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LIMIT THEOREMS FOR ABSORPTION TIMES OF GENETIC MODELS

BY S. N. ETHIER

Michigan State University

We consider a sequence of Markov chains occurring in population genetics (viz., the so-called multiallelic Wright-Fisher models) that converges weakly to a multidimensional diffusion process. Certain absorption times, which arise naturally in connection with the genetic models, are shown to also converge weakly. This extends a result of Guess. Corollaries include convergence of moments of absorption times and convergence of absorption probabilities. The latter results are used implicitly in population genetics.

1. Introduction.

Let

$$K = \{x \in R^d : x_1 \geqslant 0, \cdots, x_d \geqslant 0, \Sigma_{i=1}^d x_i \leqslant 1\},$$

where d is a fixed positive integer, and define Ω to be the space of continuous paths ω: $[0, \infty) \to K$, endowed with the topology of uniform convergence on compact sets. In Section 2, we construct a sequence $\{P^{(N)}\}$ of Borel probability measures on Ω, induced by a sequence of (suitably scaled and interpolated) discrete-parameter Markov chains occurring in population genetics, viz., the so-called multiallelic Wright-Fisher models (allowing for selection but disregarding mutation and migration). Here N represents the population size (or a fixed multiple thereof).

The problem is to investigate the asymptotic behavior as $N \to \infty$ of the $P^{(N)}$-distributions of certain absorption times that arise naturally in connection with the genetic models. Sato [11] has shown that there exists a Borel probability measure P on Ω, induced by a diffusion process in K, such that $P^{(N)} \Rightarrow P$ as $N \to \infty$ (the symbol $\Rightarrow$ denotes weak convergence). Our results include the following. For each $t \geqslant 0$, define the coordinate map $x(t) : \Omega \to K$ by $x(t)(\omega) = \omega(t)$, let $V_{(d)} = \{v^{(1)}, \cdots, v^{(d+1)}\}$ be the set of vertices of K, and define $\tau_{(d)} : \Omega \to [0, \infty]$ by $\tau_{(d)} = \inf\{t \geqslant 0 : x(t) \in V_{(d)}\}$, where $\inf \varnothing = \infty$. Then

$$P^{(N)} \circ \tau_{(d)}^{-1} \Rightarrow P \circ \tau_{(d)}^{-1} \quad \text{as} \quad N \to \infty, \tag{1.1}$$

$$\lim_{N\to\infty} P^{(N)}\text{-}E[\tau_{(d)}^k] = P\text{-}E[\tau_{(d)}^k],^1 \qquad k = 1, 2, \cdots, \tag{1.2}$$

and

$$\lim_{N\to\infty} P^{(N)}\{x(\tau_{(d)}) = v^{(i)}\} = P\{x(\tau_{(d)}) = v^{(i)}\}, \qquad i = 1, \cdots, d+1, \tag{1.3}$$

despite the fact that $\tau_{(d)}$ is P-a.s. discontinuous.

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[1]Expectations $E^P[\xi]$ are denoted by P-$E[\xi]$ in order to avoid excessive superscripts.

We note that $P^{(N)} \circ \tau_{(d)}^{-1}$ is the fixation time distribution for the Nth genetic model, where time is measured in units of N generations (see Section 3 for details). Because these distributions are rather difficult to deal with, geneticists usually approximate them by $P \circ \tau_{(d)}^{-1}$, the advantage of the latter being that expressions such as those on the right sides of equations (1.2) and (1.3) can occasionally be explicitly evaluated by solving certain differential equations subject to certain boundary conditions. The present work, which extends a result of Guess [5] (see Remark 6.1), may be regarded as providing a mathematical justification for these diffusion approximations.

2. Definitions and preliminary results. Let N be a positive integer. Put $K_N = \{\alpha/N \in K : \alpha \in Z^d\}$, where Z^d is the d-dimensional integer lattice. Define Ω_N to be the space of functions $\omega : Z_+ \to K_N$, endowed with the topology of pointwise convergence on $Z_+ = \{0, 1, \cdots\}$, and let $\mathfrak{M}^{(N)}$ be the class of Borel sets in Ω_N. For each $n \in Z_+$, define the coordinate map $X(n) : \Omega_N \to K_N$ by $X(n)(\omega) = \omega(n)$, and let $\mathfrak{M}_n^{(N)}$ be the σ-algebra of subsets of Ω_N generated by $X(m)$, $m = 0, \cdots, n$.

Given $\gamma^{(N)} : K_N \to K$ and $x \in K_N$, let $Q_x^{(N)}$ be the unique probability measure on $(\Omega_N, \mathfrak{M}^{(N)})$ such that

$$Q_x^{(N)}\{X(0) = x\} = 1 \tag{2.1}$$

and

$$Q_x^{(N)}\{X(n+1) = \alpha/N \mid \mathfrak{M}_n^{(N)}\} = N!\Pi_{i=1}^{d+1}(\alpha_i!)^{-1}[\gamma_i^{(N)}(X(n))]^{\alpha_i}, \qquad \alpha/N \in K_N,\ n \in Z_+, \tag{2.2}$$

where $\gamma_{d+1}^{(N)} = 1 - \gamma_1^{(N)} - \cdots - \gamma_d^{(N)}$ and $\alpha_{d+1} = N - \alpha_1 - \cdots - \alpha_d$; in particular, $\{X(n) : n \in Z_+\}$ is a Markov chain on the probability space $(\Omega_N, \mathfrak{M}^{(N)}, Q_x^{(N)})$. Of course, we can express (2.2) verbally by saying that, for each $n \in Z_+$, the conditional $Q_x^{(N)}$-distribution of $NX(n+1)$ given $\mathfrak{M}_n^{(N)}$ is (d-variate) multinomial with mean vector $N\gamma^{(N)}(X(n))$ and sample size N.

Recall that $\Omega = C([0, \infty), K)$ has the topology of uniform convergence on compact sets, and let $\mathfrak{M}$ be the class of Borel sets in Ω. Recall also that, for each $t \geq 0$, $x(t) : \Omega \to K$ is defined by $x(t)(\omega) = \omega(t)$, and define $x(\infty) : \Omega \to K$ by $x(\infty)(\omega) = ((d+1)^{-1}, \cdots, (d+1)^{-1})$ (then, in particular, $x(\tau_{(d)})$ in (1.3) is defined on all of Ω, not just almost surely). For each $t \geq 0$, let $\mathfrak{M}_t$ be the σ-algebra of subsets of Ω generated by $x(s)$, $0 \leq s \leq t$. Define the map $\Phi_N : \Omega_N \to \Omega$ by

$$x(t) \circ \Phi_N = X([Nt]) + (Nt - [Nt])(X([Nt]+1) - X([Nt])), \qquad t \geq 0,$$

where $[Nt]$ denotes the integral part of Nt, and note that Φ_N is Borel measurable (in fact, it is continuous).

Given $\gamma^{(N)} : K_N \to K$ and $x \in K_N$, let $P_x^{(N)}$ be the probability measure on $(\Omega, \mathfrak{M})$ defined by

$$P_x^{(N)} = Q_x^{(N)} \circ \Phi_N^{-1}, \tag{2.3}$$

where $Q_x^{(N)}$ is the probability measure on $(\Omega_N, \mathfrak{M}^{(N)})$ defined in terms of $\gamma^{(N)}$ and x

by (2.1) and (2.2). In other words, $P_x^{(N)}$ is the $Q_x^{(N)}$-distribution on $(\Omega, \mathcal{M})$ of the process obtained from $\{X(Nt) : t = 0, 1/N, 2/N, \cdots\}$ by linear interpolation.

Given $b \in C(K, R^d)$, form the degenerate elliptic operator

$$(2.4) \quad L = \tfrac{1}{2}\Sigma_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\frac{\partial^2}{\partial x_i \partial x_j} + \Sigma_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \qquad \mathfrak{D}(L) = C^2(K).$$

A solution to the martingale problem for L starting at $x \in K$ is a probability measure P_x on $(\Omega, \mathcal{M})$ for which $P_x\{x(0) = x\} = 1$ and

$$(2.5) \qquad \{f(x(t)) - \textstyle\int_0^t (Lf)(x(s))\, ds, \mathcal{M}_t : t \geq 0\}$$

is a P_x-martingale for every $f \in C^2(K)$. The following result, due to Stroock and Varadhan [12], indicates that, if the martingale problem for L is well posed, then its solution may be regarded as the diffusion process in K with generator L.

