

Then $(\mathscr{S}_J, \langle \cdot, \cdot \rangle)$ becomes an inner product space. $\lambda_2(R)$ is defined to be the completion of the inner product space $\mathscr{S}_J$ and so it is a Hilbert space. Again a typical element in $\lambda_2(R)$ is a sequence of functions convergent in norm. However formally we shall treat elements in $\lambda_2(R)$ as functions and write $\iint f(t)g(s)R(t, s)\, dt\, ds$ as the inner product $\langle f, g \rangle$.

In order to establish an isomorphism between $H(X)$ and $\lambda_2(R)$ we shall assume that X is mean square continuous which is equivalent to the continuity of the covariance function $R(t, s)$. Consider the sequence of functions $n1_{(\tau-(1/n),\tau]}(t)$ where τ is an interior point of T. It is easy to show that this sequence is a Cauchy sequence in $\lambda_2(R)$, whose limit is denoted by δ_τ, and that

$$X_\tau = \text{l.i.m.} \int n1_{(\tau-(1/n),\tau]}(t)X_t\,dt\,.$$

Then the map

$$\mathscr{S}_J \to H(X): f \mapsto \int f(t)X_t\,dt$$

preserves inner products and its range includes X_τ for all interior τ of T (which is linearly dense in $H(X)$ by mean square continuity). Hence it can be extended to an isomorphism on $\lambda_2(R)$ onto $H(X)$. Thus $\lambda_2(R) \cong H(X)$, the isomorphism is denoted by J and for $f \in \lambda_2(R)$ we define

$$\int f(t)X_t\,dt = J(f)\,.$$

A useful connection between the integrals I and J and the spaces Λ_2 and λ_2 can be established as follows. Let $Z_t = \int_{t_0}^t X_u\,du = J(1_{(t_0,t]})$ where t_0 is an arbitrary but fixed point in T. ($1_{(t_0,t]} \in \lambda_2(R)$ since R is continuous.) Then

$$\Gamma(t, s) = \mathscr{E}Z_t Z_s = \int_{t_0}^t \int_{t_0}^s R(u, v)\,du\,dv\,,$$

and we have the following result whose straightforward proof is omitted.

THEOREM 1.2. *If X is mean square continuous then $\lambda_2(R) = \Lambda_2(\Gamma)$ and for all $f \in \lambda_2(R) = \Lambda_2(\Gamma)$,*

$$\int f(t)X_t\,dt = \int f(t)\,dZ_t\,.$$

Hence $H(X) = H(Z)$.

$\lambda_2(R)$ may contain interesting classes of functions larger than $\mathscr{S}_J$. Let $\mathscr{L}_J$ be the set of all measurable functions f on T such that the following Lebesgue integrals

$$\iint |f(t)f(s)R(t, s)|\,dt\,ds < \infty\,, \qquad \iint |f(t)|1_{(a,b]}(s)|R(t, s)|\,dt\,ds < \infty$$

are finite for all $(a, b] \subset T$. We will follow the same convention (as for Λ_2) in treating functions f in $\mathscr{L}_J$ as elements of $\lambda_2(R)$ if there is a $f' \in \lambda_2(R)$ such that for all g in a dense subset of $\lambda_2(R)$,

$$\langle f', g\rangle = \iint f(t)g(s)R(t, s)\,dt\,ds\,.$$

With this convention the following is a corollary of Theorems 1.1 and 1.2.

COROLLARY 1.3. *Let $R(t, s)$ be continuous on $T \times T$. Then $\mathscr{L}_J$ is a dense subset of $\lambda_2(R)$. Also if $f_1, f_2 \in \mathscr{L}_J$ and $\iint |f_1(t)f_2(s)R(t, s)|\,dt\,ds < \infty$, then*

$$\langle f_1, f_2\rangle = \iint f_1(t)f_2(s)R(t, s)\,dt\,ds\,.$$

The spaces $\Lambda_2(R)$ and $\lambda_2(R)$ are generalizations of L_2 spaces, and in general they are larger than L_2 spaces. As an example, consider $R(t, s)$ a continuous covariance function on $[a, b] \times [a, b]$ and let $\Gamma(t, s)$ be defined as before. Every function f in $L_2([a, b], dt)$ belongs to $\lambda_2(R) = \Lambda_2(\Gamma)$ by Corollary 1.3 since

$$\int\int |f(t)f(s)R(t, s)|\, dt\, ds \leqq \max |R(t, s)| \cdot |b - a| \int f^2(t)\, dt < \infty$$

and similarly $\int\int |f(t)| 1_{(a,b]}(s) |R(t, s)|\, dt\, ds < \infty$. However, $\delta_t \in \lambda_2(R) = \Lambda_2(\Gamma)$ is not in $L_2([a, b], dt)$ since $X_t = J(\delta_t)$ for all interior points t of T implies $R(t, s) = \int\int \delta_t(u)\delta_s(v)R(u, v)\, du\, dv$.

Nevertheless, there is a special case where $\Lambda_2(R)$ reduces to an L_2 space. Let X be a zero mean process with orthogonal increments. Assume $X_{t_0} = 0$ a.s. for some fixed $t_0 \in T$. Then, $R(t, s) = F(t_0 \vee (t \wedge s)) + F(t_0 \wedge (t \vee s))$ where $F(t) = \mathscr{E}X_t^2$ if $t \geqq t_0$ and $= -\mathscr{E}X_t^2$ if $t \leqq t_0$. F is nondecreasing and thus $R(t, s)$ is of bounded variation on every finite domain of $T \times T$, and the associated measure concentrates on the diagonal $t = s$ of $T \times T$. In this case $\Lambda_2(R) = L_2(T, dF(t))$. In particular, if X is the Wiener process $\Lambda_2(R) = L_2(T, dt)$. A slightly weaker result is easily seen to be valid when $R(t, s) = \int_0^t \int_0^s k(u, v)\, du\, dv + t \wedge s$, with $k \in L_2(T \times T)$; in this case the two sets (rather than spaces) are equal, $\Lambda_2(R) = L_2(T, dt)$, and their norms are equivalent. In fact it may be shown that if R and S are two equivalent covariance functions (i.e., if the associated zero mean Gaussian measures are equivalent) then the following sets are equal, $\Lambda_2(R) = \Lambda_2(S)$ and $\lambda_2(R) = \lambda_2(S)$, and their norms are equivalent.

1.2. *Tensor products of* $\Lambda_2(R)$ *and* $\lambda_2(R)$. We now study the tensor product spaces $\otimes^p \Lambda_2(R)$ and $\otimes^p \lambda_2(R)$, $p = 2, 3, \cdots$. Consider the set $\mathscr{S}_I^p$ of all step functions $f(\mathbf{t})$ on T^p, $\mathbf{t} = (t_1, \cdots, t_p)$. Define the following function on $\mathscr{S}_I^p \times \mathscr{S}_I^p$,

$$\langle f, g\rangle = \int\int f(\mathbf{t})g(\mathbf{s})\, d^{2p}R(\mathbf{t}, \mathbf{s}),$$

where we write $d^{2p}R(\mathbf{t}, \mathbf{s})$ for $d^2R(t_1, s_1) \cdots d^2R(t_p, s_p)$, and identify f with g if $\langle f - g, f - g\rangle = 0$. Let $1_{I_1 \times \cdots \times I_p}, 1_{J_1 \times \cdots \times J_p} \in \mathscr{S}_I^p$ (i.e., I_i, J_j are bounded half open intervals in T). Then

$$\begin{aligned} \langle 1_{I_1 \times \cdots \times I_p}, 1_{J_1 \times \cdots \times J_p}\rangle &= \langle 1_{I_1}, 1_{J_1}\rangle_{\Lambda_2(R)} \cdots \langle 1_{I_p}, 1_{J_p}\rangle_{\Lambda_2(R)} \\ &= \langle 1_{I_1} \otimes \cdots \otimes 1_{I_p}, 1_{J_1} \otimes \cdots \otimes 1_{J_p}\rangle_{\otimes^p \Lambda_2(R)}. \end{aligned}$$

This implies that $(\mathscr{S}_I^p, \langle \cdot, \cdot\rangle)$ is an inner product space and we shall denote by $\Lambda_2(\otimes^p R)$ the completion of $\mathscr{S}_I^p$. Since $\{1_{I_1} \otimes \cdots \otimes 1_{I_p}\}$ is a complete set in $\otimes^p \Lambda_2(R)$, we have $\Lambda_2(\otimes^p R) \cong \otimes^p \Lambda_2(R)$.

$\lambda_2(\otimes^p R)$ can be defined in a similar manner. Let $\mathscr{S}_J^p$ be the set of functions of the form $f(t_1, \cdots, t_p) = \sum_{k=1}^N \phi_1^{(k)}(t_1) \cdots \phi_p^{(k)}(t_p)$ where the ϕ's belong to $\mathscr{S}_J$. $\mathscr{S}_J^p$ is a linear space. Define on $\mathscr{S}_J^p \times \mathscr{S}_J^p$ the function

$$\langle f, g\rangle = \mathscr{R}\int\int f(\mathbf{t})g(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})\, d\mathbf{t}\, d\mathbf{s},$$

where $R^p(\mathbf{t}, \mathbf{s}) = R(t_1, s_1) \cdots R(t_p, s_p)$, and identify f with g if $\langle f - g, f - g\rangle = 0$. With the observation that for $\phi_i, \psi_j \in \mathscr{S}_J$,

$$\begin{aligned} \langle \phi_1 \cdots \phi_p, \psi_1 \cdots \psi_p\rangle &= \langle \phi_1, \psi_1\rangle_{\lambda_2(R)} \cdots \langle \phi_p, \psi_p\rangle_{\lambda_2(R)} \\ &= \langle \phi_1 \otimes \cdots \otimes \phi_p, \psi_1 \otimes \cdots \otimes \psi_p\rangle_{\otimes^p \lambda_2(R)}, \end{aligned}$$

and with the fact that $\{\phi_1 \otimes \cdots \otimes \phi_p\}$ is a complete set in $\otimes^p \lambda_2(R)$, we conclude that $(\mathscr{S}_J{}^p, \langle \cdot, \cdot \rangle)$ is an inner product space and that its completion, which is denoted by $\lambda_2(\otimes^p R)$, is isomorphic to $\otimes^p \lambda_2(R)$.

As in the case of the spaces $\Lambda_2(R)$ and $\lambda_2(R)$ we will treat elements of $\Lambda_2(\otimes^p R)$ and $\lambda_2(\otimes^p R)$ as "formal" functions and we will write the inner products in a formal integral form. As before, under some conditions, elements of $\Lambda_2(\otimes^p R)$ and $\lambda_2(\otimes^p R)$ will be representable by functions on T^p in the corresponding sense and in this case we will identify the elements of Λ_2 and λ_2 with the functions (see Theorem 1.4 and Corollary 1.6). The important point here is that we have identified the abstract tensor product spaces $\otimes^p \Lambda_2(R)$ and $\otimes^p \lambda_2(R)$ with the (nearly) function spaces $\Lambda_2(\otimes^p R)$ and $\lambda_2(\otimes^p R)$. From now on we will make no distinction between $\otimes^p \Lambda_2(R)$ and $\Lambda_2(\otimes^p R)$, and between $\otimes^p \lambda_2(R)$ and $\lambda_2(\otimes^p R)$.

Let R be of bounded variation on every finite domain of $T \times T$ and let $\mathscr{L}_I{}^p$ be the set of all measurable functions f on T^p such that the following Lebesgue integrals

$$\iint |f(\mathbf{t})f(\mathbf{s})|\, d^{2p}|R|(\mathbf{t}, \mathbf{s}) < \infty , \qquad \iint |f(\mathbf{t})| 1_I(\mathbf{s})\, d^{2p}|R|(\mathbf{t}, \mathbf{s}) < \infty$$

are finite for all $I = I_1 \times \cdots \times I_p$ with $I_1, \cdots, I_p \subset T$ bounded half open intervals. The following theorem can be proven like Theorem 1.1 and thus its proof is omitted.

THEOREM 1.4. *Let $R(t, s)$ be of bounded variation on every finite domain of $T \times T$. Then $\mathscr{L}_I{}^p$ is a dense subset of $\Lambda_2(\otimes^p R)$. Also if $f_1, f_2 \in \mathscr{L}_I{}^p$ and $\iint |f_1(\mathbf{t})f_2(\mathbf{s})|\, d^{2p}|R|(\mathbf{t}, \mathbf{s}) < \infty$, then*

$$\langle f_1, f_2 \rangle = \iint f_1(\mathbf{t})f_2(\mathbf{s})\, d^{2p}R(\mathbf{t}, \mathbf{s}) .$$

THEOREM 1.5. *If $R(t, s)$ is continuous on $T \times T$, then $\lambda_2(\otimes^p R) = \Lambda_2(\otimes^p \Gamma)$.*

PROOF. This follows immediately from the facts that the set $\{\phi_1 \otimes \cdots \otimes \phi_p, \phi_i \in \mathscr{S}_J\}$ is complete in both $\lambda_2(\otimes^p R)$ and $\Lambda_2(\otimes^p \Gamma)$, and that the two inner products are identical on this set. □

Let $\mathscr{L}_J{}^p$ be the set of all measurable functions f on T^p such that the following Lebesgue integrals

$$\iint |f(\mathbf{t})f(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})|\, d\mathbf{t}\, d\mathbf{s} < \infty , \qquad \iint |f(\mathbf{t})| 1_I(\mathbf{s}) |R^p(\mathbf{t}, \mathbf{s})|\, d\mathbf{t}\, d\mathbf{s} < \infty$$

are finite for all $I = I_1 \times \cdots \times I_p$ with bounded half open intervals $I_i \subset T$. With the usual corresponding convention the following is a corollary of Theorems 1.4 and 1.5.

COROLLARY 1.6. *Let $R(t, s)$ be continuous on $T \times T$. Then $\mathscr{L}_J{}^p$ is a dense subset of $\lambda_2(\otimes^p R)$. Also if $f_1, f_2 \in \mathscr{L}_J{}^p$ and $\iint |f_1(\mathbf{t})f_2(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})|\, d\mathbf{t}\, d\mathbf{s} < \infty$, then*

$$\langle f_1, f_2 \rangle = \iint f_1(\mathbf{t})f_2(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})\, d\mathbf{t}\, d\mathbf{s} .$$

Finally let us consider the symmetric tensor products $\tilde{\otimes}^p \Lambda_2(R)$ and $\tilde{\otimes}^p \lambda_2(R)$.

