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# STOCHASTIC INTEGRATION AND $L^p$-THEORY OF SEMIMARTINGALES

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If $X$ is a bounded left-continuous and piecewise constant process and if $Z$ is an arbitrary process, both adapted, then the stochastic integral $\int X\,dZ$ is defined as usual so as to conform with the sure case. In order to obtain a reasonable theory one needs to put a restriction on the integrator $Z$. A very modest one suffices; to wit, that $\int X_n\,dZ$ converge to zero in measure when the $X_n$ converge uniformly or decrease pointwise to zero. Daniell's method then furnishes a stochastic integration theory that yields the usual results, including Itô's formula, local time, martingale inequalities, and solutions to stochastic differential equations. Although a reasonable stochastic integrator $Z$ turns out to be a semimartingale, many of the arguments need no splitting and so save labor. The methods used yield algorithms for the pathwise computation of a large class of stochastic integrals and of solutions to stochastic differential equations.

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**1. Introduction.** There are two situations in which the stochastic integral $\int X\,dZ$ can be defined with ease. When the integrator $Z$ is a process of finite variation the integrand $X$ can be quite arbitrary and the integral is defined as a Stieltjes integral pathwise. When the integrator is a square-integrable martingale, Itô's extension procedure, generalized by Kunita and Watanabe [KW], yields an integral with good properties provided the integrand is previsible. (For other definitions see [M2, S1].) In the search for the greatest common denominator of these two situations the notion of a semimartingale emerges: it is that kind of integrator $Z$ for which a mixture of the pathwise Stieltjes and Itô integration techniques yields a good integral. Not surprisingly, the integrand has to be previsible for the amalgamated techniques to work. An excellent account of the stochastic integral for semimartingale integrators and previsible integrands can be found in [M5] (see also [J1]).

The starting points for the present investigation are two questions arising immediately. First, does one get the most general 'reasonable' stochastic integration theory by amalgamating the two known techniques? The answer is yes. This might be considered surprising in view of the modest criterion of reasonableness adopted; to wit, that $\int X_n\,dZ$ converge to zero in measure when the sequence $X_n$ of elementary integrands either decreases pointwise to zero or converges uniformly to zero. A proof of this has recently been given by

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Dellacherie, Mokobodzki and Letta (see [M7]). We shall present an alternate proof that rests on a deep and powerful theorem of Maurey [M1] and Rosenthal [R1], which is a close relative of Grothendieck's Fundamental Theorem and yields additional information in the $L^p$-theory, $0 < p \leqq 2$.

Secondly, integrals are limits, and the question arises in which sense $\int X\, dZ$ converges. The present investigation concentrates on convergence in $L^p$-mean, $0 \leqq p < \infty$, though many of the methods developed apply as well to convergence in the topology of certain Orlicz spaces and Lorentz spaces. In the course of the investigation some of the known martingale inequalities appear in a rather natural way in the setting of stochastic integration, and a few new ones are added.

To make these remarks on our goal more precise, some notation is required. The basic data underlying everything are these: $(\Omega, \mathscr{F}_\infty, P)$ is a complete probability space; $\mathscr{F} = \{\mathscr{F}_t: t \geqq 0\}$ is an increasing and right-continuous (r.c.) family of $\sigma$-algebras with span $\mathscr{F}_\infty$; every negligible subset of $\mathscr{F}_\infty$ appears in $\mathscr{F}_0$. The word 'process' will refer to a function $Y$ on the *base space*

$$B = \Omega \times [0, \infty)$$

or to the class $\dot{Y}$ of its modifications. $Y$ is called a representative of $\dot{Y}$. All processes are also assumed to be a.s. finite at any instant and adapted:

$$Y_t \in \mathscr{L}^0(\mathscr{F}_t; P); \qquad \dot{Y}_t \in L^0(\mathscr{F}_t; P).$$

We shall use freely the results, and most of the notation, of Dellacherie's book [D2].

Let $\mathscr{T}$ denote the collection of all $\mathscr{F}$-stopping times that take only finitely many values, all of them finite; $\mathscr{A}$ the ring of subsets of $B$ generated by the stochastic intervals $(S, T]$, $S, T \in \mathscr{T}$, and the special intervals $[S_0, 0] = [S_0 = 0] \times \{0\}$ whose graphs are contained in $\Omega \times \{0\}$; and $\mathscr{R}$ the vector lattice of step functions over $\mathscr{A}$. $\mathscr{A}$ generates the previsible $\sigma$-algebra $\mathscr{P}$, and every $A \in \mathscr{A}$ can be written as a disjoint union

$$A = [S_0, 0] \cup \textstyle\bigcup_{i=1}^n (S_i, T_i]. \tag{1.1}$$

The intervals $(S_i, T_i]$ can be fixed uniquely by insisting that they be *consecutive*: $0 \leqq S_1 \leqq T_1 \leqq S_2 \leqq \cdots$ and $S_i < T_i < S_{i+1}$ on $[S_{i+1} < T_{i+1}]$. Every function $X \in \mathscr{R}$ is a left-continuous bounded process vanishing after a bounded time and has a representation[1]

$$X = r_0 \cdot [S_0, 0] + \textstyle\sum_{i=1}^n r_i \cdot (T_{i-1}, T_i]; \qquad r_i \in \mathbf{R},\ T_i \in \mathscr{T}, \tag{1.2}$$

which is unique when the intervals are chosen consecutively. (The $T_i$ are not unique, the intervals are; we shall also say the $T_i$ are chosen consecutively.) The processes in $\mathscr{R}$ are termed *elementary integrands*. One may now define, for any process $\dot{Z}$ and $X \in \mathscr{R}$ as in (1.2)

$$d\dot{Z}(X) := r_0 \cdot \dot{Z}_0 \cdot [S_0 = 0] + \textstyle\sum_{i=1}^n r_i \cdot (\dot{Z}_{T_i} - \dot{Z}_{T_{i-1}}) \tag{1.3}$$

and obtain a linear map $d\dot{Z}: \mathscr{R} \to L^0$.[2]

The questions raised above are given their precise meaning by the following notion, and amount to the problem of investigating it.

DEFINITION. Let $0 \leqq p < \infty$. The process $\dot{Z}$ is an $L^p$-integrator if $\dot{Z}_t \in L^p(\mathscr{F}_t; P)$ for all

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[1] Sets will be identified throughout with their characteristic functions; e.g., $X$ takes the value $r_0$ on $[S_0, 0]$, $r_i$ on $(T_{i-1}, T_i]$, and zero off the union of these stochastic intervals.

[2] A class $\dot{Z}$ is chosen for the integrator rather than its representative for two reasons. First, it is more general, and then the integral will be defined as a mean limit anyway. The choice is thus made for consistency—until it is abandoned for good reason in Section 2.7. The ring $\mathscr{A}$ is chosen over the ring $\mathscr{B}$ generated by the right-continuous intervals $[S, T)$ because it is 'smaller' in the sense that the $\sigma$-algebra $\mathscr{P}$ it generates is smaller than the $\sigma$-algebra $\mathscr{W}$ generated by $\mathscr{B}$: the smaller the domain of $d\dot{Z}$ the easier it is for $d\dot{Z}$ to be reasonable and thus the larger the scope of the investigation.

$t \geqq 0$ and if $d\dot{Z}: \mathscr{R} \to L^p$ has an extension satisfying the Dominated Convergence Theorem (DCT).

Replacing $L^p$ by an Orlicz space $L^\Phi$ or a Lorentz space $L^{p,q}$, etc., gives the notion of an $L^\Phi$-integrator or an $L^{p,q}$-integrator, etc. The condition that the DCT hold looks at first sight rather more stringent than the conjunction of the two mentioned above; to wit

($\mathrm{A_p}$) The set $\{d\dot{Z}(X): X \in \mathscr{R}_T, |X| \leqq 1\}$ is bounded in $L^p$ for all $T \in \mathscr{T}$,

where
$$\mathscr{R}_T = \{X \in \mathscr{R}: X = 0 \text{ on } (T, \infty)\},$$

and

($\mathrm{B_p}$) $\lim d\dot{Z}(X^n) = 0$ in $L^p$ for every decreasing sequence $X^n$ of elementary integrands with pointwise infimum zero.

However, we shall see in the next section that ($\mathrm{A_p}$) and, merely,

($\mathrm{B_0'}$) $\dot{Z}$ is right-continuous in probability

imply that $\dot{Z}$ is an $L^p$-integrator. ($\mathrm{A_p}$) simply means that $d\dot{Z}: \mathscr{R} \to L^p$ is continuous when $\mathscr{R}$ is given its natural topology, the inductive limit of the sup-norm topologies on the $\mathscr{R}_T$, $T \in \mathscr{T}$. For $p > 0$ it amounts to this:

($\mathrm{A_p'}$) $\gamma_T^p[\dot{Z}] := \sup\{\|d\dot{Z}(X)\|_{L^p}: X \in \mathscr{R}_T, |X| \leqq 1\} < \infty$ for all $T \in \mathscr{T}$.

The $\gamma_T^p[\dot{Z}]$ measure the size of $\dot{Z}$ as an $L^p$-integrator. Other quantities measuring this have been investigated. For instance, Meyer [M6] introduces numbers $\|\dot{Z}\|_{H^p}$ for $1 \leqq p \leqq \infty$. They are equivalent with our $\gamma_T^p[\dot{Z}]$ in this range of $p$'s, as was shown by Yor [Y1]. Such quantities are of great importance in the theory of stochastic differential equations, where a firm control on the size of $\dot{Z}$ is needed for the convergence arguments (see, e.g., [DM], [E1], [P2] and Section 8 below).

The idea to do the integration theory of $\int X\,dZ$ by regarding $dZ$ as a vector measure is, of course, not new. Itô's definition does just that; and it was resumed and developed by Kunita and Watanabe [KW], Meyer [M5], Pellaumail [P2, P1, MP], Yor [Y4], Metivier [M4], and Kussmaul [K2], to name but a sample. The principal difference of the present effort are the single-mindedness in the pursuit of this idea—which in our view reduces the technicality of the subject to a considerable degree—and the emphasis on the previously neglected case $0 \leqq p < 1$. In particular, all of the inequalities connecting the size $\gamma^p$, the square function and the maximal function are known in the case $p \geqq 1$—[G2] and [Y1] are good references—while most of the ones with $p < 1$ are new. Careful accounting of the quantities $\gamma_T^p[Z]$ and $\gamma_T^p[\int X\,dZ]$ yields new results on the pathwise computation of the integral (7.14) and on the pathwise solution of stochastic differential equations (8.2), and a priori estimates on the size of the solutions (8.4).

