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STRONG RATIO LIMIT THEOREMS FOR ϕ-RECURRENT MARKOV CHAINS

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Let $\{X_n; n = 0,1, \cdots\}$ be a ϕ-recurrent Markov chain on a general measurable state space $(S, \mathfrak{F})$ with transition probabilities $P(x, A)$, $x \in S$, $A \in \mathfrak{F}$. The convergence of the ratio $\lambda P^{n+m}f/\mu P^n g$ (as $n \to \infty$), where λ and μ are nonnegative measures on $(S, \mathfrak{F})$ and f and g are nonnegative measurable functions on S, is studied. We show that the ratio converges, provided that λ, μ, f and g are in a certain sense "small," and provided that for an embedded renewal sequence $\{u(n)\}$ the limit $\lim u(n+1)/u(n)$ exists.

0. Introduction. Let S be a set and $\mathfrak{F}$ a countably generated σ-field of subsets of S. Let $\{X_n; n = 0, 1, \cdots\}$ be an aperiodic ϕ-recurrent Markov chain on $(S, \mathfrak{F})$ with transition probabilities $P(x, A)$, $x \in S, A \in \mathfrak{F}$ (see, e.g., Orey (1971), Chapter 1).

The main purpose of this paper is to study the convergence of the ratio

$$\frac{\lambda P^{n+m}f}{\mu P^n g} \qquad \text{as } n \to \infty, \tag{0.1}$$

where m is a fixed integer, λ and μ are nonnegative measures on $(S, \mathfrak{F})$, and f and g are nonnegative measurable functions on S.

One of our main results is the following theorem, the proof of which is presented in Section 3. We shall denote by π the unique (up to scalar multiplication) invariant measure of the chain $\{X_n\}$. Let C be a C-set (cf. Orey's Theorem 2.1, page 7); that is, $\phi(C) > 0$ and for some integer k, some $\alpha > 0$,

$$P^k(x, \cdot) \geqslant \alpha\phi_C \qquad \text{for all} \quad x \in C, \tag{0.2}$$

where ϕ_C denotes the restriction of ϕ to C normed to a probability measure (i.e., $\phi_C(A) = \phi(A \cap C)/\phi(C)$, $A \in \mathfrak{F}$).

THEOREM 1. *The sequence* $\{u(n); n = 0, 1, \cdots\}$ *defined by*

$$u(0) = 1, \qquad u(n) = \alpha\phi_C P^{(n-1)k}(C) \qquad \text{for} \quad n \geqslant 1,$$

is a renewal sequence. If

$$\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = 1,$$

then for any probability measures λ *and* μ *on* $(S, \mathfrak{F})$ *and any nonnegative measurable*

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functions f and g on S we have:

(i) *If*

$$\limsup_{n\to\infty} \frac{\lambda P^{nk}(C)}{\phi_C P^{nk}(C)} \leq 1,$$

and if f is a small function (*see Definition* 2.1), *then*

$$\lim_{n\to\infty} \frac{\lambda P^{n+m} f}{\phi_C P^n(C)} = \frac{\pi(f)}{\pi(C)} \quad \textit{for all} \quad m \geq 0.$$

(ii) *If λ and μ are small measures, and if f and g are small functions, then*

$$\lim_{n\to\infty} \frac{\lambda P^{n+m} f}{\mu P^n g} = \frac{\pi(f)}{\pi(g)} \qquad \textit{for all} \quad m \geq 0,$$

provided that the right-hand side is well defined.

Part (ii) of this theorem generalizes the corresponding result for Markov chains on a countable state space (Orey (1961)): if S is a countable set and $\{X_n\}$ is an aperiodic irreducible recurrent Markov chain on S with transition probabilities p_{ij} and invariant measure π_i, and if for some state $a \in S$,

$$\lim_{n\to\infty} \frac{p_{aa}^{n+1}}{p_{aa}^{n}} = 1,$$

then for all $i, j, k, l \in S$

$$\lim_{n\to\infty} \frac{p_{ij}^{n+m}}{p_{kl}^{n}} = \frac{\pi_j}{\pi_l} \qquad \text{for all} \quad m \geq 0.$$

In our generalization the small functions and measures play the role of individual points of a countable state space.

