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THE ASYMPTOTIC DISTRIBUTION OF THE SCAN STATISTIC UNDER UNIFORMITY.

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The problem of testing uniformity on [0, 1] against a clustering alternative, is considered. Naus has shown that the generalized likelihood ratio test yields the scan statistic $N(d)$. The asymptotic distribution of $N(d)$ under the null hypothesis of uniformity is considered herein, and related to the version of the scan statistic defined for points from a Poisson process. An application of the above yields distributional results for the supremum of a stationary Gaussian process with a correlation function that is tent-like in shape, until it flattens out at a constant negative value.

1. Introduction. The following problem motivated the research contained in this paper. Suppose $U_1, \cdots, U_N$ is a sample of size N from the unit interval [0, 1]. We wish to test the null hypothesis of uniformity against a clustering alternative with density,

$$(1.1)\qquad \begin{aligned} f_1(x) &= 1/(1+\eta d) && 0 \leqslant x \leqslant b \\ &= (1+\eta)/(1+\eta d) && b < x \leqslant b+d \\ &= 1/(1+\eta d) && b+d < x \leqslant 1; \end{aligned}$$

i.e., a density which is constant except for a rectangular peak at position b, of width d; η is a height parameter. Orear and Cassel (1971) give a number of cases in physics where the type of departure to be expected is this "bump" type of alternative; see also Bhattacharya and Brockwell (1976), and Birnbaum (1975).

Define,

$$(1.2)\qquad N(x, h) \equiv \text{number of points in the interval } (x, x+h],$$

where $h < 1$, $0 \leqslant x \leqslant 1 - h$. We are then able to define the *scan statistic* as,

$$(1.3)\qquad N(h) \equiv \sup_{0 \leqslant x \leqslant 1-h} N(x, h).$$

On intuitive grounds, if we have some idea of the cluster width d likely to be met under the alternative, then it would make sense to slide an interval of length d over the points to see if there were any unusually large peaks of $\{N(x,d);\ 0 \leqslant x \leqslant 1-d\}$; $N(d)$ would be a natural statistic to compute. In fact, treating η and b as unknown parameters, Naus (1966b) has shown that the generalized likelihood ratio test rejects the null hypothesis of uniformity for $N(d)$ large. Saunders (1978), using size and power criteria for the more general stationary

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renewal process, also concluded that the choice of $h = d$ was optimal. So for testing purposes, we need the distribution of $N(d)$ when the sample comes from the uniform distribution on [0, 1]; all subsequent results will be under this hypothesis.

The scan statistic $N(d)$ has been studied by Naus (1965), (1966a), (1966b), Wallenstein and Naus (1974), and Huntington and Naus (1975), however only for finite N. Cressie (1977a) briefly summarises these results, as well as deriving further properties under the null hypothesis, and giving the corresponding results under the alternative hypothesis.

This paper looks at the asymptotic ($N \to \infty$) distribution of $N(d)$ under the null hypothesis of uniformity; Theorems 2.2 and 2.3 of Section 2 contain the important results, followed by two examples. In Section 3, a result of Huntington and Naus (1975) is corrected; the striking similarity between the results of Theorems 2.2 and 2.3, and a result of Shepp (1971) on the first passage time of a particular Gaussian process can then be explained. Both are derived from an identity of Karlin and McGregor (1959), although the identity is used differently in each case. In fact Shepp's result is shown able to be independently derived, using Section 2 as a starting point. The particular Gaussian process considered by Shepp is just the limiting form of the scan statistic, now defined for points coming from a Poisson process. Details are given in Section 3, while Section 4 applies the results of the previous sections to derive certain distributional results for the supremum of a stationary Gaussian distribution, adding a fifth case to the four already known.

2. The scan statistic and its asymptotic distribution. The distribution of the scan statistic $N(d)$, for finite N, and $d = 1/L$, L an integer $\geqslant 2$, was derived by Naus (1966a):

$$\Pr\{N(1/L) < m\} = \Sigma_{Q_0} N!L^{-N}\det(1/c_{ij}!), \tag{2.1}$$

where

$$\begin{aligned} c_{ij} &= (j-i)m - \Sigma_{r=i}^{j-1} m_r + m_i & 1 \leqslant i < j \leqslant L \\ &= (j-i)m + \Sigma_{r=j}^{i} m_r & L \geqslant i \geqslant j \geqslant 1. \end{aligned} \tag{2.2}$$

If $c_{ij} < 0$ or $c_{ij} > N$, put $1/c_{ij}! = 0$. Also Q_0 is the set of all partitions of N into L nonnegative integers m_i, each less than m. The extension of (2.1) to any d between 0 and 1, came from Huntington and Naus (1975): Define $L = [d^{-1}]$, the largest integer in d^{-1}, and $r = 1 - dL$. Then,

$$\Pr\{N(d) < m\} = \Sigma_{Q_r} R \det(1/h_{ij}!)\det(1/e_{ij}!), \tag{2.3}$$

where the summation extends over the set Q_r of all partitions of N into $2L + 1$ nonnegative integers m_i satisfying,

$$m_i + m_{i+1} < m \qquad i = 1, 2, \cdots, 2L,$$

and $R = N!r^M(d-r)^{N-M}$ with $M = \Sigma_{k=0}^{L} m_{2k+1}$. In the determinant of size