PROPOSITION 2.1. *Let $b \in C(K, R^d)$, and define L by* (2.4). *If the martingale problem for L starting at x has a unique solution P_x for every $x \in K$, then $\{P_x : x \in K\}$ is a homogeneous, strong Markov family.*

We will be concerned specifically with the case in which b is determined by a sequence of functions $\gamma^{(N)} : K_N \to K$ (defined for $N = 1, 2, \cdots$) as follows:

$$(2.6) \qquad \lim_{N\to\infty} \sup_{x \in K_N} |N(\gamma^{(N)}(x) - x) - b(x)| = 0.$$

It should be noted that in this case b satisfies the inward drift condition

$$(2.7) \qquad b_i(x) \geq 0 \text{ if } x \in K \text{ and } x_i = 0, \qquad i = 1, \cdots, d+1,$$

where

$$(2.8) \qquad x_{d+1} = 1 - \Sigma_{i=1}^{d} x_i, \qquad x \in K,$$

and

$$(2.9) \qquad b_{d+1} = -\Sigma_{i=1}^{d} b_i, \qquad b \in C(K, R^d).$$

The conventions (2.8) and (2.9) are used throughout.

THEOREM 2.1. *For $N = 1, 2, \cdots$, let $\gamma^{(N)} : K_N \to K$, and define the family $\{P_x^{(N)} : x \in K_N\}$ of probability measures on $(\Omega, \mathcal{M})$ in terms of $\gamma^{(N)}$ by* (2.3). *Suppose that $b \in C(K, R^d)$ satisfies* (2.6), *define L by* (2.4), *and let $x \in K$. If the martingale problem for L starting at x has a unique solution P_x, and if $x_N \in K_N$ for $N = 1, 2, \cdots$ and $x_N \to x$ as $N \to \infty$, then $P_{x_N}^{(N)} \Rightarrow P_x$ as $N \to \infty$.*

PROOF. For $N = 1, 2, \cdots$, define the family $\{Q_x^{(N)} : x \in K_N\}$ of probability measures on $(\Omega_N, \mathcal{M}^{(N)})$ in terms of $\gamma^{(N)}$ by (2.1) and (2.2). Using (2.6), it is easily verified that

$$\lim_{N\to\infty} \sup_{x \in K_N} |NQ_x^{(N)}\text{-}E[X_i(1) - x_i] - b_i(x)| = 0,$$

$$\lim_{N\to\infty} \sup_{x \in K_N} |NQ_x^{(N)}\text{-}E[(X_i(1) - x_i)(X_j(1) - x_j)] - x_i(\delta_{ij} - x_j)| = 0,$$

$$\lim_{N\to\infty} \sup_{x \in K_N} NQ_x^{(N)}\text{-}E[(X_i(1) - x_i)^4] = 0,$$

for $i, j = 1, \cdots, d$. The third of these equations implies that

$$\lim_{N\to\infty} \sup_{x\in K_N} NQ_x^{(N)}\{|X(1) - x| > \varepsilon\} = 0$$

for every $\varepsilon > 0$, so the theorem is a consequence of the invariance principle of Stroock and Varadhan [12] (cf. [10]).

The next result is useful in connection with Theorem 2.1. The proof is given in [3].

PROPOSITION 2.2. *Suppose that $b \in C^4(K, R^d)$ satisfies (2.7),[2] and define L by (2.4). Then the martingale problem for L starting at x has a unique solution for every $x \in K$.*

PROPOSITION 2.3. *Given $\sigma \in C(K, R^{d+1})$, define $b : K \to R^d$ by*

$$b_i(x) = x_i\big(\sigma_i(x) - \textstyle\sum_{j=1}^{d+1} x_j\sigma_j(x)\big) \tag{2.10}$$

and L by (2.4). Then the martingale problem for L starting at x has a unique solution P_x for every $x \in K$. Define

$$L_0 = \tfrac{1}{2}\textstyle\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\dfrac{\partial^2}{\partial x_i \partial x_j}, \qquad \mathfrak{D}(L_0) = C^2(K), \tag{2.11}$$

and for each $x \in K$, let Q_x be the unique solution to the martingale problem for L_0 starting at x. Then $P_x \ll Q_x$ on $\mathfrak{M}_t$ for each $x \in K$ and $t \geqslant 0$.

PROOF. If we define $a : K \to S_d$ by $a_{ij}(x) = x_i(\delta_{ij} - x_j)$ (S_d is the set of symmetric, nonnegative definite, $d \times d$ matrices) and $c : K \to R^d$ by $c_j(x) = \sigma_j(x) - \sigma_{d+1}(x)$, then

$$\begin{aligned} b_i(x) &= x_i\big(\sigma_i(x) - \textstyle\sum_{j=1}^{d} x_j\sigma_j(x) - \big(1 - \sum_{j=1}^{d} x_j\big)\sigma_{d+1}(x)\big) \\ &= \textstyle\sum_{j=1}^{d} x_i(\delta_{ij} - x_j)\big(\sigma_j(x) - \sigma_{d+1}(x)\big), \\ &= \textstyle\sum_{j=1}^{d} a_{ij}(x)c_j(x), \qquad x \in K,\ i = 1, \cdots, d. \end{aligned}$$

Therefore the conclusions follow from Proposition 2.2 and the Cameron-Martin formula (cf. [12]).

The class of genetic models considered in this paper is described by the following theorem, which is an immediate consequence of Theorem 2.1; in fact, the theorem is a special case of a result of Sato [11]. The interpretation of these genetic models is indicated in Section 3.

THEOREM 2.2. *Let $\sigma \in C(K, R^{d+1})$. For $N = 1, 2, \cdots$, define $\gamma^{(N)} : K_N \to K$ by*

$$\gamma_i^{(N)}(x) = x_i\left(1 + \frac{\sigma_i(x)}{N \vee N_0}\right) \Big/ \textstyle\sum_{j=1}^{d+1} x_j\left(1 + \dfrac{\sigma_j(x)}{N \vee N_0}\right), \tag{2.12}$$

where N_0 is the smallest positive integer N such that $1 + N^{-1}\sigma_i(x) > 0$ for all $x \in K$ and $i = 1, \cdots, d + 1$, and let $\{P_x^{(N)} : x \in K_N\}$ be the family of probability measures on $(\Omega, \mathfrak{M})$ defined in terms of $\gamma^{(N)}$ by (2.3). Define $b : K \to R^d$ by (2.10)

[2] In fact, it suffices to assume that $b : K \to R^d$ is Lipschitz continuous and satisfies (2.7).

and L by (2.4), and for each $x \in K$, let P_x be the unique solution to the martingale problem for L starting at x. Suppose that $x_N \in K_N$ for $N = 1, 2, \cdots, x \in K$, and $x_N \to x$ as $N \to \infty$. Then $P_{x_N}^{(N)} \Rightarrow P_x$ as $N \to \infty$.

We conclude this section by defining the hitting times to be considered. For $i = 1, \cdots, d+1$, let

$$V_i = \{x \in R^d : x_i = 0\}, \quad v^{(i)} = (\delta_{i1}, \cdots, \delta_{id}) \in K,$$

and define $\tau_i : \Omega \to [0, \infty]$ by

$$\tau_i = \inf\{t \geq 0 : x(t) \in V_i \text{ or } x(t) = v^{(i)}\}. \tag{2.13}$$

For $j = 1, \cdots, d$, let

$$V_{(j)} = \cup \{V_{i_1} \cap \cdots \cap V_{i_j} : 1 \leq i_1 < \cdots < i_j \leq d+1\},$$

and define $\tau_{(j)} : \Omega \to [0, \infty]$ by

$$\tau_{(j)} = \inf\{t \geq 0 : x(t) \in V_{(j)}\}. \tag{2.14}$$

Of course, $\tau_1, \cdots, \tau_{d+1}$ and $\tau_{(1)}, \cdots, \tau_{(d)}$ are all Borel measurable. We note also that each of these hitting times is in fact an absorption time with respect to each of the probability measures $P_x^{(N)}$ on $(\Omega, \mathfrak{M})$ defined in the statement of Theorem 2.2.

3. Genetic interpretation. Here we consider two special cases of the class of genetic models described by Theorem 2.2. Let $s = (s_{ij})$ be a real symmetric $(d+1) \times (d+1)$ matrix (resp. $s \in R^{d+1}$), and define $\sigma : K \to R^{d+1}$ by

$$\sigma_i(x) = \Sigma_{j=1}^{d+1} s_{ij} x_j \qquad (\text{resp. } \sigma_i(x) = s_i).$$

Let N be an even positive integer (resp. a positive integer) such that $1 + N^{-1}\sigma_i(x) > 0$ for each $x \in K$ and $i = 1, \cdots, d+1$, and define $\gamma^{(N)} : K_N \to K$ by (2.12). Note that

$$\gamma_i^{(N)}(x) = \Sigma_{j=1}^{d+1} x_i x_j \left(1 + \frac{s_{ij}}{N}\right) \Big/ \Sigma_{k,l=1}^{d+1} x_k x_l \left(1 + \frac{s_{kl}}{N}\right)$$

$$\left(\text{resp. } \gamma_i^{(N)}(x) = x_i\left(1 + \frac{s_i}{N}\right) \Big/ \Sigma_{j=1}^{d+1} x_j \left(1 + \frac{s_j}{N}\right)\right)$$

for all $x \in K_N$ and $i = 1, \cdots, d$. Let $x \in K_N$, and define the probability measure $Q_x^{(N)}$ on $(\Omega_N, \mathfrak{M}^{(N)})$ by (2.1) and (2.2).