For other works on the strong ratio limit property (SRLP) of Markov chains the reader is referred to Kingman and Orey (1964), Pruitt (1965), Jain (1969), Orey (1971) and Lin (1976). The bibliographies of Orey's book and Lin's paper also contain a great number of other works on this subject.

Section 1 of the present paper deals with the preliminaries. In particular, we shall formulate the minorization assumption (M) and a useful decomposition of the iterates of the transition probability function P. In Section 2 we study the so-called small functions and measures, give them some characterization results and formulate some lemmas needed in Section 3. Section 3 contains the main results (Theorems 2, 3 and 4). The preceding Theorem 1 is a direct corollary of Theorems 2 and 3. Theorem 4 deals with the case when the ratio (0.1) converges for all λ and μ, and all small f and g. Finally, in Section 4 we shall briefly discuss the generalization to noncontractive positive operators, and as an application we study the existence of the I-type quasi-stationary distribution for an R-null recurrent chain.

1. Notation and preliminaries. We write $\mathbb{N} = \{0, 1, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$. For λ a probability measure on $(S, \mathfrak{F})$ we denote by $\mathbb{P}_\lambda$ the canonical probability measure on the product space $(S^\infty, \mathfrak{F}^\infty)$ corresponding to the initial distribution λ of X_0 and to the transition kernel P. $\mathbb{E}_\lambda$ denotes the corresponding expectation. We write $\mathbb{P}_{\varepsilon_x} = \mathbb{P}_x$, where ε_x is the probability measure assigning unit mass to the point $x \in S$.

We denote by $\mathfrak{F}_+$ (resp. $\mathfrak{M}_+$) the set of nonnegative measurable functions (σ-finite measures) on $(S, \mathfrak{F})$. For any $A \in \mathfrak{F}$ we denote by 1_A the indicator of A and by I_A the kernel defined by

$$I_A(x, E) = 1_{A \cap E}(x), \qquad x \in S, E \in \mathfrak{F}.$$

Let $f \in \mathfrak{F}_+$ and $\lambda \in \mathfrak{M}_+$. We use the notation $f \otimes \lambda$ for the kernel

$$f \otimes \lambda(x, A) = f(x)\lambda(A), \qquad x \in S, A \in \mathfrak{F}.$$

The transition kernel P induces in the well-known manner two operators, one on the set $\mathfrak{F}_+$, and the other on $\mathfrak{M}_+$:

$$f \mapsto Pf = \int P(\cdot, dy)f(y),$$
$$\lambda \mapsto \lambda P = \int \lambda(dx)P(x, \cdot).$$

Denote

$$\psi = \Sigma_{n=1}^{\infty} 2^{-n}\tilde{\phi}P^n,$$

where $\tilde{\phi}$ is a probability measure equivalent to ϕ. It is well known that the invariant measure π is equivalent to ψ and that the chain is ψ-recurrent.

We define

$$\mathfrak{F}^+ = \{f \in \mathfrak{F}_+; \pi(f) > 0\} = \{f \in \mathfrak{F}_+; \psi(f) > 0\}.$$

For A a set, we write $A \in \mathfrak{F}^+$ to indicate that $1_A \in \mathfrak{F}^+$.

We shall make the following basic assumption, called *minorization assumption* (M).

(M): There exist $k \in \mathbb{N}_+$, $h \in \mathfrak{F}^+$ and a probability measure ν such that

$$P^k \geqslant h \otimes \nu. \tag{1.1}$$

Henceforth k, h and ν will be fixed and will exclusively denote those quantities satisfying (1.1).