$(L+1) \times (L+1)$,

$$\begin{aligned} h_{ij} &= \Sigma_{k=2j-1}^{2i-1} m_k - (i-j)m \qquad & L+1 \geqslant i \geqslant j \geqslant 1 \\ &= -\Sigma_{k=2i}^{2j-2} m_k + (j-i)m \qquad & 1 \leqslant i < j \leqslant L+1, \end{aligned}$$

and in the determinant of size $L \times L$,

$$\begin{aligned} e_{ij} &= \Sigma_{k=2j}^{2i} m_k - (i-j)m \qquad & L \geqslant i \geqslant j \geqslant 1 \\ &= -\Sigma_{k=2i+1}^{2j-1} m_k + (j-i)m \qquad & 1 \leqslant i < j \leqslant L, \end{aligned}$$

where $1/\nu! = 0$, if $\nu < 0$, $\nu > N$.

In this section we allow $N \to \infty$, and derive corresponding asymptotic results for the above. The details will be given for $d = 1/L$, and results merely stated for any $0 < d < 1$; there is no real difference in approach, only in complication. Define,

$$k \equiv \frac{m - N.d}{N^{\frac{1}{2}}} \quad \text{and} \quad S_N(d) \equiv \frac{N(d) - N.d}{N^{\frac{1}{2}}}. \tag{2.4}$$

Then

$$\Pr\{N(1/L) < m\} = \Pr\{S_N(1/L) < k\}.$$

Now

$$Q_0 = \{(m_1, \cdots, m_L) : 0 \leqslant m_i < m \forall i, \quad \text{and } \Sigma_{i=1}^L m_i = N\}.$$

Therefore write $m_i \equiv N/L + (1 - x_i)k(N^{\frac{1}{2}})$, where $x_i \geqslant 0$, $(i = 1, \cdots, L)$ and $\Sigma_{i=1}^L x_i = L$. We then substitute these expressions into (2.2), giving

$$(c_{ij}) = \left(\frac{N}{L} + k(N^{\frac{1}{2}})d_{ij}\right),$$

where

$$\begin{aligned} d_{ij} &= 1 + \Sigma_{r=i}^{j-1} x_r - x_i \qquad & 1 \leqslant i < j \leqslant L \\ &= 1 - \Sigma_{r=j}^{i} x_r \qquad & L \geqslant i \geqslant j \geqslant 1, \end{aligned} \tag{2.5}$$

remembering that $\Sigma_{r=1}^L x_r = L$.

Now use Stirling's formula: $n! = (2\pi n)^{\frac{1}{2}} n^n e^{-n+w(n)/12}$ where $(n+\frac{1}{2})^{-1} < w(n) < n^{-1}$, to evaluate $\det(1/c_{ij}!)$, for large N. The resulting algebra is extremely messy, however tractable if we exploit the fact that $\Sigma_{i=1}^L d_{i\sigma(i)} = 0$, for every permutation σ, on the integers $\{1, 2, \cdots, L\}$. Hence $\Pi_{i=1}^L \exp(c.d_{i\sigma(i)}) = 1$, and we are able to use this to great advantage when looking for the asymptotic expansion of

$$N!L^{-N} \det(1/c_{ij}!) = N!L^{-N} \Sigma_\sigma \varepsilon(\sigma) \Pi_{i=1}^L (1/c_{i\sigma(i)}!),$$

where the summation is over all permutations σ, and $\varepsilon(\sigma)$ is the sign of the permutation. The $L = 2$ case illustrates the approach:

$$N!2^{-N}\det(1/c_{ij}!)$$

$$= \left(\frac{N}{2}\right)^N \frac{\left(N/2 + (1 - x_1)k(N^{\frac12})\right)^{-N/2-(1-x_1)k(N^{\frac12})}}{\left\{\left(N/2 + (1 - x_1)k(N^{\frac12})\right)\left(N/2 - (1 - x_1)k(N^{\frac12})\right)2\pi/N\right\}^{\frac12}}$$

$$\times \frac{\left(N/2 - (1 - x_1)k(N^{\frac12})\right)^{-N/2+(1-x_1)k(N^{\frac12})}}{\left\{\left(N/2 + (1 - x_1)k(N^{\frac12})\right)\left(N/2 - (1 - x_1)k(N^{\frac12})\right)2\pi/N\right\}^{\frac12}}$$

$$- \left(\frac{N}{2}\right)^N \frac{\left(N/2 - k(N^{\frac12})\right)^{-N/2+k(N^{\frac12})}\left(N/2 + k(N^{\frac12})\right)^{-N/2-k(N^{\frac12})}}{\left\{\left(N/2 - k(N^{\frac12})\right)\left(N/2 + k(N^{\frac12})\right)2\pi/N\right\}^{\frac12}} + O\left(\frac{1}{N}\right)$$

$$= \frac{2}{(2\pi N)^{\frac12}}\left\{e^{-2(1-x_1)^2k^2} - e^{-2k^2} + O\left(\frac{1}{N^{\frac12}}\right)\right\}$$

$$= \frac{2}{(2\pi N)^{\frac12}}\left\{\det\begin{bmatrix} e^{-k^2(1-x_1)^2} & e^{-k^2(1)^2} \\ e^{-k^2(-1)^2} & e^{-k^2(-(1-x_1))^2}\end{bmatrix} + O\left(\frac{1}{N^{\frac12}}\right)\right\}.$$

We are thus able to prove:

THEOREM 2.1. *For c_{ij} given by* (2.2), *and large N,*

$$L^{-N}N!\det(1/c_{ij}!) = \frac{L^{L/2}}{(2\pi N)^{(L-1)/2}}\left\{\det\left(\exp\left(-\frac{L}{2}k^2d_{ij}^2\right)\right) + O\left(\frac{1}{N^{\frac12}}\right)\right\};$$

$$k > 0,$$

where the d_{ij}'s are given by (2.5).