For $f \in \Lambda_2(\otimes^p R)$ define $\tilde{f}(t_1, \cdots, t_p) = (p!)^{-1} \sum_\pi f(t_{\pi_1}, \cdots, t_{\pi_p})$ where the sum is over all permutations $\pi = (\pi_1, \cdots, \pi_p)$ of $(1, \cdots, p)$, and $\tilde{f}$ is called the symmetric version of f. $\tilde{f}$ is well defined since f is a "function." Indeed, $\tilde{f}$ is first defined for $f \in \mathscr{S}_I{}^p$, and then, using the easily verified fact that $\|\tilde{f}\|^2_{\Lambda_2(\otimes^p R)} \leqq p!\|f\|^2_{\Lambda_2(\otimes^p R)}$, the definition is extended by continuity to $\Lambda_2(\otimes^p R)$. If $f = \tilde{f}$ then f is said to be a symmetric "function." Let $\Lambda_2(\tilde{\otimes}^p R)$ be the subspace of all symmetric "functions" in $\Lambda_2(\otimes^p R)$. Then it is easy to show that $\Lambda_2(\tilde{\otimes}^p R)$ is a Hilbert space and $\tilde{\otimes}^p \Lambda_2(R) \cong \Lambda_2(\tilde{\otimes}^p R)$ under the correspondence $f_1 \tilde{\otimes} \cdots \tilde{\otimes} f_p \leftrightarrow (f_1(t_1) \cdots f_p(t_p))^\sim$. Similarly, let $\lambda_2(\tilde{\otimes}^p R)$ be the subspace of all symmetric "functions" in $\lambda_2(\otimes^p R)$. Then we can show that $\tilde{\otimes}^p \lambda_2(R) \cong \lambda_2(\tilde{\otimes}^p R)$ (under the natural correspondence). As before, we shall hereon identify $\tilde{\otimes}^p \Lambda_2(R)$ with $\Lambda_2(\tilde{\otimes}^p R)$, and $\tilde{\otimes}^p \lambda_2(R)$ with $\lambda_2(\tilde{\otimes}^p R)$.

1.3. *Fourier transform on $\Lambda_2(\otimes^p R)$ and $\lambda_2(\otimes^p R)$.* Consider the covariance function $R(t, s)$ of a zero mean, mean square continuous process $X = (X_t, t \in \mathbb{R} = (-\infty, \infty))$ with (wide sense) stationary increments. For convenience such R is said to have stationary increments. Let

$$R(t_1, s_1; t_2, s_2) = \mathscr{E}(X_{t_1} - X_{s_1})(X_{t_2} - X_{s_2}) . \tag{1.1}$$

Then it is well known (Doob (1953), page 552) that

$$R(t_1, s_1; t_2, s_2) = \int_{-\infty}^{\infty} (e^{it_1\lambda} - e^{is_1\lambda})(e^{-it_2\lambda} - e^{-is_2\lambda}) \frac{1 + \lambda^2}{\lambda^2} \, dF(\lambda) \tag{1.2}$$

$$X_t - X_s = \int_{-\infty}^{\infty} (e^{it\lambda} - e^{is\lambda}) \frac{(1 + \lambda^2)^{\frac{1}{2}}}{i\lambda} \, dV_\lambda \tag{1.3}$$

where $dF(\lambda)$ is a finite measure on $\mathscr{B}(\mathbb{R})$ and $V = \{V_\lambda, -\infty < \lambda < \infty\}$ is a process with orthogonal increments, $\mathscr{E}|dV_\lambda|^2 = dF(\lambda)$, and $H(\Delta X) = H(\Delta V)$, where ΔX denotes the set of increments of the process X.

Define the Fourier transform of $f \in \mathscr{S}_I{}^p$ by

$$\hat{f}(\boldsymbol{\lambda}) = \int_{-\infty}^{\infty} e^{i\mathbf{t}\boldsymbol{\lambda}} f(\mathbf{t}) \, d\mathbf{t} .$$

The program is to define the Fourier transform $\hat{f}$ of every f in $\Lambda_2(\otimes^p R)$. (For this reason, it is convenient to extend $\Lambda_2(\otimes^p R)$ from a real Hilbert space to a complex Hilbert space.) From (1.2) it follows easily that $\hat{f} \in L_2(\mathbb{R}^p, \mu^p)$, for $f \in \mathscr{S}_I{}^p$, where the measure μ is defined by $d\mu(\lambda) = (1 + \lambda^2) \, dF(\lambda)$, and $\langle f, g \rangle_{\Lambda_2(\otimes^p R)} = \langle \hat{f}, \hat{g} \rangle_{L_2(\mathbb{R}^p, \mu^p)}$. Since the map $\mathscr{F}: \mathscr{S}_I{}^p \to L_2(\mathbb{R}^p, \mu^p) : f \mapsto \hat{f}$ is linear and preserves inner products, it can be extended to an isomorphism on $\Lambda_2(\otimes^p R)$. We now show that $\mathscr{F}$ is onto $L_2(\mathbb{R}^p, \mu^p)$. It is sufficient to show that $\hat{1}_{(\mathbf{a},\mathbf{b}]}(\boldsymbol{\lambda}) = \prod_{i=1}^p \hat{1}_{(a_i,b_i]}(\lambda_i)$ form a complete set in $L_2(\mathbb{R}^p, \mu^p)$; or equivalently that $\hat{1}_{(a,b]}(\lambda) = (e^{ib\lambda} - e^{ia\lambda})/(i\lambda)$ form a complete set in $L_2(\mathbb{R}, \mu)$. Since $L_2(\mathbb{R}, dF) \cong H(\Delta V)$ under the correspondence $h \leftrightarrow \int h \, dV$, it follows from (1.3) and $H(\Delta X) = H(\Delta V)$ that $\{(e^{it\lambda} - e^{is\lambda})(1 + \lambda^2)^{\frac{1}{2}}/(i\lambda), s < t\}$ is complete in $L_2(\mathbb{R}, dF)$, and hence $\{\hat{1}_{(a,b]}, a < b\}$ is complete in $L_2(\mathbb{R}, \mu)$. We thus have the following

THEOREM 1.7. *The map* $\mathscr{F}: \mathscr{S}_I^p \to L_2(\mathbb{R}^p, \mu^p): f \mapsto \hat{f}$ *has a unique extension to an isomorphism from* $\Lambda_2(\otimes^p R)$ *onto* $L_2(\mathbb{R}^p, \mu^p)$.

The extended map is again denoted by $\mathscr{F}$ and is called the *Fourier transform* on $\Lambda_2(\otimes^p R)$. We also write $\hat{f}$ for $\mathscr{F}(f)$. If we let X be the Wiener process, i.e., $R(t, s) = t \wedge s$, then $\Lambda_2(\otimes^p R) = L_2(\mathbb{R}^p, dt^p)$, $d\mu(\lambda) = (1/2\pi)\, d\lambda$. Therefore $\mathscr{F}$ reduces to the ordinary Fourier transform on $L_2(\mathbb{R}^p, dt^p)$.

Suppose now that $R(t, s)$ is stationary (which implies that R has stationary increments). Then by Bochner's theorem we have

$$R(t, s) = \int e^{i(t-s)\lambda}\, d\nu(\lambda) \tag{1.4}$$

with ν a finite measure on $\mathscr{B}(\mathbb{R})$. It is plain to deduce from (1.2) and (1.4) that $d\nu(\lambda) = \lambda^{-2}\, d\mu(\lambda)$. Thus $\Lambda_2(\otimes^p R) \cong_{\mathscr{F}} L_2(\mathbb{R}^p, d^p(\lambda^2\nu(\lambda)))$. Now we define the Fourier transform on $\lambda_2(\otimes^p R)$. Let $\Gamma(t, s) = \int_0^t \int_0^s R(u, v)\, du\, dv$. Then the covariance Γ has stationary increments and $\Lambda_2(\otimes^p \Gamma) = \lambda_2(\otimes^p R)$. The Fourier transform on $\lambda_2(\otimes^p R)$ is defined to be the Fourier transform $\mathscr{F}$ on $\Lambda_2(\otimes^p \Gamma)$. It is a simple matter to verify that the spectral measure of Γ is $(1 + \lambda^2)^{-1}\, d\nu(\lambda)$ and thus $\lambda_2(\otimes^p R) \cong_{\mathscr{F}} L_2(\mathbb{R}^p, \nu^p)$.

THEOREM 1.8. *If* $R(t, s)$ *is a continuous stationary covariance function, then* $L_1(\mathbb{R}^p)$ *is a dense subspace of* $\lambda_2(\otimes^p R)$, *and* $\mathscr{F}$ *restricted to* $L_1(\mathbb{R}^p)$ *is the ordinary Fourier transform.*

PROOF. Let $f \in L_1(\mathbb{R}^p)$. Then

$$\iint |f(\mathbf{t})\bar{f}(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})|\, d\mathbf{t}\, d\mathbf{s} \leqq \nu(\mathbb{R})^p \|f\|^2_{L_1(\mathbb{R}^p)} < \infty$$

since $|R(t, s)| \leqq \nu(\mathbb{R})$. Similarly, the second condition in the definition of $\mathscr{L}_J^p$ is verified. Thus $f \in \lambda_2(\otimes^p R)$ by Corollary 1.6. Since $\mathscr{S}_I^p \subset L_1(\mathbb{R}^p)$ is dense in $\Lambda_2(\otimes^p \Gamma) = \lambda_2(\otimes^p R)$, it follows that $L_1(\mathbb{R}^p)$ is dense in $\lambda_2(\otimes^p R)$.

To prove the second assertion it is sufficient to prove that for $f \in L_1(\mathbb{R}^p)$ the ordinary Fourier transform $\hat{f}$ belongs to $L_2(\mathbb{R}^p, \nu^p)$ and $\|\hat{f}\|^2_{L^2(\mathbb{R}^p, \nu^p)} = \|f\|_{\lambda_2(\otimes^p R)}$. But $|\hat{f}| \leqq \|f\|_{L_1(\mathbb{R}^p)}$ implies $\hat{f} \in L_2(\mathbb{R}^p, \nu^p)$. We have from Corollary 1.6 that $\|f\|^2_{\lambda_2(\otimes^p R)} = \iint f(\mathbf{t})\bar{f}(\mathbf{s})R^p(\mathbf{t}, \mathbf{s})\, d\mathbf{t}\, d\mathbf{s}$. Substituting R and interchanging the order of integration by Fubini's theorem, we obtain $\|f\|^2_{\lambda_2(\otimes^p R)} = \|\hat{f}\|^2_{L_2(\mathbb{R}^p, \nu^p)}$. The proof is now complete. □

Let R be again a covariance function having stationary increments and let $f \in \Lambda_2(\otimes^p R)$. We define the translation f^τ of f by $\boldsymbol{\tau} \in R^p$ as follows. Pick a sequence of step functions ϕ_n such that $\lim \phi_n = f$ in $\Lambda_2(\otimes^p R)$ and let $\phi_n^\tau(\mathbf{t}) = \phi_n(\mathbf{t} + \boldsymbol{\tau})$. Clearly $\|\phi_n^\tau\|^2_{\Lambda_2(\otimes^p R)} = \|\phi_n\|^2_{\Lambda_2(\otimes^p R)}$. This implies that $\{\phi_n^\tau\}$ is a Cauchy sequence in $\Lambda_2(\otimes^p R)$, and f^τ is defined to be $\lim \phi_n^\tau$. A simple argument shows that the definition of f^τ does not depend on the choice of the approximating sequence $\{\phi_n\}$ and f becomes the usual translation if f is indeed a function. When R is stationary, the translation f^τ of $f \in \lambda_2(\otimes^p R)$ can be defined similarly (or via the identity $\lambda_2(\otimes^p R) = \Lambda_2(\otimes^p \Gamma)$).

THEOREM 1.9. *If R is a continuous covariance with stationary increments, then $\Lambda_2(\otimes^p R)$ is invariant under translations, and for $f, g \in \Lambda_2(\otimes^p R)$ we have*

$$\langle f^\tau, g^\sigma\rangle_{\Lambda_2(\otimes^p R)} = \langle f, g_{\tau-\sigma}\rangle_{\Lambda_2(\otimes^p R)}, \qquad \hat{f}^\tau(\boldsymbol{\lambda}) = e^{-i\tau\lambda}\hat{f}(\boldsymbol{\lambda}).$$

PROOF. Since R has stationary increments, all these assertions hold for $f \in \mathscr{S}_I^p$. Hence they hold for all $f \in \Lambda_2(\otimes^p R)$ by continuity (cf. Theorem 1.7). □

The corresponding theorem for translations on $\lambda_2(\otimes^p R)$ also holds.

2. Multiple Wiener integrals. We shall define the multiple Wiener integrals (MWI's) of the following two types:

$$I_p(f) = \int \cdots \int f(t_1, \cdots, t_p)\, dX_{t_1} \cdots dX_{t_p} = \int f(\mathbf{t})\, dX_{\mathbf{t}}^p$$
$$J_p(f) = \int \cdots \int f(t_1, \cdots, t_p) X_{t_1} \cdots X_{t_p}\, dt_1 \cdots dt_p = \int f(\mathbf{t}) X_{\mathbf{t}}^p\, d\mathbf{t}$$

where $p = 1, 2, \cdots$. While dealing with integrals I_p, resp. J_p, we will assume (I), resp. (J),

$$(I): X_{t_0} = 0 \quad \text{a.s.} \qquad \text{for some} \quad t_0 \in T,$$
$$(J): X \quad \text{is mean square continuous.}$$

For X a Wiener process f is taken to be a function in $L_2(T^p, dt^p)$ and $(p!)^{-\frac{1}{2}} I_p$ is an isomorphism on $\tilde{L}_2(T^p, dt^p)$ (the Hilbert space of all symmetric functions in $L_2(T^p, dt^p)$) into $L_2(X)$. The major step in generalizing the notion of the MWI I_p to a Gaussian process other than the Wiener process is to determine a proper Hilbert space of functions on which I_p will be defined. Clearly I_1 should be defined as the isomorphism I from $\Lambda_2(R)$ onto $H(X)$. Now for $p > 1$, in accordance with the Wiener process case, it is reasonable to expect that functions $f(t_1, \cdots, t_p)$ of the form $\phi_1(t_1) \cdots \phi_p(t_p)$, $\phi_i \in \Lambda_2(R)$, are admissible integrands, and their integral $I_p(f)$ is the iterated integral $I(\phi_1) \cdots I(\phi_p)$ when $\phi_1, \cdots, \phi_p$ are orthogonal. This suggests that $\Lambda_2(\otimes^p R)$ is the proper class of integrands for the MWI I_p; and similarly $\lambda_2(\otimes^p R)$ is the proper class of integrands for the MWI J_p. In defining the MWI's we will use the following result on the structure of the nonlinear space of a Gaussian process X (see Kakutani (1961), Neveu (1968), Kallianpur (1970)). Here H_{p,σ^2} is the Hermite polynomial of degree $p = 0, 1, 2, \cdots$ with parameter σ^2 defined as follows: $\{H_{p,\sigma^2}(X), p = 0, 1, 2, \cdots\}$ is obtained by applying the Gram–Schmidt procedure to orthogonalize the sequence of rv's $\{X^p, p = 0, 1, 2, \cdots\}$ in $L_2(X)$ where X is a Gaussian variable with mean 0 and variance σ^2.