REMARK. By Orey's C-set theorem there is no loss of generality in assuming (M): we can choose

$$h = \alpha 1_C, \qquad \nu = \phi_C \qquad \text{(cf. (0.2))}.$$

We shall use the notation

$$Q = \Sigma_{n=0}^{\infty}(P^k - h \otimes \nu)^n.$$

Let U be any kernel such that $0 \leqslant U \leqslant P$ and $\pi U(S) > 0$. The same calculations as in the case $U = PI_A(A \in \mathcal{F}^+)$ lead to the identity

$$\Sigma_{n=0}^{\infty}(P - U)^n U1 \equiv 1,$$

and to the following expression for the invariant measure π: if $\mu \in \mathfrak{M}_+$ is such that $\mu\Sigma_{n=0}^{\infty}(P - U)^n U = \mu$, then

$$\mu\Sigma_{n=0}^{\infty}(P - U)^n = \pi.$$

Replacing P by P^k (note that $\{X_{nk};\ n \in \mathbb{N}\}$ is a ϕ-recurrent Markov chain by Lemma 2.1 of Nummelin (1978)) and writing $U = h \otimes \nu$, we obtain

$$Qh \equiv 1, \tag{1.2}$$

$$\nu Q = \pi. \tag{1.3}$$

The following simple algebraic lemma turns out to be useful.

LEMMA 1.1. *Let a and b be two elements of a ring. Denote $c_0 = b$, $c_n = ba^{n-1}b$ for $n \in \mathbb{N}_+$. Then*

(i) $a^n = (a - b)^n + \Sigma_{m=1}^{n} a^{n-m} b (a - b)^{m-1}$ *and*

(ii) $a^n = (a - b)^n + \Sigma_{i=0}^{n-1}\Sigma_{j=1}^{n-i}(a - b)^i c_{n-i-j}(a - b)^{j-1}$.

PROOF. Elementary combinatorics or induction. □

Applying Lemma 1.1 with $a = P^k$, $b = h \otimes \nu$, denoting

$$u(0) = 1,\ u(n) = \nu P^{(n-1)k} h \qquad (n \in \mathbb{N}_+), \tag{1.4}$$

$$f(n) = \nu(P^k - h \otimes \nu)^{n-1} h \qquad (n \in \mathbb{N}_+),$$

and using (1.2) we obtain

COROLLARY 1.2. (i) *The sequence $\{u(n)\}$ is a renewal sequence satisfying the renewal equation*

$$u(n) = \Sigma_{i=1}^{n} f(i) u(n - i).$$

(ii) *For any $\lambda \in \mathfrak{M}_+$, $f \in \mathcal{F}_+$, $n \in \mathbb{N}_+$*

(1.5)

$$\lambda P^{nk} f = \lambda(P^k - h \otimes \nu)^n f + \Sigma_{i=0}^{n-1}\Sigma_{j=1}^{n-i}\left[\lambda(P^k - h \otimes \nu)^i h\right]\left[\nu(P^k - h \otimes \nu)^{j-1} f\right] u(n - i - j).$$

REMARK. For a probabilistic interpretation of the renewal sequence $\{u(n)\}$ and of the decomposition (1.5) the reader is referred to Nummelin (1978): the sequence $\{u(n)\}$ is the renewal sequence associated with the atom of the "split chain of $\{X_n\}$," and (1.5) corresponds to the first-entrance last-exit decomposition of the split chain w.r.t. the atom.

2. On small functions and measures. The following definition of small functions is the same as in Lin (1976). In an obvious way we also define the concept of small measures.

DEFINITION 2.1. A function $f \in \mathcal{F}_+$ (resp. a measure $\lambda \in \mathfrak{M}_+$) is called *small*, provided that for all $A \in \mathcal{F}^+$ there exist $\delta > 0$, $N \in \mathbb{N}_+$ such that

$$\Sigma_{n=0}^{N} P^n 1_A \geqslant \delta f \left(\text{resp. } \Sigma_{n=0}^{N} \pi I_A P^n \geqslant \delta\lambda\right). \tag{2.1}$$

Note that a small function f is bounded and π-integrable, and that a small measure λ is finite and absolutely continuous w.r.t. π having a bounded Radon-Nikodym derivative $d\lambda/d\pi$.

The following lemmas are formulated only for small functions. There exist, of course, the corresponding dual results for small measures.

LEMMA 2.2. (i) *If f is small, then for any $m \in \mathbb{N}_+$, $P^m f$ is small.*

(ii) *For any $m \in \mathbb{N}_+$, f is small, if and only if it is small* w.r.t. *the m-step chain* $\{X_{nm}; n \in \mathbb{N}\}$.