The reader has noticed the close analogy of the definition of an $L^p$-integrator with that of a distribution function $z = z(t)$ on the half-line: (a) the set function $dz$ must be bounded on the ring $\mathscr{a}$ generated by the left-continuous[1] intervals and (b) $z$ must be right-continuous. The quantitative expression of (a) is (a′): the variation $\gamma_t[z]$ of $z$ on every interval $[0, t]$ is finite. In this case, and in this case only (apart from changing right for left) does $dz$ have a good, Lebesgue type integration theory.

Our first goal is to show that the same holds in the stochastic case.

## 2. Characterization of $L^p$-integrators and their extension theory.

2.1 THEOREM. *Let $0 \leq p < \infty$. The process $\dot{Z}$ is an $L^p$-integrator if and only if* ($\mathrm{A_p}$) *and* ($\mathrm{B_p}$) *are satisfied.*

The necessity of the conditions is obvious. To show the sufficiency really amounts to an exercise in vector valued Daniell integration. We shall give a sketch in order to establish notation needed later and also because there seems to be no account in the literature of the case most interesting to us, namely that of a nonlocally compact base space $B$ and a nonlocally convex range ($0 \leq p < 1$) of the vector measure.[3] We indicate the steps in the most economical order, leaving the straightforward details to the reader.

*Step* 1. Construction of the upper integral. To begin with, we choose a translation-invariant metric $\rho$ for $L^p$, say the usual

$$\rho_p(f) = \|f\|_{L^p}^{p\wedge 1} = \left(\int |f|^p \, dP\right)^{1\wedge 1/p} \qquad \text{if } p > 0,$$

$$\rho_0(f) = \inf\{c: P[|f| > c] < c\} \qquad \text{if } p = 0.$$

The Daniell upper integral $G = G_p^Z$ for the measure $d\dot{Z}$ is defined first on processes $H$ that are pointwise suprema of *sequences* in $\mathscr{R}$ by

$$G(H) = \sup\{\rho(d\dot{Z}(X)): X \in \mathscr{R}, |X| \leq H\}$$

and then on arbitrary functions $F: B \to \bar{\mathbf{R}}$ by

$$G(F) = \inf\{G(H): |F| \leq H; \qquad H \text{ as above}\}.$$

Conditions ($A_p$), ($B_p$) and the subadditivity of $\rho$ translate into the following four properties of the pair ($\mathscr{R}$, $G$);

($\alpha$) $\qquad G$ is finite on $\mathscr{R}$; i.e., $\lim_{\lambda\to 0} G(\lambda X) = 0 \qquad$ for $X \in \mathscr{R}$.

This is immediate from ($A_p$). For $p > 0$ one has, more quantitatively,

($\alpha_p$) $\qquad G_p(\lambda X) = |\lambda|^{1\wedge p} G_p(X), \qquad X \in \mathscr{R}$.

($\beta$) $\qquad G$ is solid: $|F| \leq |F'|$ implies $G(F) \leq G(F')$.

($\gamma$) $\qquad G$ is countably subadditive.

To see this, note that ($B_p$) implies in the presence of ($A_p$) that $\lim d\dot{Z}(X_n) = d\dot{Z}(X)$ for any increasing sequence $X_n$ in $\mathscr{R}$ with pointwise supremum $X \in \mathscr{R}$. Then use a standard argument.

($\delta$) $\lim G(X_n) = 0$ for every sequence $(X_n)$ of elementary integrands $X_n \geq 0$ satisfying

$$\lim_{\lambda\to 0} \sup_{N\in\mathbf{N}} G(\lambda \textstyle\sum_{n=1}^N X_n) = 0.$$

If $p > 0$, the last property reads: $G(X_n) \to 0$ if $G(\{\sum X_n: n \leq N\})$ is bounded. For all $p \geq 0$ it means this: *If the finite partial sums of the sequence $X_n \geq 0$ form a set bounded in G-mean then necessarily $X_n \to 0$ in G-mean.* It is the most crucial of the four properties in the sense that it distinguishes $G$ from an $\mathscr{L}^\infty$- norm, which shares the other three properties; and it derives from the fact that the spaces $L^p$ have the analogous topological feature. Here is a sketch of its provenience:

Let $r_n$ denote the Rademacher functions. Khintchine's inequalities (cf., e.g., [Z1]) say that the map $(a_n) \to \sum a_n r_n$ is a homeomorphism of $l^2$ into $L^p(0, 1)$, for any $p \in [0, \infty)$. In particular, there are constants $k_p$ so that

$$(\textstyle\sum a_n^2)^{1/2} \leq k_p \|\sum a_n r_n\|_{L^p(0,1)}, \qquad \text{for } 0 < p < \infty.$$

Let then $X'_n \in \mathscr{R}$ with $|X'_n| \leq |X_n|$. For all $N \in \mathbf{N}$,

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[3] For a detailed treatment see [B2]; also [T1] for locally compact base spaces. The method of Metivier and Pellaumail [MP] is very close to the one used here in the case $p = 0$.

$$\| (\textstyle\sum_{n=1}^{N} | d\dot{Z}(X'_n) |^2)^{1/2} \|_{L^p(P)} \le k_p \left( \int | \sum_{n=1}^{N} d\dot{Z}(X'_n)(\omega) r_n(t) |^p \, dP(\omega) \, dt \right)^{1/p} .$$

Since $| \sum^N X'_n r_n(t) | \le \sum^N | X_n |$ for all $N$ and $t \in (0, 1)$, the right-hand side is majorized by

$$k_p (G_p^Z(\textstyle\sum^N | X_n|))^{p \wedge 1} \le \text{const} < \infty$$

and so

$$G_p^Z(X_n) = \sup\{\| d\dot{Z}(X'_n) \|_{L^p(P)}^{p\wedge 1} : | X'_n | \le | X_n | \} \to 0.$$

The slight alterations in the case $p = 0$ are left to the reader. Incidentally, this estimate applied to $X'_0 = [0], X'_n = \pm ( T_{n-1}, T_n]^1$ gives the following very useful corollary (established for $p \geqq 1$ by Yor [Y2] in this very way).

2.2 Proposition. *Let $\dot{Z}$ be any process and $0 < p < \infty$. Then*

$$\| (\dot{Z}_0^2 + \textstyle\sum_{n=1}^{N} (\dot{Z}_{T_n} - \dot{Z}_{T_{n-1}})^2)^{1/2} \|_{L^p} \le k_p \cdot \gamma_T^p[\dot{Z}] \tag{2.2}$$

*for any sequence $0 \le T_1 \le T_2 \cdots \le T$ of consecutive stopping times in $\mathcal{T}$. For $p = 0$, the corresponding statement reads: if $\dot{Z}$ is an $L^0$-integrator the family*

$$(\dot{Z}_0^2 + \textstyle\sum_{n=1}^{N} (\dot{Z}_{T_n} - \dot{Z}_{T_{n-1}})^2)^{1/2}$$

*stays bounded in $L^0$ as $0 \le T_1 \le \cdots \le T_N \le T$ vary, with $T \in \mathcal{T}$ fixed.*

We return to the main theme. Couples $(\mathcal{R}, G)$ with properties $(\alpha)$ -$(\gamma)$ are termed *upper gauges* (cf. [B2]). The upper gauge $G_p^Z$ constructed above is the smallest upper gauge $G$ satisfying

$$\rho_p(d\dot{Z}(X)) \le G(X), \qquad X \in \mathcal{R}, \tag{$\epsilon$}$$

but not always the easiest one to handle. It should therefore be kept in mind that the integration theory of $(\mathcal{R}, G)$, sketched in the next step, does not use the provenience of $G$ but merely properties $(\alpha)$ -$(\delta)$.

*Step* 2. The integration theory of $(\mathcal{R}, G)$.

2.3.1. A function $F: B \to \bar{\mathbf{R}}$ is *G-negligible*: $G(F) = 0$, if and only if $G([F \neq 0]) = 0$, i.e., iff $F$ vanishes $G$-a.e.

2.3.2. The functions *finite for G* are defined as

$$\mathcal{F}[G] = \{ F: B \to \bar{\mathbf{R}}; \lim_{\lambda \to 0} G(\lambda F) = 0\},$$

which equals $\{ F: G(F) < \infty\}$ when $G = G_p^Z$, $0 < p < \infty$. $\mathcal{F}[G]$ is a complete pseudometric vector lattice under the distance $G(F - F')$. Every $G$-mean convergent sequence $F_n \in \mathcal{F}[G]$ *has a G-a.e. convergent subsequence* $F_{n(k)}$ such that $\sum_k G(F_{n(k+1)} - F_{n(k)}) < \infty$.

2.3.3. *The $(\mathcal{R}, G)$-integrable functions* are defined as the $G$-mean closure of the elementary integrands $\mathcal{R}$ in $\mathcal{F}[G]$ and are denoted by $\mathcal{L}^1(\mathcal{R}, G)$. When the provenience of $G$ is from $d\dot{Z}$ as above we also write $\mathcal{L}^1(\mathcal{R}, G) = \mathcal{L}^1(d\dot{Z}; p, P)$ and define $\int \cdot d\dot{Z}$ on $\mathcal{L}^1(d\dot{Z}; p, P)$ as the unique extension by $G$-mean continuity of $d\dot{Z}$: $\mathcal{R} \to L^p(P)$. In any case, $\mathcal{L}^1(\mathcal{R}, G)$ is a complete and order complete pseudometric vector lattice.

2.3.4. *The* DCT *holds*: if $X_n \in \mathcal{L}^1(\mathcal{R}, G)$ converges $G$-a.e. to $X$ and if $\sup | X_n |$ is finite for $G$ then $X \in \mathcal{L}^1(\mathcal{R}, G)$ and $X_n \to X$ in $G$-mean. To prove this one shows first that the pair $(\mathcal{L}^1(\mathcal{R}, G), G)$ is an upper gauge, derives the Monotone Convergence Theorem from that, and obtains the DCT with a standard argument.