The Markov chain $\{X(n) : n \in Z_+\}$ on the probability space $(\Omega_N, \mathfrak{M}^{(N)}, Q_x^{(N)})$ is known as a Wright-Fisher model and has the following genetic interpretation (see [4] for definitions). Consider a single locus, with alleles $A_1, \cdots, A_{d+1}$, in a randomly mating monoecious diploid (resp. haploid) population in which there are exactly $N/2$ (resp. N) individuals in each generation. Suppose that the genotype A_iA_j has fitness $1 + N^{-1}s_{ij}$ for $i, j = 1, \cdots, d+1$ (resp. A_i has fitness $1 + N^{-1}s_i$ for $i = 1, \cdots, d+1$), and that neither mutation nor migration is present. Then, for $i = 1, \cdots, d$ and $n \in Z_+$, $X_i(n)$ represents the relative frequency of A_i genes among the N genes in the nth generation (at the locus under consideration).

As mentioned earlier, the probability measure $P_x^{(N)}$ on $(\Omega, \mathfrak{M})$ defined by (2.3) is just the $Q_x^{(N)}$-distribution of the process obtained from $\{X(Nt) : t = 0, 1/N, 2/N, \cdots\}$ by linear interpolation. The important point here is that, on the probability space $(\Omega, \mathfrak{M}, P_x^{(N)})$, time is measured in units of N generations. The relevance of the absorption times defined at the end of Section 2 should now be clear. For $i = 1, \cdots, d+1$, τ_i represents the first time at which the allele A_i is either lost or fixed, while for $j = 1, \cdots, d$, $\tau_{(j)}$ represents the first time at which at least j alleles are lost; in particular, $\tau_{(d)}$ represents the fixation time.

4. Some properties of the limiting diffusion. In this section we derive several properties of the limiting diffusion process of Theorem 2.2.

PROPOSITION 4.1. *Given $\sigma \in C(K, R^{d+1})$, put $\lambda = \max(\beta^2/2\eta, 2\beta)$, where $\beta = \sup_{x \in K} \Sigma_{j=1}^{d+1} |\sigma_j(x)|$ and $\eta e^{\eta} = 3/32$, and let $f_0 \in C([0, 1]) \cap C^2((0, 1))$ be the unique solution to the differential equation*

$$\tag{4.1} \tfrac{1}{2}u(1-u)f_0''(u) + \lambda u(1-u)(1-2u)f_0'(u) = -2, \qquad 0 < u < 1,$$

with boundary conditions $f_0(0) = f_0(1) = 0$. Define $b : K \to R^d$ by (2.10) and L by (2.4), and for each $x \in K$, let P_x be the unique solution to the martingale problem for L starting at x. Then, for $i = 1, \cdots, d+1$,

$$\tag{4.2} P_x\text{-}E[\tau_i] \leqslant f_0(x_i), \qquad x \in K,$$

where $\tau_i : \Omega \to [0, \infty]$ is defined by (2.13).

PROOF. Let $H = \{h \in C([0, 1]) : h(u) = h(1-u), 0 \leqslant u \leqslant 1\}$. Given $g \in H$ with $0 \leqslant g \leqslant 1$, let $f \in C([0, 1]) \cap C^2((0, 1))$ be the unique solution to the differential equation

$$\tfrac{1}{2}u(1-u)f''(u) + \lambda u(1-u)(1-2u)f'(u) = -2g(u), \quad 0 < u < 1,$$

with boundary conditions $f(0) = f(1) = 0$. It is then easily verified that $f = Bg$, where B is the linear operator on $C([0, 1])$ defined by

$$(Bh)(u) = \int_0^u e^{-2\lambda v(1-v)} \int_v^{\frac{1}{2}} \frac{4h(w)}{w(1-w)} e^{2\lambda w(1-w)}\, dw\, dv.$$

In particular, $f' \geqslant 0$ on $(0, \frac{1}{2}]$, $f' \leqslant 0$ on $[\frac{1}{2}, 1)$, and

$$\tag{4.3} \sup_{|u-\frac{1}{2}|<\varepsilon} |f'(u)| \leqslant 4\varepsilon e^{2\varepsilon^2\lambda}\left(\tfrac{1}{4} - \varepsilon^2\right)^{-1}$$

for $0 \leqslant \varepsilon < \frac{1}{2}$. Fix $i \in \{1, \cdots, d+1\}$. By (2.8) and (2.9),

$$|b_i(x)| \leqslant x_i\big((1-x_i)|\sigma_i(x)| + \Sigma_{j=1;\, j\neq i}^{d+1} x_j|\sigma_j(x)|\big) \leqslant \beta x_i(1-x_i)$$

for all $x \in K$, so we conclude from (4.3) with $\varepsilon = \beta/2\lambda$ ($\varepsilon = 0$ if $\lambda = 0$) that

$$\begin{aligned}
&\tfrac{1}{2}x_i(1-x_i)f''(x_i) + b_i(x)f'(x_i) + 2g(x_i) \\
&\qquad = x_i(1-x_i)\left[\frac{b_i(x)}{x_i(1-x_i)} - \lambda(1-2x_i)\right]f'(x_i) \\
(4.4)\qquad &\qquad \leq \tfrac{1}{4}(\beta + 2\varepsilon\lambda)4\varepsilon e^{2\varepsilon^2\lambda}\left(\tfrac{1}{4} - \varepsilon^2\right)^{-1} \\
&\qquad \leq 2\eta e^{\eta}/\tfrac{3}{16} = 1, \qquad x \in K, 0 < x_i < 1.
\end{aligned}$$

We remark that the special case $g \equiv 1$ yields

$$(4.5)\qquad \tfrac{1}{2}x_i(1-x_i)f_0''(x_i) + b_i(x)f_0'(x_i) \leq -1, \quad x \in K, 0 < x_i < 1,$$

a result that is needed in Section 5.

Now choose a sequence $\{g_n\}_{n\geq 1} \subset H \cap C^1([0, 1])$ such that $0 \leq g_n \leq \chi_{(0,1)}$ for each n and $g_n \uparrow \chi_{(0,1)}$ as $n \uparrow \infty$. ($\chi_{(0,1)}$ is the indicator function of the interval (0, 1).) Define the sequence $\{f_n\}_{n\geq 1} \subset C([0, 1])$ by $f_n = Bg_n$, and note that $f_n \in C^2([0, 1])$ for each n and $f_n \uparrow f_0$ as $n \uparrow \infty$. Finally, define $\{g_n^i\}_{n\geq 1} \subset C(K)$, $\{f_n^i\}_{n\geq 1} \subset C^2(K)$, and $f_0^i \in C(K)$ by $g_n^i(x) = g_n(x_i)$, $f_n^i(x) = f_n(x_i)$, and $f_0^i(x) = f_0(x_i)$, and observe that

$$(Lf_n^i)(x) = \tfrac{1}{2}x_i(1-x_i)f_n''(x_i) + b_i(x)f_n'(x_i) \leq 1 - 2g_n^i(x)$$

for all $x \in K$ and $n \geq 1$; here we are using (2.8), (2.9), and (4.4). Thus, by the optional stopping theorem,

$$\begin{aligned}
P_x\text{-}E\left[f_n^i(x(\tau_i \wedge t))\right] &= f_n^i(x) + P_x\text{-}E\left[\textstyle\int_0^{\tau_i \wedge t}(Lf_n^i)(x(s))\,ds\right] \\
&\leq f_n^i(x) + P_x\text{-}E\left[\textstyle\int_0^{\tau_i \wedge t}\left(1 - 2g_n^i(x(s))\right)ds\right]
\end{aligned}$$

for each $x \in K$, $t \geq 0$, and $n \geq 1$. Letting $n \to \infty$, we find that

$$P_x\text{-}E\left[f_0^i(x(\tau_i \wedge t))\right] \leq f_0^i(x) - P_x\text{-}E\left[\tau_i \wedge t\right]$$

for each $x \in K$ and $t \geq 0$. It follows that $P_x\{\tau_i < \infty\} = 1$ for each $x \in K$, so since $f_0(0) = f_0(1) = 0$, we obtain (4.2) by letting $t \to \infty$.