PROOF. (i) In order to get the assertion, let P^m operate on both sides of (2.1).

(ii) From (2.1) immediately follows the sufficiency. In order to prove the necessity, assume that f is small and $A \in \mathcal{F}^+$ is arbitrary. Let $C \in \mathcal{F}^+$ be a C-set. Since by assumption $\{X_n\}$ is aperiodic, there exists $q \in \mathbb{N}_+$ such that

$$\gamma = \inf_{i=0,\cdots,m-1;\, x\in C} P^{qm-i}(x, A) > 0.$$

Let $N \in \mathbb{N}_+$ and $\delta > 0$ be defined so that

$$\Sigma_{n=0}^{N} P^n 1_C \geqslant \delta f.$$

Then

$$\begin{aligned}\Sigma_{n=0}^{N+q} P^{nm} 1_A &\geqslant m^{-1} \Sigma_{n=0}^{N} \Sigma_{i=0}^{m-1} P^{nm+i} P^{qm-i} 1_A \\ &\geqslant \gamma m^{-1} \Sigma_{n=0}^{mN} P^n 1_C \\ &\geqslant \gamma m^{-1} \delta f,\end{aligned}$$

which shows that f is small w.r.t. $\{X_{nm}; n \in \mathbb{N}\}$. □

LEMMA 2.3. (i) *Let $g \in \mathcal{F}^+$ be small. Then for any $f \in \mathcal{F}_+$, f is small, if and only if there exist $N \in \mathbb{N}_+$ and $\delta > 0$ such that*

$$\Sigma_{n=0}^{N} P^n g \geqslant \delta f. \tag{2.2}$$

(ii) *The function h appearing in* (M) *is small.*

COROLLARY 2.4. *A function $f \in \mathcal{F}_+$ is small, if and only if there exist $N \in \mathbb{N}_+$, $\delta > 0$ such that*

$$\Sigma_{n=0}^{N} P^n h \geqslant \delta f.$$

PROOF OF LEMMA 2.3. Let $A \in \mathcal{F}^+$ be arbitrary.

(i) Assume that (2.2) holds for some N, δ and small $g \in \mathcal{F}^+$. Increase N and decrease δ such that also

$$\Sigma_{n=0}^{N} P^n 1_A \geqslant \delta g.$$

Then

$$\Sigma_{n=0}^{2N} P^n 1_A \geqslant (N+1)^{-1} \Sigma_{m=0}^{N} \Sigma_{n=0}^{N} P^{m+n} 1_A \geqslant (N+1)^{-1} \delta f,$$

which shows that f is small.

The converse is trivial.

(ii) By irreducibility we can find for any $A \in \mathcal{F}^+$ an integer i such that $\nu P^i(A) > 0$. By (M)

$$P^{i+k} 1_A \geqslant \nu P^i(A) h,$$

which gives the assertion. □

REMARK. The dual result of Corollary 2.4 states that a measure λ is small, if and only if there exist $N \in \mathbb{N}_+$, $\delta > 0$ such that

$$\Sigma_{n=0}^{N} \nu P^n \geqslant \delta \lambda.$$

3. Strong ratio limit theorems. In this section we shall study the convergence of the ratio (0.1) for an aperiodic, ϕ-recurrent Markov chain $\{X_n\}$ on $(S, \mathcal{F})$ satisfying the minorization assumption (M). Of course our results are interesting only in the case when the chain is null recurrent; that is, when $\pi(S)$ is infinite. If the chain is positive recurrent, then Orey's convergence theorem tells us that, for all probability measures λ, the sequence λP^n converges in total variation norm to the invariant probability measure π.

We shall henceforth assume that for the embedded renewal sequence, defined by (1.4), *the limit of the ratio* $u(n+1)/u(n)$ $(n \to \infty)$ *exists.* By ϕ-recurrence then necessarily

$$\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = 1. \tag{3.1}$$

The reader is referred to Chapter 3 of Orey's book for a thorough discussion on the strong ratio limit property of renewal sequences. Let in the following $\lambda, \mu \in \mathfrak{M}_+$, $f, g \in \mathcal{F}_+$ be arbitrary and fixed. The following proposition is in some ways similar to Lemma 1.2 of Orey (1971), page 73 (cf. also Jain (1969)).