There exists a unique isomorphism Φ from $\bigoplus_{p\geq 0} H^{\tilde{\otimes} p}(X)$ (where the space for $p = 0$ is the set of all constant rv's *in $L_2(X)$) onto $L_2(X)$ such that*

$$\Phi(e^{\tilde{\otimes}\xi}) = e^{\xi - \frac{1}{2}\mathscr{E}\xi^2}$$

where $e^{\tilde{\otimes}\xi} = \sum_{p\geq 0} (p!)^{-\frac{1}{2}} \xi^{\tilde{\otimes} p}$, $\xi \in H(X)$. If $\xi_1, \cdots, \xi_k \in H(X)$ are orthogonal then

$$\Phi(\xi_1^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \xi_k^{\tilde{\otimes} p_k}) = (p!)^{-\frac{1}{2}} \prod_{j=1}^k H_{p_j, \mathscr{E}\xi_j^2}(\xi_j)$$

where $p = p_1 + \cdots + p_k$. If $\{\xi_\gamma, \gamma \in \Gamma\}$ (Γ linearly ordered) is a CONS *in $H(X)$*

then the family

$$\left(\frac{p!}{p_{\gamma_1}! \cdots p_{\gamma_k}!}\right)^{\frac{1}{2}} \Phi(\xi_{\gamma_1}^{\tilde{\otimes} p_{\gamma_1}} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_{\gamma_k}}) = \prod_{j=1}^{k} (p_{\gamma_j}!)^{-\frac{1}{2}} H_{p_{\gamma_j}, \mathscr{E}\xi_{\gamma_j}^2}(\xi_{\gamma_j}),$$

$p \geqq 0$, $k \geqq 1$, $p_{\gamma_1} + \cdots + p_{\gamma_k} = p$, $\gamma_1 < \cdots < \gamma_k$, *is a* CONS *in* $L_2(X)$.

We now define I_p, $p \geqq 1$. Since $\Lambda_2(R)$ is isomorphic to $H(X)$ under the isomorphism $I: f \mapsto \int f\,dX$, $\Lambda_2(\tilde{\otimes}^p R) = \tilde{\otimes}^p \Lambda_2(R)$ is isomorphic to $H^{\tilde{\otimes} p}(X)$. Denote this isomorphism by $I^{\tilde{\otimes} p}$. For $\phi_1, \cdots, \phi_p$ orthogonal in $\Lambda_2(R)$ we have

$$\begin{aligned}(\Phi \circ I^{\tilde{\otimes} p})(\phi_1 \tilde{\otimes} \cdots \tilde{\otimes} \phi_p) &= \Phi(\textstyle\int \phi_1\,dX \tilde{\otimes} \cdots \tilde{\otimes} \int \phi_p\,dX) \\ &= (p!)^{-\frac{1}{2}} \textstyle\int \phi_1\,dX \cdots \int \phi_p\,dX,\end{aligned}$$

which suggests the following definition of $I_p: \Lambda_2(\tilde{\otimes}^p R) \to L_2(X)$ (in fact onto $\Phi(H^{\tilde{\otimes} p}(X))$),

$$I_p = (p!)^{\frac{1}{2}} \Phi \circ I^{\tilde{\otimes} p}.$$

Furthermore we define $I_p(f) = I_p(\tilde{f})$ for $f \in \Lambda_2(\otimes^p R)$, where $\tilde{f}$ is the symmetric tensor of f. The following results are then immediate consequences of the fact that $I^{\tilde{\otimes} p}$ is an isomorphism.

THEOREM 2.1. *Let X be a zero mean Gaussian process satisfying (I). Then the* MWI's I_p, $p \geqq 1$, *have the following properties* ($f, g \in \Lambda_2(\otimes^p R)$)

$$\begin{aligned} I_p(af + bg) &= aI_p(f) + bI_p(g), \quad a, b \in \mathbb{R}, \\ I_p(f) &= I_p(\tilde{f}), \\ \langle I_p(f), I_p(g)\rangle_{L_2(X)} &= p!\,\langle \tilde{f}, \tilde{g}\rangle_{\Lambda_2(\otimes^p R)}, \\ \langle I_p(f), I_q(g)\rangle_{L_2(X)} &= 0 \quad \text{if } p \neq q, \\ I_p(\phi_1^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \phi_k^{\tilde{\otimes} p_k}) &= \prod_{j=1}^{k} H_{p_j, \|\phi_j\|^2}(\textstyle\int \phi_j\,dX),\end{aligned}$$

where $\{\phi_1, \cdots, \phi_k\}$ *is an orthogonal set in* $\Lambda_2(R)$ *and* $p_1 + \cdots + p_k = p$. *Also every L_2-functional θ of X, $\theta \in L_2(X)$, has an orthogonal development*

$$\theta = \mathscr{E}(\theta) + \textstyle\sum_{p \geq 1} I_p(f_p), \quad f_p \in \Lambda_2(\otimes^p R),$$

and if $\theta - \mathscr{E}(\theta) = \sum_{p \geq 1} I_p(f_p) = \sum_{p \geq 1} I_p(g_p)$ *then* $\tilde{f}_p = \tilde{g}_p$, $p \geqq 1$.

In exactly the same way we can define $J_p(f)$ for $f \in \lambda_2(\otimes^p R)$, and $(p!)^{-\frac{1}{2}} J_p$ restricted to $\lambda_2(\tilde{\otimes}^p R)$ is an isomorphism onto $\Phi(H^{\tilde{\otimes} p}(X))$. The corresponding Theorem 2.1 also holds for the MWI's J_p, $p \geqq 1$. Both MWI's I_p and J_p can be evaluated from the sample paths of X, but we will not discuss this here.

3. Stochastic integrals. In this section we define integrals of the form $\int f(t)\,dX_t$ with $f(t)$ a stochastic process appropriately defined. We first generalize the notion of Λ_2 spaces and define an integral denoted by $\int f(t) \otimes dX_t$. The details are omitted since the argument is analogous to that in Section 1.1.

Let H and K be Hilbert spaces. Let X_t be an K-valued function on an interval T, and let $R(t, s) = \langle X_t, X_s\rangle_K$. Then R is a nonnegative definite function (i.e., a covariance function). Consider the set $\mathscr{S}_{I;H}$ of all H-valued step functions on

T, $f(t) = \sum_1^N f_n 1_{(a_n, b_n]}$, $(a_n, b_n] \subset T$, $f_n \in H$. $\mathscr{S}_{I;H}$ equipped with the binary function

$$\langle f, g \rangle = \iint \langle f(t), g(s) \rangle_H \, d^2R(t, s)$$

is an inner product space. The Hilbert space $\Lambda_{2;H}(R)$ is defined to be the completion of $\mathscr{S}_{I;H}$. Note that $\Lambda_{2;H}(R)$ reduces to $\Lambda_2(R)$ when $H = \mathbb{R}$.

For $f = \sum_1^N f_n 1_{(a_n, b_n]} \in \mathscr{S}_{I;H}$, define

$$\int f(t) \otimes dX_t = \sum_1^N f_n \otimes (X_{b_n} - X_{a_n}) . \tag{3.1}$$

Then

$$I_\otimes : \mathscr{S}_{I;H} \to H \otimes K : f \mapsto \int f \otimes dX \tag{3.2}$$

is a norm-preserving linear map since

$$\begin{aligned} \|\textstyle\int f \otimes dX\|^2_{H \otimes K} &= \textstyle\sum_1^N \sum_1^N \langle f_n, f_m \rangle_H \langle X_{b_n} - X_{a_n}, X_{b_m} - X_{a_m} \rangle_K \\ &= \textstyle\iint \langle f(t), f(s) \rangle_H \, d^2R(t, s) \\ &= \|f\|^2_{\Lambda_{2;H}(R)} . \end{aligned}$$

Thus $I_\otimes$ has a unique extension to an isomorphism on $\Lambda_{2;H}(R)$ into $H \otimes K$. It is clear that the range of $I_\otimes$ is $H \otimes H(\Delta X)$, where $H(\Delta X)$ denotes the closed subspace of K spanned by the increments of X.

We remark that Theorem 1.1 is valid for the present general case with the proviso that one should read absolute values and usual products as norms and inner products respectively. Also, if the K-valued function X has orthogonal increments and $dF(t) = \|dX_t\|_K^2$, then $\Lambda_{2;H}(R) = L_{2;H}(dF)$, the Hilbert space of all dF-square integrable K-valued functions (for integration of K-valued functions see, e.g., Lang (1969)). The following simple fact will be used in the sequel.

LEMMA 3.1. *If G_p, $p \geqq 0$, are closed subspaces of H and $H = \bigoplus_{p \geqq 0} G_p$, then*

$$H \otimes K = \textstyle\bigoplus_{p \geqq 0} (G_p \otimes K) , \tag{3.3}$$

$$\Lambda_{2;H}(R) = \textstyle\bigoplus_{p \geqq 0} \Lambda_{2;G_p}(R) . \tag{3.4}$$

PROOF. (3.3) is clear, and (3.4) follows from the following facts $\mathscr{S}_{I;G_p} \subset \mathscr{S}_{I;H}$, $\mathscr{S}_{I;G_p} \perp \mathscr{S}_{I;G_q}$ for $p \neq q$, and $\mathscr{S}_{I;H} = \overline{\bigcup_{p \geqq 0} \mathscr{S}_{I;G_p}}$. □

Now let $X = \{X_t, t \in T\}$ be a zero mean Gaussian process with covariance R, and assume as in Section 2 that $X_{t_0} = 0$ a.s. for some $t_0 \in T$. Let $H = L_2(X)$ and $K = H(X) (= H(\Delta X))$. Then for $f \in \Lambda_{2;L_2(X)}(R)$ the integral $I_\otimes(f) = \int f(t) \otimes dX_t$ is defined and belongs to $L_2(X) \otimes H(X)$. The program is to identify as many elements in $L_2(X) \otimes H(X)$ as possible with elements in $L_2(X)$ through a suitable (unbounded) linear map Ψ and then define the stochastic integral of f with respect to X by

$$\mathscr{I}(f) = \int f(t) \, dX_t = \Psi\left(\int f(t) \otimes dX_t\right) . \tag{3.5}$$

Note that $f \in \Lambda_{2;L_2(X)}(R)$ may be viewed as a "second order stochastic process," and each $f(t)$ as an "L_2-functional" of the entire process X; thus such f's need not be adapted functionals of X.

We first define for each $p \geqq 1$ a bounded linear map

$$\Psi_p : H^{\tilde{\otimes} p}(X) \otimes H(X) \to H^{\tilde{\otimes} p+1}(X) \tag{3.6}$$

as follows. Pick a CONS $\{\xi_\gamma, \gamma \in \Gamma\}$, Γ linearly ordered, in $H(X)$. Then $S_p = \{(\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k}) \otimes \xi_\gamma : k \geqq 0, p_1 + \cdots + p_k = p; \gamma, \gamma_1, \cdots, \gamma_k \in \Gamma; \gamma_1 < \cdots < \gamma_k\}$ is a complete orthogonal set in $H^{\tilde{\otimes} p}(X) \otimes H(X)$. Define Ψ_p on S_p by

$$\Psi_p[(\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k}) \otimes \xi_\gamma] = (p+1)^{\frac{1}{2}} \xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k} \tilde{\otimes} \xi_\gamma . \tag{3.7}$$

Writing $\zeta_p = \xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k}$ and using the facts that $\|\theta \otimes \xi\|_{H \otimes K} = \|\theta\|_H \|\xi\|_K$ and $\|\zeta_p\|^2_{H^{\tilde{\otimes} p}(X)} = (p_1! \cdots p_k!)/p!$, we obtain

$$\|\zeta_p \otimes \xi_\gamma\|^2_{H^{\tilde{\otimes} p}(X) \otimes H(X)} = \frac{p_1! \cdots p_k!}{p!}$$

$$\begin{aligned} \|\Psi_p(\zeta_p \otimes \xi_\gamma)\|^2_{H^{\tilde{\otimes} p+1}(X)} &= \frac{p_1! \cdots p_k!}{p!} && \text{if } \gamma \neq \gamma_j \\ &= \frac{p_1! \cdots (p_j+1)! \cdots p_k!}{p!} && \text{if } \gamma = \gamma_j . \end{aligned}$$

It follows that

$$\|\zeta_p \otimes \xi_\gamma\| \leqq \|\Psi_p(\zeta_p \otimes \xi_\gamma)\| = q^{\frac{1}{2}} \|\zeta_p \otimes \xi_\gamma\|$$

for some $q \in \{2, \cdots, p+1\}$. Note that all elements on the right-hand side of (3.7) form a complete orthogonal set in $H^{\tilde{\otimes} p+1}(X)$, and for each such element there are k or $k+1$ ($\leqq p+1$) elements in S_p corresponding to it depending on whether $\gamma = \gamma_j$ for some j or not. It is now clear that Ψ_p can be extended uniquely to a bounded linear map with norm $(p+1)^{\frac{1}{2}}$ from $H^{\tilde{\otimes} p}(X) \otimes H(X)$ onto $H^{\tilde{\otimes} p+1}(X)$, and that its definition is independent of the choice of a CONS in $H(X)$. It is also clear that given any $\zeta_{p+1} \in H^{\tilde{\otimes} p+1}(X)$ one can find $\eta_p \in H^{\tilde{\otimes} p}(X) \otimes H(X)$ such that $\Psi_p(\eta_p) = \zeta_{p+1}$ and

$$\|\eta_p\| \leqq \|\Psi_p(\eta_p)\| \, (\leqq (p+1)^{\frac{1}{2}} \|\eta_p\|) . \tag{3.8}$$

Notice that, since Ψ_p is a many-to-one map, (3.8) need not be true for all $\eta_p \in H^{\tilde{\otimes} p}(X) \otimes H(X)$.

Now let Ψ_0 be the natural isomorphism between $\mathbb{R} \otimes H(X)$ and $H(X)$ ($a \otimes \xi \to a\xi$). We then define $\Psi^* = \bigoplus_{p \geq 0} \Psi_p$ to be the map from $\{\bigoplus_{p \geq 0} H^{\tilde{\otimes} p}(X)\} \otimes H(X)$ onto $\bigoplus_{p \geq 1} H^{\tilde{\otimes} p}(X)$ whose restriction to each $H^{\tilde{\otimes} p}(X) \otimes H(X)$ is Ψ_p. Since $\|\Psi_p\| = (p+1)^{\frac{1}{2}}$ is unbounded in p, Ψ^* is an unbounded densely defined linear map with domain

$$\mathscr{H}^* = \{\phi \in \{\bigoplus_{p \geq 0} H^{\tilde{\otimes} p}(X)\} \otimes H(X) : \textstyle\sum_{p \geq 0} \|\Psi_p(\phi_p)\|^2 < \infty\}$$

where $\phi = \sum_{p \geq 0} \phi_p$, ϕ_p being the projection of ϕ onto $H^{\tilde{\otimes} p}(X) \otimes H(X)$. It is easily seen that $\mathscr{H}^*$ is a dense subspace of $\{\bigoplus_{p \geq 0} H^{\tilde{\otimes} p}(X)\} \otimes H(X)$. Also it follows from (3.8) (left-hand side inequality) that the range of Ψ^* is $\bigoplus_{p \geq 1} H^{\tilde{\otimes} p}(X)$.