2.4. *$(\mathscr{R}, G)$-measurability.* Let us equip the base space $B$ with the initial uniformity of the collection $\mathscr{R}$ of functions on $B$. (Since each $X \in \mathscr{R}$ takes its values in a compact set, the completion $\mathscr{B}$ of $B$ is compact.) A collection $\mathscr{K}$ of $(\mathscr{R}, G)$-integrable sets[1] is termed *$(\mathscr{R}, G)$-dense* if for every integrable set $A \subset B$ and every $\epsilon > 0$ there is a $K \in \mathscr{K}$ with $K \subset A$ and $G(A - K) < \epsilon$. A function $F$ on $B$ is defined to be *$(\mathscr{R}, G)$-measurable* if the collection $\mathscr{U}(F)$ of integrable sets $K \subset B$ on which $F$ is uniformly continuous is dense.

2.4.1. *Egoroff's Theorem.* The pointwise limit $X$ of a $G$-a.e. convergent sequence $X_n$ of $G$-measurable functions is $G$-measurable; moreover, the collection of integrable sets on which the convergence is uniform is dense. Thus every previsible process is $(\mathscr{R}, G)$-measurable.

2.4.2. *A function $F$ on $B$ is $(\mathscr{R}, G)$-integrable* iff *it is $(\mathscr{R}, G)$-measurable and finite for $G$.*

Every $G_p^Z$-measurable function $X$ on $B$ is equal $G_p^Z$-a.e. to a previsible process, so knowing the $dZ$-integrable processes is equivalent with having a manageable expression for $G_p^Z$ or a manageable replacement of it. This will be attempted in Section 7. In the meantime, here is a little information:

2.4.3. For any $G_p^Z$-measurable process $F$,

$$G_p^Z(F) = \sup\left\{\rho_p\left(\int X\, d\dot{Z}\right) : X \ d\dot{Z}\text{-}p\text{-}integrable, \qquad |X| \leq |F|\right\}$$
$$= \sup\left\{\rho_p\left(\int KF\, d\dot{Z}\right) : K \in \mathscr{U}(F)\right\}$$

2.5 PROPOSITION. *Suppose $\dot{Z}$ is an $L^p$-integrator, for some $p \in [0, \infty)$. Then there exists a representative $Z \in \dot{Z}$, unique up to indistinguishability, whose paths are right continuous with left limits* (r.c.l.l.) *and bounded on every bounded interval.*

PROOF. Let $Z'$ be a representative of $\dot{Z}$. Fix an instant $t > 0$, a finite set $S = \{0 = s_1 < s_2 < \cdots < s_n = t\}$ of consecutive rationals, and two rationals $a < b$. Set $T_0 = 0$ and

$$T_{2k+1} = t \wedge \inf\{s \in S: s > T_{2k} \quad, Z'_s > b\}$$
$$T_{2k} = t \wedge \inf\{s \in S: s > T_{2k-1}, Z'_s < a\}, \qquad k = 0, 1, \cdots.$$

Let $U_S^{[a,b]}$ denote the number of upcrossings of $[a, b]$ performed by the path $s \to Z'_s$ on $S$. Clearly[1]

$$n[U_S^{[a,b]} \geq n] \leq \frac{1}{b-a}\left(\textstyle\sum_{k=1}^{n} (Z'_{T_{2k+1}} - Z'_{T_{2k}}) + (a - Z'_t)_+\right).$$

Due to $(A_p)$, the right-hand side stays bounded in $L^p$ independently of $S \subset \mathbf{Q}$. Thus $[U_Q^{[a,b]} = \infty]$ is negligible. Taking the union over $a < b$ rational shows that $Z'$ has a.s. right and left limits through rationals. By $(B_p)$, $d\dot{Z}$ is not changed if $Z'$ is replaced by $Z$ with

$$Z_t = \lim\{Z'_q: t < q \in \mathbf{Q}\}.$$

To see that the r.c.l.l. representative $Z \in \dot{Z}$ is a.s. bounded on bounded intervals, consider the *maximal process* $Z^*$ of $Z$ defined by

$$Z_t^* := \sup_{s \leq t} |Z_s| = \sup \cdots_{s \leq t, s \in \mathbf{Q}}.$$

Fix an instant $t$, a $c > 0$, and let $T$ be the first time $|Z|$ exceeds $c$. Then $[Z_t^* > c] \subset [T \leq t]$, on which set $|Z_T| \geq c$. Hence[1] $c[Z_t^* > c] \leq |Z_T|[T \leq t] \leq |Z_T| = |dZ([0, T])|$ and consequently

(2.5.1) $\rho_p(c[Z_t^* > c]) \leq G_p^Z([0, t])$ and $P[Z_t^* = \infty] = 0.$

The last result of this section shows that ($B_p$) is equivalent with ($B_0$) and with the existence of a r.c.l.l. modification and shows that ($A_p$) alone characterizes $L^p$-integrators.

2.6 THEOREM. *Let $\dot{Z}$ be a process right-continuous in probability and let $0 \leq p < \infty$. Then $\dot{Z}$ is an $L^p$-integrator if and only if $d\dot{Z}$: $\mathscr{R} \to L^p$ is continuous. For $p > 0$ this condition reads $\gamma_T^p[\dot{Z}] < \infty$ for all $T \in \mathscr{T}$.*

PROOF. Only the sufficiency of this condition needs to be established, and by 2.1 it suffices to show that it implies ($B_p$). If $p = 0$ this is easy. Suppose

$$X^n = r_0^n \cdot [S_0^n, 0] + \textstyle\sum_{k=1}^{K(n)} r_k^n \cdot (S_k^n, T_k^n] \in \mathscr{R} \qquad \text{cf. (1.2)}$$

decrease to zero, and let $\epsilon > 0$. By the stochastic right continuity there are stopping times $\bar{S}_k^n \in \mathscr{T}$ with $S_k^n < \bar{S}_k^n$ on $[S_k^n < T_k^n]$ such that

$$\textstyle\sum_n \sum_{k=1}^{K(n)} \rho_0(r_k^n \,|\, Z_{\bar{S}_k^n} - Z_{S_k^n}|) < \epsilon.$$

The *$d\dot{Z}$-integral* of

$$\bar{X}^n = (X^n - \textstyle\sum_{m=1}^{n} \sum_{k=1}^{K(m)} r_k^m \cdot (S_k^m, \bar{S}_k^m])_+ \in \mathscr{R}$$

then differs from that of $X^n$ by less than $\epsilon$, if measured with $\rho_0$. But $\bar{X}^n$ is majorized by the process

$$\tilde{X}^n = (X^n - \textstyle\sum_{m=1}^{n} \sum_{k=1}^{K(m)} r_k^m \cdot (S_k^m, \bar{S}_k^m))_+ \leqq X^n,$$

which has upper semicontinuous trajectories. By Dini's theorem, $|\tilde{X}^n|_\infty^* \downarrow 0$. For $n$ so large that $P[|\bar{X}^n|_\infty^* > \epsilon] < \epsilon$, $\rho_0\,(d\dot{Z}(\bar{X}^n - \bar{X}^n \wedge \epsilon)) < \epsilon$. Since $\rho_0(d\dot{Z}(\bar{X}^n \wedge \epsilon)) \to 0$ as $\epsilon \to 0$ by ($A_0$), $d\dot{Z}(X^n) \to 0$ in $L^0$ as claimed.

For the remaining case $p > 0$ a little trick is needed. Recall that the completion $\mathscr{B}$ of $B$ is compact (2.4). For every $X \in \mathscr{R}$ let $\hat{X}$: $\mathscr{B} \to \mathbf{R}$ be its extension by uniform continuity. The collection $\hat{\mathscr{R}}$ so obtained is a vector lattice of continuous functions on $\mathscr{B}$, all vanishing at a point $\infty \in \mathscr{B} \setminus B$, dense in $C_{00}(\mathscr{B} \setminus \{\infty\})$ by the theorem of Stone-Weierstrass. We define a continuous linear map $d\hat{Z}$: $\hat{\mathscr{R}} \to L^p$ by $d\hat{Z}(\hat{X}) = dZ(X)$, $X \in \mathscr{R}$, and extend it to a continuous linear map $U$: $C_{00}(\mathscr{B} \setminus \{\infty\}) \to L^p$ by continuity. Note here that the natural topology of $C_{00}(\mathscr{B} \setminus \{\infty\})$ coincides, on $\hat{\mathscr{R}}$, with the natural topology of $\mathscr{R}$ commented on at the end of Section 1. The map $U$ satisfies ($A_p$), and ($B_p$) is now automatically satisfied by Dini's theorem. Following the extension procedure outlined in 2.1–2.3 one arrives at an extension $\int \cdot\, dU$ of $U$ satisfying the DCT. Thus, if the $X^n \in \mathscr{R}$ decrease to zero, $\lim dZ(X^n) = \lim U(\hat{X}^n) = \int \inf \hat{X}^n\, dU$ exists in $L^p$. Now observe that $Z$ is also an $L^0$-integrator, for which as shown above $\lim dZ(X^n) = 0$ exists in $L^0$. Hence $\lim dZ(X^n) = 0$ in $L^p$.

We point out again that every result in this section persists if the spaces $L^p$ are replaced by Lorentz-spaces $L^{p,q}$, $p \neq \infty$, or Orlicz spaces $L^\Phi$, where $\Phi$ satisfies a $\Delta_2$-condition (this restriction is needed to arrive at ($\delta$)).

2.7. We close the section establishing some notation. Note first that the generality attempted by admitting *classes* $\dot{Z}$ as $L^p$-integrators is illusory, by 2.5. Prompted by 2.5 and 2.6 we set forth the notion we shall be investigating throughout the remainder of the paper.

DEFINITION. A *numerical* process $Z$ is an $L^p$-integrator, $0 \leqq p < \infty$, if it is right-continuous with left limits a.s. and if $dZ: \mathscr{R} \to \mathscr{L}^p$ is continuous.

