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A NOTE ON A MIXING CONDITION[1]

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The only strictly stationary random sequences satisfying a certain mixing condition are the sequences of independent, identically distributed rv's.

Let $(X_k, k = \cdots, -1, 0, 1, \cdots)$ be a sequence of random variables in a probability space $(\Omega, \mathfrak{F}, P)$. For $-\infty \leqslant J \leqslant L \leqslant \infty$ let $\mathfrak{F}_J^L$ be the Borel field generated by $(X_k, J \leqslant k \leqslant L)$. Consider the mixing condition

$$(1) \qquad \lim_{n\to\infty}\left(\sup_J\left(\sup\left\{|P(C|A \cap B) - P(C|B)| : A \in \mathfrak{F}_J^J,\right.\right.\right.$$
$$\left.\left.\left. B \in \mathfrak{F}_{J+1}^{J+n-1}, C \in \mathfrak{F}_{J+n}^{J+n}, P(A \cap B) > 0\right\}\right)\right) = 0.$$

In several articles, condition (1) or stronger conditions are said to be "of Markov type" and used in limit theorems on random sequences. M. Rosenblatt (1979) mentioned examples of these articles and showed, in essence, that (1) cannot be satisfied by strictly stationary Markov chains other than i.i.d. sequences. He also conjectured to this author that of all strictly stationary sequences, only the i.i.d. ones satisfy (1). His conjecture turns out to be correct, which will be shown here for an apparently weaker condition.

THEOREM 1. *If* $(X_k, k = \cdots, -1, 0, 1, \cdots)$ *is strictly stationary and satisfies*

$$(2) \quad \lim_{n\to\infty}\left(\sup\left\{|P(A \cap B \cap C) \cdot P(B) - P(A \cap B)\right.\right.$$
$$\left.\left. \cdot P(B \cap C)| : A \in \mathfrak{F}_0^0, B \in \mathfrak{F}_1^{n-1}, C \in \mathfrak{F}_n^n\right\}\right) = 0$$

then (X_k) *is a sequence of independent, identically distributed* rv's.

PROOF. The proof is a modification of the argument of Rosenblatt (1979). Suppose (X_k) is strictly stationary and satisfies (2). Let T be the measure-preserving shift operator on $\mathfrak{F}_{-\infty}^{\infty}$ for which $T(\{c < X_k < d\}) = \{c < X_{k+1} < d\}$.

LEMMA. (X_k) *is mixing; that is,*

$$\forall A, B \in \mathfrak{F}_{-\infty}^{\infty}, \lim_{n\to\infty}\left[P(A \cap T^n B) - P(A)P(B)\right] = 0.$$

PROOF. Suppose the Lemma is false. Then there are events A and B and an $\varepsilon > 0$ such that

$$\limsup_{n\to\infty}\left[P(A \cap T^n B) - P(A)P(B)\right] \geqslant \varepsilon.$$

For $-\infty < J < L < \infty$, let $\mathfrak{D}_J^L$ be the set of all events of the form $\cap_{k=J}^L D_k$, $D_k \in \mathfrak{F}_k^k$, and let $\mathfrak{E}_J^L$ be the set of all events of the form $\cup_{m=1}^M E_m$, $E_m \in \mathfrak{D}_J^L$.

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In the metric on $\mathcal{F}_{-\infty}^{\infty}$ defined by $d(D, E) = P(D\Delta E) \equiv P(D - E) + P(E - D)$, any event in $\mathcal{F}_{-\infty}^{\infty}$ can be approximated arbitrarily closely by some event in $\mathcal{E}_J^L$ for some integers $J < L$. It follows that for some $J < L$ and some $A', B' \in \mathcal{E}_J^L$,

$$\limsup_{n\to\infty}\left[P(A' \cap T^nB') - P(A')P(B')\right] \geqslant \varepsilon/2:$$

It follows that for some $\varepsilon* > 0$, some integers $J < L$, and some $A*$, $B* \varepsilon \mathfrak{D}_J^L$,

$$\limsup_{n\to\infty}\left[P(A* \cap T^nB*) - P(A*)P(B*)\right] \geqslant \varepsilon*. \tag{3}$$

Denote $A* = \cap_{k=J}^L A_k$, $A_k \in \mathcal{F}_k^k$, and $B* = \cap_{k=J}^L B_k$, $B_k \in \mathcal{F}_k^k$. For $J \leqslant q \leqslant L$ let $A(q) = \cap_{k=q}^L A_k$ and $B(q) = \cap_{k=J}^q B_k$. Let $A(L + 1) = B(J - 1) = \Omega$. Also, let $S = \{n > 0 : P(A* \cap T^nB*) - P(A*)P(B*) \geqslant \varepsilon*/2\}$; S is infinite, by (3).