Since $\bigoplus_{p\geq 0} H^{\tilde{\otimes} p}(X) \cong_{\Phi} L_2(X)$, we have

$$\{\bigoplus_{p\geq 0} H^{\tilde{\otimes} p}(X)\} \otimes H(X) \cong_{\Phi_0} L_2(X) \otimes H(X)$$

denoting this isomorphism by Φ_0. Finally we let $\mathscr{H} = \Phi_0(\mathscr{H}^*) \subset L_2(X) \otimes H(X)$ and we define Ψ on $\mathscr{H}$ to $L_2(X)$ by

$$\Psi = \Phi \circ \Psi^* \circ \Phi_0^{-1} . \tag{3.9}$$

Then the stochastic integral is defined by (3.5), i.e., $\int f(t)\, dX_t = \mathscr{I}(f) = \Psi(\int f(t) \otimes dX_t) = \Psi \circ I_{\otimes}(f)$ for all $f \in \Lambda_{2;L_2(X)}(R)$ such that $I_{\otimes}(f) = \int f \otimes dX \in \mathscr{H}$. The set of all such f's, denoted by $\Lambda^*_{2;L_2(X)}(R)$, is a dense subspace of $\Lambda_{2;L_2(X)}(R)$. We should point out that the fact that the stochastic integral is not defined for every f in $\Lambda_{2;L_2(X)}(R)$ is a consequence of the critical choice of the constant in (3.7). We will see that the constant $(p+1)^{\frac{1}{2}}$ is the logical one and that $\Lambda^*_{2;L_2(X)}(R)$ is large enough to include most integrands of interest.

For this we need to introduce the following notation. Let $\mathscr{P}$ be the set of all polynomials in the elements of $H(X)$. For each $p \geqq 0$ let $\mathscr{P}_p$ be the set of all polynomials in $\mathscr{P}$ with degree no greater than p ($\mathscr{P}_0$ is the set of constants). For each $p \geqq 1$ let $\mathscr{Q}_p$ be the set of polynomials in $\mathscr{P}_p$ which are orthogonal to $\mathscr{P}_{p-1}$, and let $\mathscr{Q}_0 = \mathscr{P}_0$. The closure $\bar{\mathscr{Q}}_p$ of $\mathscr{Q}_p$ in $L_2(X)$ is called the pth homogeneous chaos. The following are then clear or well known (e.g., Kallianpur (1970), Neveu (1968))

$$\begin{aligned} &\bar{\mathscr{Q}}_p \perp \bar{\mathscr{Q}}_q \quad \text{for} \quad p \neq q \\ &\bar{\mathscr{P}}_p = \textstyle\bigoplus_{q=0}^{p} \bar{\mathscr{Q}}_q \\ &L_2(X) = \bar{\mathscr{P}} = \textstyle\bigoplus_{p\geq 0} \bar{\mathscr{Q}}_p \\ &H^{\tilde{\otimes} p}(X) \cong_{\Phi} \bar{\mathscr{Q}}_p \end{aligned} \tag{3.10}$$

and the following is a CONS in each $\bar{\mathscr{Q}}_p$, $p \geqq 1$,

$$\{(p_1! \cdots p_k!)^{-\frac{1}{2}} H_{p_1}(\xi_{\gamma_1}) \cdots H_{p_k}(\xi_{\gamma_k}) : \\ p_1 + \cdots + p_k = p;\ k = 1, \cdots, p;\ \gamma_1, \cdots, \gamma_k \in \Gamma\}$$

where $\{\xi_\gamma, \gamma \in \Gamma\}$ is a CONS in $H(X)$. Lemma 3.1 and (3.10) imply that

$$\Lambda_{2;L_2(X)}(R) = \textstyle\bigoplus_{p\geq 0} \Lambda_{2;\bar{\mathscr{Q}}_p}(R) , \qquad \Lambda_{2;\bar{\mathscr{P}}_p}(R) = \textstyle\bigoplus_{q=0}^{p} \Lambda_{2;\bar{\mathscr{Q}}_q}(R) . \tag{3.11}$$

The basic properties of the stochastic integral $\mathscr{I} = \Psi \circ I_{\otimes} = \Phi \circ \Psi^* \circ \Phi_0^{-1} \circ I_{\otimes}$ follow from the following structure:

(3.12)

$$\begin{array}{ccccccccc} \Lambda_{2;L_2(X)}(R) & \overset{I_\otimes}{\cong} & L_2(X) \otimes H(X) & \overset{\Phi_0^{-1}}{\cong} & (\bigoplus_{p\geq 0} H^{\tilde{\otimes} p}(X)) \otimes H(X) & & & & \\ \Lambda^*_{2;L_2(X)}(R) & \cong & \mathscr{H} & \cong & \mathscr{H}^* & \underset{\text{onto}}{\overset{\Psi^*}{\rightarrow}} & \bigoplus_{p\geq 1} H^{\tilde{\otimes} p}(X) & \overset{\Phi}{\cong} & L_2(X) \ominus \mathscr{P}_0 \\ \Lambda_{2;\bar{\mathscr{Q}}_p}(R) & \cong & \bar{\mathscr{Q}}_p \otimes H(X) & \cong & H^{\tilde{\otimes} p}(X) \otimes H(X) & \rightarrow & H^{\tilde{\otimes} p+1}(X) & \cong & \bar{\mathscr{Q}}_{p+1} \end{array}$$

and are given in the following

THEOREM 3.2. *The stochastic integral* $\mathscr{I}: \Lambda_{2;L_2(X)}(R) \to L_2(X)$ *is an unbounded densely defined closed linear map with domain* $\Lambda^*_{2;L_2(X)}(R)$ *and range* $L_2^0(X) = L_2(X) \ominus \mathscr{P}_0$, *the set of all zero mean* rv's *in* $L_2(X)$. *Hence every* L_2*-functional* θ *of* X *admits the representation*

$$\theta = \mathscr{E}(\theta) + \int f(t)\, dX_t$$

for some (nonunique) $f \in \Lambda^*_{2;L_2(X)}(R)$. *For each* $p \geq 0$, $\Lambda_{2;\bar{\mathscr{Q}}_p}(R) \subset \Lambda^*_{2;L_2(X)}(R)$, *and hence* $\Lambda_{2;\bar{\mathscr{P}}_p}(R) \subset \Lambda^*_{2;L_2(X)}(R)$, *and the restriction of the stochastic integral* $\mathscr{I}$ *to* $\Lambda_{2;\bar{\mathscr{Q}}_p}(R)$ *is a bounded linear map onto* $\bar{\mathscr{Q}}_{p+1}$ *with norm* $(p+1)^{\frac{1}{2}}$. *If* $f \in \Lambda_{2;L_2(X)}(R)$ *and* $f = \sum_{p\geq 0} f_p$, $f_p \in \Lambda_{2;\bar{\mathscr{Q}}_p}(R)$, *then* $f \in \Lambda^*_{2;L_2(X)}(R)$ *if and only if* $\sum_{p\geq 0} \|\mathscr{I}(f_p)\|^2 < \infty$, *and if* $f \in \Lambda^*_{2;L_2(X)}(R)$ *then* $\mathscr{I}(f) = \sum_{p\geq 0} \mathscr{I}(f_p)$.

PROOF. It is clear from (3.12) and the fact that for each $p \geq 0$, $H^{\tilde{\otimes} p}(X) \otimes H(X) \subset \mathscr{H}^*$ that $\Lambda_{2;\bar{\mathscr{Q}}_p}(R) \subset \Lambda^*_{2;L_2(X)}(R)$. Since Ψ_p is a bounded linear map with norm $(p+1)^{\frac{1}{2}}$, so is the restriction of $\mathscr{I}$ to $\Lambda_{2;\bar{\mathscr{Q}}_p}(R)$ and clearly, again from (3.12), $\mathscr{I}(\Lambda_{2;\bar{\mathscr{Q}}_p}(R)) = \bar{\mathscr{Q}}_{p+1}$.

Since Ψ^* is onto $\bigoplus_{p\geq 1} H^{\tilde{\otimes} p}(X)$, and $\Phi(\bigoplus_{p\geq 1} H^{\tilde{\otimes} p}(X)) = L_2(X) \ominus \bar{\mathscr{Q}}_0 = L_2(X) \ominus \mathscr{P}_0 = L_2^0(X)$, it follows that $\mathscr{I}$ maps its domain onto $L_2^0(X)$.

We now prove the claim in the last sentence of the theorem. Let $f \in \Lambda_{2;L_2(X)}(R)$ and write $f = \sum_{p\geq 0} f_p$, $f_p \in \Lambda_{2;\bar{\mathscr{Q}}_p}(R)$. Then $f \in \Lambda^*_{2;L_2(X)}(R)$ is equivalent to $\Phi_0^{-1} \circ I_{\otimes}(f) \in \mathscr{H}^*$, and since $\Phi_0^{-1} \circ I_{\otimes}(f) = \sum_{p\geq 0} \Phi_0^{-1} \circ I_{\otimes}(f_p)$ and each $\Phi_0^{-1} \circ I_{\otimes}(f_p)$ belongs to $H^{\tilde{\otimes} p}(X) \otimes H(X)$, by the definition of $\mathscr{H}^*$, this is in turn equivalent to

$$\sum_{p\geq 0} \|\Psi_p \circ \Phi_0^{-1} \circ I_{\otimes}(f_p)\|^2 < \infty .$$

But now $\|\Psi_p \circ \Phi_0^{-1} \circ I_{\otimes}(f_p)\| = \|\Phi \circ \Psi_p \circ \Phi_0^{-1} \circ I_{\otimes}(f_p)\| = \|\mathscr{I}(f_p)\|$. It follows that $f \in \Lambda^*_{2;L_2(X)}(R)$ if and only if $\sum_{p\geq 0} \|\mathscr{I}(f_p)\|^2 < \infty$. Assuming now that $f \in \Lambda_{2;L_2(X)}(R)$ it follows from the definition of Ψ^* that

$$\mathscr{I}(f) = \Phi \circ \Psi^* \circ \Phi_0^{-1} \circ I_{\otimes}(f) = \sum_{p\geq 0} \Phi \circ \Psi_p \circ \Phi_0^{-1} \circ I_{\otimes}(f_p) = \sum_{p\geq 0} \mathscr{I}(f_p) .$$

In order to complete the proof of the theorem we need to show that $\mathscr{I}$ is closed. Let $f_n \in \Lambda^*_{2;L_2(X)}(R) = \mathscr{D}(\mathscr{I})$, $f_n \to f$ in $\Lambda_{2;L_2(X)}(R)$, and $\mathscr{I}(f_n) \to \theta$ in $L_2(X)$. We will show that $f \in \mathscr{D}(\mathscr{I})$ and $\mathscr{I}(f) = \theta$. Write

$$f = \sum_{p\geq 0} f_p , \qquad f_n = \sum_{p\geq 0} f_{n,p} ; \quad f_p, f_{n,p} \in \Lambda_{2;\bar{\mathscr{Q}}_p}(R) .$$

Since $f_n \in \mathscr{D}(\mathscr{I})$, by the last claim of the theorem (just shown) we have $\mathscr{I}(f_n) = \sum_{p\geq 0} \mathscr{I}(f_{n,p})$. Also $f_n \to f$ implies that for each fixed $p \geq 0$, $f_{n,p} \to_n f_p$ in $\Lambda_{2;\bar{\mathscr{Q}}_p}(R)$, and since $\mathscr{I}$ restricted to $\Lambda_{2;\bar{\mathscr{Q}}_p}(R)$ is bounded we have $\mathscr{I}(f_{n,p}) \to_n \mathscr{I}(f_p)$. By Fatou's lemma we have

$$\begin{aligned} \sum_{p\geq 0} \|\mathscr{I}(f_p)\|^2 &= \sum_{p\geq 0} \lim_n \|\mathscr{I}(f_{n,p})\|^2 \\ &\leq \liminf_n \sum_{p\geq 0} \|\mathscr{I}(f_{n,p})\|^2 = \liminf_n \|\mathscr{I}(f_n)\| < \infty \end{aligned}$$

since $\mathscr{I}(f_n) \to \theta$, showing (by the last claim of the theorem) that $f \in \mathscr{D}(\mathscr{I})$. Thus $\mathscr{I}(f) = \sum_{p\geq 0} \mathscr{I}(f_p)$ and writing $\theta = \sum_{p\geq 0} \theta_{p+1}$, $\theta_p \in \bar{\mathscr{Q}}_p$, we have, again by Fatou's lemma,

$$\begin{aligned}\sum_{p\geq 0}\|\mathscr{I}(f_p)-\theta_{p+1}\|^2 &= \sum_{p\geq 0}\lim_n\|\mathscr{I}(f_{n,p})-\theta_{p+1}\|^2\\ &\leq \liminf_n \sum_{p\geq 0}\|\mathscr{I}(f_{n,p})-\theta_{p+1}\|^2\\ &= \liminf_n\|\mathscr{I}(f_n)-\theta\| = 0\end{aligned}$$

showing that $\mathscr{I}(f)=\theta$, which concludes the proof. □

The same argument can be applied to define spaces $\lambda_{2;L_2(X)}(R)$; $\lambda^*_{2;L_2(X)}(R)$ and the stochastic integral $\mathscr{I}(f)=\int f(t)X_t\,dt$ for $f\in\lambda^*_{2;L_2(X)}(R)$, where we assume accordingly that X is a zero mean, mean square continuous Gaussian process. It can also be shown that $\lambda^*_{2;L_2(X)}(R)=\Lambda^*_{2;L_2(Z)}(\Gamma)$ and $\int f(t)X_t\,dt=\int f(t)\,dZ_t$ where Γ and Z are related to R and X as in Section 1.1.

We now consider some of the properties of the stochastic integral. First we show that Itô's integral is a special case of the general stochastic integral defined here. The proof is based on (i) of the following lemma which also will be useful later.

LEMMA 3.3. (i) *If $\theta\in L_2(X)$ and $\eta\in H(X)$ are independent then $\Psi(\theta\otimes\eta)=\theta\eta$.*
(ii) *If $\theta,\eta\in H(X)$ then $\Psi(\theta\otimes\eta)=\theta\eta-\mathscr{E}(\theta\eta)$.*

PROOF. (i) Assume without loss of generality that $\mathscr{E}(\eta^2)=1$, and let $\{\eta,\xi_\gamma,\gamma\in\Gamma\}$, Γ linearly ordered, be a CONS in $H(X)$. By the Cameron–Martin representation of $\theta\in L_2(X)$ there is a countable subset Γ' of Γ such that

$$\theta=\sum_{p\geq 0}\sum_{p_0+p_1+\cdots+p_k=p;k\geq 1}\sum_{\gamma_1,\cdots,\gamma_k\in\Gamma';\gamma_1<\cdots<\gamma_k} a^{p_0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k}\times H_{p_0}(\eta)H_{p_1}(\xi_{\gamma_1})\cdots H_{p_k}(\xi_{\gamma_k}).$$

Thus θ is a function of the rv's $\{\eta,\xi_\gamma,\gamma\in\Gamma'\}$, and since η is independent of the rv's $\{\theta,\xi_\gamma,\gamma\in\Gamma'\}$ it follows by an elementary property of conditional expectations that θ is a function of the rv's $\{\xi_\gamma,\gamma\in\Gamma'\}$ only and in fact $\theta=\mathscr{E}(\theta/\xi_\gamma,\gamma\in\Gamma')$. It then follows from the series expansion of θ and $\mathscr{E}(H_{p_0}(\eta)/\xi_\gamma,\gamma\in\Gamma')=\mathscr{E}(H_{p_0}(\eta))=0$, $p_0\geq 1$, that

$$\begin{aligned}\theta &= \sum_{p\geq 0}\sum_{p_1+\cdots+p_k=p;k\geq 1}\sum_{\gamma_1,\cdots,\gamma_k\in\Gamma';\gamma_1<\cdots<\gamma_k} a^{0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k}\\ &\qquad\times H_{p_1}(\xi_{\gamma_1})\cdots H_{p_k}(\xi_{\gamma_k})\\ &= \sum_{p\geq 0}\theta_p\end{aligned}\tag{3.13}$$

and we have $\theta_p\in\bar{\mathscr{Q}}_p$, $p\geq 0$.