PROPOSITION 4.2. *Let $b \in C(K, R^d)$, and define L by* (2.4). *If the martingale problem for L starting at x has a unique solution P_x for every $x \in K$, then, for $0 < \varepsilon < \frac{1}{2}$ and $i = 1, \cdots, d+1$, the stopping time $\tau_i^{(\varepsilon)} : \Omega \to [0, \infty]$, defined by $\tau_i^{(\varepsilon)} = \inf\{t \geq 0 : x(t) \notin G_i(\varepsilon)\}$, where*

$$(4.6)\qquad G_i(\varepsilon) = \{x \in R^d : \varepsilon < x_i < 1 - \varepsilon\},$$

is P_x-a.s. continuous for every $x \in K$.

PROOF. Fix $i \in \{1, \cdots, d+1\}$ and $0 < \varepsilon < \frac{1}{2}$, and define $\bar{\tau}_i^{(\varepsilon)} : \Omega \to [0, \infty]$ by $\bar{\tau}_i^{(\varepsilon)} = \inf\{t \geq 0 : x(t) \notin \overline{G_i}(\varepsilon)\}$, where $\overline{G_i}(\varepsilon)$ denotes the closure of $G_i(\varepsilon)$. Clearly, $\tau_i^{(\varepsilon)}$ is continuous at each path $\omega \in \Omega$ for which $\tau_i^{(\varepsilon)}(\omega) = \bar{\tau}_i^{(\varepsilon)}(\omega)$. By Proposition 2.1,

$$P_x\{\tau_i^{(\varepsilon)} = \bar{\tau}_i^{(\varepsilon)}\} = P_x\text{-}E\left[\chi_{\{\tau_i^{(\varepsilon)} < \infty\}}P_{x(\tau_i^{(\varepsilon)})}\{\bar{\tau}_i^{(\varepsilon)} = 0\}\right] + P_x\{\tau_i^{(\varepsilon)} = \infty\}$$

for every $x \in K$, so it suffices to show that $P_x\{\bar{\tau}_i^{(\varepsilon)} = 0\} = 1$ for each $x \in K \cap \partial G_i(\varepsilon)$.

Fix such an x, and suppose that

$$P_x\{\bar{\tau}_i^{(\varepsilon)} > 0\} > 0. \tag{4.7}$$

We now show that this leads to a contradiction. Since (2.5) is a P_x-martingale for every $f \in C^2(K)$, it is easily verified that

$$\{f(x(t)) - \textstyle\int_0^t (Lf)(x(s))\,ds,\ \mathfrak{M}_{t+} : t \geqslant 0\}$$

is also a P_x-martingale for every $f \in C^2(K)$. But $\bar{\tau}_i^{(\varepsilon)}$ is an $\{\mathfrak{M}_{t+}\}$ stopping time, so by the optional stopping theorem,

$$P_x\text{-}E\big[f(x(t \wedge \bar{\tau}_i^{(\varepsilon)}))\big] = f(x) + P_x\text{-}E\big[\textstyle\int_0^{t\wedge\bar{\tau}_i^{(\varepsilon)}}(Lf)(x(s))\,ds\big]$$

for every $f \in C^2(K)$ and $t \geqslant 0$. Consequently, if $f \in C^2(K)$ satisfies

$$f(x) \geqslant \sup_{y \in K \cap \bar{G}_i(\varepsilon)} f(y), \tag{4.8}$$

then

$$P_x\text{-}E\big[\textstyle\int_0^{t\wedge\bar{\tau}_i^{(\varepsilon)}}(Lf)(x(s))\,ds\big] \leqslant 0$$

for all $t \geqslant 0$; dividing by t and letting $t \downarrow 0$, we find that $(Lf)(x)P_x\{\bar{\tau}_i^{(\varepsilon)} > 0\} \leqslant 0$, which, by (4.7), implies that

$$(Lf)(x) \leqslant 0. \tag{4.9}$$

However, there exist functions $f \in C^2(K)$ satisfying (4.8) but not (4.9). To see this, define $\varphi : K \to R^1$ by $\varphi(y) = (y_i - \varepsilon)(1 - \varepsilon - y_i)$, where $y_{d+1} = 1 - y_1 - \cdots - y_d$, and choose $h \in C^2(R^1)$ such that $\chi_{(-\infty,1/2]} \leqslant h \leqslant \chi_{(-\infty,1]}$. Then, for each $\delta > 0$, the function $f_\delta \in C^2(K)$ given by $f_\delta = (\varphi^2 - \delta\varphi)h(\varphi/\delta)$ satisfies (4.8), but $(Lf_\delta)(x) = \varepsilon(1-\varepsilon)(1-2\varepsilon)^2 - \delta(L\varphi)(x)$, and this is positive for δ sufficiently small.

Recall that K, Ω, $x(\cdot)$, L_0, V_i, $V_{(j)}$, and $\tau_{(j)}$ all depend implicitly upon d, which is a fixed positive integer. In the remainder of this section, it will be convenient to indicate this dependence with a superscript. Thus, for $n = 1, \cdots, d$, we denote by $K^{(n)}$, $\Omega^{(n)}$, $x^{(n)}(\cdot)$, $L_0^{(n)}$, $V_i^{(n)}$, $V_{(j)}^{(n)}$, and $\tau_{(j)}^{(n)}$ the corresponding objects with n replacing d. (The absence of a superscript will imply that $n = d$.) Given $n \in \{1, \cdots, d\}$ and integers $1 \leqslant i_1 < \cdots < i_n \leqslant d+1$, define $\pi_{i_1 \cdots i_n} : K \to K^{(n)}$ by $\pi_{i_1\cdots i_n}(x) = (x_{i_1}, \cdots, x_{i_n})$ (recall (2.8)) and $\hat{\pi}_{i_1\cdots i_n} : \Omega \to \Omega^{(n)}$ by $\hat{\pi}_{i_1\cdots i_n}(\omega) = \pi_{i_1\cdots i_n} \circ \omega$.

LEMMA 4.1. *Let $n \in \{1, \cdots, d\}$ and $1 \leqslant i_1 < \cdots < i_n \leqslant d+1$. Define L_0 by (2.11), and let $x \in K$. If Q_x solves the martingale problem for L_0 starting at x, then $Q_x \circ \hat{\pi}_{i_1\cdots i_n}^{-1}$ solves the martingale problem for $L_0^{(n)}$ starting at $\pi_{i_1\cdots i_n}(x)$.*

PROOF. The result follows from the easily verified fact that $L_0(f \circ \pi_{i_1\cdots i_n}) = (L_0^{(n)}f) \circ \pi_{i_1\cdots i_n}$ for every $f \in C^2(K^{(n)})$.

PROPOSITION 4.3. *Given* $\sigma \in C(K, R^{d+1})$, *define* $b : K \to R^d$ *by* (2.10) *and* L *by* (2.4), *and for each* $x \in K$, *let* P_x *be the unique solution to the martingale problem for* L *starting at* x. *Then the hitting times* $\tau_1, \cdots, \tau_{d+1}$ *and* $\tau_{(1)}, \cdots, \tau_{(d)}$ *defined at the end of Section* 2 *are in fact absorption times with respect to* P_x *for every* $x \in K$. *Moreover, for every* $x \in K$, *each of these hitting times either has a (proper) continuous* P_x*-distribution or is* P_x-a.s. *equal to zero.*

REMARK. The conclusion that $\tau_1, \cdots, \tau_{d+1}$ and $\tau_{(1)}, \cdots, \tau_{(d)}$ are P_x-a.s. finite for every $x \in K$ clearly implies that each of these hitting times is P_x-a.s. discontinuous for every $x \in K$.

PROOF. Define L_0 by (2.11), and for each $x \in K$, let Q_x be the unique solution to the martingale problem for L_0 starting at x. Fix $i \in \{1, \cdots, d+1\}$. By Lemma 4.1, $\{Q_x \circ \hat{\pi}_i^{-1} : x \in K\}$ may be regarded as the diffusion process in [0, 1] with generator $L_0^{(1)}$, so

$$1 = Q_x \circ \hat{\pi}_i^{-1}\{x^{(1)}(t) = 0 \text{ or } 1\}$$
$$= Q_x\{x(t) \in V_i \cup \{v^{(i)}\}\} = P_x\{x(t) \in V_i \cup \{v^{(i)}\}\}$$

for every $x \in K \cap (V_i \cup \{v^{(i)}\})$ and $t \geq 0$, and

$$0 = Q_x \circ \hat{\pi}_i^{-1}\{\tau_1^{(1)} = t\} = Q_x\{\tau_i = t\} = P_x\{\tau_i = t\}$$

for every $x \in K$ and $t > 0$, where, in both cases, the third equality follows from Proposition 2.3 together with the first two. By Proposition 4.1, $P_x\{\tau_i < \infty\} = 1$ for all $x \in K$, so Proposition 2.1 yields

$$P_x\{x(\tau_i + t) \in V_i \cup \{v^{(i)}\} \quad \text{for all} \quad t \geq 0\} = 1$$

for every $x \in K$.