That is to say, the class $\dot{Z}$ is an $L^p$-integrator and $Z$ is its 'unique' r.c.l.l. representative. We write $\gamma_T^p[Z] = \gamma_T^p[\dot{Z}]$, $G_p^Z = G_p^{\dot{Z}}$ etc. Suppose $X$ is a $dZ$-$p$-integrable process—meaning, of course, that it is $d\dot{Z}$-$p$-integrable. The very extension procedure produces a *class* $\int X\, dZ$

$:= \int X\, d\dot{Z}$ for the integral. Note, however, that clearly the process $\dot{Y}$ defined by

$$\dot{Y}_t := \int [0, t]\cdot X\, d\dot{Z}, \qquad t \in [0, \infty),$$

is an $L^p$-integrator. As such it has a r.c.l.l. representative, unique up to indistinguishability, which will be denoted by

$$X * Z \quad \cdot \quad \int_{[0}^{T]} X\, dZ$$

is defined as $(X*Z)_T$, the value of this representative at $T$. If $S \leqq T$ are any two stopping times one defines

$$\int_{(S}^{T]} X\, dZ := (X*Z)_T - (X*Z)_S$$

and checks easily that it is a member of the class $\int X\cdot(S, T]\, dZ$.

The following relation on the size of $Y = X*Z$ is useful in subsequent computations and is easily verified from 2.4.3.

$$\gamma_T^p[X*Z] = (G_p^Z(X\cdot[0, T]))^{1 \vee 1/p} \qquad \text{for } p > 0. \tag{2.7}$$

Lastly, for any r.c.l.l. process $Y$, $Y_-$ is the process with values

$$Y_{-t}(\omega) = \lim_{s\uparrow t} Y_s(\omega) \qquad \text{for } t > 0,\ Y_{-0} = 0.$$

Abusing the language, we call $Y_-$ the left-continuous version of $Y$. The value of

$$\Delta Y := Y - Y_-$$

at any time $T$ is denoted $\Delta_T Y$.

## 3. Examples.

3.1. A right-continuous process $I$ (adapted and a.s. finite as all processes are by convention) is called an *increasing process*, if a.s. its paths $t \to I_t(\omega)$ are increasing functions. It is clearly an $L^0$-integrator: For every $T \in \mathcal{T}$, the set $\{\int X\, dI: X \in \mathcal{R}_T, |X| \leq 1\}$ is order-bounded in $\mathcal{L}^0$—by $\int [0, T]\, dI = I_T$—and so is topologically bounded. It is an $L^p$-integrator, $p \in (0, \infty)$, if and only if

$$\gamma_T^p[I] = \|I_T\|_{L^p}$$

is finite for all bounded times $T$; in fact it is easy to see (cf. 2.4.3) that

$$G_p^I(F) = \left(\int \left| \int |F_t(\omega)|\, dI_t(\omega) \right|^p dP(\omega)\right)^{1 \wedge 1/p}$$

for $F$ $G_p^I$-measurable. In any event, the right-hand side defines an upper gauge on all processes $F$, which majorizes $dI$ (cf. 2.1$\epsilon$) and is just as good as $G_p^I$ for the purpose of extending $dI$ to an integral. The theorem of Fubini in a suitably general form ([B1], page 228) says that $\int X\, dI$ can be evaluated pathwise as an ordinary Stieltjes integral ($X \in \mathcal{L}^1(\mathcal{R}, G_p^I)$, $p \in [0, \infty)$).

In fact, the pathwise Stieltjes integral $\int X\, dI$ is defined for any suitably smooth integrand $X$; for instance, when $X$ is progressively measurable, $Y_t(\omega) = \int_0^t X_s(\omega)\, dI_s(\omega)$ defines an adapted process $Y = X*Z$.

3.2. A r.c. process $V$ is called a *process of finite variation* if a.s. its paths are functions of finite variation. From the usual definition of the variation $\int |dV|$ of $V$,

$$\int_0^t |dV| = \sup_n \textstyle\sum_{k=1}^{\infty} |V_{t\wedge k2^{-n}} - V_{t\wedge (k-1)2^{-n}}| \tag{3.2.1}$$

which yields an increasing r.c. process $\int |dV|$, we see that $V$ can be written as the difference of two increasing processes:

$$V = \frac{1}{2}\left(\int |dV| + V\right) - \frac{1}{2}\left(\int |dV| - V\right) = V^+ - V^-.$$

Conversely, a difference of two increasing processes is clearly a process of finite variation. Such a process is therefore an $L^0$-integrator as well. It is not so easy to determine when $V$ is an $L^p$-integrator if $p > 0$. A sufficient condition is evidently that $\int |dV|$ or both $V^+$ and $V^-$ be $L^p$-integrators. In this case

(3.2.2) $$G_p^{\int |dV|}(F) = \left\| \int |F|\,|dV| \right\|_{L^p}^{1\wedge p} \geq G_p^V(F)$$

is an upper gauge, reasonably manageable, that majorizes $dV$ on $\mathscr{R}$ and can be used to extend it to an integral. However, $G_p^{\int |dV|}$ is not in general equivalent with $G_p^V$. No easy necessary and sufficient condition for $V$ to be an $L^p$-integrator, $p > 0$, seems to be known, and no manageable equivalent of $G_p^V$ either. The reason is, of course, that the natural domain for $dV$ are the well-measurable sets, and that $\mathscr{A}$ is too small a subdomain to distinguish the behaviour of $dV$.

3.3. The situation improves when $V$ is previsible. We cite a most useful criterion for previsibility [D2; page 85 & page 105]. It is, incidentally, also the most advanced result of Dellacherie's book that we shall need here.

3.3.1 LEMMA. *Let $X$ be a* r.c.l.l. *adapted process. Then $X$ is previsible* iff $\Delta_T X = 0$ a.s. *at all totally inaccessible stopping times $T$ and $X_T \cdot [T < \infty] \in \mathscr{F}_{T-}$ at all predictable stopping times $T$; in this instant, $X_T \cdot [T < \infty] \in \mathscr{F}_{T-}$ at all times $T$. Suppose $X$ is increasing and integrable: $X_t \in \mathscr{L}^1\ \forall\ t$. Then $X$ is previsible* iff

$$E(M_t X_t - M_0 X_0) = E \int_0^t M_-\, dX$$

*for all bounded positive* r.c. *martingales $M$.*

Let then $V$ be a previsible process of finite variation. (3.2.1) shows that the jump of $\int |dV|$ at any time $T$ equals $|\Delta_T V| \in \mathscr{F}_{T-}$. Thus $\int |dV|$ and $V^+$, $V^-$ are previsible as well.

3.3.2 PROPOSITION. *Suppose $V$ is previsible and of finite variation, and $0 < p < \infty$. Then $V$ is an $L^p$-integrator* iff $\int |dV|$ *is; in fact*

$$\gamma_T^p[V] = \gamma_T^p\left[\int |dV|\right] \quad \text{and} \quad G_p^V(F) = G_p^{\int |dV|}(F) = \left\| \int |F|\,|dV| \right\|_{L^p}^{1\wedge p}$$

*for $F$ previsible.*

PROOF. Let $T$ be any stopping time such that $\int |dV|$ is bounded on $[0, T]$. Arbitrarily large such times exist:[4] Choosing $K$ sufficiently big, $S = \inf\{s: \int_0^s |dV| \geq K\}$ can be made arbitrarily large (2.5) and is predictable [D2, page 74]. Any $T$ predicting $S$ will do. Denote by $\mu_V$ the measure $A \mapsto E \int A dV^T$ on the previsible $\sigma$-algebra, and let $D$ be a previsible derivative $d\mu_V / d\mu_{\int |dV|}$. Then $D^2 = 1$, $D * V = \int |dV|$, and so $\gamma_T^p[\int |dV|] \leq \gamma_T^p[V]$. The reverse inequality being trivial, the statement follows from (2.6).

[4] i.e., for any $t$ and $\epsilon > 0$, a stopping time $T$ can be found that satisfies this description and $P[T < t] < \epsilon$.

3.4 DOOB PROCESSES. The scalar measure $\mu$ attached to $V$ in the previous argument is worth some attention. First note that it can be defined for any r.c. integrable process $Z$:

$$m_Z(A) = E\,dZ(A), \qquad A \in \mathscr{A}.$$

Our present aim is to characterize those processes $Z$ for which $m_Z$ behaves well.

DEFINITION. A right continuous integrable ($Z_t \in \mathscr{L}^1\ \forall\ t$) process $Z$ is a *Doob process* if it satisfies these two conditions:

(D1) There exists an increasing function $f$ on $\mathscr{T}$ such that $|E(Z_T - Z_S)| \le f(T) - f(S)$ for $S \le T$ in $\mathscr{T}$.

(D2) For any $T \in \mathscr{T}$, the collection $\{Z_S: T \ge S \in \mathscr{T}\}$ is uniformly integrable.

Condition (D1) is met by submartingales, supermartingales and linear combinations thereof. It is but another way of saying $m_Z$ has finite variation.

PROPOSITION (Pellaumail [P2]). *Let $Z$ be a right continuous integrable process. The following are equivalent.*

(1) *$Z$ is a Doob process.*
(2) *$m_Z$ has finite variation, is $\sigma$-additive, and vanishes on evanescent sets.*
(3) *There exist a* r.c. *previsible process $\langle Z\rangle$ with integrable variation and with $\langle Z\rangle_0 = 0$ and a* r.c. *martingale $\tilde{Z}$ such that $Z = \tilde{Z} + \langle Z\rangle$.*

*Moreover, the decomposition* (3) *with the stated properties is unique in the sense of indistinguishability; it is termed the Doob-decomposition of the Doob process Z.*

(The term Doob-Meyer decomposition is more appropriate but longer.)