The main result of this section then follows:

THEOREM 2.2. *As $N \to \infty$,*

$$\Pr\{S_N(1/L) < k\}$$

$$= \frac{L^{L/2}k^{L-1}}{(2\pi)^{(L-1)/2}}\int_S \det\left(\exp\left(-\frac{L}{2}k^2d_{ij}^2\right)\right)dx_1\cdots dx_{L-1} + O\left(\frac{1}{N^{\frac12}}\right);$$

$$k > 0,$$

where $S_N(1/L)$ is given by (2.4), *$x_L \equiv L - (x_1 + \cdots + x_{L-1})$, and the region*

$$S = \left\{(x_1,\cdots,x_{L-1}) : x_i \geqslant 0\ \forall i, \quad \text{and } \Sigma_{i=1}^{L-1}x_i \leqslant L\right\}.$$

PROOF. From (2.1), we see that

$$\Pr\{S_N(1/L) < k\} = \Sigma_{Q_0} N!L^{-N}\det(1/c_{ij}!)$$
$$= \Sigma_{Q_0}\frac{L^{L/2}}{(2\pi N)^{(L-1)/2}}\left\{\det\left(\exp\left(-\frac{L}{2}k^2 d_{ij}^2\right)\right) + O\left(\frac{1}{N^{\frac{1}{2}}}\right)\right\};$$
$$k > 0,$$

by Theorem 2.1. Now the sum can be approximated by an integral, and since $m_i - 1 = N/L + (1 - (x_i + 1/k(N^{\frac{1}{2}})))k(N^{\frac{1}{2}})$, then $1 = \Delta m_i = k(N^{\frac{1}{2}})\Delta x_i$. Therefore the above becomes

$$\left(k(N^{\frac{1}{2}})\right)^{L-1}\int_S \frac{L^{L/2}}{(2\pi N)^{(L-1)/2}}\left\{\det\left(\exp\left(-\frac{L}{2}k^2 d_{ij}^2\right)\right)\right\}dx_1\cdots dx_{L-1} + O\left(\frac{1}{N^{\frac{1}{2}}}\right),$$

which is the required result. □

We can prove the analogous results to Theorems 2.1, 2.2 for general $0 < d < 1$: Now $Q_r = \{(m_1, \cdots, m_{2L+1}) : m_i \geq 0\ \forall i,\ m_i + m_{i+1} < m\ (i = 1, \cdots, 2L)$, and $\Sigma_{i=1}^{2L+1} m_i = N\}$. Write

$$m_{2i-1} = Nr + \left(\frac{r}{d} - x_i\right)k(N^{\frac{1}{2}}) \qquad i = 1, \cdots, L+1$$
$$m_{2i} = N(d-r) + \left(\frac{d-r}{d} - y_i\right)k(N^{\frac{1}{2}}) \qquad i = 1, \cdots, L,$$

where $x_i + y_i \geq 0,\ y_i + x_{i+1} \geq 0\ (i = 1, \cdots, L)$ and $\Sigma_{i=1}^{L+1}x_i + \Sigma_{i=1}^{L}y_i = 1/d$. Define

$$l_{ij} \equiv r/d - \Sigma_{k=j}^{i}(x_k + y_k) + y_i \qquad L+1 \geq i \geq j \geq 1$$
$$\equiv r/d + \Sigma_{k=i}^{j-1}(x_k + y_k) - x_i \qquad 1 \leq i < j \leq L+1, \tag{2.6}$$

and

$$f_{ij} \equiv (d-r)/d - \Sigma_{k=j}^{i}(x_k + y_k) + x_j \qquad L \geq i \geq j \geq 1$$
$$\equiv (d-r)/d + \Sigma_{k=i+1}^{j}(x_k + y_k) - y_j \qquad 1 \leq i < j \leq L. \tag{2.7}$$

Note that $\Sigma_{i=1}^{L+1} l_{i\sigma(i)} = (L+1)r/d - \Sigma_{i=1}^{L+1}x_i$ and $\Sigma_{i=1}^{L} f_{i\tau(i)} = L(d-r)/d - \Sigma_{i=1}^{L}y_i$, for any permutations σ, τ; which implies $\Sigma_{i=1}^{L+1} l_{i\sigma(i)} + \Sigma_{i=1}^{L} f_{i\tau(i)} = 0$. We use this in proving:

THEOREM 2.3. *For $k > 0$, $\Pr\{S_N(d) < k\}$*

$$= \frac{k^{2L}}{(2\pi)^L r^{(L+1)/2}(d-r)^{L/2}}\int_T \det\left(\exp\left(-\frac{k^2}{2r}l_{ij}^2\right)\right)$$
$$\cdot\det\left(\exp\left(-\frac{k^2}{2(d-r)}f_{ij}^2\right)\right)dx_1 dy_1\cdots dx_L dy_L + O\left(\frac{1}{N^{\frac{1}{2}}}\right),$$

where $S_N(d)$ is given by (2.4), $1/d$ *is not an integer, and the region*

$$T = \{(x_1, y_1, \cdots, x_L, y_L, x_{L+1}) : x_i + y_i \geqslant 0, y_i + x_{i+1} \geqslant 0$$
$$(i = 1, \cdots, L), \quad \textit{and} \;\; \Sigma_{i=1}^{L+1} x_i + \Sigma_{i=1}^{L} y_i = 1/d.$$

Theorems 2.2 and 2.3 are the exact asymptotic results for the scan statistic. As L gets larger, the integration becomes intractable. For $d = 1/L$, $L = 2$, Theorem 2.2 gives a closed form result:

$$\text{(2.8)} \qquad \lim_{N\to\infty} \Pr\{S_N(1/2) < k\} = \{2\Phi(2k) - 1\} - 4k\phi(2k),$$

where $\phi(x) = \dfrac{1}{(2\pi)^{\frac{1}{2}}} e^{-x^2/2}$ and $\Phi(y) = \int_{-\infty}^{y} \phi(x)\, dx$. For $L = 3$:

$$\lim_{N\to\infty} \Pr\{S_N(1/3) < k\} = \frac{3k}{2.\pi^{\frac{1}{2}}} \Bigg\{ \int_0^3 \left(e^{-9k^2(1-x)^2/4} + e^{-9k^2(1+x)^2/4}\right)$$
$$\times \left(2\Phi((3-x)(3/2)^{\frac{1}{2}}k) - 1\right) dx$$
$$\text{(2.9)} \qquad - 3e^{-9k^2/4}\left(2\Phi((3/2)^{\frac{1}{2}}3k) - 1\right) + \frac{9.3^{\frac{1}{2}}k}{2.\pi^{\frac{1}{2}}} e^{-9k^2}$$
$$- \frac{3^{\frac{1}{2}}k}{\pi^{\frac{1}{2}}} e^{-9k^2}\left(\frac{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}{k}\Phi(3.6^{\frac{1}{2}}.k) - \frac{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}{2k} + \frac{e^{-27k^2}}{6k^2} - \frac{1}{6k^2}\right)\Bigg\},$$

which cannot be put into closed form. Now consider the asymptotic distribution for large k; i.e., in the tail.

A result of Naus (1965) says that for $d \leqslant \frac{1}{2}$,

$$1 - \Pr\{N(d) < m\} = 2\Sigma_{i=m+1}^{N} \binom{N}{i} d^i (1-d)^{N-i}$$
$$\text{(2.10)} \qquad + \{d(1-d) - N + m + 1\}\binom{N}{m} d^m (1-d)^{N-m},$$

provided $m \geqslant \left[\dfrac{N+1}{2}\right]$. This is an incomplete solution to the distribution of $N(d)$, since we need results for $m > Nd$. Equation (2.3) fills the gap, but with a complicated expression. With the help of Stirling's formula, and using the definitions of k and $S_N(d)$ given in (2.4), we see that

$$\text{(2.11)} \qquad 1 - \Pr\{S_N(d) < k\} \simeq k(d(1-d))^{-\frac{1}{2}}\phi\left(k(d(1-d))^{-\frac{1}{2}}\right)/d$$
$$+ 2\left[1 - \Phi\left(k(d(1-d))^{-\frac{1}{2}}\right)\right],$$

provided $k \geqslant \left(\left[\dfrac{N+1}{2}\right] - Nd\right) N^{-\frac{1}{2}}$.

It does not seem unreasonable to hope that (2.11) would be accurate enough to give a quick and easy critical level for the distribution of $S_N(d)$, in much the same way as in Watson (1967). Kuiper (1960) did a similar thing, but also kept the leading term of $O(1/N^{\frac{1}{2}})$. Unfortunately, from extensive numerical studies using the table of exact probabilities given by Wallenstein and Naus (1974), we concluded that the approximation was not good enough. We believe this is due mainly to the heavy skewness of distribution, for small d. A similar result to (2.11) for the scan statistic on the circle may be obtained; see Cressie (1977b). For $d = \frac{1}{2}$, this shows agreement with the *first term* of the exact asymptotic result given by Ajne (1968).