Finally, the latter result implies that the conclusions for $\tau_{(1)}, \cdots, \tau_{(d)}$ follow from those for $\tau_1, \cdots, \tau_{d+1}$.

LEMMA 4.2. *Define* $L_0^{(2)}$ *by* (2.11), *and for each* $x \in K^{(2)}$, *let* Q_x *be the unique solution to the martingale problem for* $L_0^{(2)}$ *starting at* x. *Then*

$$Q_x\{x^{(2)}(\tau_{(1)}^{(2)}) = (0, 0)\} = 0 \tag{4.10}$$

for each $x \in K^{(2)}$ *except* $x = (0, 0)$, *where* $\tau_{(1)}^{(2)} : \Omega^{(2)} \to [0, \infty]$ *is defined by* (2.14).

PROOF. Let $K_0^{(2)}$ denote the interior of $K^{(2)}$. It is known that there exists a nonnegative $q \in C((0, \infty) \times K_0^{(2)} \times K_0^{(2)})$ such that

$$Q_x\{x^{(2)}(t) \in \Gamma\} = \int_\Gamma q(t, x, y)\, dy$$

for each Borel set $\Gamma \subset K_0^{(2)}$, $x \in K_0^{(2)}$, and $t > 0$. In fact, an explicit eigenfunction expansion for q has been obtained by Kimura [6]. However, here it will suffice to note that

$$\sup_{y \in K_0^{(2)}} \int_\delta^\infty q(s, x, y)\, ds < \infty$$

for every $x \in K_0^{(2)}$ and $\delta > 0$.

For each $\varepsilon \in (0, 1)$, define $h_\varepsilon \in C^2(K^{(2)})$ by $h_\varepsilon(x) = \log(x_1 + x_2 + \varepsilon)/\log \varepsilon$, and note that $(L_0^{(2)}h_\varepsilon)(y) \leqslant (2\log(1/\varepsilon))^{-1}(y_1 + y_2)^{-1}$ for every $y \in K_0^{(2)}$. By the optional stopping theorem and the fact that $\int_{K_0^{(2)}}(y_1 + y_2)^{-1}\,dy = 1$,

$$\begin{aligned}
&Q_x\text{-}E\big[h_\varepsilon(x^{(2)}(\tau_{(1)}^{(2)} \vee \delta))\big] - Q_x\text{-}E\big[h_\varepsilon(x^{(2)}(\delta))\big] \\
&= Q_x\text{-}E\big[\textstyle\int_\delta^{\tau_{(1)}^{(2)}\vee\delta}(L_0^{(2)}h_\varepsilon)(x^{(2)}(s))\,ds\big] \\
&= Q_x\text{-}E\big[\textstyle\int_\delta^\infty(L_0^{(2)}h_\varepsilon)(x^{(2)}(s))\chi_{K_0^{(2)}}(x^{(2)}(s))\,ds\big] \\
&= \textstyle\int_\delta^\infty\int_{K_0^{(2)}}(L_0^{(2)}h_\varepsilon)(y)q(s, x, y)\,dy\,ds \\
&\leqslant \left(2\log\frac{1}{\varepsilon}\right)^{-1}\sup_{y\in K_0^{(2)}}\textstyle\int_\delta^\infty q(s, x, y)\,ds
\end{aligned}$$

for each $x \in K_0^{(2)}$, $\delta > 0$, and $0 < \varepsilon < 1$. Letting $\varepsilon \downarrow 0$, we obtain

$$Q_x\{x^{(2)}(\tau_{(1)}^{(2)} \vee \delta) = (0, 0)\} = Q_x\{x^{(2)}(\delta) = (0, 0)\}$$

for each $x \in K_0^{(2)}$ and $\delta > 0$. Letting $\delta \downarrow 0$, we conclude that (4.10) holds for each $x \in K_0^{(2)}$, which suffices for the proof.

Proposition 4.4. *Given $\sigma \in C(K, R^{d+1})$, define $b : K \to R^d$ by* (2.10) *and L by* (2.4), *and for each $x \in K$, let P_x be the unique solution to the martingale problem for L starting at x. If $d \geqslant 2$, then, for $j = 1, \cdots, d-1$,*

$$P_x\{x(\tau_{(j)}) \in V_{(j+1)}\} = 0, \qquad x \in K \cap V_{(j+1)}^c, \tag{4.11}$$

where $\tau_{(j)} : \Omega \to [0, \infty]$ is defined by (2.14).

Proof. Define L_0 by (2.11), and for each $x \in K$, let Q_x be the unique solution to the martingale problem for L_0 starting at x. Note that, for $j = 1, \cdots, d-1$, (4.11) holds if and only if $P_x\{x(\tau_{(j)} \wedge t) \in V_{(j+1)}\} = 0$ for every $x \in K \cap V_{(j+1)}^c$ and $t \geqslant 0$. Therefore, by Proposition 2.3, it suffices to show that, for $j = 1, \cdots, d-1$,

$$Q_x\{x(\tau_{(j)}) \in V_{(j+1)}\} = 0, \qquad x \in K \cap V_{(j+1)}^c. \tag{4.12}$$

We proceed inductively. Given integers $1 \leqslant i_1 < i_2 \leqslant d+1$,

$$\{x(\tau_{(1)}) \in V_{i_1} \cap V_{i_2}\} \subset \hat{\pi}_{i_1 i_2}^{-1}\{x^{(2)}(\tau_{(1)}^{(2)}) = (0, 0)\},$$

so Lemmas 4.1 and 4.2 imply that (4.12) holds for $j = 1$. If $d \geqslant 3$, fix $k \in \{2, \cdots, d-1\}$ and suppose that (4.12) holds for $j = k-1$. To show that

$$Q_x\{x(\tau_{(k)}) \in V_{(k+1)}\} = 0 \tag{4.13}$$

for every $x \in K \cap V_{(k+1)}^c$, it is enough to consider $x \in K \cap V_{(k)}^c$. But for such x, $x(\tau_{(k-1)}) \in V_{(k-1)} \cap V_{(k)}^c$ Q_x-a.s. by the induction hypothesis, and

$$Q_x\{x(\tau_{(k)}) \in V_{(k+1)}\} = Q_x\text{-}E\Big[Q_{x(\tau_{(k-1)})}\{x(\tau_{(k)}) \in V_{(k+1)}\}\Big]$$

by Proposition 2.1. Thus, we need only verify (4.13) for each $x \in V_{(k-1)} \cap V_{(k)}^c$. Fix such an x, and suppose that $x_i \neq 0$ for $i = i_1, \cdots, i_{n+1}$, where $1 \leqslant i_1$

$< \cdots < i_{n+1} \leqslant d + 1$ and $n = d + 1 - k$. It is then clear that

(4.14) $$Q_x\{x(\tau_{(k)}) \in V_{(k+1)}\} = Q_x \circ \hat{\pi}_{i_1 \cdots i_n}{}^{-1}\{x^{(n)}(\tau_{(1)}^{(n)}) \in V_{(2)}^{(n)}\}.$$

But the right side of (4.14) is zero by Lemma 4.1 and by (4.12) with $j = 1$ and $d = n$.

We conclude this section by observing that Proposition 4.4 yields the following result of Littler [8].

COROLLARY 4.1. *Under the conditions of Lemma* 4.2,

(4.15) $$Q_x\{x^{(2)}(\tau_{(1)}^{(2)}) \in V_3^{(2)}\} = x_1x_2[(1 - x_1)^{-1} + (1 - x_2)^{-1}]$$

for each $x \in K^{(2)}$ *except* $x = (0, 1)$ *and* $x = (1, 0)$.

PROOF. For $0 \leqslant \delta < 1$, let $J_\delta = \{x \in R^2 : x_1 < 1 - \delta, x_2 < 1 - \delta\}$, and define $\tau_\delta : \Omega^{(2)} \to [0, \infty]$ by $\tau_\delta = \inf\{t \geqslant 0 : x^{(2)}(t) \notin J_\delta\}$. The function $f \in C^2(K^{(2)} \cap J_0)$ defined by $f(x) = x_1x_2[(1 - x_1)^{-1} + (1 - x_2)^{-1}]$ satisfies $L_0^{(2)}f = 0$, so by the optional stopping theorem,

$$Q_x\text{-}E[f(x^{(2)}(\tau_{(1)}^{(2)} \wedge \tau_\delta))] = f(x)$$

for each $x \in K^{(2)} \cap J_0$ and $0 < \delta < 1$. Noting that $Q_x\{\tau_0 \leqslant \tau_{(1)}^{(2)}\} = 0$ by Proposition 4.4 with $d = 2$ and $\sigma \equiv 0$ (or by a slight extension of Lemma 4.2), we obtain (4.15) for each $x \in K^{(2)} \cap J_0$ by letting $\delta \downarrow 0$.