We first show that $\theta\otimes\eta\in\mathscr{H}$ or equivalently that $\Phi_0^{-1}(\theta\otimes\eta)=\Phi^{-1}(\theta)\otimes\eta\in\mathscr{H}^*$. Since $\Phi^{-1}(\theta)=\sum_{p\geq 0}\Phi^{-1}(\theta_p)$, $\Phi^{-1}(\theta_p)\in H^{\otimes p}(X)$, this is equivalent to

$$\sum_{p\geq 0}\|\Psi_p(\Phi^{-1}(\theta_p)\otimes\eta)\|^2<\infty\,.$$

It follows from the expression of each θ_p given in (3.13) that

$$\Phi^{-1}(\theta_p)=(p!)^{\frac{1}{2}}\sum a^{0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k}\xi_{\gamma_1}^{\tilde{\otimes}p_1}\tilde{\otimes}\cdots\tilde{\otimes}\xi_{\gamma_k}^{\tilde{\otimes}p_k}$$

and, by (3.7), that

$$\Psi_p(\Phi^{-1}(\theta_p)\otimes\eta)=\{(p+1)!\}^{\frac{1}{2}}\sum a^{0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k}\xi_{\gamma_1}^{\tilde{\otimes}p_1}\tilde{\otimes}\cdots\tilde{\otimes}\xi_{\gamma_k}^{\tilde{\otimes}p_k}\tilde{\otimes}\eta\,.\tag{3.14}$$

Thus we have

$$\|\Psi_p(\Phi^{-1}(\theta_p)\otimes\eta)\|^2 = (p+1)!\sum (a^{0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k})^2\frac{p_1!\cdots p_k!}{(p+1)!}$$
$$= \|\Phi^{-1}(\theta_p)\|^2 = \|\theta_p\|^2$$

and hence $\sum_{p\geq 0}\|\Psi_p(\Phi^{-1}(\theta_p)\otimes\eta)\|^2 = \sum_{p\geq 0}\|\theta_p\|^2 = \|\theta\|^2 < \infty$. It follows that $\theta\otimes\eta\in\mathscr{H}$.

Now from the definitions of Ψ, (3.9), and Ψ^* we have

$$\begin{aligned}\Psi(\theta\otimes\eta) &= \Phi\circ\Psi^*\circ\Phi_0^{-1}(\theta\otimes\eta) = \Phi\circ\Psi^*(\Phi^{-1}(\theta)\otimes\eta)\\ &= \Phi(\textstyle\sum_{p\geq 0}\Psi_p[\Phi^{-1}(\theta_p)\otimes\eta])\\ &= \textstyle\sum_{p\geq 0}\Phi(\Psi_p[\Phi^{-1}(\theta_p)\otimes\eta]).\end{aligned}$$

For each $p\geqq 0$, using (3.14) we obtain

$$\begin{aligned}\Phi(\Psi_p[\Phi^{-1}(\theta_p)\otimes\eta]) &= \sum a^{0,p_1,\cdots,p_k}_{\gamma_1,\cdots,\gamma_k}H_{p_1}(\xi_{\gamma_1})\cdots H_{p_k}(\xi_{\gamma_k})H_1(\eta)\\ &= \theta_p\eta\end{aligned}$$

and thus $\Psi(\theta\otimes\eta) = \sum_{p\geq 0}\theta_p\eta = \theta\eta$ as desired.

(ii) Now let $\theta,\eta\in H(X)$, assume again that $\mathscr{E}(\eta^2)=1$, and write $\theta = \mathscr{E}(\theta\eta)\eta+\zeta$ where $\zeta = \theta - \mathscr{E}(\theta\eta)\eta$ is independent of η. Then $\theta\otimes\eta = \mathscr{E}(\theta\eta)\eta^{\otimes 2}+\zeta\otimes\eta$ and by (i), $\Psi(\theta\otimes\eta) = \mathscr{E}(\theta\eta)\Psi(\eta^{\otimes 2})+\zeta\eta$. But $\Psi(\eta^{\otimes 2}) = \Phi\circ\Psi^*\circ\Phi_0^{-1}(\eta\otimes\eta) = \Phi\circ\Psi_1(\eta\otimes\eta) = 2^{\frac{1}{2}}\Phi(\eta\tilde{\otimes}\eta) = H_2(\eta) = \eta^2-1$. Hence

$$\Psi(\theta\otimes\eta) = \mathscr{E}(\theta\eta)(\eta^2-1)+\{\theta-\mathscr{E}(\theta\eta)\eta\}\eta = \theta\eta-\mathscr{E}(\theta\eta). \qquad \square$$

THEOREM 3.4. *Itô's integral for the Wiener process is a special case of the stochastic integral* $\mathscr{I}$.

PROOF. Let X be the Wiener process, i.e., $R(t,s) = t\wedge s$. Then Itô's integral, denoted by $\mathscr{I}^*$, defines an isomorphism from M_2 onto $L_2^0(X)$, where M_2 is the Hilbert subspace of $L_2(\Omega\times\mathbb{R}, \mathscr{B}(X)\times\mathscr{B}(\mathbb{R}), dP\times dt) = L_2(dP\times dt)$ consisting of all elements adapted to X. Note that $M_2\subset L_2(dP\times dt) = L_{2;L_2(X)}(dt) = \Lambda_{2;L_2(X)}(R)$. Let M be the set of all elements of the form $\sum_1^N f_n 1_{(a_n,b_n]}$, where $f_n\in L_2(X)$ and f_n is $\mathscr{B}(X_u, u\leqq a_n)$-measurable. M is a dense subspace of M_2.

We first show that $M\subset L^*_{2;L_2(X)}(dt) = \Lambda^*_{2;L_2(X)}(R)$ and that $\mathscr{I}=\mathscr{I}^*$ on M. Let $f = \sum_1^N f_n 1_{(a_n,b_n]}\in M$. Then

$$\mathscr{I}^*(f) = \textstyle\sum_1^N f_n(X_{b_n}-X_{a_n}).$$

Since each f_n is $\mathscr{B}(X_u, u\leqq a_n)$-measurable, it is independent of $X_{b_n}-X_{a_n}$, and by Lemma 3.3(i) $f_n\otimes(X_{b_n}-X_{a_n})\in\mathscr{H}$ and $\Psi(f_n\otimes(X_{b_n}-X_{a_n})) = f_n(X_{b_n}-X_{a_n})$. It follows that $\int f\otimes dX = \sum_1^N f_n\otimes(X_{b_n}-X_{a_n})\in\mathscr{H}$, and thus $f\in L^*_{2;L_2(X)}(dt)$, and

$$\mathscr{I}(f) = \Psi(\textstyle\int f\otimes dX) = \sum_1^N f_n(X_{b_n}-X_{a_n}).$$

Hence $\mathscr{I}(f) = \mathscr{I}^*(f)$.

Finally we show that $M_2\subset L^*_{2;L_2(X)}(dt)$ and that $\mathscr{I}=\mathscr{I}^*$ on M_2. Let $f\in M_2$. Then for some $f_n\in M\subset L^*_{2;L_2(X)}(dt)$, $f_n\to f$. It follows from the properties of

Itô's integral that $\mathscr{I}^*(f_n) \to \mathscr{I}^*(f)$ and since, as it was just shown, $\mathscr{I}^*(f_n) = \mathscr{I}(f_n)$, we have $\mathscr{I}(f_n) \to \mathscr{I}^*(f)$. Since $\mathscr{I}$ is closed it follows that $f \in L^*_{2;L_2(X)}(dt)$ and $\mathscr{I}(f) = \mathscr{I}^*(f)$. Clearly M_2 is a smaller class than $L^*_{2;L_2(X)}(dt) = \Lambda^*_{2;L_2(X)}(R)$ and thus $\mathscr{I}$ provides an extension of $\mathscr{I}^*$. □

We now consider the problem of calculating the stochastic integral for specific integrands, starting with the simplest possible case where $f(t) = \theta\phi(t)$ with $\theta \in L_2(X)$ and $\phi \in \Lambda_2(R)$.

THEOREM 3.5. (i) *If $\theta \in L_2(X)$, $\phi \in \Lambda_2(R)$, and θ and $\int \phi(t)\, dX_t$ are independent then*

$$\int \theta\phi(t)\, dX_t = \theta \int \phi(t)\, dX_t \,.$$

(ii) *If $\theta \in H(X)$ and $\phi \in \Lambda_2(R)$ then*

$$\int \theta\phi(t)\, dX_t = \theta \int \phi(t)\, dX_t - \mathscr{E}\Big(\theta \int \phi(t)\, dX_t\Big) \,.$$

(iii) *If for some $u \in T$, $F(x) \in L_2(\mathbb{R}, \exp(-x^2/2R(u,u))\, dx)$ and if $F(x)$ has an $L_2(\mathbb{R}, \exp(-x^2/2R(u,u))\, dx)$-derivative denoted by $F'(x)$, then*

$$\int F(X_u)\phi(t)\, dX_t = F(X_u) \int \phi(t)\, dX_t - F'(X_u)\mathscr{E}\Big(X_u \int \phi(t)\, dX_t\Big) \,.$$

PROOF. We first show that if $\theta \in L_2(X)$ and $\phi \in \Lambda_2(R)$ then

$$I_\otimes(\theta\phi) = \int (\theta\phi(t)) \otimes dX_t = \theta \otimes \int \phi(t)\, dX_t \,.$$

Indeed, if ϕ is a simple function $\phi = \sum_1^N c_n 1_{(a_n, b_n]}$ we have

$$I_\otimes(\theta\phi) = I_\otimes\Big(\sum_1^N \theta c_n 1_{(a_n,b_n]}\Big) = \sum_1^N (\theta c_n) \otimes (X_{b_n} - X_{a_n}) = \theta \otimes \int \phi\, dX \,,$$

and since $I_\otimes$ is an isomorphism the same is valid for all $\phi \in \Lambda_2(R)$. It follows that

$$\mathscr{I}(\theta\phi) = \Phi \circ \Psi^* \circ \Phi_0^{-1} \circ I_\otimes(\theta\phi) = \Phi \circ \Psi^*\Big(\Phi^{-1}(\theta) \otimes \int \phi\, dX\Big) \,.$$

(i) Let $f(t) = \theta\phi(t)$. We will show that $f \in \Lambda^*_{2;L_2(X)}(R) = \mathscr{D}(\mathscr{I})$ and $\mathscr{I}(f) = \theta \int \phi\, dX$. Write $\theta = \sum_{p\geq 0} \theta_p$, $\theta_p \in \bar{\mathscr{Q}}_p$. Then $f = \sum_{p\geq 0} \theta_p \phi$ in $\Lambda_{2;L_2(X)}(R)$ with each $f_p(t) = \theta_p\phi(t)$ in $L_{2;\bar{\mathscr{Q}}_p}(R)$. We first calculate $\mathscr{I}(\theta_p\phi(t))$. Note that it is clear from the proof of Lemma 3.3(i) that the independence of θ and $\int \phi\, dX$ ($= \eta \in H(X)$) implies the independence of each θ_p and $\int \phi\, dX$. Thus by Lemma 3.3(i)

$$\mathscr{I}(\theta_p\phi) = \Psi \circ I_\otimes(\theta_p\phi) = \Psi\Big(\theta_p \otimes \int \phi\, dX\Big) = \theta_p \int \phi\, dX \,.$$

Now the independence of θ_p and $\int \phi\, dX$ implies $\|\mathscr{I}(\theta_p\phi)\| = \|\theta_p\|\|\int \phi\, dX\|$ and thus

$$\sum_{p\geq 0} \|\mathscr{I}(f_p)\|^2 = \sum_{p\geq 0} \|\mathscr{I}(\theta_p\phi)\|^2 = \sum_{p\geq 0} \|\theta_p\|^2\Big\|\int \phi\, dX\Big\|^2 = \|\theta\|^2\Big\|\int \phi\, dX\Big\|^2 < \infty \,.$$

It follows from Theorem 3.2 that $f \in \mathscr{D}(\mathscr{I})$ and that $\mathscr{I}(f) = \sum_{p\geq 0} \mathscr{I}(f_p) = \sum_{p\geq 0} \theta_p \int \phi\, dX$, and again by independence we have $\mathscr{I}(f) = \theta \int \phi\, dX$.

(ii) If $\theta \in H(X)$ and $\phi \in \Lambda_2(R)$ then $\theta\,\phi \in \Lambda_{2;\bar{\mathscr{A}}_1}(R)$, and by Lemma 3.3(ii)

$$\mathscr{I}(\theta\phi) = \Psi \circ I_{\otimes}(\theta\phi) = \Psi(\theta \otimes \textstyle\int \phi\, dX) = \theta \int \phi\, dX - \mathscr{E}(\theta \int \phi\, dX)\,.$$

(iii) First let $F(X_u) = H_{p,\sigma_u^2}(X_u)$, $p \geqq 1$, where $\sigma_u{}^2 = \mathscr{E}(X_u{}^2) = R(u, u)$. Then letting $\int \phi\, dX = \eta$ and noting that $\Phi^{-1}(H_{p,\sigma_u^2}(X_u)) = (p!)^{\frac{1}{2}} X_u^{\tilde{\otimes} p}$ we have

$$\int H_{p,\sigma_u^2}(X_u)\phi(t)\, dX_t = \Phi \circ \Psi^*((p!)^{\frac{1}{2}} X_u^{\tilde{\otimes} p} \otimes \eta)\,.$$

Write $\eta = \sigma_u{}^{-2}\mathscr{E}(X_u\eta)X_u + \zeta$ where $\zeta = \eta - \sigma_u{}^{-2}\mathscr{E}(X_u\eta)X_u \perp X_u$. Then

$$\Psi_p(X_u^{\tilde{\otimes} p} \otimes \eta) = (p+1)^{\frac{1}{2}}\{\sigma_u{}^{-2}\mathscr{E}(X_u\eta)X_u^{\tilde{\otimes} p+1} + X_u^{\tilde{\otimes} p} \tilde{\otimes}\, \zeta\}$$

and thus

$$\begin{aligned}\int H_{p,\sigma_u^2}(X_u)\phi(t)\, dX_t &= \sigma_u{}^{-2}\mathscr{E}(X_u\eta)H_{p+1,\sigma_u^2}(X_u) + H_{p,\sigma_u^2}(X_u)\zeta \\ &= H_{p,\sigma_u^2}(X_u)\eta + \sigma_u{}^{-2}\mathscr{E}(X_u\eta)\{H_{p+1,\sigma_u^2}(X_u) - X_u H_{p,\sigma_u^2}(X_u)\} \\ &= H_{p,\sigma_u^2}(X_u)\eta - p\mathscr{E}(X_u\eta)H_{p-1,\sigma_u^2}(X_u)\,,\end{aligned}$$

which is the desired relationship since $dH_{p,\sigma^2}(x)/dx = pH_{p-1,\sigma^2}(x)$.