For each q, $r\varepsilon\{J, J + 1, \cdots, L\}$ we have by (2) and (3),

$$\lim_{n\to\infty;\, n\in S}\left[P(T^nB_r|A(q) \cap T^nB(r - 1)) - P(T^nB_r|A(q + 1) \cap T^nB(r - 1))\right]$$
$$\times P(A(q) \cap T^nB(r - 1))\cdot P(A(q + 1) \cap T^nB(r - 1)) = 0,$$

and hence by (3),

$$\lim_{n\to\infty;\, n\in S}\left[P(T^nB_r|A(q) \cap T^nB(r - 1)) - P(T^nB_r|A(q + 1) \cap T^nB(r - 1))\right] = 0.$$

Hence for each r, $J \leqslant r \leqslant L$,

$$\lim_{n\to\infty;\, n\in S}\left[P(T^nB_r|A* \cap T^nB(r - 1)) - P(T^nB_r|T^nB(r - 1))\right] = 0.$$

Hence $\lim_{n\to\infty;\, n\in S}[P(T^nB*|A*) - P(B*)] = 0$, which contradicts (3). Therefore the Lemma is proved.

Now suppose (X_k) is not an i.i.d. sequence. There are integers $J < L$ and events $A \in \mathcal{F}_{J-1}^{J-1}$, $F \in \mathfrak{D}_J^L$, $G \in \mathfrak{D}_J^L$, and $C \in \mathcal{F}_{L+1}^{L+1}$ such that $P(A \cap F) - P(A)P(F) > 0$ and $P(G \cap C) - P(G)P(C) > 0$. For each n let $B_n = (F \cap T^nG) \cup (F^c \cap T^nG^c)$. By (2),

$$\lim_{n\to\infty}\left[P(A \cap B_n \cap T^nC)P(B_n) - P(A \cap B_n)P(B_n \cap T^nC)\right] = 0.$$

That is,

$$\lim_{n\to\infty}\Big(\left[P(A \cap F \cap T^nG \cap T^nC) + P(A \cap F^c \cap T^nG^c \cap T^nC)\right]$$
$$\times\left[P(F \cap T^nG) + P(F^c \cap T^nG^c)\right]$$
$$-\left[P(A \cap F \cap T^nG) + P(A \cap F^c \cap T^nG^c)\right]$$
$$\times\left[P(F \cap T^nG \cap T^nC) + P(F^c \cap T^nG^c \cap T^nC)\right]\Big) = 0.$$

By the Lemma,

$$\Big(\left[P(A \cap F)P(G \cap C) + P(A \cap F^c)P(G^c \cap C)\right]$$
$$\times\left[P(F)P(G) + P(F^c)P(G^C)\right]$$
$$-\left[P(A \cap F)P(G) + P(A \cap F^c)P(G^c)\right]$$
$$\times\left[P(F)P(G \cap C) + P(F^c)P(G^c \cap C)\right]\Big) = 0.$$

That is,

$$\begin{aligned}0 &= [P(A \cap F)P(F^c) - P(A \cap F^c)P(F)] \\ &\quad \times [P(G \cap C)P(G^c) - P(G^c \cap C)P(G)] \\ &= [P(A \cap F) - P(A)P(F)][P(G \cap C) - P(G)P(C)] > 0,\end{aligned}$$

a contradiction. Thus Theorem 1 is proved.

Under slightly stronger conditions of the same general nature, even non-stationary sequences are "almost" independent sequences. Defining for any σ-fields $\mathcal{A}$ and $\mathcal{B}$,

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A) \cdot P(B)|$$

we have the following theorem:

THEOREM 2. *If* $(X_k, k = \cdots, -1, 0, 1, \cdots)$ *is a sequence of random variables for which*

(4)

$$\lim_{n\to\infty}(\sup_J(\sup\{|P(A \cap B \cap C) \cdot P(B) - P(A \cap B) \cdot P(B \cap C)|: A \in \mathcal{F}_{-\infty}^J, B \in \mathcal{F}_{J+1}^{J+n-1}, C \in \mathcal{F}_{J+n}^{\infty}\})) = 0$$

then

$$\lim_{t, u\to\infty;\, t<u} \alpha(\mathcal{F}_t^{u-1}, \mathcal{F}_u^{\infty}) = 0,$$

and

$$\lim_{t, u\to -\infty;\, t<u} \alpha(\mathcal{F}_{-\infty}^{t}, \mathcal{F}_{t+1}^{u}) = 0.$$

PROOF. First, (4) implies the strong mixing condition of Rosenblatt:

$$\lim_{n\to\infty}(\sup_J \alpha(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^{\infty})) = 0.$$

If Theorem 2 does not hold, then without losing generality we may assume that (i) there are integers $q < r$ and events $A \in \mathcal{F}_{-\infty}^q$, $F \in \mathcal{F}_{q+1}^r$ such that $P(A \cap F) - P(A) \cdot P(F) > 0$, and (ii) there is an $\varepsilon > 0$ such that for each t, there exists an integer $u(t) > t$ and events $G_t \in \mathcal{F}_t^{u(t)-1}$ and $C_t \in \mathcal{F}_{u(t)}^{\infty}$ such that $P(G_t \cap C_t) - P(G_t) \cdot P(C_t) \geqslant \varepsilon$. Defining for each t, $B_t = (F \cap G_t) \cup (F^c \cap G_t^c)$, one uses (4) to prove a contradiction, mimicking the proof of Theorem 1, and thereby Theorem 2 is proved.

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