PROPOSITION 3.1. *We have*

$$\liminf_{n\to\infty} \frac{\lambda P^{nk} f}{\nu P^{nk} h} \geqslant \frac{\lambda(S)\pi(f)}{\pi(h)}.$$

PROOF. It suffices to prove the assertion in the case when $k = 1$, since the general case then easily follows by considering the k-step chain $\{X_{nk}\}$.

We shall use the decomposition of Corollary 1.2 (ii) and write the r.h.s. of (1.5) into two parts (cf. Orey (1971), page 74):

$$\lambda P^n f = s_{n,N}(\lambda, f) + r_{n,N}(\lambda, f),$$

where

$$s_{n,N}(\lambda, f) = \Sigma_{i=0}^{N} \Sigma_{j=1}^{N} \left[\lambda(P - h \otimes \nu)^i h\right]\left[\nu(P - h \otimes \nu)^{j-1} f\right] u(n-i-j),$$

and $r_{n,N}(\lambda, f) \geqslant 0$. Hence, similarly as in Orey (1971), we can conclude that for all N

$$\text{(3.2)} \qquad \liminf_{n\to\infty} \frac{\lambda P^n f}{\nu P^n h} = \liminf_{n\to\infty} \frac{\lambda P^n f}{u(n+1)} \geqslant \lim_{n\to\infty} \frac{s_{n,N}(\lambda, f)}{u(n+1)}$$

$$= \Sigma_{i=0}^{N}\Sigma_{j=1}^{N}\left[\lambda(P - h\otimes\nu)^i h\right]\left[\nu(P - h\otimes\nu)^{j-1} f\right]\left[\lim_{n\to\infty}\frac{u(n-i-j)}{u(n+1)}\right]$$

$$= \Sigma_{i=0}^{N}\lambda(P - h\otimes\nu)^i h \Sigma_{j=0}^{N-1}\nu(P - h\otimes\nu)^j f.$$

Letting $N \to \infty$ and recalling (1.2) and (1.3) we get the assertion. □

The following three theorems are the main results of this paper.

THEOREM 2. (i) *If*

$$\text{(3.3)} \qquad \limsup_{n\to\infty} \frac{\lambda P^{nk} h}{\nu P^{nk} h} \leqslant \lambda(S),$$

and if f is small, then

$$\text{(3.4)} \qquad \lim_{n\to\infty} \frac{\lambda P^{n+m} f}{\nu P^n h} = \frac{\lambda(S)\pi(f)}{\pi(h)} \quad \textit{for all} \quad m \in \mathbb{N}.$$

(ii) *If λ is small and*

$$\text{(3.5)} \qquad \limsup_{n\to\infty} \frac{\nu P^{nk} f}{\nu P^{nk} h} \leqslant \frac{\pi(f)}{\pi(h)},$$

then we have (3.4).

PROOF. We prove only (i), since the proof of (ii) is similar. Assume first that $k = 1$ in (M). Writing $f = P^i h$ in (3.2) we get

$$\lim_{N\to\infty}\lim_{n\to\infty} \frac{s_{n,N}(\lambda, P^i h)}{u(n+1)} = \lambda(S)\frac{\pi(P^i h)}{\pi(h)} = \lambda(S) \qquad \text{for any} \quad i.$$

Since by assumptions $\limsup_{n\to\infty} \lambda P^{n+i} h/\nu P^n h \leqslant \lambda(S)$, we have

$$\lim_{N\to\infty}\lim_{n\to\infty} \frac{r_{n,N}(\lambda, P^i h)}{u(n+1)} = 0.$$