3. The Poisson process and Shepp's result. Suppose $\{K(t), t \geq 0\}$ is a Poisson process with $E(K(t)) = \lambda t$, $\lambda > 0$. Then define,

$$K(t, t+h] \equiv K(t+h) - K(t), \tag{3.1}$$

to be the number of points of the process in the interval $(t, t+h]$. We will hold h fixed, and allow t to vary. Define,

$$W_{m,h} \equiv \begin{array}{l}\text{time to the first occurrence of an interval of length } h,\\ \text{which contains } m \text{ points of the Poisson process.}\end{array}$$

$$K_T(h) \equiv \sup_{0<t<T} K(t, t+h].$$

Then,

$$\{W_{m,h} > T\} = \{K_T(h) < m\},$$

and,

$$\Pr\{K_T(h) < m\}$$

$$= \Sigma_{N=0}^{\infty} \Pr\{K_T(h) < m \mid K(0, T+h) = N\} \Pr\{K(0, T+h) = N\} \tag{3.2}$$

$$= \Sigma_{N=0}^{\infty} \frac{e^{-\lambda(T+h)}[\lambda(T+h)]^N}{N!} \Pr\left\{N\left(\frac{h}{T+h}\right) < m \mid N \text{ points in } [0,1]\right\}.$$

Note that $K_T(h)$ is the version of the scan statistic on the positive half line; it is also the maximum queue length of an $M/D/\infty$ queue, where the service time is deterministic of length h. Hence we wish to find its distribution. Since we know the distribution of $N(d)$ from (2.3), the distribution of $K_T(h)$ can be found from (3.2). A similar expression is given in Huntington and Naus (1975) as a corollary to their main theorem, however their result is in error. It should read,

$$\Pr\{W_{m,h} \leq T\} = 1 - \Sigma_{Q^*} R^* \det(1/h_{ij}!) \det(1/e_{ij}!), \tag{3.3}$$

where Q^* is the set of all $2L+1$ nonnegative integers m_i satisfying $m_i + m_{i+1} < m$ $(i = 1, \cdots, 2L)$, *and where* $R^* = \mathrm{R}e^{-\lambda(T+h)}[\lambda(T+h)]^N/N!$; R is obtained from (2.3) *with* $d = h/(T+h)$.

It is a very simple matter to show that

$$\left\{\frac{K(0,t] - \lambda t}{\lambda^{\frac{1}{2}}}; 0 < t \leq T\right\} \to_w \{W(t); 0 < t \leq T\}, \quad \text{as } \lambda \to \infty,$$

where $W(t)$ is the standard Weiner process, and "$\to_w$" means "converges weakly to." Then by methods similar to those of Cressie (1977a),

$$\left\{\frac{K(t, t+h] - \lambda h}{(\lambda h)^{\frac{1}{2}}}\right\} \to_w \{X_h(t)\}, \tag{3.4}$$

where $X_h(t) \equiv (W(t+h) - W(t))h^{-\frac{1}{2}}$. The process $X_h(t)$ is a stationary, separable Gaussian process with mean zero and covariance function,

$$\begin{aligned} r_h(s, t) &= 1 - |s-t|/h \qquad (|s-t| \leqslant h) \\ &= 0 \qquad\qquad\qquad\quad (|s-t| > h), \end{aligned}$$

and from (3.4), $\dfrac{K_T(h) - \lambda h}{(\lambda h)^{\frac{1}{2}}} \to_{\mathfrak{D}} \sup_{0<t<T} X_h(t)$, as $\lambda \to \infty$, where "$\to_{\mathfrak{D}}$" means "converges in distribution to."

Now $\sup_{0<t<T} X_h(t) \equiv \sup_{0<t<T/h} X_1(t)$, and the supremum of $X_1(t)$ has been studied by Slepian (1961), and Shepp (1966), for $T \leqslant 1$. Finally, Shepp (1971) gave the general result for all T:

THEOREM 3.1. (Shepp).

$$\begin{aligned} Q_a(T/h) &\equiv \Pr\{\sup_{0<t<T/h} X_1(t) < a\} \\ &= \int_{D'} \cdots \int \det \phi(y_i - y_{j+1} + a) dy_1 \cdots dy_{n+1}, \end{aligned}$$

where $D' = \{0 < y_1 < y_2 < \cdots < y_{n+1}\}$, $T/h = n$, *an integer, and the determinant is of size* $(n+1) \times (n+1)$, $0 \leqslant i, j \leqslant n$, *with* $y_0 \equiv 0$.

PROOF. Equation (2.15) from Shepp (1971) gives $\Pr\{\sup_{0<t<T/h} X_1(t) < a \mid X_1(0) = x\}$. The starting value may be integrated out to give the required result. □

Note that when T/h is *not* an integer, a similar result to the above theorem may be obtained from Shepp (1971), expression (2.25). We will now give a direct derivation of the distribution of $K_T(h)$ when $T/h = n$, showing that we do indeed arrive at the distribution given in Theorem 3.1.

THEOREM 3.2.

$$\lim_{\lambda\to\infty} \Pr\left\{\frac{K_T(h) - \lambda h}{(\lambda h)^{\frac{1}{2}}} < a\right\} = \int_{D'} \cdots \int \det \phi(y_i - y_{j+1} + a) dy_1 \cdots dy_{n+1}.$$