Now if $F(X_u)$ is as in the statement of the theorem we have

$$F(X_u) = \textstyle\sum_{p\geqq 0} a_p H_{p,\sigma_u^2}(X_u)\,, \qquad F'(X_u) = \sum_{p\geqq 1} pa_p H_{p-1,\sigma_u^2}(X_u)$$

with both series converging in $L_2(X)$. Since for each $p \geqq 0$, $\|\mathscr{I}(H_{p,\sigma_u^2}(X_u)\phi)\| \leqq (p+1)^{\frac{1}{2}}\|H_{p,\sigma_u^2}(X_u)\phi\| = (p+1)^{\frac{1}{2}}\|H_{p,\sigma_u^2}(X_u)\|\|\phi\|$, we have

$$\begin{aligned}\textstyle\sum_{p\geqq 0}\|\mathscr{I}(a_p H_{p,\sigma_u^2}(X_u)\phi)\|^2 &\leqq \textstyle\sum_{p\geqq 0}(p+1)a_p{}^2\|H_{p,\sigma_u^2}(X_u)\|^2 \circ \|\phi\|^2 \\ &= (\|F'(X_u)\|^2 + \|F(X_u)\|^2)\|\phi\|^2 < \infty\,.\end{aligned}$$

It follows by Theorem 3.2 that $F(X_u)\phi \in \mathscr{D}(\mathscr{I})$ and

$$\begin{aligned}\int F(X_u)\phi(t)\, dX_t &= \textstyle\sum_{p\geqq 0}\int a_p H_{p,\sigma_u^2}(X_u)\phi(t)\, dX_t \\ &= \textstyle\sum_{p\geqq 0}\{a_p H_{p,\sigma_u^2}(X_u)\int\phi\, dX - pa_p H_{p-1,\sigma_u^2}(X_u)\mathscr{E}(X_u \int\phi\, dX)\} \\ &= F(X_u)\textstyle\int\phi\, dX - F'(X_u)\mathscr{E}(X_u\int\phi\, dX)\,.\end{aligned}$$ □

(iii) includes the cases of Hermite polynomials, $H_{p,\sigma_u^2}(X_u)$, and exponentials, $\exp(X_u - \frac{1}{2}\sigma_u{}^2)$, and it admits a natural generalization to F's of the form $F(X_{u_1}, \cdots, X_{u_k})$. As an illustration we write the following simple integral

$$\int X_u X_v \phi(t)\, dX_t = X_u X_v \int\phi\, dX - \mathscr{E}(X_u\int\phi\, dX)X_v - \mathscr{E}(X_v\int\phi\, dX)X_u\,.$$

Before evaluating some less trivial stochastic integrals we consider the following interesting result. Let $T = [a, b]$ and $a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b$, $n = 1, 2, \cdots$, be a sequence of partitions of T whose mesh goes to zero, $\max_i (t_{i,n} - t_{i-1,n}) \to 0$. The mean square quadratic variation of X on T along such a sequence of partitions is defined as the mean square limit of $\sum_{i=1}^n (X_{t_{i,n}} - X_{t_{i-1,n}})^2$ whenever the latter exists.

THEOREM 3.6. *Let $X = \{X_t,\ t \in [a, b]\}$ be a zero mean Gaussian process with continuous covariance R of bounded variation on $[a, b] \times [a, b]$ (the signed measure on $[a, b] \times [a, b]$ corresponding to R is denoted again by R). Then the mean square* *quadratic variation* $V_a{}^b$ *of* X *on* $[a, b]$ *along any sequence of partitions whose mesh goes to zero exists and is given by*

$$V_a{}^b = R(D_a{}^b)$$

where $D_a{}^b$ *is the diagonal of* $[a, b] \times [a, b]$.

PROOF. By the mean square continuity of X and the bounded variation of R we have $\int\int |\langle X_t, X_s\rangle|\, d^2|R|(t, s) < \infty$ and $\int\int |\langle X_t, \theta 1_{(\alpha,\beta]}(s)\rangle|\, d^2|R|(t, s) < \infty$ for all $(\alpha, \beta] \subset [a, b]$ and $\theta \in L_2(X)$. It then follows from the extended version of Theorem 1.1 that $X_t \in \Lambda_{2;H(X)}(R) \subset \mathscr{D}(\mathscr{I})$ and thus the stochastic integral $\int X_t\, dX_t$ is defined.

Let $a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b$ be any sequence of partitions with mesh tends to zero. If $X_t{}^{(n)}$ is defined by $X_t{}^{(n)} = X_{t_{i-1}}$ on each $(t_{i-1}, t_i]$, then $X^{(n)} \to X$ in $\Lambda_{2;H(X)}(R)$ (by the mean square continuity of X and the bounded variation of R) and hence $\int X_t{}^{(n)}\, dX_t \to \int X_t\, dX_t$ in $L_2(X)$. Thus, by Theorem 3.5(ii),

$$\begin{aligned}(3.15)\qquad \int_a^b X_t\, dX_t &= \lim_n \sum_{i=1}^n \int_a^b X_{t_{i-1}} 1_{(t_{i-1},t_i]}(t)\, dX_t \\ &= \lim_n \sum_{i=1}^n \{X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \mathscr{E}[X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})]\},\end{aligned}$$

and similarly, by defining $X_t'^{(n)} = X_{t_i}$ on $(t_{i-1}, t_i]$, we have

$$(3.15')\qquad \int_a^b X_t\, dX_t = \lim_n \sum_{i=1}^n \{X_{t_i}(X_{t_i} - X_{t_{i-1}}) - \mathscr{E}[X_{t_i}(X_{t_i} - X_{t_{i-1}})]\}.$$

Subtracting (3.15) from (3.15′) gives

$$0 = \lim_n \{\textstyle\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2 - \sum_{i=1}^n \mathscr{E}[(X_{t_i} - X_{t_{i-1}})^2]\}$$

and since the second term has limit $R(D_a{}^b)$, the result follows. □

Notice that by adding (3.15) and (3.15′) we obtain

$$\int_a^b X_t\, dX_t = \tfrac{1}{2}(X_b{}^2 - X_a{}^2 - \sigma_b{}^2 + \sigma_a{}^2)$$

where $\sigma_t{}^2 = \mathscr{E}(X_t{}^2) = R(t, t)$. A similar approach leads to the following result.

THEOREM 3.7. *Let* X *be as in Theorem* 3.6. *Then*

$$\int_a^b H_{p,\sigma_t^2}(X_t)\, dX_t = \frac{1}{p+1}\{H_{p+1,\sigma_b^2}(X_b) - H_{p+1,\sigma_a^2}(X_a)\}, \quad p \geqq 0$$

$$\int_a^b \exp(X_t - \tfrac{1}{2}\sigma_t{}^2)\, dX_t = \exp(X_b - \tfrac{1}{2}\sigma_b{}^2) - \exp(X_a - \tfrac{1}{2}\sigma_a{}^2).$$

PROOF. It is shown as in the proof of Theorem 3.6 that $H_{p,\sigma_t^2}(X_t) \in \Lambda_{2;\bar{\mathscr{O}}_p}(R)$. Letting $a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b$ be any refining sequence of partitions with mesh going to zero, and using the (uniform) mean square continuity of X and the bounded variation of R, we have (writing t_i for $t_{i,n}$ for simplicity)

$$\begin{aligned}\int_a^b H_{p,\sigma_t^2}(X_t)\, dX_t &= \Phi \circ \Psi^* \circ \Phi_0^{-1}(\textstyle\int_a^b H_{p,\sigma_t^2}(X_t) \otimes dX_t) \\ &= \Phi \circ \Psi_p \circ \Phi_0^{-1}\{\lim_n \textstyle\sum_i H_{p,\sigma_{t_{i-1}}^2}(X_{t_{i-1}}) \otimes (X_{t_i} - X_{t_{i-1}})\} \\ &= \{(p+1)!\}^{\frac{1}{2}}\Phi\{\lim_n \textstyle\sum_i X_{t_{i-1}}^{\bar{\otimes} p} \tilde{\otimes} (X_{t_i} - X_{t_{i-1}})\}\end{aligned}$$

$$= \{(p+1)!\}^{\frac{1}{2}}\Phi\left\{\lim_n \sum_i \frac{1}{p+1}\sum_{m=0}^p X_{t_{i-1}}^{\otimes m} \tilde{\otimes} X_{t_i}^{\otimes p-m} \tilde{\otimes} (X_{t_i} - X_{t_{i-1}})\right\}$$

$$= \frac{\{(p+1)!\}^{\frac{1}{2}}}{p+1}\Phi\{\lim_n \sum_i \sum_{m=0}^p (X_{t_{i-1}}^{\otimes m} \tilde{\otimes} X_{t_i}^{\otimes p+1-m} - X_{t_{i-1}}^{\otimes m+1} \tilde{\otimes} X_{t_i}^{\otimes p-m})\}$$

$$= \frac{\{(p+1)!\}^{\frac{1}{2}}}{p+1}\Phi\{\lim_n \sum_i (X_{t_i}^{\otimes p+1} - X_{t_{i-1}}^{\otimes p+1})\}$$

$$= \frac{\{(p+1)!\}^{\frac{1}{2}}}{p+1}\Phi(X_b^{\otimes p+1} - X_a^{\otimes p+1})$$

$$= \frac{1}{p+1}\{H_{p+1,\sigma_b^2}(X_b) - H_{p+1,\sigma_a^2}(X_a)\}\,.$$

The second result follows from the first and

$$\exp(X_t - \tfrac{1}{2}\sigma_t^2) = \sum_{p \geq 0}\frac{1}{p!}H_{p,\sigma_t^2}(X_t)\,. \qquad \square$$

Theorem 3.7 shows that Hermite polynomials $H_{p,\sigma_t^2}(X_t)$ play the role of customary powers, X_t^p, and $\exp(X_t - \frac{1}{2}\sigma_t^2)$ the role of the customary exponential, $\exp(X_t)$, in this stochastic calculus.

4. Iterated, adapted and future increments independent integrals. Throughout this section we assume that X is as in Theorem 3.6. We first explore the connection between the MWI's and the stochastic integral. We want to establish that each MWI can be written as an iterated integral, i.e., that for $f_p \in \Lambda_2(\tilde{\otimes}^p R)$,

$$(4.1) \qquad \int_{T^p} f_p(\mathbf{t})\, dX_t^p = \int_T (\int_T (\cdots (\int_T f_p(t_1, \cdots, t_p)\, dX_{t_1}) \cdots dX_{t_{p-1}})\, dX_{t_p})$$

where of course the iterated integral remains to be defined.

Let H be a Hilbert space and $\mathscr{S}_{I;H}^p$ the set of all H-valued step functions $f(\mathbf{t})$ on T^p. Then $\mathscr{S}_{I;H}^p$ is an inner product space with inner product

$$\langle f, g\rangle = \int\int \langle f(\mathbf{t}), g(\mathbf{s})\rangle\, d^{2p}R(\mathbf{t}, \mathbf{s})$$

and its completion is denoted by $\Lambda_{2;H}(\otimes^p R)$. It is easily seen that $\Lambda_{2;H}(\otimes^p R) \cong \Lambda_2(\otimes^p R) \otimes H$ under the correspondence $(\phi_1 \otimes \cdots \otimes \phi_p)\xi \leftrightarrow (\phi_1 \otimes \cdots \otimes \phi_p) \otimes \xi$. Thus each element in $\Lambda_{2;H}(\otimes^p R)$ has an orthogonal development of the form

$$\sum a_{\gamma_1,\cdots,\gamma_p}^\alpha (\phi_{\gamma_1} \otimes \cdots \otimes \phi_{\gamma_p})\xi_\alpha$$

where $\{\phi_\gamma, \gamma \in \Gamma\}$ and $\{\xi_\alpha, \alpha \in A\}$ are CONS's in $\Lambda_2(R)$ and H respectively.

Consider the following chain of maps

$$\Lambda_2(\otimes^p R) \to_{\pi_1} \Lambda_{2;\mathscr{E}_1}(\otimes^{p-1} R) \to_{\pi_2} \cdots \to_{\pi_{p-1}} \Lambda_{2;\mathscr{E}_{p-1}}(R) \to_{\pi_p} \bar{\mathscr{Q}}_p$$

defined first by

$$\begin{aligned}\phi_{\gamma_1} \otimes \cdots \otimes \phi_{\gamma_p} &\to_{\pi_1} (\int \phi_{\gamma_1}\, dX)\phi_{\gamma_2} \otimes \cdots \otimes \phi_{\gamma_p} \\ &\to_{\pi_2} 2^{\frac{1}{2}}\Phi(\int \phi_{\gamma_1}\, dX \tilde{\otimes} \int \phi_{\gamma_2}\, dX)\phi_{\gamma_3} \otimes \cdots \otimes \phi_{\gamma_p} \to \cdots \\ &\to_{\pi_{p-1}} \{(p-1)!\}^{\frac{1}{2}}\Phi(\int \phi_{\gamma_1}\, dX \tilde{\otimes} \cdots \tilde{\otimes} \int \phi_{\gamma_{p-1}}\, dX)\phi_{\gamma_p} \\ &\to_{\pi_p} (p!)^{\frac{1}{2}}\Phi(\int \phi_{\gamma_1}\, dX \tilde{\otimes} \cdots \tilde{\otimes} \int \phi_{\gamma_p}\, dX)\,.\end{aligned}$$

Then by the same argument used for defining Ψ_p, each π_q can be extended to a bounded linear onto (not one to one) map with norm $q^{\frac{1}{2}}$. It is important to note that π_p is the stochastic integral, and that on $\bar{\mathscr{Q}}_{q-1}$-valued step functions π_q acts like the stochastic integral by fixing the "extra" variables. The iterated integral in (4.1) is now defined to be $\pi_p \circ \cdots \circ \pi_1(f_p)$ and the equation follows.