Since f is small, there exist γ, $I < \infty$ such that $P^m f \leqslant \gamma\Sigma_{i=0}^{I} P^i h$. Hence

$$\lim_{N\to\infty}\limsup_{n\to\infty} \frac{r_{n,N}(\lambda, P^m f)}{u(n+1)} = 0$$

and

$$\lim_{n\to\infty} \frac{\lambda P^{n+m} f}{\nu P^n h} = \lim_{N\to\infty}\lim_{n\to\infty} \frac{s_{n,N}(\lambda, P^m f)}{u(n+1)} = \frac{\lambda(S)\pi(f)}{\pi(h)}.$$

Now allow k in (M) to be arbitrary. By Lemma 2.2, for any $q \in \mathbb{N}$, the function $P^q f$ is small w.r.t. the k-step chain $\{X_{nk}\}$. From the proof of the case $k = 1$ it now follows that for all $q \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\lambda P^{nk+q} f}{\nu P^{nk} h} = \frac{\lambda(S)\pi(P^q f)}{\pi(h)} = \frac{\lambda(S)\pi(f)}{\pi(h)}. \tag{3.6}$$

Since $\lambda = \nu$ clearly satisfies (3.3), and h is small, we have also (3.6) with $\lambda = \nu$, $f = h$. This and (3.6) then give us (3.4). ▯

THEOREM 3. *If λ, μ, f and g all are small, then*

$$\lim_{n\to\infty} \frac{\lambda P^{n+m} f}{\mu P^n g} = \frac{\lambda(S)\pi(f)}{\mu(S)\pi(g)} \quad \textit{for all } m \in \mathbb{N} \tag{3.7}$$

(*provided that the* r.h.s. *is well defined*).

PROOF. The inequality (3.3) is trivially satisfied with $\lambda = \nu$. Hence, by Theorem 2 (i)

$$\lim_{n\to\infty} \frac{\nu P^n f}{\nu P^n h} = \pi(f),$$

and so (3.5) holds. By Theorem 2 (ii) we get (3.4). A similar result holds for μ and g, from which the assertion easily follows. ▯

The following corollary shows us that in this context small measures and functions (for general chains) correspond to individual points of a countable state space.

COROLLARY 3.2. (Orey (1961)). *Let S be a countable set, and assume that $\{X_n\}$ is an aperiodic irreducible recurrent Markov chain on S. If for some state, say $a \in S$,*

$$\lim_{n\to\infty} \frac{P^{n+1}(a, a)}{P^n(a, a)} = 1 \tag{3.8}$$

(*we omit the brackets in the notation of singletons*), *then for any states $x, y, z, u \in S$, any $m \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{P^{n+m}(x, y)}{P^n(z, u)} = \frac{\pi(y)}{\pi(u)}.$$

PROOF. This result is easy to deduce from our general result of Theorem 3, since in the case of countable S, the minorization assumption (M) is automatically satisfied with $k \in \mathbb{N}_+$ such that $\alpha = P^k(a, a) > 0$, $h = \alpha 1_{\{a\}}$, $\nu = \varepsilon_a$; and clearly for any $b \in S$, the indicator $1_{\{b\}}$ is a small function as well as ε_b is a small probability measure. ▯

Finally we shall briefly discuss the case when the limit in (3.7) exists for *all probability measures* λ and μ, and for all *small functions* f and g. The proof follows closely that of Orey (1971), Theorem 1.3 on page 78. Lin's (1976) Corollary 2.3 is also close to ours dealing with the case $f = g$.

THEOREM 4. *The following two conditions are equivalent*:

(i)

$$\limsup_{n\to\infty} \left\| \frac{P^{nk}h}{\nu P^{nk}h} \right\|_\infty < \infty;$$

(ii) (3.7) *holds for some* (*or, equivalently, for all*) *m, for all probability measures* λ *and* μ, *and for all small functions f and g.*

PROOF. (i) $\Rightarrow$ (ii): Let $N \in \mathbb{N}_+$, $\gamma < \infty$ be such that

$$\gamma = \sup_{n\geq N} \left\| \frac{P^{nk}h}{\nu P^{nk}h} \right\|_\infty < \infty.$$

For any probability measure λ, any fixed $m \in \mathbb{N}$, we have (cf. Orey (1971), page 79):

$$\limsup_{n\to\infty} \left| \frac{\lambda P^{nk}h - \nu P^{nk}h}{\nu P^{nk}h} \right|$$

$$= \limsup_{n\to\infty} \left| (\lambda P^{mk} - \nu P^{mk}) \frac{P^{(n-m)k}h}{\nu P^{(n-m)k}h} \frac{u(n-m+1)}{u(n+1)} \right|$$

$$\leq \gamma \| \lambda P^{mk} - \nu P^{mk} \|.$$

Letting $m \to \infty$, we immediately get (3.3). Theorem 2 (i) now gives (3.7).