5. A property of the sequence of Markov chains. In this section we prove essentially the analogue of Proposition 4.1 for the sequence of Markov chains of Theorem 2.2.

PROPOSITION 5.1. *Given* $\sigma \in C(K, R^{d+1})$, *define* $f_0 \in C([0, 1])$ *as in the statement of Proposition* 4.1. *For* $N = 1, 2, \cdots$, *define* $\gamma^{(N)} : K_N \to K$ *by* (2.12), *and let* $\{P_x^{(N)} : x \in K_N\}$ *be the family of probability measures on* $(\Omega, \mathcal{M})$ *defined in terms of* $\gamma^{(N)}$ *by* (2.3). *Then there exist positive integers* ν *and* N_1, *depending only on* σ, *such that, for* $i = 1, \cdots, d + 1$,

(5.1) $$P_x^{(N)}\text{-}E[\tau_i] \leqslant \nu f_0(x_i), \qquad x \in K_N, N \geqslant N_1,$$

where $\tau_i : \Omega \to [0, \infty]$ *is defined by* (2.13).

PROOF. Fix $i \in \{1, \cdots, d + 1\}$. For $N = 1, 2, \cdots$, let $\{Q_x^{(N)} : x \in K_N\}$ be the family of probability measures on $(\Omega_N, \mathcal{M}^{(N)})$ defined in terms of $\gamma^{(N)}$ by (2.1) and (2.2), and define the linear operator L_N on $C(K_N)$ by

$$(L_Nf)(x) = N\{Q_x^{(N)}\text{-}E[f(X(1))] - f(x)\}.$$

In addition, define $f_0^i \in C(K)$ by $f_0^i(x) = f_0(x_i)$, and for $0 \leqslant \varepsilon < \frac{1}{2}$, define $G_i(\varepsilon) \subset R^d$ by (4.6). The crux of the proof is to show that there exist positive integers ν and N_1, depending only on σ, such that

(5.2) $$\sup_{x \in K_N \cap G_i(0)}(L_Nf_0^i)(x) \leqslant -1/\nu, \qquad N \geqslant N_1.$$

Let us first prove that

(5.3) $\limsup_{m\to\infty} \limsup_{N\to\infty} \sup_{x\in K_N\cap G_i(m/N)}(L_Nf_0^i)(x) \leqslant -1.$

A fourth order Taylor expansion yields

$$(L_Nf_0^i)(x) = N\Big\{\Sigma_{k=1}^3 \frac{1}{k!} Q_x^{(N)}\text{-}E\big[(X_i(1) - x_i)^k\big] f_0^{(k)}(x_i) + \frac{1}{3!} Q_x^{(N)}\text{-}E\big[(X_i(1) - x_i)^4 \textstyle\int_0^1 (1-t)^3 f_0^{(4)}(x_i + t(X_i(1) - x_i))\, dt\big]\Big\}$$

for all $x \in K_N \cap G_i(0)$ and $N = 1, 2, \cdots$, where $X_{d+1}(1) = 1 - \Sigma_{j=1}^d X_j(1)$. (We note that the integral under the fourth expectation exists, as does the expectation itself.) Expanding each of these moments about $\gamma_i^{(N)}(x)$, which we denote temporarily by γ_i (of course, $\gamma_{d+1}^{(N)} = 1 - \gamma_1^{(N)} - \cdots - \gamma_d^{(N)}$), we obtain

$$\begin{aligned}(L_Nf_0^i)(x) &= N(\gamma_i - x_i)f_0'(x_i) + \frac{N}{2}\left[\frac{\gamma_i(1-\gamma_i)}{N} + (\gamma_i - x_i)^2\right] f_0''(x_i) \\ &\quad + \frac{N}{6}\left[\frac{\gamma_i(1-\gamma_i)(1-2\gamma_i)}{N^2} + \frac{3\gamma_i(1-\gamma_i)(\gamma_i - x_i)}{N} + (\gamma_i - x_i)^3\right] f_0^{(3)}(x_i) \\ &\quad + \theta\frac{N}{6}\left[\frac{3\gamma_i^2(1-\gamma_i)^2}{N^2} + \frac{\gamma_i(1-\gamma_i)(1-6\gamma_i+6\gamma_i^2)}{N^3}\right. \\ &\qquad + \frac{4\gamma_i(1-\gamma_i)(1-2\gamma_i)(\gamma_i - x_i)}{N^2} \\ &\qquad \left. + \frac{6\gamma_i(1-\gamma_i)(\gamma_i - x_i)^2}{N} + (\gamma_i - x_i)^4\right] \\ &\qquad \cdot \int_0^1 (1-t)^3 \sup_{0\leqslant u\leqslant 1}|f_0^{(4)}(x_i + t(u - x_i))|\, dt\end{aligned}$$

for all $x \in K_N \cap G_i(0)$ and $N = 1, 2, \cdots$, where $\theta = \theta_{x,N} \in [-1, 1]$. Now one can easily check that

$$N(\gamma_i^{(N)}(x) - x_i) = b_i(x)(1 + O(N^{-1}))$$

and

$$\gamma_i^{(N)}(x)(1 - \gamma_i^{(N)}(x)) = x_i(1-x_i)(1 + O(N^{-1}))$$

as $N \to \infty$, uniformly over $x \in K_N$; here we are using (2.8) and (2.9). Also, by direct calculation, there exist constants M_k ($k = 1, 2, 3, 4$), depending only on σ, such that

$$|f_0'(u)| \leqslant M_1 \log[u(1-u)]^{-1}, \quad |f_0^{(k)}(u)| \leqslant M_k[u(1-u)]^{-(k-1)},$$

for $0 < u < 1$ and $k = 2, 3, 4$. Finally, we note that

$$\min_{0\leqslant u\leqslant 1}[x_i + t(u - x_i)][1 - x_i - t(u - x_i)] \geqslant (1-t)x_i(1-x_i)$$

for each $x \in K$ and $0 \leqslant t \leqslant 1$ since the minimum occurs at $u = 0$ or $u = 1$, and therefore

$$\int_0^1 (1-t)^3 \sup_{0 \leqslant u \leqslant 1} |f_0^{(4)}(x_i + t(u - x_i))|\, dt \leqslant M_4[x_i(1-x_i)]^{-3}$$

for all $x \in K \cap G_i(0)$. Combining these results, we conclude that

$$\begin{aligned}(L_N f_0^i)(x) \leqslant{} & \tfrac{1}{2}x_i(1-x_i)f_0''(x_i) + b_i(x)f_0'(x_i) \\ & + \left(\tfrac{1}{6}M_3 + \tfrac{1}{2}M_4\right)[Nx_i(1-x_i)]^{-1} + \tfrac{1}{6}M_4[Nx_i(1-x_i)]^{-2} + O(N^{-1})\end{aligned}$$

as $N \to \infty$, uniformly over $x \in K_N \cap G_i(0)$, which, by (4.5), implies (5.3).

Next, we show that

$$\limsup_{N\to\infty} \sup_{x \in K_N;\, x_i = m/N} (L_N f_0^i)(x) < 0, \qquad m = 1, 2, \cdots. \tag{5.4}$$

A second order Taylor expansion, in conjunction with (4.1), yields

$$\begin{aligned}(L_N f_0^i)(x) &= N\left\{Q_x^{(N)}\text{-}E[X_i(1) - x_i]f_0'(x_i) + Q_x^{(N)}\text{-}E\left[(X_i(1)-x_i)^2\int_0^1(1-t)f_0''(Z_i)\,dt\right]\right\} \\ &= NQ_x^{(N)}\text{-}E[X_i(1)-x_i]f_0'(x_i) \\ &\quad -2\lambda NQ_x^{(N)}\text{-}E\left[(X_i(1)-x_i)^2\int_0^1(1-t)(1-2Z_i)f_0'(Z_i)\,dt\right] \\ &\quad -4NQ_x^{(N)}\text{-}E\left[(X_i(1)-x_i)^2\int_0^1(1-t)Z_i^{-1}(1-Z_i)^{-1}\,dt\right]\end{aligned}$$

for all $x \in K_N \cap G_i(0)$ and $N = 1, 2, \cdots$, where $Z_i = x_i + t(X_i(1) - x_i)$. Since, for $m = 1, 2, \cdots$, the first two terms on the right are $O(N^{-1}\log N)$ as $N \to \infty$, uniformly over $x \in K_N$ with $x_i = m/N$, (5.4) is equivalent to