Letting $T = [a, b]$ and noting that $I_p(f_p) = I_p(\tilde{f}_p)$, we should expect to obtain from (4.1)

$$\int_a^b f_p(\mathbf{t})\, dX_{\mathbf{t}}^p = p! \int_a^b \left(\int_a^{t_p} \cdots \left(\int_a^{t_2} \tilde{f}_p(t_1, \cdots, t_{p-1}, t_p)\, dX_{t_1}\right) \cdots dX_{t_{p-1}}\right) dX_{t_p}$$
$$= \int_a^b h_p(t_p)\, dX_{t_p}$$

where h_p is adapted to X. This will now be made precise (in the proof of Theorem 4.2). The following definition will be used. A step function $f = \sum_1^N f_n 1_{(a_n, b_n]}$ in $\Lambda_{2;L_2(X)}(R)$ is called *adapted* if each f_n is $B(X_t, a \leqq t \leqq a_n)$-measurable. The closed subspace of $\Lambda_{2;L_2(X)}(R)$ generated by the adapted simple functions is denoted by $\Lambda_{2;L_2(X)}^{\mathrm{ad}}(R)$ and its elements are called adapted. We also let

$$\Lambda_{2;L_2(X)}^{\mathrm{ad}*}(R) = \Lambda_{2;L_2(X)}^{\mathrm{ad}}(R) \cap \Lambda_{2;L_2(X)}^{*}(R) .$$

LEMMA 4.1. *If $f \in \Lambda_2(\bigotimes^p R)$ is a step function then $g(t_p) = \int_0^{t_p}(\cdots(\int_0^{t_2} f(t_1, \cdots, t_{p-1}, t_p)\, dX_{t_1}) \cdots)\, dX_{t_{p-1}}$ is an adapted step function and*

$$\text{(4.2)} \qquad \int_a^b \int_a^{t_p} \cdots \int_a^{t_2} f(\mathbf{t})\, dX_{\mathbf{t}}^p$$
$$= \int_a^b \left(\int_a^{t_p} \cdots \left(\int_a^{t_2} f(t_1, \cdots, t_{p-1}, t_p)\, dX_{t_1}\right) \cdots dX_{t_{p-1}}\right) dX_{t_p}$$

where both integrals are defined in the usual way as the corresponding integrals over the entire interval of $f(t_1, \cdots, t_p) 1_{(t_1 < \cdots < t_p)}$.

PROOF. For ease of exposition we only consider the case $p = 2$, the case of $p > 2$ being similar. It is then sufficient to prove the assertions for f of the form (i) $1_{(\alpha,\beta]}(t_1) 1_{(\gamma,\delta]}(t_2)$, (ii) $1_{(\alpha,\beta]}(t_1) 1_{(\alpha,\beta]}(t_2)$, (iii) $1_{(\gamma,\delta]}(t_1) 1_{(\alpha,\beta]}(t_2)$ where $\alpha < \beta < \gamma < \delta$. Then $g(t_2)$ equals (i) $(X_\beta - X_\alpha) 1_{(\gamma,\delta]}(t_2)$, (ii) $(X_{t_2} - X_\alpha) 1_{(\alpha,\beta]}(t_2)$, (iii) 0 and is thus an adapted step function. Using Theorems 3.5(ii) and 3.7 we find that the right-hand side of (4.2) equals

(i) $$\int_a^b (X_\beta - X_\alpha) 1_{(\gamma,\delta]}(t_2)\, dX_{t_2}$$
$$= (X_\beta - X_\alpha)(X_\delta - X_\gamma) - \mathscr{E}[(X_\beta - X_\alpha)(X_\delta - X_\gamma)]$$

(ii) $$\int_a^b (X_{t_2} - X_\alpha) 1_{(\alpha,\beta]}(t_2)\, dX_{t_2}$$
$$= \int_\alpha^\beta X_{t_2}\, dX_{t_2} - \int_\alpha^\beta X_\alpha\, dX_{t_2}$$
$$= \tfrac{1}{2}\{X_\beta^2 - X_\alpha^2 - \sigma_\beta^2 + \sigma_\alpha^2\} - \{X_\alpha(X_\beta - X_\alpha) - \mathscr{E}[X_\alpha(X_\beta - X_\alpha)]\}$$
$$= \tfrac{1}{2}\{(X_\beta - X_\alpha)^2 - \mathscr{E}[(X_\beta - X_\alpha)^2]\}$$

(iii) $$\int_a^b 0\, dX_{t_2} = 0 .$$

On the other hand the left-hand side of (4.2) equals

(i) $$I_2(1_{(\alpha,\beta]}(t_1) 1_{(\gamma,\delta]}(t_2) 1_{(t_1 < t_2)})$$
$$= 2^{\frac{1}{2}} \Phi\{(X_\beta - X_\alpha) \tilde{\otimes} (X_\delta - X_\gamma)\}$$
$$= (X_\beta - X_\alpha)(X_\delta - X_\gamma) - \mathscr{E}[(X_\beta - X_\alpha)(X_\delta - X_\gamma)]$$

(ii) $$I_2(1_{(\alpha,\beta]}(t_1)1_{(\alpha,\beta]}(t_2)1_{(t_1<t_2)}) = \tfrac{1}{2}I_2(1_{(\alpha,\beta]}(t_1)1_{(\alpha,\beta]}(t_2)) = 2^{-\frac{1}{2}}\Phi\{(X_\beta - X_\alpha)^{\tilde{\otimes} 2}\} = \tfrac{1}{2}\{(X_\beta - X_\alpha)^2 - \mathscr{E}[(X_\beta - X_\alpha)^2]\}$$

(iii) $$I_2(1_{(\gamma,\delta]}(t_1)1_{(\alpha,\beta]}(t_2)1_{(t_1<t_2)}) = I_2(0) = 0$$

and the proof of the lemma is complete. □

THEOREM 4.2. *$\mathscr{I}(\Lambda^{\mathrm{ad}*}_{2;L_2(X)}(R)) = L_2^0(X)$ and thus each L_2 functional θ of X admits the stochastic integral representation*

$$\theta - \mathscr{E}(\theta) = \int f(t)\, dX_t$$

where (the not necessarily unique) f is adapted ($f \in \Lambda^{\mathrm{ad}*}_{2;L_2(X)}(R)$).

PROOF. It suffices to prove the second assertion of the theorem. Assume first that $\theta \in \bar{\mathscr{Q}}_p$ so that by Theorem 2.1, $\theta = I_p(f_p) = I_p(\tilde{f}_p)$ for some $f_p \in \Lambda_2(\otimes^p R)$.

If ϕ is a step function in $\Lambda_2(\otimes^p R)$ it is easily checked that

$$\tilde{\phi} = \textstyle\sum_{\pi\in\Pi} \tilde{\phi} 1_{(t_{\pi_1}<\cdots<t_{\pi_p})}$$

and

$$I_p(\tilde{\phi}) = p!\, I_p(\tilde{\phi} 1_{(t_1<\cdots<t_p)})$$

where $\pi = (\pi_1, \cdots, \pi_p)$ is a permutation of $(1, \cdots, p)$ and Π is the set of all such permutations. Now let $\{\phi_n\}$ be a sequence of step functions in $\Lambda_2(\otimes^p R)$ with $\phi_n \to f_p$. Then

$$\|\tilde{\phi}_n 1_{(t_1<\cdots<t_p)} - \tilde{\phi}_m 1_{(t_1<\cdots<t_p)}\| = \frac{1}{p!}\|\tilde{\phi}_n - \tilde{\phi}_m\|$$

implies that $\{\tilde{\phi}_n 1_{(t_1<\cdots<t_p)}\}$ is Cauchy and we denote its limit by $\tilde{f}_p 1_{(t_1<\cdots<t_p)}$. Then

$$I_p(\tilde{f}_p) = \lim_n I_p(\tilde{\phi}_n) = \lim_n p!\, I_p(\tilde{\phi}_n 1_{(t_1<\cdots<t_n)}) = p!\, I_p(\tilde{f}_p 1_{(t_1,\cdots,t_p)}) .$$

If we let $g_n = \pi_{p-1} \circ \cdots \circ \pi_1(p!\, \tilde{\phi}_n 1_{(t_1<\cdots<t_n)})$, then g_n is clearly a step function in $\Lambda_{2;\bar{\mathscr{Q}}_{p-1}}(R)$ adapted to X by Lemma 4.1, and by the continuity of π_q's

$$g_n \to \pi_{p-1} \circ \cdots \circ \pi_1(p!\, \tilde{f}_p 1_{(t_1<\cdots<t_p)}) = h_p .$$

It follows that $h_p \in \Lambda_{2;\bar{\mathscr{Q}}_{p-1}}(R)$ is adapted to X and satisfies

$$I_p(\tilde{f}_p) = \lim_n \pi_p(g_n) = \pi_p(h_p) = \mathscr{I}(h_p) .$$

For a general $\theta \in L_2(X)$ we have by Theorem 2.1 and the above

$$\theta - \mathscr{E}(\theta) = \textstyle\sum_{p\ge1} I_p(f_p) = \sum_{p\ge1} \mathscr{I}(h_p) = \mathscr{I}(\sum_{p\ge1} h_p)$$

where $h = \sum_{p\ge1} h_p$ belongs to $\Lambda^*_{2;L_2(X)}(R)$ (by Theorem 3.2, since $\sum_{p\ge1}\|\mathscr{I}(h_p)\|^2 = \sum_{p\ge1}\|I_p(f_p)\|^2 < \infty$) and is also adapted to X since each h_p is. □

It is clear from the definition of $\tilde{f}_p(t_1, \cdots, t_p)1_{(t_1<\cdots<t_p)}$ in the proof of Theorem 4.2 and from Lemma 4.1 that equality (4.2) is valid for all $f_p \in \Lambda_2(\otimes^p R)$

where both integrals in (4.2) are defined as the corresponding integrals of $\tilde{f}_p(t_1, \cdots, t_p)1_{(t_1<\cdots<t_p)}$.

We finally consider the stochastic integral of future increments independent functions. A step function $f = \sum_1^N f_n 1_{(a_n, b_n]}$ in $\Lambda_{2;L_2(X)}(R)$ is called *future increments independent* if each f_n is independent of the increments of X after a_n. The closed subspace of $\Lambda_{2;L_2(X)}(R)$ which is generated by the future increments independent step functions is denoted by $\Lambda^{\text{fii}}_{2;L_2(X)}(R)$ and its elements are called future increments independent (fii) functions.

THEOREM 4.3. *$\Lambda^{\text{fii}}_{2;L_2(X)}(R) \subset \Lambda^*_{2;L_2(X)}(R)$ and the stochastic integral restricted to $\Lambda^{\text{fii}}_{2;L_2(X)}(R)$ is norm preserving.*

PROOF. Let $f = \sum_1^N f_n 1_{(a_n, b_n]}$ be a fii step function in $\Lambda_{2;L_2(X)}(R)$. Since for each n, f_n is independent of $X_{b_n} - X_{a_n}$, if follows from Theorem 3.5 (i) that

$$\mathscr{I}(f) = \int f(t)\, dX_t = \textstyle\sum_1^N f_n(X_{b_n} - X_{a_n}) .$$

Then

$$\|\mathscr{I}(f)\|^2 = \textstyle\sum_{n,m=1}^N \mathscr{E}\{f_n f_m (X_{b_n} - X_{a_n})(X_{b_m} - X_{a_m})\} .$$

Put $\Delta_n X = X_{b_n} - X_{a_n}$. When $n = m$ we have $\mathscr{E}\{f_n^2(\Delta_n X)^2\} = \mathscr{E}(f_n^2)\mathscr{E}\{(\Delta_n X)^2\}$. When $a_n < a_m$ and $\mathscr{E}\{(\Delta_n X)^2\} \neq 0$ we can write

$$\Delta_n X = \frac{\mathscr{E}\{\Delta_n X \cdot \Delta_m X\}}{\mathscr{E}\{(\Delta_m X)^2\}}\, \Delta_m X + \eta$$

where η is independent of $\Delta_m X$, and since $\mathscr{E}(f_n f_m \eta \Delta_m X) = \mathscr{E}(f_n f_m \eta)\mathscr{E}(\Delta_m X) = 0$ we have

$$\mathscr{E}(f_n f_m \Delta_n X \Delta_m X) = \mathscr{E}\left\{f_n f_m \frac{\mathscr{E}(\Delta_n X \Delta_m X)}{\mathscr{E}\{(\Delta_m X)^2\}} (\Delta_m X)^2\right\} = \mathscr{E}(f_n f_m)\mathscr{E}(\Delta_n X \Delta_m X) .$$

It follows that

$$\begin{aligned} \|\mathscr{I}(f)\|^2 &= \textstyle\sum_{n,m=1}^N \mathscr{E}(f_n f_m)\mathscr{E}[(X_{b_n} - X_{a_n})(X_{b_m} - X_{a_m})] \\ &= \textstyle\iint \langle f(t), f(s)\rangle\, d^2R(t, s) = \|f\|^2 \end{aligned}$$

and thus the stochastic integral is norm-preserving for step fii functions.

Now let $f \in \Lambda^{\text{fii}}_{2;L_2(X)}(R)$. We will show that $f \in \Lambda^*_{2;L_2(X)}(R)$ and $\|\mathscr{I}(f)\| = \|f\|$. By definition there is a sequence of step fii functions f_n such that $f_n \to f$. Write

$$f = \textstyle\sum_{p\geqq 0} f_p , \qquad f_n = \sum_{p\geqq 0} f_{n,p} ; \quad f_p, f_{n,p} \in \Lambda_{2;\mathscr{Q}_p}(R) .$$

Then $f_{n,p} \to_n f_p$ and thus $\mathscr{I}(f_{n,p}) \to_n \mathscr{I}(f_p)$. It follows by Fatou's lemma that

$$\begin{aligned} \textstyle\sum_{p\geqq 0} \|\mathscr{I}(f_p)\|^2 &= \textstyle\sum_{p\geqq 0} \lim_n \|\mathscr{I}(f_{n,p})\|^2 \leqq \liminf_n \sum_{p\geqq 0} \|\mathscr{I}(f_{n,p})\|^2 \\ &= \textstyle\liminf_n \sum_{p\geqq 0} \|f_{n,p}\|^2 = \liminf_n \|f_n\|^2 = \|f\|^2 < \infty . \end{aligned}$$

Hence by Theorem 3.2, $f \in \Lambda^*_{2;L_2(X)}(R)$. Now $\|\mathscr{I}(f_n) - \mathscr{I}(f_m)\| = \|\mathscr{I}(f_n - f_m)\| = \|f_n - f_m\|$ implies that the sequence $\mathscr{I}(f_n)$ converges, and since $\mathscr{I}$ is closed we have $\mathscr{I}(f) = \lim \mathscr{I}(f_n)$ and thus $\|\mathscr{I}(f)\| = \lim_n \|\mathscr{I}(f_n)\| = \lim_n \|f_n\| = \|f\|$. Hence $\Lambda^{\text{fii}}_{2;L_2(X)}(R) \subset \Lambda^*_{2;L_2(X)}(R)$ and $\mathscr{I}$ restricted to $\Lambda^{\text{fii}}_{2;L_2(X)}(R)$ is a norm preserving map into $L_2^0(X)$. □

Note that $\mathscr{I}(\Lambda^{\text{fii}}_{2;L_2(X)}(R))$ is a closed subspace of $L_2^0(X)$ and it would be of interest to know how large it is in general, and under what conditions we have $\mathscr{I}(\Lambda^{\text{fii}}_{2;L_2(X)}(R)) = L_2^0(X)$ or equivalently that each L_2-functional of X has a fii stochastic integral representation

$$\theta - \mathscr{E}(\theta) = \int f(t)\, dX_t$$

where f is fii. We conjecture that this would be the case if the germ σ-fields of X are trivial.