(ii) $\Rightarrow$ (i): The proof is completely similar to that of Orey (1971), Theorem 1.3, page 79. $\square$

4. *R*-null recurrent chains and quasi-stationary distributions. In fact the preceding results could as well have been formulated for a larger class of positive operators. *Let us now drop the assumptions that P is stochastic and ϕ-recurrent.* We only need to assume:

(i) $\phi(A) > 0 \Rightarrow \sum_{n=0}^{\infty} P^n(x, A) \equiv \infty$;

(ii) there exists $0 \leq e < \infty$, $e \not\equiv 0(\psi\text{-a.e.})$ such that $Pe \leq e(\psi\text{-a.e.})$.

In Tweedie's (1974a) terminology (cf. Theorems 1, 3 and 8 of Tweedie) this means that the operator P is 1-recurrent. Condition (i) implies in particular that P is ϕ-irreducible, and hence by Orey's C-set theorem the minorization assumption (M) is satisfied for some k, h and ν. Similarly as in the case when P is ϕ-recurrent, we can prove that the measure $\pi = \nu Q$ is σ-finite, equivalent to ψ and invariant, and that the function Qh is finite (ψ-a.e.), invariant (that is: $Qh(x) = PQh(x)$ for all $x \in S$) and equal to e (ψ-a.e. and up to scalar multiplication).

As an illustration we formulate Theorem 3 in this wider context:

THEOREM 3′. *Assume the conditions* (i) *and* (ii) *above, and that the limit* $\lim_{n\to\infty} u(n+1)/u(n)$ *exists* (*the value of this limit is necessarily equal to* 1). *Then for any small* λ, μ, f *and* g

$$\lim_{n\to\infty} \frac{\lambda P^{n+m}f}{\mu P^n g} = \frac{\lambda(e)\pi(f)}{\mu(e)\pi(g)} \qquad \textit{for all} \quad m \in \mathbb{N}.$$

PROOF. Similar to that of Theorem 3. □

If, for some $R > 0$, the kernel RP satisfies the conditions (i) and (ii), then P is called *R-recurrent* (see Tweedie (1974a)). Theorem 3′ automatically gives us a strong ratio limit theorem for R-recurrent chains.

Assume from now on that the Markov chain $\{X_n\}$ *is R-recurrent and that* $\lim_{n\to\infty} u(n+1)/u(n) = R^{-k}$ *exists.* We shall denote by π (resp. e) the unique invariant measure (resp. function) of the operator RP. Recall from Tweedie (1974a) the definitions of R-positive and R-null recurrence: the chain is R-positive (resp. R-null), if $\pi(e)$ is finite (resp. infinite).

Fix a set $E \in \mathfrak{F}^+$ and ask the following general question: given that at time n X_n belongs to E, what is the limiting conditional distribution of X_n on $(E, E \cap \mathfrak{F})$ as $n \to \infty$?

If E is chosen to be equal to the whole state space S, then the condition $X_n \in S$ means that the chain has not absorbed outside the state space up to time n (note that the transition kernel P is now allowed to be substochastic). In this case the preceding limit distribution (if it exists) is called the I-type *quasi-stationary distribution* of $\{X_n\}$. The reader is referred to Seneta and Vere-Jones (1966) (countable state space) and to Tweedie (1974b) (general state space) for a thorough discussion of the quasi-stationary distributions of Markov chains. However, in all papers dealing with the quasi-stationarity a basic assumption is that the chain is *R-positive recurrent.* We shall state results which hold true also in the R-null ($\pi(e) = \infty$) case. As is easily seen, the analysis of this case requires the existence of some kind of strong ratio limit theorem. Here we shall apply our Theorem 3 and we immediately get the following corollaries.