PROOF. (1) $\Rightarrow$ (2): It has to be shown that $m_Z(A^{(n)}) \to 0$ if $A^{(n)}$ is a decreasing sequence in $\mathscr{A}$ with evanescent intersection. Let $\epsilon > 0$ be given, and let $T \in \mathscr{T}$ be such that $A^{(1)} \subset [0, T] = A$. Let $m_Z = m^+ - m^-$ be the Jordan decomposition of $m_Z$. There exist sets $A^\pm \in \mathscr{A}$ with $A^+ + A^- = A$ and $m^+(A^-) + m^-(A^+) < \epsilon$. Reviving the notation of 2.5, let

$$A^{(n)} = [S_0^{(n)}, 0] \cup \bigcup_{k=1}^{K(n)} (S_k^{(n)}, T_k^{(n)}].$$

Clearly $m_Z(A^{(n)} \cap [0]) = m_Z([S_0^{(n)}, 0]) = E(Z_0 \cdot [S_0^{(n)} = 0]) \to 0$. We may assume

$$A^{(n)} = \bigcup_{k=1}^{K(n)} (S_k^{(n)}, T_k^{(n)}].$$

Consider the stopping time $(S_k^{(n)} + 1/p) \wedge T_k^{(n)} = \bar{S}_k^{(n)}$ between $S_k^{(n)}$ and $T_k^{(n)}$. It decreases to $S_k^{(n)}$ as $p$ increases, by right continuity $Z_{\bar{S}_k^{(n)}} \to Z_{S_k^{(n)}}$, and (D2) ensures that $E|Z_{\bar{S}_k^{(n)}} - Z_{S_k^{(n)}}| \le \epsilon 2^{-n-k}$ for the proper choice of $p$. The $m_Z$-measure of the set

$$\bar{A}^{(n)} = A^{(n)} \backslash \bigcup_{m=1}^{n} \bigcup_{k=1}^{K(m)} (S_k^{(m)}, \bar{S}_k^{(m)}] \in \mathscr{A}$$

then differs from that of $A^{(n)}$ by less than $\epsilon$. But $\bar{A}^{(n)}$ is contained in the set

$$\tilde{A}^{(n)} = \bigcup_{k=1}^{K(n)} [\bar{S}_k^{(n)}, T_k^{(n)}] \subset A^{(n)}$$

which has compact sections, whose intersection over all $n$ is void a.s. In other words,

$$R^n = T \wedge \inf\{t: \tilde{A}_t^{(n)} > 0\}^1$$

increases to $T$, being eventually equal to $T$ a.s. We estimate:

$$\begin{aligned} m_Z(A^{(n)}) &\le \epsilon + m_Z(\bar{A}^{(n)}) = \epsilon + m^+(\bar{A}^{(n)}) - m^-(\bar{A}^{(n)}) \\ &\le \epsilon + m^+((R^n, T]) \le 2\epsilon + m^+((R^n, T] \cap A^+) \\ &\le 3\epsilon + m((R^n, T] \cap A^+). \end{aligned}$$

Now $A^+$ is of the form $\bigcup_{k=1}^K (S_k^+; T_k^+]$, and so

$$m((R^n, T] \cap A^+) = \sum_{k=1}^K \int_{[T_k > S_k^+ \vee R^n]} (Z_{T_k^+ \wedge T} - Z_{S_k^+ \vee R^n})\, dP.$$

The integrands converge to zero a.s., staying uniformly integrable, and so $m_Z(A^{(n)}) \leq 4\epsilon$ eventually. We repeat the argument with $-Z$ replacing $Z$ and find $m_Z(A^{(n)}) \geq -4\epsilon$ eventually. Thus $m_Z(A^{(n)}) \to 0$.

(2) $\Rightarrow$ (3) This is a lemma due to Pellaumail [P2]. Here is a sketch of the proof. For any $T \in \mathcal{T}$ and $g \in \mathcal{L}^\infty(\mathcal{F}_T)$ let $M^g$ denote the r.c. martingale $M^g = E(g \mid \mathcal{F})$ and set $m_T^\pm(g) = m^\pm(M_-^g \cdot (0, T])$. If $g_n \downarrow 0$ a.s. then $M_-^{g_n}$ decreases to an evanescent process by Doob's maximal inequality, and so $m_T^\pm(g_n) \to 0$. Hence $m_T^\pm \ll P$ and there is a derivative $\dot{A}_T^\pm$. Let $\langle Z \rangle^\pm$ be a r.c. representative and $\langle Z \rangle = \langle Z \rangle^+ - \langle Z \rangle^-$. One checks easily that $m_{\langle Z \rangle} = m_Z$, so that the difference $\tilde{Z} = Z - \langle Z \rangle$ is a martingale. Since by the definition of $\langle Z \rangle$, $EM_T\langle Z \rangle_T = E \int_0^+ M_-\, d\langle Z \rangle$ for all bounded martingales, $\langle Z \rangle$ is previsible (3.3.1). This property also implies the uniqueness stated.

(3) $\Rightarrow$ (1) is obvious from (3.3.2) and the fact that the $\tilde{Z}_S = E(\tilde{Z}_T \mid \mathcal{F}_S)$, $T \geq S \in \mathcal{T}$, form a uniformly integrable family.

3.5. *A* r.c. *square-integrable martingale $M$ is an $L^2$-integrator.*

Let $X = r_0[S_0, 0] + \sum r_i(T_{i-1}, T_i]$ be an element of $\mathcal{R}_T$ with $|X| \leq 1$. $T \in \mathcal{T}$ is fixed, $0 \leq T_1 \leq \cdots \leq T_n \leq T$ consecutive, and $|r_i| \leq 1$. Then, using an elementary version of Doob's optional sampling theorem and the fact that $M^2$ is a submartingale,

$$\begin{aligned}
\| dM(X) \|_{L^2}^2 &= E(r_0 M_0[S_0 = 0] + \textstyle\sum_i r_i(M_{T_i} - M_{T_{i-1}}))^2 \\
&\leq E(r_0^2 M_0^2 + 2r_0 M_0[S_0 = 0] \textstyle\sum_i \cdots + (\sum_i)^2) \\
&\leq EM_0^2 + E \textstyle\sum_{i,j} r_i r_j (M_{T_i} - M_{T_{i-1}})(M_{T_j} - M_{T_{j-1}}) \\
&= EM_0^2 + E \textstyle\sum r_i^2 (M_{T_i} - M_{T_{i-1}})^2 \leq EM_0^2 + E \sum (M_{T_i}^2 - 2M_{T_i} M_{T_{i-1}} + M_{T_{i-1}}^2) \\
&= EM_0^2 + E \textstyle\sum (M_{T_i}^2 - M_{T_{i-1}}^2) = EM_{T_n}^2 \leq \| M_T \|_{L^2}^2.
\end{aligned}$$

Hence $\gamma_T^2[M] = \| M_T \|_{L^2}$. As opposed to the last example, the fact that the integrand $X$ is previsible was crucial to the computation.

3.6 Lemma. *Let $Z$ be a positive bounded* r.c. *supermartingale that vanishes at the stopping time $T$. Then $Z$ is an $L^2$-integrator and*

$$\gamma_T^2[Z] = \gamma_\infty^2[Z^T] \leq 2(2 \cdot \sup Z \cdot E(Z_0))^{1/2}.$$

Proof. Clearly $Z$ is a Doob process and so has a Doob decomposition $Z = \tilde{Z} + \langle Z \rangle$, with $\langle Z \rangle$ decreasing, $\langle Z \rangle_0 = 0$. Let $X \in \mathcal{R}_T$ with $|X| \leq 1$, and estimate, with $c = \sup Z$:

$$E\tilde{Z}_T^2 = E\langle Z \rangle_T^2 = 2E\langle Z \rangle_T^2 - E \int (\langle Z \rangle + \langle Z \rangle_-)\, d\langle Z \rangle \qquad \text{see 5.10}$$

$$\leq 2E \int_0^T (\langle Z_T \rangle - \langle Z \rangle)\, d\langle Z \rangle = 2E \int_0^T (Z_T - Z)\, d\langle Z \rangle \qquad \text{see 3.3.1}$$

$$= 2E \int_0^T Z\, d(-\langle Z \rangle) \leq -2cE\langle Z \rangle_T = 2cE\tilde{Z}_T = 2cE\tilde{Z}_0 = 2cEZ_0.$$

By 3.5 and 3.3.2 the processes $\tilde{Z}$ and $\langle Z \rangle$ both have $\gamma_T^2[\,\cdot\,] \leq (2cEZ_0)^{1/2}$, and 3.6 follows.

3.7. *A right-continuous martingale $M$ is an $L^{1,\infty}$-integrator.* (It is not, in general, an $L^1$-integrator; those that are form the class $H^1$, to be characterized in 7.2 below.)

Again, we need to show only that $dM$: $\mathscr{R} \to \mathscr{L}^{1,\infty}$ is continuous; i.e., that for fixed $T \in \mathscr{T}$ the numbers

$$\| dM(X) \|_{L^{1,\infty}} = \sup_c cP[|X*M|_T > c]$$

stay bounded as $X$ ranges over the unit ball of $\mathscr{R}_T$. We could quote a result of Burkholder-Davis-Gundy [BDG] to the effect that actually

$$\sup_c cP[|X*M|_T^* > c] \le C \cdot \| M_T \|_{L^1}. \tag{3.7.1}$$

The constant $C$ can be chosen to be 2, which is sharp [B4]. For completeness' sake, we prove the following inequality, which certainly serves the purpose.

$$cP[|X*M|_T > c] \le 18 \| M_T \|_{L^1}, \tag{3.7.2}$$

for any $c > 0$ and any $X \in \mathscr{R}_T$, $|X| \le 1$.

PROOF. Up to time $T \in \mathscr{T}$, $M$ is the difference of two positive martingales. We may therefore reduce the problem to establishing (3.7.2) for $M \ge 0$, with 9 replacing 18. To do this, let $S$ be the first time that $M$ exceeds $c$, and set $Z = M \cdot [0, S)$. Note that $Z$ is a bounded positive r.c. supermartingale, and that the processes $X*M$ and $X*Z$ agree on $[S > T]$ up to time $T$. Doob's maximal inequality and 3.6 yield

$$\begin{aligned} cP[|X*M|_T > c] &\le cP[S \le T] + cP[|X*Z|_T > c; S > T] \\ &\le cP[|M|_T^* \ge c] + c^{-1}E\left(\int_0^T X\,dZ\right)^2 \le EM_T + c^{-1}(\gamma_T^2[Z])^2 \\ &\le EM_T + c^{-1} \cdot 8 \cdot c \cdot EZ_0 \le EM_T + 8EM_0 = 9 \| M_T \|_{L^1}. \end{aligned}$$

3.8. *A* r.c. *$p$-integrable martingale $M$ is an $L^p$-integrator*, $1 < p < \infty$.