(5.5)

$$\liminf_{N\to\infty} \inf_{x \in K_N;\, x_i = m/N} NQ_x^{(N)}\text{-}E\left[(X_i(1)-x_i)^2\int_0^1(1-t)Z_i^{-1}(1-Z_i)^{-1}\,dt\right] > 0$$

for $m = 1, 2, \cdots$, where $Z_i = x_i + t(X_i(1) - x_i)$. Fix a positive integer m. Clearly, $\gamma_i^{(N)}(x) = m/N + O(N^{-2})$ as $N \to \infty$, uniformly over $x \in K_N$ with $x_i = m/N$, so for each $l \in Z_+$,

$$\lim_{N\to\infty} \sup_{x\in K_N;\, x_i = m/N} \left|\binom{N}{l}\gamma_i^{(N)}(x)^l(1-\gamma_i^{(N)}(x))^{N-l} - \frac{m^l}{l!}e^{-m}\right| = 0;$$

of course, this is the familiar Poisson approximation of the binomial distribution. Since the expectation in (5.5) can be expressed as

$$\begin{aligned}\sum_{l=0}^N \left(\frac{l}{N} - \frac{m}{N}\right)^2 \int_0^1 (1-t)\left[\frac{m}{N} + t\left(\frac{l}{N} - \frac{m}{N}\right)\right]^{-1}\left[1 - \frac{m}{N} - t\left(\frac{l}{N} - \frac{m}{N}\right)\right]^{-1} dt \\ \cdot \binom{N}{l}\gamma_i^{(N)}(x)^l(1-\gamma_i^{(N)}(x))^{N-l},\end{aligned}$$

an application of Fatou's lemma shows that the left side of (5.5) is at least as large as

$$\sum_{l=0}^\infty (l-m)^2\int_0^1(1-t)[m + t(l-m)]^{-1}\,dt\,\frac{m^l}{l!}e^{-m},$$

which of course is positive. This proves (5.4), and a similar argument yields

$$\lim\sup_{N\to\infty}\sup_{x\in K_N;\, x_i=1-m/N}(L_N f_0^i)(x) < 0, \quad m = 1, 2, \cdots. \tag{5.6}$$

The combination of (5.3), (5.4), and (5.6) implies that there exist positive integers ν and N_1, depending only on σ, such that (5.2) holds.

Now for $N = 1, 2, \cdots$, define $T_i : \Omega_N \to Z_+ \cup \{\infty\}$ by

$$T_i = \inf\{n \in Z_+ : X(n) \in V_i \text{ or } X(n) = v^{(i)}\}, \tag{5.7}$$

and note that by (2.2),

$$\left\{ f_0^i(X(n)) - \frac{1}{N}\Sigma_{m=0}^{n-1}(L_N f_0^i)(X(m)), \mathfrak{M}_n^{(N)} : n \in Z_+ \right\}$$

is a $Q_x^{(N)}$-martingale for every $x \in K_N$. Thus, by the (discrete parameter) optional stopping theorem and (5.2),

$$Q_x^{(N)}\text{-}E[f_0^i(X(T_i \wedge n))] = f_0^i(x) + Q_x^{(N)}\text{-}E\left[\frac{1}{N}\Sigma_{m=0}^{(T_i\wedge n)-1}(L_N f_0^i)(X(m))\right]$$

$$\leqslant f_0^i(x) - \frac{1}{\nu} Q_x^{(N)}\text{-}E\left[\frac{1}{N}(T_i \wedge n)\right]$$

for each $x \in K_N$, $n \in Z_+$, and $N \geqslant N_1$. It follows that $Q_x^{(N)}\{T_i < \infty\} = 1$ for every $x \in K_N$ and $N \geqslant N_1$, so since $f_0(0) = f_0(1) = 0$, we obtain

$$Q_x^{(N)}\text{-}E\left[\frac{1}{N}T_i\right] \leqslant \nu f_0(x_i), \qquad x \in K_N,\ N \geqslant N_1,$$

by letting $n \to \infty$. Of course, this is precisely (5.1).

COROLLARY 5.1. *Under the conditions of Proposition* 5.1, *there is a constant* $\mu > 0$, *depending only on* σ, *such that, for* $i = 1, \cdots, d+1$,

$$\sup_{N\geqslant 1}\sup_{x\in K_N} P_x^{(N)}\text{-}E[e^{\mu\tau_i}] < \infty.$$

PROOF. Fix $i \in \{1, \cdots, d+1\}$. Choose $f_0 \in C([0, 1])$, $\nu \geqslant 1$, and $N_1 \geqslant 1$ as in the statement of Proposition 5.1. Then

$$\sup_{x\in K_N} P_x^{(N)}\{\tau_i > t\} \leqslant t^{-1}\sup_{x\in K_N} P_x^{(N)}\text{-}E[\tau_i] \leqslant t^{-1}\nu\sup_{0\leqslant u\leqslant 1} f_0(u)$$

for all $N \geqslant N_1$ and $t > 0$, so $\sup_{N\geqslant 1}\sup_{x\in K_N} P_x^{(N)}\{\tau_i > t\} < 1$ for all t sufficiently large. The result therefore follows from standard considerations (cf. [2], page 112).

6. Limit theorems for absorption times.

THEOREM 6.1. *Let* κ *be the identity map of* Ω *into* Ω, *and define* $(\kappa, \tau_1, \cdots, \tau_{d+1}) : \Omega \to \Omega \times [0, \infty]^{d+1}$ *by* (2.13). ($[0, \infty]^{d+1}$ *denotes the* $(d+1)$-*fold product space* $[0, \infty] \times \cdots \times [0, \infty]$.) *Then, under the conditions of Theorem* 2.2,

$$P_{x_N}^{(N)} \circ (\kappa, \tau_1, \cdots, \tau_{d+1})^{-1} \Rightarrow P_x \circ (\kappa, \tau_1, \cdots, \tau_{d+1})^{-1} \quad \textit{as} \quad N \to \infty. \tag{6.1}$$

The proof of this theorem depends on the following proposition, which, as observed by Lindvall [7], is a consequence of a result of Billingsley ([1], Theorem 4.2).

PROPOSITION 6.1. *Let $P^{(N)}$, $N = 1, 2, \cdots$, and P be Borel probability measures on a metric space S such that $P^{(N)} \Rightarrow P$ as $N \to \infty$. Let S' be a separable metric space with metric ρ, and suppose that ξ_k, $k = 1, 2, \cdots$, and ξ are Borel measurable mappings of S into S' such that*

(i) ξ_k *is P-a.s. continuous for $k = 1, 2, \cdots$,*

(ii) $\xi_k \to \xi$ *as $k \to \infty$ P-a.s.,*

(iii) $\lim_{k\to\infty} \limsup_{N\to\infty} P^{(N)}\{\rho(\xi_k, \xi) > \delta\} = 0$ *for every $\delta > 0$.*

Then $P^{(N)} \circ \xi^{-1} \Rightarrow P \circ \xi^{-1}$ as $N \to \infty$.

PROOF OF THEOREM 6.1. We apply Proposition 6.1 with $\{P^{(N)}\} = \{P_{x_N}^{(N)}\}$, $P = P_x$, $S = \Omega$, $S' = \Omega \times [0, \infty]^{d+1}$, $\{\xi_k\} = \{(\kappa, \tau_1^{(\varepsilon)}, \cdots, \tau_{d+1}^{(\varepsilon)})\}_{\varepsilon = \frac{1}{3}, \frac{1}{4}, \cdots}$ (see Proposition 4.2), and $\xi = (\kappa, \tau_1, \cdots, \tau_{d+1})$. Further, we take ρ to be the metric on $\Omega \times [0, \infty]^{d+1}$ defined by

$$\rho((\omega, t_1, \cdots, t_{d+1}), (\omega', t_1', \cdots, t_{d+1}')) = \rho_0(\omega, \omega') + \max_{1 \leqslant i \leqslant d+1} |\tan^{-1} t_i - \tan^{-1} t_i'|,$$

where ρ_0 is some metric on Ω compatible with its topology.