COROLLARY 4.1. *Let* $E \in \mathfrak{F}^+$ *be a small set* (*which means that* 1_E *is a small function*), *and let* λ *be a small probability measure. Then* $0 < \pi(E) < \infty$ *and for any measurable set* $A \subset E$

$$\lim_{n\to\infty} \mathbb{P}_\lambda\{X_n \in A | X_n \in E\} = \frac{\pi(A)}{\pi(E)}.$$

In particular, if the whole state space S is small (*which is, by Corollary* 2.4, *equivalent to the condition*

$$\inf_{x\in S} \Sigma_{n=0}^{N} P^n h(x) > 0 \qquad \textit{for some } N),$$

then $\pi(S) < \infty$, *and given that we start with a small probability measure* λ, *the I-type quasi-stationary distribution exists*:

$$\lim_{n\to\infty} \mathbb{P}_\lambda\{X_n \in A | X_n \in S\} = \frac{\pi(A)}{\pi(S)}. \tag{4.1}$$

Since the preceding corollary yields a new result even in the case when S is countable we also formulate it in that case.

COROLLARY 4.2. *Assume that S is countable, $\{X_n\}$ is irreducible R-recurrent, and that for some state $a \in S$*

$$\lim_{n\to\infty} \frac{P^{n+1}(a, a)}{P^n(a, a)} = R^{-1}.$$

Then for any small set $E \subset S$, $i \in S$, $A \subset E$

$$\lim_{n\to\infty} \mathbb{P}_i\{X_n \in A | X_n \in E\} = \frac{\pi(A)}{\pi(E)}.$$

In particular, if S is small, that is, for some $j \in S$, $N \in \mathbb{N}$

$$\inf_{i\in S} \Sigma_{n=0}^N P^n(i, j) > 0,$$

then the I-type quasi-stationary distribution exists: for all $i \in S$, $A \subset S$

$$\lim_{n\to\infty} \mathbb{P}_i\{X_n \in A | X_n \in S\} = \frac{\pi(A)}{\pi(S)}. \tag{4.2}$$

EXAMPLE. If $\{X_n\}$ is 1-recurrent and the state space S is small, then it is easily seen that the chain is necessarily 1-positive. One might suspect that this would hold also for $R > 1$, in which case our quasi-stationarity results (4.1) and (4.2) would contribute nothing to the earlier theory. However, we can construct an example which shows that this conjecture is false.

Let $S = \mathbb{N}$, and define for all $i, j \in \mathbb{N}$ the transition probabilities $P(i, j)$ as follows:

$$\begin{aligned} P(i, 0) &= \frac{3}{\pi^2}, \\ P(i, i+1) &= \frac{(i+1)^2}{2(i+2)^2}, \\ P(i, j) &= 0 \qquad \text{otherwise.} \end{aligned} \tag{4.3}$$

The condition (4.3) immediately implies that the whole state space $\mathbb{N}$ is small. It is also easily seen that the chain is 2-null recurrent. Define $u(n) = P^n(0, 0)$ and

$$\begin{aligned} f_n &= \mathbb{P}_0\{X_i \neq 0 \text{ for } 1 \leqslant i < n, X_n = 0\} \\ &= \mathbb{P}_0\{X_i = i \text{ for } 1 \leqslant i < n, X_n = 0\} \\ &= \frac{6}{\pi^2 2^n n^2}. \end{aligned}$$

Hence $\lim_{n\to\infty} f_{n+1}/f_n = \frac{1}{2}$, from which we can easily conclude by using Theorem 4.1 of Orey (1971), page 96, that $\lim_{n\to\infty} u(n+1)/u(n) = \frac{1}{2}$. By Corollary 4.2, for this particular chain the I-type quasi-stationary distribution (4.2) exists and is equal to the left invariant vector of the matrix $2P$:

$$\frac{\pi(i)}{\pi(\mathbb{N})} = \frac{6}{\pi^2(i+1)^2}, \qquad i \in \mathbb{N}.$$

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