PROOF. From (3.2),

$$\begin{aligned} \Pr\left\{\frac{K_T(h) - \lambda h}{(\lambda h)^{\frac{1}{2}}} < a\right\} = \Sigma_{N=0}^{\infty} \phi\left(\frac{N - \lambda(T+h)}{(\lambda(T+h))^{\frac{1}{2}}}\right) \cdot \frac{1}{(\lambda(T+h))^{\frac{1}{2}}} \\ \cdot \Pr\left\{\frac{N(d) - N.d}{N^{\frac{1}{2}}} < \frac{\lambda h - N.h/(T+h) + a(\lambda h)^{\frac{1}{2}}}{N^{\frac{1}{2}}}\right\} \\ + O\left(\frac{1}{\lambda^{\frac{1}{2}}}\right), \end{aligned}$$

where $d = h/(T + h)$; see Johnson and Kotz (1969) page 99, for justification of the remainder term. Now put $z = \frac{N - \lambda(T + h)}{(\lambda(T + h))^{\frac{1}{2}}}$; i.e., $N = N(z;\ \lambda) \equiv \lambda(T + h) + z(\lambda(T + h))^{\frac{1}{2}}$; and approximate the infinite sum by an integral with respect to z. Hence the above becomes,

$$\int_{-(\lambda(T+h))^{\frac{1}{2}}}^{\infty}\phi(z)\cdot \Pr\left\{ S_{N(z;\ \lambda)}(d) < \left(-\left(\frac{h}{T+h}\right)z + \left(\frac{h}{T+h}\right)^{\frac{1}{2}}a\right)\left(1 + \frac{z}{(\lambda(T+h))^{\frac{1}{2}}}\right)\right\}dz + O\left(\frac{1}{\lambda^{\frac{1}{2}}}\right).$$

The upper limit can be modified to $a((T + h)/h)^{\frac{1}{2}}$, since the integrand is nonzero only when $a(\lambda h)^{\frac{1}{2}} + \lambda h > Nh/(T + h)$. Now since the integrand is positive and bounded, we can use dominated convergence to show that as $\lambda \to \infty$ the above converges to

$$\int_{-\infty}^{a((T+h)/h)^{\frac{1}{2}}}\phi(z)\cdot \lim_{\lambda\to\infty} \Pr\left\{ S_N(d) < -\frac{zh}{T+h} + a\left(\frac{h}{T+h}\right)^{\frac{1}{2}}\right\} dz.$$

Recall that $T/h = n$, and so $h/(T + h) = 1/(n + 1)$; also make the change of variables $w = -z/(n + 1) + z/(n + 1)^{\frac{1}{2}}$, and use Theorem 2.2 with $L = n + 1$ to give,

$$(n+1)\int_0^{\infty}\phi\left(w(n+1) - a(n+1)^{\frac{1}{2}}\right)\frac{(n+1)^{(n+1)/2}}{(2\pi)^{n/2}}w^n \times \int_S \det\left(\exp\left(-\frac{n+1}{2}w^2 d_{ij}^2\right)\right)dx_1 \cdots dx_n dw.$$

Make the change of variables

$$y_i = \Sigma_{l=0}^{i}(n+1)^{\frac{1}{2}}wx_l \qquad (i = 1, \cdots, n+1),$$

yielding as Jacobian, $\{(n + 1)^{(n+3)/2}w^n\}^{-1}$, and hence obtain

$$\int_{D'} \cdots \int \det \phi(y_i - y_{j+1} + a)dy_1 \cdots dy_{n+1}. \qquad \square$$

Note that when $T/h = n + \theta$, $0 < \theta < 1$, we can in a similar manner, obtain

$$\Pr\left\{\frac{K_T(h) - \lambda h}{(\lambda h)^{\frac{1}{2}}} < a\right\} \to$$

$$\int_{D''} \det(\phi_\theta(r_i - s_j)) \det(\phi_{1-\theta}(s_i - r_{j+1} + a))dr_1 \cdots dr_{n+1}\, ds_0 \cdots ds_{n+1}, \quad \text{as } \lambda \to \infty,$$

where $\phi_\theta(u) \equiv (2\pi\theta)^{-\frac{1}{2}}e^{-u^2/2\theta}$, $D'' \equiv \{0 < r_1 < \cdots < r_{n+1}, s_0 < s_1 < \cdots < s_{n+1}\}$, $r_0 \equiv 0$, and the first determinant is of size $(n + 2) \times (n + 2)$, $0 \leqslant i,\ j \leqslant n + 1$ while the second is of size $(n + 1) \times (n + 1)$, $0 \leqslant i,\ j \leqslant n$.

It is not surprising that the asymptotic scan distribution of Theorem 2.2, and Shepp's result of Theorem 3.1 look so similar and, as Theorem 3.2 shows, are in fact related. They are essentially both derived from an identity of Karlin and McGregor (1959), although the identity is used differently in each case. Theorem 3.2 also provides a very useful check on the validity of our results, as well as an independent derivation of Shepp's result.

4. The supremum of another Gaussian process. Let $X(t)$ be a stationary Gaussian process with $\Sigma(X(t)) = 0$ and $\Sigma(X(t)X(t')) = \rho(t - t')$. Define

$$P_X(a, T|x) \equiv \Pr\{X(t) < a; 0 \leqslant t \leqslant T|X(0) = x\}. \tag{4.1}$$

In a recent paper, Shepp and Slepian (1976) stated that to the best of their knowledge, P_X was only known for the three cases:

(i) $\rho(\tau) = e^{-|\tau|}$,

(ii) $\rho(\tau) = 1 - |\tau|$ for $|\tau| \leqslant 1$, $\quad = 0$ for $|\tau| \geqslant 1$,

(iii) $\rho(\tau) = 3/2 \exp(-|\tau|/3)(1 - \exp(-2|\tau|/3^{\frac{1}{2}})/3)$.