To see this, fix a $T \in \mathscr{T}$ and $X \in \mathscr{R}_T$ with $|X| \le 1$, and consider the following map $U$ from $L^\infty(\mathscr{F}_T, P)$ to $L^\infty(\mathscr{F}_T, P)$:

$$U(f) = dM^f(X),$$

where $M^f$ is the r.c. version of the martingale $E(f|\mathscr{F})$. By (3.7), $U$ is of weak type (1, 1), and by (3.5) it is of strong type (2, 2). A trivial calculation shows that it is self-adjoint on $L^2$. By interpolation, it is of strong type $(p, p)$ for $1 \quad p < \infty$ with constants $C_p$ depending only on $c^{(3.7.1)} = 2$ and $c^{(3.5)} = 1$; and this fact reads

$$\| dM(X) \|_{L^p} \le C_p \| M_T \|_{L^p} \quad \text{or} \quad \gamma_T^p[M] \le C_p \| M_T \|_p, \qquad 1 < p < \infty. \tag{3.8}$$

3.9. A process $Z$ is said to have a property $Q$ *locally* if there exist arbitrarily large[4] stopping times $T$ such that the stopped process $Z^T$ has the property $Q$. $T$ is then said to *reduce* $Z$ to a process with the property in question. A *semimartingale* is a process which is locally the sum of a r.c. process of finite variation and a r.c. *uniformly integrable* martingale. A *local martingale* is a process that is locally a *uniformly integrable* martingale. A process that is locally an $L^p$-integrator is also called a *local $L^p$-integrator*. $Z$ is a *global* $L^p$-integrator if $dZ$: $\mathscr{R} \to L^p$ is continuous in the uniform topology on $\mathscr{R}$.

3.9.1 PROPOSITION. *A semimartingale $Z$ is an $L^0$-integrator.*

(See 7.6 for the converse.)

PROOF. By 3.2 and 3.7, the reduced processes $Z^T$ are $L^0$-integrators. Let $T$ be any a.s. finite stopping time. Given $\epsilon > 0$, find a reducing time $S$ with $P[S < T] < \epsilon$. Now $dZ$ is bounded on the unit ball of $\mathscr{R}_S$. If $X \in \mathscr{R}_T$ with $|X| \le 1$, write

$$X = X \cdot [0, S] + X \cdot (S, \infty) = X_1 + X_2.$$

Then $dZ(X_1)$ lies in a fixed bounded subset of $L^0$, while $\rho_0(dZ(X_2)) \leq P[S < T] < \epsilon$: $dZ$ is continuous on $\mathscr{R}_T$.

3.9.2 COROLLARY. *If Z is an $L^0$-integrator then $dZ$: $\mathscr{R}_T \to L^0$ is bounded for any* a.s. *finite (not necessarily bounded) stopping time T.*

3.9.3 PROPOSITION. *A local martingale M is a local $L^1$-integrator.*

PROOF. We may evidently assume that $M$ is a uniformly integrable martingale, and positive. There are arbitrarily large times $T$ such that $M$ is bounded on $[0, T)$, e.g., $T = \inf\{t: |M_t| \geq K\}$. Write

$$M^T = M \cdot [0, T) + M_T[T, \infty) = Z_1 + Z_2.$$

Then $Z_2$ is a process of integrable total variation $|M_T|$ and thus is an $L^1$-integrator (3.2), while $Z_1$ is an $L^2$-integrator by 3.6.

3.9.4 COROLLARY. *Let Z be a* r.c. *process. The following are equivalent*

(1) *Z is locally a Doob process;*

(2) *Z is a local $L^1$-integrator;*

(3) *Z has a decomposition $Z = \tilde{Z} + \langle Z \rangle$ into a local martingale $\tilde{Z}$ and a* r.c. *previsible process $\langle Z \rangle$ of finite variation with $\langle Z \rangle_0 = 0$. Moreover, this decomposition is then unique. It is termed the Doob decomposition of Z.*

The processes described in 3.9.4 are thus the 'special semimartingales' of Meyer [M5].

PROOF. (1) ⟹ (2) Suppose $T$ reduces $Z$ to a Doob process, with $Z^T = M + V$ its Doob decomposition. Let $S^n$ be a sequence announcing the previsible time $\inf\{t: \int_0^t |dV| \geq K\}$ and $S'$ a time reducing $M$ to an $L^1$-integrator (3.9.3). Clearly $S = T \wedge S^n \wedge S'$ can be made arbitrarily large[4] and reduces both $M$ and $V$, and thus $Z$, to a (global) $L^1$-integrator.

(2) ⟹ (3) If $T$ reduces $Z$, the measure $m_{Z^T}$ evidently satisfies (2) of proposition 3.4. By uniqueness, the resulting decompositions $Z^T = (Z^T)^\sim + \langle Z^T \rangle$ are compatible as $T \to \infty$ and yield the decomposition of (3).

(3) ⟹ (1) is evident from 3.4.

## 4. Properties of the integral.

4.1 PROPOSITION. *Suppose Z is an $L^p$-integrator, $0 \leqq p < \infty$, and X is dZ-p-integrable.*

(1) *If $X^n \in \mathscr{L}^1(dZ, p)$ converges to X in $G_p^Z$-mean then $X^n * Z \to X * Z$ uniformly, in measure.*

(We shall see later (7.4) that actually $\| |X^n * Z - X * Z|_\infty^* \|_{L^p} \to 0$.)

(2) *There exists a sequence $X^n \in \mathscr{R}$ converging to X in $G_p^Z$-mean and so that $X^n * Z \to X * Z$ uniformly* a.s.

PROOF. By 2.5.1 and (2.7), $\rho_p(c[|X^n * Z - X * Z|_\infty^* > c]) \leqq G_p^{(X^n - X)*Z}([0, \infty)) = G_p^Z(X^n - X) \to 0$ for any $c > 0$. This proves (1). As for (2), first define inductively a sequence $X_k$ so that $G_p^Z(2^k(X - X_1 - X_2 - \cdots - X_k)) \leqq 2^{-k}$, then set $X^n = X_1 + \cdots + X_n$. Then $\sum G_p^Z(2^k X_k) < \infty$, and $P[\limsup |(X - X^n) * Z|_\infty^* > 2^{-m}] \leqq \inf_N P(\bigcup_{n>N} [\sum_{k>n} |X_k * Z|_\infty^* > 2^{-m}]) \leqq \inf_N \sum_{k>N} P[2^k |X_k * Z|_\infty^* > 1] \leqq \inf_N \sum_{k>N} G_p^Z(2^k X_k) = 0$ for any $m$.

4.2. Despite its definition as a limit in $p$-mean, $(X, Z) \to X * Z$ is local in nature:

PROPOSITION. *Suppose Z, Z′ are $L^p$-integrators, $0 \leq p < \infty$, and X, X′ are processes integrable for both. Let $\Omega_0 \subset \Omega$ and assume the paths of X equal those of X′ and the paths* *of $Z$ equal those of $Z'$, almost surely on $\Omega_0$ up to time $T$. Then the paths of $X*Z$ and $X'*Z'$ coincide* a.s. *on $\Omega_0$ up to time $T$.*

PROOF. It suffices to treat the cases $X \equiv X'$ and $Z \equiv Z'$. In the former case, evidently $X*Z = X*Z'$ on $\Omega_0$ up to time $T$ for all elementary integrands $X$, and then by 4.1 for all $X$ integrable for both $Z$ and $Z'$.

In the latter case it suffices to prove this: If $X = 0$ a.s. on $\Omega_0$ up to time $T$ then $\int_0^S XdZ = 0$ on $\Omega_0$ for any bounded time $S \le T$. Let $B_0 = (\Omega_0 \times [0, \infty)) \cap [0, S]$, and denote by $\mathscr{A}_0$ the trace of the ring $\mathscr{A}$ on $B_0$. Define an $L^0$-valued set function $\mu$ on $\mathscr{A}_0$ by

$$\mu(A \cap B_0) = \dot{\Omega}_0 \cdot d\dot{Z}(A), \qquad A \in \mathscr{A}.$$

Clearly $\mu$ is well-defined and satisfies $(A_0)$ and $(B_0)$ of Section 2. Therefore, $\mu$ has an extension $\int \cdot d\mu$ satisfying the DCT. Note that $B_0$ is $\mu$-integrable; in fact $B_0 \in \mathscr{A}_0$. Let $\mathscr{H}$ denote the set of all bounded previsible processes $X$ such that

$$\int XB_0 \, d\mu = \dot{\Omega}_0 \int_0^S X \, d\dot{Z}.$$

Clearly $\mathscr{R} \subset \mathscr{H}$. By the DCT, $\mathscr{H}$ is closed under pointwise limits of dominated sequences and so contains all bounded previsible processes. By truncation, the equality is true for all $d\dot{Z}$-integrable $X$. In particular, if $X = 0$ a.s. on $B_0$ then $\int_0^S X \, dZ = 0$ a.s. on $\Omega_0$, as was to be proved. We use here the fact that $\int_0^S X \, dZ = (X*Z)_S$ is a member of the class $\int [0, S]X \, d\dot{Z} \in L^0$. This is true for constant $S$, then for $S$ taking countably many values, then for the limits of sequences of such, i.e., for all stopping times $S$.

So far the argument was carried out as if 'up to time $T$' meant 'up to and including time $T$'. What happens if $X = X'$ and $Z = Z'$ a.s. on $\Omega_0$ up to and excluding time $T$, i.e., on $B_0' = (\Omega_0 \times [0, \infty)) \cap [0, T)$? We may then discard, for every rational $q$, from $\Omega_0 \cap [q < T]$ a negligible set $\Omega_q'$, leaving an equivalent set $\Omega_q$ such that $X*Z = X'*Z'$ on $(\Omega_q \times [0, \infty)) \cap [0, T \wedge q]$. Then clearly $X'*Z' = X*Z$ on $\Omega_0 \cup \Omega_q'$ up to and excluding time $T$.