According to Proposition 4.2, condition (i) of Proposition 6.1 is satisfied here. Moreover, for $i = 1, \cdots, d+1$, $\tau_i^{(\varepsilon)} \to \tau_i$ pointwise as $\varepsilon \downarrow 0$, so condition (ii) is also satisfied. Thus it suffices to verify that

(6.2)

$$\lim_{\varepsilon\downarrow 0} \limsup_{N\to\infty} P_{x_N}^{(N)}\{\rho((\kappa, \tau_1^{(\varepsilon)}, \cdots, \tau_{d+1}^{(\varepsilon)}), (\kappa, \tau_1, \cdots, \tau_{d+1})) > \delta\} = 0, \ \delta > 0.$$

For $N = 1, 2, \cdots, 0 < \varepsilon < \frac{1}{2}$, and $i = 1, \cdots, d+1$, define $T_i^{(\varepsilon)} : \Omega_N \to Z_+ \cup \{\infty\}$ by

$$T_i^{(\varepsilon)} = \inf\{n \in Z_+ : X(n) \notin G_i(\varepsilon)\},$$

where $G_i(\varepsilon)$ is given by (4.6), define $T_i : \Omega_N \to Z_+ \cup \{\infty\}$ by (5.7), and let $\{Q_x^{(N)} : x \in K_N\}$ be the family of probability measures on $(\Omega_N, \mathfrak{M}^{(N)})$ defined in terms of $\gamma^{(N)}$ by (2.1) and (2.2). Further, choose $f_0 \in C([0, 1])$, $\nu \geqslant 1$, and $N_1 \geqslant 1$ as in the statement of Proposition 5.1. Then, using the inequality $|\tan^{-1} u - \tan^{-1}(u + v)| \leqslant v$ $(u, v \geqslant 0)$ and the (discrete parameter) strong Markov property, we obtain

$$\begin{aligned}
P_x^{(N)}\{|\tan^{-1} \tau_i^{(\varepsilon)} - \tan^{-1} \tau_i| > \delta\}
&\leqslant Q_x^{(N)}\{|\tan^{-1}(T_i^{(\varepsilon)}/N) - \tan^{-1}(T_i/N)| > \delta - N^{-1}\} \\
&\leqslant Q_x^{(N)}\text{-}E\left[\chi\{T_i^{(\varepsilon)} < \infty\} Q_{X(T_i^{(\varepsilon)})}^{(N)}\{T_i/N > \delta - N^{-1}\}\right] \\
&\leqslant \sup_{x \in K_N \cap G_i(\varepsilon)^c} P_x^{(N)}\{\tau_i > \delta - N^{-1}\} \\
&\leqslant (\delta - N^{-1})^{-1} \sup_{x \in K_N \cap G_i(\varepsilon)^c} P_x^{(N)}\text{-}E[\tau_i] \\
&\leqslant (\delta - N^{-1})^{-1} \nu \sup_{x \in K_N \cap G_i(\varepsilon)^c} f_0(x_i)
\end{aligned}$$

for each $\delta > 0$, $N \geq N_1 \vee \delta^{-1}$, $0 < \varepsilon < \frac{1}{2}$, and $i = 1, \cdots, d+1$. Since $f_0(0) = f_0(1) = 0$, this implies (6.2) and completes the proof.

THEOREM 6.2. *Let κ be the identity map of Ω into Ω, and define $(\kappa, \tau_{(1)}, \cdots, \tau_{(d)}) : \Omega \to \Omega \times [0, \infty]^d$ by* (2.14). *Then, under the conditions of Theorem* 2.2,

$$(6.3) \quad P_{x_N}^{(N)} \circ (\kappa, \tau_{(1)}, \cdots, \tau_{(d)})^{-1} \Rightarrow P_x \circ (\kappa, \tau_{(1)}, \cdots, \tau_{(d)})^{-1} \quad \text{as} \quad N \to \infty.$$

PROOF. Define $\zeta : \Omega \times [0, \infty]^{d+1} \to \Omega \times [0, \infty]^d$ by $\zeta(\omega, t_1, \cdots, t_{d+1}) = (\omega, t_{[1]}, \cdots, t_{[d]})$, where $t_{[1]}, \cdots, t_{[d+1]}$ are the order statistics based on the sample $t_1, \cdots, t_{d+1}$ (e.g., $t_{[1]} = \min(t_1, \cdots, t_{d+1})$). Clearly, ζ is continuous, so by (6.1),

$$P_{x_N}^{(N)} \circ (\kappa, \tau_1, \cdots, \tau_{d+1})^{-1} \circ \zeta^{-1} \Rightarrow P_x \circ (\kappa, \tau_1, \cdots, \tau_{d+1})^{-1} \circ \zeta^{-1} \quad \text{as } N \to \infty.$$

But observe that $\zeta \circ (\kappa, \tau_1, \cdots, \tau_{d+1}) = (\kappa, \tau_{(1)}, \cdots, \tau_{(d)})$ $P_{x_N}^{(N)}$-a.s. for $N = 1, 2, \cdots$ and P_x-a.s., so this is precisely (6.3).

COROLLARY 6.1. *Under the conditions of Theorem* 2.2,

$$\lim_{N\to\infty} P_{x_N}^{(N)}\text{-}E[\tau_i^k] = P_x\text{-}E[\tau_i^k]$$

and

$$\lim_{N\to\infty} P_{x_N}^{(N)}\text{-}E[\tau_{(j)}^k] = P_x\text{-}E[\tau_{(j)}^k]$$

for $i = 1, \cdots, d+1, j = 1, \cdots, d$, and $k = 1, 2, \cdots$.

PROOF. For $j = 1, \cdots, d$, we note that $\tau_{(j)} \leq \tau_1 + \cdots + \tau_{d+1}$ $P_{x_N}^{(N)}$-a.s. for $N = 1, 2, \cdots$. Therefore, by Corollary 5.1, both τ_i^k and $\tau_{(j)}^k$ are uniformly integrable with respect to $\{P_{x_N}^{(N)}\}_{N=1,2,\cdots}$ for $i = 1, \cdots, d+1$, $j = 1, \cdots, d$, and $k = 1, 2, \cdots$. Thus, the results follow from (6.1) and (6.3).

COROLLARY 6.2. *Under the conditions of Theorem* 2.2,

$$(6.4) \quad \lim_{N\to\infty} P_{x_N}^{(N)}\{\tau_{(j)} \leq t, x(\tau_{(j)}) \in V_{i_1} \cap \cdots \cap V_{i_j}\} = P_x\{\tau_{(j)} \leq t, x(\tau_{(j)}) \in V_{i_1} \cap \cdots \cap V_{i_j}\}$$

for $j = 1, \cdots, d$, $0 < t \leq \infty$, and $1 \leq i_1 < \cdots < i_j \leq d+1$, provided that $x \notin V_{(j+1)}$ (where $V_{(d+1)} \equiv \varnothing$).

PROOF. Fix $j \in \{1, \cdots, d\}$, and define the map $\psi_j : \Omega \times [0, \infty]^d \to [0, \infty] \times K$ by $\psi_j(\omega, t_1, \cdots, t_d) = (t_j, x(t_j)(\omega))$. Then ψ_j is $P_x \circ (\kappa, \tau_{(1)}, \cdots, \tau_{(d)})^{-1}$-a.s. continuous, so by (6.3),

$$(6.5) \quad P_{x_N}^{(N)} \circ (\tau_{(j)}, x(\tau_{(j)}))^{-1} \Rightarrow P_x \circ (\tau_{(j)}, x(\tau_{(j)}))^{-1} \quad \text{as} \quad N \to \infty.$$

Now fix $0 < t \leq \infty$ and $1 \leq i_1 < \cdots < i_j \leq d+1$. Then there is a closed set F in K such that $V_{(j)} \cap F = V_{i_1} \cap \cdots \cap V_{i_j}$ and $V_{(j)} \cap \partial F \subset V_{(j+1)}$, so

$$P_x\{(\tau_{(j)}, x(\tau_{(j)})) \in \partial([0, t] \times F)\} \leq P_x\{\tau_{(j)} = t\} + P_x\{x(\tau_{(j)}) \in V_{(j+1)}\} = 0$$

by Propositions 4.3 and 4.4. Consequently, (6.5) implies that

$$\lim_{N\to\infty} P_{x_N}^{(N)}\{(\tau_{(j)}, x(\tau_{(j)})) \in [0, t] \times F\}$$
$$= P_x\{(\tau_{(j)}, x(\tau_{(j)})) \in [0, t] \times F\},$$

and this is precisely (6.4).

REMARK 6.1. The special case of Corollary 6.2 in which $d = 1$ and $t = \infty$ is known ([9], page 260; [5]). A second special case of Corollary 6.2, that in which $d = 1$ and $i_1 = 1$, was proved by Guess [5] under the further condition that $\sigma : [0, 1] \to R^2$ is defined by $\sigma_i(x) = s_{i1}x + s_{i2}(1 - x)$, where $s = (s_{ij})$ is a real symmetric 2×2 matrix satisfying $s_{11} \leqslant s_{12} \leqslant s_{22} = 0$ (cf. Section 3). The question of the necessity of this "directional selection" hypothesis provided the original motivation for the present work.

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DEPARTMENT OF STATISTICS AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824