They then added a fourth case to the above:

(iv) $\rho(\tau) = \rho(-\tau) = \rho(\tau + 2)$; $\quad \rho(\tau) = 1 - \alpha\tau$ for $0 \leqslant \tau \leqslant 1$, $\quad = 1 + \alpha(\tau - 2)$ for $1 \leqslant \tau \leqslant 2$. Also $0 \leqslant \alpha \leqslant 2$.

Using the results of Section 2, we will now derive expressions for (4.1) when the covariance function is,

$$\begin{aligned} \rho_X(\tau) &= 1 - |\tau|/(1 - \beta) && (|\tau| \leqslant 1) \\ &= -\beta/(1 - \beta) && (1 \leqslant |\tau| < (1 - \beta)/\beta, \end{aligned} \tag{4.2}$$

where $0 < \beta \leqslant \frac{1}{2}$. The range of the covariance is necessarily restricted, in order to retain positive-definiteness. Thus the covariance has the same initial "tent" shape as cases (i) and (iv), but after going negative, it stays negative, and constant. If we put $\beta = 0$ in (4.2), we obtain case (ii).

Define,

$$Y(t) \equiv X(t/\beta); \qquad 0 \leqslant t \leqslant 1 - \beta. \tag{4.3}$$

Then $\{Y(t); 0 \leqslant t \leqslant 1 - \beta\}$ has covariance function,

$$\begin{aligned} \rho_Y(\tau) &= 1 - |\tau|/\beta(1 - \beta) && (|\tau| \leqslant \beta) \\ &= -\beta/(1 - \beta) && (\beta \leqslant |\tau| \leqslant 1 - \beta), \end{aligned}$$

and $P_X(a, \beta^{-1} - 1|x) = P_Y(a, 1 - \beta|x)$. Recall from (1.2), the definition of $N(t, \beta)$. Then Cressie (1977) has proved that $\{(N(t, \beta) - N.\beta)(N\beta(1 - \beta))^{-\frac{1}{2}}; 0 \leqslant t \leqslant 1 - \beta\}$ converges weakly (as $N \to \infty$) to $\{Y(t); 0 \leqslant t \leqslant 1 - \beta\}$, on the Skorohod metric space of functions that are right continuous and have left hand limits. Therefore,

$$\Pr\{\sup_{0 \leqslant t \leqslant 1-\beta} N(t, \beta) < m|N(0, \beta) = m_1\} \to P_Y(a, 1 - \beta|x), \quad \text{as } N \to \infty, \tag{4.4}$$

where $m = (N\beta(1 - \beta))^{\frac{1}{2}}a + N\beta$ and $m_1 = (N\beta(1 - \beta))^{\frac{1}{2}}x + N\beta$. Let us consider for the moment $\beta = 1/L$, L an integer. With a little modification to Naus (1966), we easily have, for $0 \leqslant m_1 < m$, $[N/L] < m \leqslant N$,

$$\Pr\{\sup_{0<t<1-1/L} N(t, 1/L) < m | N(0, 1/L) = m_1\}$$

$$= \left[\binom{N}{m_1}(1/L)^{m_1}(1 - 1/L)^{N-m_1}\right]^{-1} \Sigma_{Q'} N! L^{-N} \det(1/c_{ij}!),$$

where c_{ij} is given by (2.2), and $Q' = \{(m_2, \cdots, m_L) : 0 \leqslant m_i < m; \Sigma_{i=2}^{L} m_i = N - m_1\}$. Thus via an identical proof to that of Theorem 2.2, we can prove,

THEOREM 4.1. *For $L \geqslant 2$, $a > 0$,*

$$P_X(a, L-1|(1-x_1)a) = P_Y(a, 1-1/L|(1-x_1)a)$$

$$= \frac{I(0 \leqslant x_1 \leqslant L) a^{L-2} L^{\frac{1}{2}} (1 - 1/L)^{(L-1)/2}}{(2\pi)^{(L-2)/2} \exp(-a^2(1-x_1)^2/2)}$$

$$\times \int_{S'} \det\left(\exp\left(-\tfrac{1}{2}(1 - 1/L) a^2 d_{ij}^2\right)\right) dx_2 \cdots dx_{L-1},$$

where $I(A)$ is the indicator function of the set A, d_{ij} is given by (2.5), $\Sigma_{r=1}^{L} x_r = L$, and $S' = \{(x_2, \cdots, x_L) : x_i \geqslant 0; \Sigma_{i=2}^{L} x_i = L - x_1\}$. When $L = 2$, the integrand is a constant; here interpret $\int_{S'} dx_2 \cdots dx_{L-1}$ to be equal to 1.

By multiplying the above expression by $(2\pi)^{-\frac{1}{2}} a \cdot \exp(-a^2(1-x_1)^2/2)$, and integrating out over x_1, we get,

$$P_X(a, L-1) = P_Y(a, 1-1/L) = \text{the expression given in Theorem 2.2},$$

where $k = a(1/L(1 - 1/L))^{\frac{1}{2}}$, which is what the theory predicts. A further check is got by repeating almost line by line the argument of Section 3; we can then derive independently the distribution of $\Pr\{\sup_{0<t<L-1} X_1(t) < a | X_1(0) = x\} = P_{X_1}(a, L - 1|x)$, for X_1 the Gaussian process of case (ii). The formula is originally due to Shepp (1971) (see his equation (2.15)).