4.3. Where and by how much does $X*Z$ jump? Recall that $Z_{-t} = \lim\{Z_s \colon s < t\}$, with $Z_{-0} = 0$ by convention, and that $\Delta Z = Z - Z_-$.

PROPOSITION. *Suppose $X$ is $dZ$-0-integrable. For any stopping time $T$,*

$$\Delta_T(X*Z) = X_T \cdot \Delta_T Z \quad \text{a.s.}$$

PROOF. This is true by inspection for $X \in \mathscr{R}$. For arbitrary integrable $X$ we find a sequence $X^n \in \mathscr{R}$ as in proposition 4.1(2). Since $X = \lim X^n$ $G_0^Z$-a.e., $\Delta_T(X*Z) = \lim \Delta_T(X^n * Z) = \lim X_T^n \cdot \Delta_T Z = X_T \cdot \Delta_T Z$ a.s.

4.4 PROPOSITION. *Let $Z$ be an $L^0$-integrator, $T$ an* a.s. *finite time, and $X$ a previsible process with $|X|_T^* < \infty$* a.s. *Then $X \cdot [0, T]$ is $dZ$-0-integrable.*

PROOF. Let $\epsilon > 0$. Choose $n$ so large that $P[|X|_T^* > n] < \epsilon$, and write $X = X \cdot [|X| \le n] + X \cdot [|X| > n] = X_1 + X_2$. By 3.9.2 and 2.4.2, $X_1 \cdot [0, T]$ is $dZ$-0-integrable. Now $G_0^Z(X_2 \cdot [0, T]) < \epsilon$. For if $Y$ is bounded and $G_0^Z$-measurable with $|Y| \le |X_2|[0, T]$, then $Y$ vanishes up to time $T$ on $[|X|_T^* \le n]$ and so $\int_0^T Y \, dZ$ vanishes on this set (4.2); by 2.4.3, $G_0^Z(X_2 \cdot [0, T]) < \epsilon$ as claimed. That is to say, $X \cdot [0, T]$ differs arbitrarily little from an integrable process and so is integrable itself.

The merit of this simple result is that the expressions $\int_0^T X \, dZ$ and $X*Z$ make sense under very general circumstances and can be written down fearlessly. We shall do this in the future without checking explicitly the rather weak conditions on their existence when they are clear from inspection.

4.5. PROPOSITION. *Suppose $Z$ is an $L^p$-integrator, $0 \leqq p < \infty$, and the* r.c.l.l. *process $X$ has $dZ$-$p$-integrable left continuous version $X_-$. Then*

$$\int X_- \, dZ = X_0 \cdot Z_0 + \lim \textstyle\sum_{n=1}^{N} X_{T_{n-1}} \cdot (Z_{T_n} - Z_{T_{n-1}}) \text{ in } L^p, \tag{4.5}$$

*where the limit is taken as the partition $\tau$: $0 \leqq T_0 \leqq T_1 \leqq \cdots \leqq T_N$ of $[0, \infty)$ is refined.*

(See the proof about how to refine partitions.)

PROOF. Let $\epsilon > 0$. The DCT provides an instant $u$ so that $G_p^Z(|X_-| \cdot (u, \infty)) < \epsilon/2$. With $\delta > 0$ so that $G_p^Z(\delta \cdot [0, u]) < \epsilon$, define a partition $\sigma'$: $0 = S_{-1} \leqq S_0 \leqq S_1 \leqq \cdots$ by $S_{m+1} = u \wedge \inf\{t > S_m : |X_t - X_{S_m}| > \delta\}$. Then $S_m \uparrow u$ a.s., and by the DCT once again there will be an $M$ with $G_p^Z(|X_-| \cdot (S_M, \infty)) < \epsilon$. Suppose then that $\tau: 0 \leqq T_0 \leqq T_1 \leqq \cdots \leqq T_N$ is a refinement of $\sigma: 0 \leqq S_0 \leqq S_1 \leqq \cdots \leqq S_M$. That is to say, each of the intervals $(T_n, T_{n+1}]$ is entirely contained in one of the $(S_m, S_{m+1}]$. Set

$$X^\tau = X_0 \cdot [0, T_0] + \textstyle\sum_{n=1}^{N} X_{T_{n-1}} \cdot (T_{n-1}, T_n].$$

Then $|X^\tau - X_-| \cdot [0, S_M] \leqq \delta \cdot [0, u]$, and so $G_p^Z(X^\tau - X_-) \leqq \epsilon + G_p^Z(|X_-| \cdot (S_M, \infty)) \leqq 2\epsilon$. Now (4.5) follows from the observation that the expression under the limit is just $\int X^\tau \, dZ$. Thus $\| \int X_- \, dZ - \int X^\tau \, dZ \|_{L^p} \leqq 2\epsilon$ for all refinements $\tau$ of $\sigma$.

We take the occasion to draw attention to theorem 7.14 below. It says that $\int X_- \, dZ$ can actually be evaluated pathwise, thus considerably strengthening the locality property 4.2. An $L^p$-version of 4.4 for $0 < p < \infty$ can be found in 7.11 (See also Yor [Y1]).

REMARK. Since every bounded stopping time $T$ is the infimum of a sequence $S_k$ in $\mathscr{T}$ and $Z_{S_k} \to Z_T$ in $L^p$, the $T_n$ themselves can be chosen in $\mathscr{T}$. There will then be a refinement of $\tau$ consisting of sure times. It is not hard to see that, thus,

$$\int X_- \, dZ = \lim \, (X_0 Z_0 + \textstyle\sum_{n=1}^{N} X_{t_{n-1}} \cdot (Z_{t_n} - Z_{t_{n-1}}) \text{ in } L^p, \tag{4.5'}$$

where the limit is taken as the sure partition $0 \leqq t_0 \leqq t_1 \leqq \cdots \leqq t_N$ is refined.

**5. Functions of $\mathbf{L^0}$-integrators.** The question addressed in this section is this: For which functions $F$ is $F \circ Z$ an $L^0$-integrator when $Z$ is? The answer is, roughly: For functions $F$ as smooth as a convex function. The corresponding question for $p > 0$: Suppose $Z$ is an $L^p$-integrator, when is $F(Z)$ an $L^q$-integrator? is open. A few results concerning this question appear below.

We return to $p = 0$. For some applications later on, results in dimension $d \geq 1$ are needed. Accordingly, consider an open convex subset $D$ of $\mathbf{R}^d$, a function $F$: $D \to \mathbf{R}$, and a vector

$$\vec{Z}, = (Z^1, \cdots, Z^d)$$

of $L^0$-integrators taking values in $D$. The partial derivatives of $F$, when they exist, will be denoted by

$$F'_a = \partial F / dx^a, \qquad F''_{ab} = \partial^2 F / dx^a \, dx^b \qquad \text{etc.}$$

and the Einstein summation convention will be used.

5.1 THEOREM. *Assume that $F$ is convex and of class $C^1$ and that the path of $Z$ and $Z_-$ stays* a.s. *in $D$.*

*Then $F(Z)$ is an $L^0$-integrator. There exists a* r.c.l.l. *increasing process $A^F = A^F[Z]$*

*such that*

$$F(Z_T) - F(Z_0) = \int_{(0}^{T]} F'_a(Z_-)\, dZ^a + A_T^F \qquad \text{a.s.}$$

*for all* a.s. *finite stopping times* $T$. $A^F$ *is the sum* $A^F = C^F + J^F$ *of a continuous increasing process* $C^F = C^F[Z]$ *and an increasing pure jump process* $J^F = J^F[Z]$ *which is given by*

$$J_t^F = \textstyle\sum_{0\le s\le t} \{F(Z_s) - F(Z_{s-}) - F'_a(Z_{-s})(Z_s^a - Z_{s-}^a)\},$$

a.s. *absolutely convergent sum of positive terms. For any* a.s. *finite time* $T$

$$A^F[Z^T] = (A^F[Z])^T, \qquad C^F[Z^T] = (C^F[Z])^T, \qquad \text{etc.}$$

PROOF. The proof rests on the simple observation that $F(Z_2) - F(Z_1) - F'_a(Z_1)(Z_2^a - Z_1^a) \ge 0$ for any two points $Z_1, Z_2 \in D$, a consequence of the convexity. Let $\gamma: 0 = S_0 \le S_1 \le \cdots \le S_N = T$ be any partition of $(0, T]$ and set

$$\begin{aligned} {}^{\gamma}A_T^F &= F(Z_T) - F(Z_0) - \textstyle\sum_{i=1}^N F'_a(Z_{S_{i-1}})(Z_{S_i}^a - Z_{S_{i-1}}^a) \\ &= \textstyle\sum_{i=1}^N (F(Z_{S_i}) - F(Z_{S_{i-1}}) - F'_a(Z_{S_{i-1}})(Z_{S_i}^a - Z_{S_{i-1}}^a)), \end{aligned}$$

a positive random variable. The assumption on $Z$ has the effect that the maximal function $(F'_a(Z_-))_t^*$ is finite a.s., so that the l.c.r.l. process[1] $F'_a(Z_-)\cdot(0, T]$ is $dZ^a$-integrable (4.4). By (4.5), the sums on the right of the first equality converge in $L^0$ to $\int_0^T F'_a(Z_-)\, dZ$ as $\gamma$ is refined. Hence ${}^{\gamma}A_T^F$ converges in $L^0$ to $A_T^F = F(Z_T) - F(Z_0) - (F'_a(Z_-) * Z^a)_T$. From the second equality above it is seen that the r.c.l.l. process $A^F$ so defined (2.7) is increasing. Writing $dA^F = dF(Z) + F'_a(Z_-)\, dZ^a$, the jump part of $A^F$ is identified as stated using 4.3. The last statement is obvious.

5.1′ REMARK. The theorem stays if $F$ is, instead, a difference of convex functions of class $C^1$, except that the processes $A$, $C$, $J$ are now of finite variation, with the expression for $J^F$ converging absolutely.