The notions of "adapted" and of "future increments independent" introduced above are of course identical when X is Wiener process.

5. Nonlinear noise. A (strictly) stationary process $Y = (Y_t, -\infty < t < \infty)$ with $\mathscr{E}Y_t = 0$ and $\mathscr{E}Y_t^2 < \infty$ is called noise. Quoting from McKean (1973), Wiener liked to think of such a noise as the output of a "black box" θ: you put in a white noise $\mathring{W} = (\mathring{W}_t, -\infty < t < \infty)$ (formally the derivative of a Wiener process W) and you get $Y_0 = \theta(\mathring{W}_t, -\infty < t < \infty)$ out; the noise $(Y_t, -\infty < t < \infty)$ is produced by shifting the input by the flow of the white noise $\mathring{W}(\circ) \to \mathring{W}(\circ + t)$. In order for Y to be a noise we require that $\mathscr{E}\theta = 0$ and $\mathscr{E}\theta^2 < \infty$. Since θ has the orthogonal development

$$\theta = \sum_{p \geq 1} \int_{\mathbf{R}^p} f_p(\mathbf{t})\, dW_{\mathbf{t}}^p\,, \tag{5.1}$$

where $f_p \in L_2(R^p)$, the noise Y obtained by shifting the incoming white noise through the nonlinear device θ can be expressed as

$$Y_t = \sum_{p \geq 1} \int_{\mathbf{R}^p} f_p(\mathbf{u} - t)\, dW_{\mathbf{u}}^p\,, \tag{5.2}$$

where $t = (t, \cdots, t)$, and the covariance function of Y is readily seen to be $(\tau = t - s)$

$$\mathscr{E}Y_t Y_s = \sum_{p \geq 1} p! \int_{\mathbf{R}^p} \tilde{f}_p(\mathbf{u}) \tilde{f}_p(\mathbf{u} - \tau)\, d\mathbf{u}\,.$$

Wiener's theory of nonlinear noise starts from this idea. He also proved a profound theorem which was clarified by Nisio (1960) and which states that every ergodic noise can be approximated in law by noises of the form (5.2). Note that not every ergodic noise has the representation (5.2), and a necessary condition is strong mixing (McKean (1973)).

Here generally, instead of sending white noise (or Wiener process) through a nonlinear device θ, we may send a Gaussian process with stationary increments $X = (X_t, -\infty < t < \infty)$, with say $X_0 = 0$ a.s. and covariance R. Then the noise Y obtained by shifting the incoming Gaussian noise X can be expressed as

$$Y_t = \sum_{p \geq 1} \int_{\mathbf{R}^p} f_p(\mathbf{u} - t)\, dX_{\mathbf{u}}^p \tag{5.3}$$

where $f_p \in \Lambda_2(\otimes^p R)$, and the covariance function of Y is again readily seen to be

$$\mathscr{E}Y_t Y_s = \sum_{p \geq 1} p! \langle \tilde{f}_p(\circ), \tilde{f}_p(\circ - \tau) \rangle_{\Lambda_2(\otimes^p R)} \tag{5.4}$$

where $\tau = t - s$ (cf. Theorem 2.1). Although (5.2) and (5.3) are intuitively clear, they require proof. The proof of (5.3) follows from the following property.

LEMMA 5.1. *If X is a zero mean Gaussian process with stationary increments and covariance R then, for $f \in \Lambda_2(\otimes^p R)$,*

$$\int_{\mathbf{R}^p} f(\mathbf{u})\, dX_{\mathbf{u}+t}^p = \int_{\mathbf{R}^p} f(\mathbf{u} - t)\, dX_{\mathbf{u}}^p .$$

PROOF. Both integrals are well defined since X has stationary increments. Pick a CONS $\{\phi_\gamma, \gamma \in \Gamma\}$ in $\Lambda_2(R)$. Since $\{\phi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \phi_{\gamma_k}^{\tilde{\otimes} p_k} : \gamma_1, \cdots, \gamma_k \in \Gamma, p_1 + \cdots + p_k = p, k \geqq 0\}$ is complete in $\Lambda_2(\otimes^p R)$ and $I_p(f) = I_p(\tilde{f})$, it suffices to prove this assertion for $f = \phi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes} \phi_{\gamma_k}^{\tilde{\otimes} p_k}$. But for such f, the assertion becomes

$$\prod_{i=1}^k H_{p_i, \|\phi_i\|^2}\left(\int \phi_i(u_i)\, dX_{u_i+t}\right) = \prod_{i=1}^k H_{p_i, \|\phi_i\|^2}\left(\int \phi_i(u_i - t)\, dX_{u_i}\right)$$

and thus we need only to show that

$$\int \phi(u)\, dX_{u+t} = \int \phi(u - t)\, dX_u .$$

This is true for $\phi \in \mathscr{S}_I$ and hence for $\phi \in \Lambda_2(R)$. The proof is now complete. □

When Y has representation (5.3), we say that Y is X-presentable. Note that X is always X-presentable since $X_t = \int_0^t dX_u$. As McKean (1973) showed, if Y is not strongly mixing then Y is not Wiener process-presentable. The same property is true for X-presentable processes.

THEOREM 5.2. *Let X be a mean square continuous Gaussian process with stationary increments, $X_0 = 0$ a.s., and with absolutely continuous spectral distribution. Then every X-presentable noise Y is strongly mixing.*

Having introduced the Fourier transform on $\Lambda_2(\otimes^p R)$ in Section 1.3, the proof is identical to McKean's proof for X Wiener process.

We now show the analogue of Wiener–Nisio's theorem using Nisio's approach as simplified (for convergence in law) by McKean (1973). X will be a zero mean sample continuous ergodic Gaussian process with stationary increments which satisfies $X_0 = 0$ a.s. and the following condition

$$\text{(S)} \qquad \Pr\{\Delta_t X > 0, 0 \leqq t \leqq n; \Delta_{-n} X < 0\} > 0 \qquad \text{for all} \quad n \geqq 1$$

where $\Delta_t X = X_{t+1} - X_{t-1}$.

THEOREM 5.3. *Every measurable ergodic noise Y (defined on any probability space) is the limit in law of a sequence of X-presentable noises.*

PROOF. Examining McKean's (1973) proof for X the Wiener process we see that the argument remains valid for the present general case if it can be shown that there exists a sequence of nonnegative functionals a_n on the paths of X, such that the probability distribution of each a_n is absolutely continuous and its density function is constant on $[0, n]$ and decreasing on (n, ∞). We proceed to construct such a_n's. For simplicity we suppose that X is a coordinate process, i.e., $(\Omega, \mathscr{B}, P) = (\mathbb{R}^{\mathbb{R}}, \mathscr{B}(\mathbb{R}^{\mathbb{R}}), P)$ and $X_t(\omega) = \omega(t)$. Define sets $S(\omega)$, $S_n(\omega)$ and a rv $f(\omega)$ as in Nisio (1960), pages 210–211. It follows from the ergodicity of X that $S_n(\omega)$ is nonempty a.s. if $\mathscr{E}f > 0$. But $\mathscr{E}f = \Pr\{\Delta_t X > 0, 0 \leqq t \leqq n; \Delta_{-n} X < 0\} > 0$ by assumption (S). Thus $S_n(\omega)$ is nonempty a.s. We now define $a_n(\omega) = n + \inf S_n(\omega)$. Note that $0 \leqq a_n < \infty$ a.s. The same argument as on page 211 of Nisio (1960) shows that a_n has the desired probability distribution. Thus the proof is complete. □

We now give a discussion of assumption (S). We believe that (S) always holds when X is a zero mean sample continuous ergodic Gaussian process with stationary increments, yet we are not able to prove it. Instead, we have the following sufficient condition for (S), which indicates that (S) is a mild assumption (if it is a restriction at all),

$$\text{(S}_1\text{)}\quad \begin{array}{l} \text{For each } n \geqq 1 \text{ there is an } f_n \in \mathscr{R}(C)\,, \text{ the reproducing} \\ \text{kernel Hilbert space of the covariance } C \text{ of } X\,, \text{ such that} \\ \Delta_t f_n > 0 \text{ for } t \in [0, n] \text{ and } \Delta_{-n} f_n < 0\,. \end{array}$$

LEMMA 5.4. (S_1) *implies* (S).

PROOF. We owe this proof to Loren D. Pitt. Note that X is mean square continuous, since it is a sample continuous Gaussian process. Then C is continuous and so is every $f \in \mathscr{R}(C)$. Assume (S_1). Then, by the sample continuity of X and the continuity of f,

$$\{\omega \in \Omega : \Delta_t(X + cf) > 0, 0 \leqq t \leqq n; \Delta_{-n}(X + cf) < 0\} \uparrow \Omega$$

as $c \uparrow \infty$, and hence there exists $c > 0$ such that

$$\Pr\{\Delta_t(X + cf) > 0, 0 \leqq t \leqq n, \Delta_{-n}(X + cf) < 0\} > 0\,.$$

(S) now follows from the equivalence of the Gaussian processes X and $X + cf$ (since $cf \in \mathscr{R}(C)$). □

We next show that (S_1) is satisfied by all processes with stationary increments having rational spectral densities. This implies in particular that (S_1) is satisfied by the Wiener process, by stationary processes with rational spectral densities, and by (indefinite) integrals of stationary processes with rational spectral densities. The proof is based on the following result which is of independent interest.

LEMMA 5.5. *A mean square continuous process $X = \{X_t, -\infty < t < \infty\}$ with zero mean and covariance $C(t, s)$ has (wide sense) stationary increments (with spectral measure dF) if and only if there is a mean square continuous measurable (wide sense) stationary process $Y = \{Y_t, -\infty < t < \infty\}$ with zero mean and covariance $r(t, s)$ (with spectral measure dF) such that for each t and s*

$$X_t - X_s = Y_t - Y_s - \textstyle\int_s^t Y_u\, du \quad \text{a.s.} \tag{5.5}$$

and $H(\Delta X) = H(Y)$. Also if $X_0 = 0$ a.s., then $f \in \mathscr{R}(C)$, the reproducing kernel

Hilbert space of C, if and only for some $g \in \mathscr{R}(r)$ and all t

$$f(t) = g(t) - g(0) - \textstyle\int_0^t g(u)\, du\,. \tag{5.6}$$

PROOF. The sufficiency of (5.5) being obvious, we only show its necessity. Suppose that X has the spectral representation given by (1.2) and (1.3). Then we have

$$X_t = \int \frac{e^{it\lambda} - 1}{i\lambda}(1 + \lambda^2)^{\frac{1}{2}}\, dV_\lambda$$

and $H(\Delta X) = H(\Delta V)$. Define the process with orthognal increments $U = \{U_\lambda, -\infty < \lambda < \infty\}$ by $(\lambda + i)\, dU_\lambda = i^{-1}(1 + \lambda^2)^{\frac{1}{2}}\, dV_\lambda$. Then $\mathscr{E}|dU_\lambda|^2 = \mathscr{E}|dV_\lambda|^2 = dF(\lambda)$, and $H(\Delta U) = H(\Delta V)$. Define the process $Y = \{Y_t, -\infty < t < \infty\}$ by $Y_t = \int e^{it\lambda}\, dU_\lambda$. Then Y is (wide sense) stationary and mean square continuous, and hence it has a measurable modification, denoted again by Y. First we see that $H(Y) = H(\Delta U) = H(\Delta V) = H(\Delta X)$. Also for each fixed t and s we have the following equalities in L_2 and thus also a.s.,

$$\begin{aligned} X_t - X_s &= \int \frac{e^{it\lambda} - e^{is\lambda}}{\lambda}(\lambda + i)\, dU_\lambda = \int \left\{(e^{it\lambda} - e^{is\lambda}) - \frac{e^{it\lambda} - e^{is\lambda}}{i\lambda}\right\} dU_\lambda \\ &= Y_t - Y_s - \textstyle\int_{-\infty}^{\infty} (\int_s^t e^{iu\lambda}\, du)\, dU_\lambda = Y_t - Y_s - \int_s^t Y_u\, du\,. \end{aligned}$$

The last equality is justified by the following equality for all v,

$$\begin{aligned} \mathscr{E}\textstyle\int_{-\infty}^{\infty} (\int_s^t e^{iu\lambda}\, du)\, dU_\lambda \cdot \bar{Y}_v &= \textstyle\int_{-\infty}^{\infty} \int_s^t e^{i(u-v)\lambda}\, du\, dF(\lambda) = \int_s^t \int_{-\infty}^{\infty} e^{i(u-v)\lambda}\, dF(\lambda)\, du \\ &= \textstyle\int_s^t \mathscr{E}(Y_u \bar{Y}_v)\, du = \mathscr{E}(\int_s^t Y_u\, du \cdot \bar{Y}_v) \end{aligned}$$

where Fubini's theorem has been repeatedly applied, its justification being quite clear.

For the second claim notice that for all $\eta \in H(X) = H(Y)$ and t we have by (5.5)

$$\mathscr{E}(X_t\bar{\eta}) = \mathscr{E}(Y_t\bar{\eta}) - \mathscr{E}(Y_0\bar{\eta}) - \textstyle\int_0^t \mathscr{E}(Y_u\bar{\eta})\, du\,. \tag{5.7}$$

If $f \in \mathscr{R}(C)$ then $f(t) = \mathscr{E}(X_t\bar{\eta})$ for some $\eta \in H(X)$ and (5.6) follows from (5.7) with $g(t) = \mathscr{E}(Y_t\bar{\eta}) \in \mathscr{R}(r)$. Conversely, if $g \in \mathscr{R}(r)$ then $g(t) = \mathscr{E}(Y_t\bar{\eta})$ for some $\eta \in H(Y)$. Thus if f satisfies (5.6) it follows from (5.7) that $f(t) = \mathscr{E}(X_t\bar{\eta})$ and thus $f \in \mathscr{R}(C)$. □

LEMMA 5.6. *Let X be a zero mean mean square continuous process with (wide sense) stationary increments having a rational spectral density, and with covariance C. Then condition* (S_1) *is satisfied.*

PROOF. Thus X is as in Lemma 5.5, and dF has a rational density. It is then well known that $\mathscr{R}(r) = W_2^m$, the set of all functions possessing on every finite interval absolutely continuous derivatives up to order $m - 1$ and square integrable mth derivatives (with $2m$ = degree of denominator—degree of numerator of the polynomials of the rational spectral density). Now it is clear that for each fixed $n \geq 1$, by a suitable choice of $g \in W_2^m$, f defined by (5.6) will have the desired property stated in (S_1). Indeed, for fixed $n \geq 1$ we may choose

f in W_2^m satisfying (S_1) and $f(0) = 0$. Then a simple calculation shows that g defined by $g(t) = e^t \int_0^t e^{-u} f'(u)\, du$ belongs to W_2^m and satisfies (5.6). Hence, by Lemma 5.5, $f \in \mathscr{R}(C)$. Since f was chosen to satisfy (S_1) the proof is complete. □

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