Suppose now that $\beta = d \neq 1/L$, but define $L = [d^{-1}]$, the largest integer in d^{-1}, and $r = 1 - dL$. Then by an argument similar to that which led to Theorem 4.1, we can prove,

THEOREM 4.2. *For $0 < d \leqslant \frac{1}{2}$, $a > 0$,*

$$P_X(a, d^{-1} - 1|(1-z)a) = P_Y(a, 1-d|(1-z)a)$$

$$= \frac{I(0 \leqslant z \leqslant d^{-1}) \cdot a^{2L-1}(d(1-d))^L}{(2\pi)^{L-1/2} r^{(L+1)/2} (d-r)^{L/2} \exp(-a^2(1-z)^2/2)}$$

$$\int_{T'} \det\left(\exp\left(-\frac{a^2 d(1-d) l_{ij}^2}{2r}\right)\right)$$

$$\cdot \det\left(\exp\left(-\frac{a^2 d(1-d) f_{ij}^2}{2(d-r)}\right)\right) dy_1 dx_2 dy_2 dx_3 \cdots dx_L dy_L,$$

where l_{ij}, f_{ij} *are given by* (2.6), (2.7), *and the region of integration is* $T' = \{(x_1, y_1, x_2, y_2, \cdots, x_L, y_L, x_{L+1}) : x_i + y_i \geqslant 0, y_i + x_{i+1} \geqslant 0 \ (i = 1, \cdots, L); x_1 + y_1 = z; \Sigma_{i=1}^{L+1} x_i + \Sigma_{i=1}^{L} y_i = d^{-1}\}$.

We will now illustrate the results of this section for $\beta = \frac{1}{2}$ and $\beta = \frac{1}{3}$; i.e., $\beta = 1/L$ for $L = 2, 3$. This enables us to use the simpler formula of Theorem 4.1. When $L = 2$, the integrand of Theorem 4.1 is a constant. Substituting into the formula, and interpreting $\int_{S'} dx_2 \cdots dx_{L-1} = 1$, we get,

$$P_X(a, 1|(1 - x_1)a) =$$

$$I(0 \leqslant x_1 \leqslant 2) \cdot e^{a^2(1-x_1)^2/2} \cdot \det \begin{bmatrix} e^{-a^2(1-x_1)^2/4} & e^{-a^2/4} \\ e^{-a^2/4} & e^{-a^2(1-x_1)^2/4} \end{bmatrix} \tag{4.5}$$

$$= I(0 \leqslant x_1 \leqslant 2)\left\{1 - e^{-a^2/2}/e^{-a^2(1-x_1)^2/2}\right\}; \qquad a > 0,$$

where X has covariance function, $\rho_X(\tau) = 1 - 2|\tau|$ for $|\tau| \leqslant 1$. Note that the presence of negative correlation gives the Gaussian process an interesting behaviour. As the initial starting point x, goes negative and tends to $-a$, the probability that $\{X(t); 0 \leqslant t \leqslant 1\}$ remains *below* $+a$, gets *smaller* and tends to zero. Then, for all $x < -a$, this probability is exactly zero.

When $L = 3$, X has covariance function,

$$\begin{aligned} \rho_X(\tau) &= 1 - 3|\tau|/2 && (|\tau| \leqslant 1) \\ &= -1/2 && (1 \leqslant |\tau| \leqslant 2); \end{aligned}$$

i.e., the covariance function has the initial "tent" shape, goes negative, and at $|\tau| = 1$ "flattens out" to stay negative. After some algebra,

$$\begin{aligned} P_X(a, 2|(1 - x_1)a) = I(0 \leqslant x_1 \leqslant 3) \cdot (2\pi)^{-\frac{1}{2}} e^{a^2(1-x_1)^2/2} \Big[& e^{-2a^2}\Big\{2a(3 - x_1)3^{-\frac{1}{2}} \\ & - \Phi\big(2a(3 - x_1)3^{-\frac{1}{2}}\big) + \tfrac{1}{2}\Big\} \\ & - e^{-2a^2(x_1^2 - 3x_1 + 3)/3}\Big\{2a(3 - x_1)3^{-\frac{1}{2}} - e^{a^2 x_1^2/6} \\ & \Big\{\Phi\big(2a(3 - x_1/2)3^{-\frac{1}{2}}\big) - \Phi\big(ax_1 3^{-\frac{1}{2}}\big)\Big\}\Big\}\Big], \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variate. Note that at $x_1 = 0, 3$; i.e., $x = (1 - x_1)a = a, -2a$; the formula gives P_X to be zero, and for $x_1 \geqslant 3$; i.e., $x \leqslant -2a$; P_X is identically zero. Once again this says that in order for $\{X(t); 0 \leqslant t \leqslant 2\}$ to stay *below* $+a$ with positive probability, the starting value x *cannot be too large negative*.

In conclusion then, we have used the asymptotic techniques developed in the previous sections, to calculate the supremum-type probability (4.1), for the particular case of $T = (1 - \beta)/\beta$, and the Gaussian process given by (4.2). Ideally we would like to calculate $P_X(a, T|x)$, for all $T \leqslant (1 - \beta)/\beta$, under the assumption that X has covariance (4.2).

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