5.2 THE SQUARE FUNCTIONS. Particularly important is the simple case that $d = 2$ and that $F(y, z) = y\cdot z = ((y + z)^2 - (y^2 + z^2))/2$. For any two $L^0$-integrators $Y$ and $Z$ one defines the brackets

$$[Y; Z] := A^F[Y, Z] \qquad \text{and} \qquad \{Y; Z\} := C^F[Y, Z]$$

and computes the difference

$$[Y; Z]_t^j = J_t^F = \textstyle\sum_{0\le s\le t} \Delta_s Y\cdot\Delta_s Z.$$

Let $Y = Z$. The increasing process

$$S[Z] := [Z; Z]^{1/2}$$

is termed the *square function* of $Z$. It has the value $S_T[Z]$ at $T$. Of equal interest is the *continuous square function*

$$\sigma[Z] := \{Z; Z\}^{1/2}$$

It is well to keep in mind that these brackets are defined by the equations

$$Y\cdot Z = Y_- * Z + Z_- * Y + [Y; Z]; \qquad [Y; Z]_T = \{Y; Z\}_T + \textstyle\sum_{0\le s\le T} \Delta_s Y\cdot\Delta_s Z.$$

The importance of the square functions for martingale theory and stochastic integration is well illuminated in, e.g., [KW], [BDG], [B3], [G2], and [M5]. Meyer's definition reads differently from ours; but its equivalence with ours is implicit in [M5]. The proof of Theorem 5.1, with $F(z) = z^2$, exhibits $S_T[Z]$ as $\lim(\sum(Z_{S_i} - Z_{S_{i-1}})^2)^{1/2}$. An appeal to Fatou's lemma and (2.2) gives the following important quantitative information. (As usual, the case $1 \leqq p$ is well known, e.g., [Y1], while the case $0 < p < 1$ seems new, inasmuch as the $\gamma^p[Z]$ have not been considered before in this range of $p$'s.)

5.3 PROPOSITION. *There are constants $C_p$, depending only on $p$, $0 < p < \infty$, such that for any stopping time $T$*

$$\| \sigma_T[Z] \|_{L^p} \leq \| S_T[Z] \|_{L^p} \leq C_p \gamma_T^p[Z]. \tag{5.3}$$

5.4. Suppose now that both $Y$ and $Z$ are local $L^2$-integrators. Then, by 5.3, so are $S[Y]$ and $S[Z]$. The first inequality of the proposition below yields upon integration $E \int_0^T |d[Y; Z]| \leq E(S_T[Y] \cdot S_T[Z]) \leq \| S_T[Y] \|_{L^2} \cdot \| S_T[Z] \|_{L^2}$, so that $[Y; Z]$ is a local $L^1$-integrator: when $T$ reduces both $Y$ and $Z$ to global $L^2$-integrators, the expression is finite by 5.3. As such, it has a Doob decomposition

$$[Y; Z] = [Y; Z]^{\sim} + \langle Y; Z \rangle,$$

the oblique bracket being defined as $\langle [Y; Z] \rangle$. For $Y = Z$, one obtains the *previsible square function*

$$s[Z] = \langle Z; Z \rangle^{1/2}.$$

PROPOSITION. (Inequalities of Kunita-Watanabe [KW]). *For any two $L^0$-integrators $Y, Z$, any two $\mathcal{F}_\infty \times$ Borel $([0, \infty))$-measurable processes $U$, $V$, and any stopping time $T$,*

$$\int_0^T UV| d[Y, Z] | \leq \left( \int_0^T U^2 d[Y; Y] \right)^{1/2} \cdot \left( \int_0^T V^2 d[Z, Z] \right)^{1/2} \quad \text{a.s.,}$$

$$\int_0^T UV| d\{Y; Z\} | \leq \left( \int_0^T U^2 d\{Y; Y\} \right)^{1/2} \cdot \left( \int_0^T V^2 d\{Z; Z\} \right)^{1/2} \quad \text{a.s.;}$$

*and*

$$\int_0^T UV| d\langle Y; Z \rangle | \leq \left( \int_0^T U^2 d\langle Y; Y \rangle \right)^{1/2} \cdot \left( \int_0^T V^2 \cdot d\langle Z, Z \rangle \right)^{1/2}$$

*when $Y, Z$ are local $L^2$-integrators.*

PROOF. [M5]. Abbreviate $[Y; Z]_S^T = [Y; Z]_T - [Y; Z]_S$. Expressing the fact that the polynomial $p(\lambda) = [Y + \lambda Z; Y + \lambda Z]_s^t$ is positive for any two rational instances $s$, $t$, yields

$$[Y, Z]_s^t \leq ([Y; Y]_s^t)^{1/2} ([Z; Z]_s^t)^{1/2}.$$

For $U = \sum u_i \cdot (t_i, t_{i+1}]$ and $V = \sum v_i \cdot (t_i, t_{i+1}]$ with $u_i, v_i \in \mathcal{F}_\infty$, summation and Schwarz' inequality yield the first inequality for these special $U$, $V$. But then these generate $\mathcal{F}_\infty \times$ Borel$[0, \infty)$, so the first inequality holds in general by DCT. The other two inequalities are proved the same way.

COROLLARY. *$S[Y + Z] \leq S[Y] + S[Z]$ and similarly for $s$ and $\sigma$. Also, for $1/r = 1/p + 1/q$*

$$\left\| \int_0^T | d[Y; Z] | \right\|_{L^r} \leq \| S_T[Y] \|_{L^p} \| S_T[Z] \|_{L^q}$$

*and similarly for $s$ and $\sigma$.*

In preparation of Itô's formula, we establish a technical result;

5.5 LEMMA. *Let $T$ be an* a.s. *finite time and $X$ a* l.c.r.l. adapted process such that

$|X|^*_T < \infty$ a.s. *Then*

$$\int_0^T Xd[Y;Z] = \lim \textstyle\sum_{i=0}^n X_{S_{i-1}} (Y_{S_i} - Y_{S_{i-1}})(Z_{S_i} - Z_{S_{i-1}}), \tag{5.5.1}$$

$$\int_0^T Xd\{Y;Z\} = \lim \textstyle\sum_{i=0}^n X_{S_{i-1}} (Y_{-S_i} - Y_{S_{i-1}})(Z_{-S_i} - Z_{S_{i-1}}). \tag{5.5.2}$$

The limits exist in $L^0$ and are taken as the random partition $\gamma$: $0 = S_0 \le S_1 \le \cdots \le S_n \le T$ of $[0, T]$ is refined. The convention $X_{S_{-1}} = X_0$, $Y_{S_{-1}} = Y_{-0} = Z_{S_{-1}} = Z_{-0} = 0$ is used.

PROOF. Note that the integrals on the left exist (4.4). Polarization reduces the situation to the case $Y = Z$. As $d(Z^2) = -2Z_- \, dZ + d[Z, Z]$, 4.5 yields

$$\begin{aligned}\int_0^T Xd[Z,Z] &= \int_0^T X(d(Z^2) - 2Z_- \, dZ)\\ &= \lim \textstyle\sum X_{S_{i-1}} (Z^2_{S_i} - Z^2_{S_{i-1}} - 2Z_{S_{i-1}}(Z_{S_i} - Z_{S_{i-1}}))\\ &= \lim \textstyle\sum X_{S_{i-1}} (Z_{S_i} - Z_{S_{i-1}})^2.\end{aligned}$$

This proves (5.5.1). The proof of (5.5.2) is more complicated. Write

$$\begin{aligned}\textstyle\sum_i X_{S_{i-1}} \cdot (Z_{S_i} - Z_{S_{i-1}})^2 &= \textstyle\sum_i \{X_{S_{i-1}} \cdot \sum_{S_{i-1}<s\le S_i} (\Delta_s Z)^2\} + \sum_i X_{S_{i-1}}(Z_{-S_i} - Z_{S_{i-1}})^2\\ &\quad + 2\textstyle\sum_i X_{S_{i-1}} \cdot (Z_{-S_i} - Z_{S_{i-1}})(Z_{S_i} - Z_{-S_i})\\ &\quad - \textstyle\sum_i \{X_{S_{i-1}} \cdot \sum_{S_{i-1}<s<S_i} (\Delta_s Z)^2\}\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

The first sum converges to $\int_0^T Xd[Z,Z]^j$ by 4.5. The second sum is the sum in question, so it is left to be shown that III and IV converge to zero in $L^0$ as the partition is refined. This is easily seen for IV: let $T_1, T_2, \cdots$, be stopping times such that all the jumps of $Z$ before time $T$ occur on the union of their graphs. As $[Z,Z]^j_T = \sum (\Delta_{T_n} Z)^2 < \infty$ a.s., there is an $N$ such that

$$\rho_0((X^*_T + X^{*2}_T) \cdot \textstyle\sum \{(\Delta_s Z)^2 : s \ne T_1, \ldots, s \ne T_N\}) < \epsilon. \tag{$*$}$$

We may assume that $\gamma_0$: $0 \le T_1 \le T_2 \le \cdots \le T_N \le T$. For any partition $\gamma$ refining $\gamma_0$, $\rho_0(\mathrm{IV}) < \epsilon$. The remaining sum III can be estimated with Schwartz' inequality by

$$\begin{aligned}|\mathrm{III}| &\le 2\Big(\textstyle\sum \{X^2_{S_{i-1}} (Z_{-S_i} - Z_{S_{i-1}})^2 : S_i \in \gamma_0\}\Big)^{1/2} \Big(\sum \{(Z_{S_i} - Z_{-S_i})^2 : S_i \in \gamma_0\}\Big)^{1/2}\\ &\quad + 2\Big(\textstyle\sum (Z_{-S_i} - Z_{S_{i-1}})^2\Big)^{1/2} \Big(\sum \{(Z_{S_i} - Z_{-S_i})^2 : S_i \notin \gamma_0\}\Big)^{1/2} = \mathrm{III}_1 + \mathrm{III}_2.\end{aligned}$$

The first factor of $\mathrm{III}_2^2$ is dominated by $4([Z,Z]_T + \sum (Z_{S_i} - Z_{S_{-1}})^2)$ and so stays bounded in $L^0$ (2.2). The second factor goes to zero as sufficiently many of the $T_n$ are included in $\gamma_0$, by $(*)$. Say $T_1, \ldots, T_N$ are needed to make $\rho_0(\mathrm{III}_2) < \epsilon$. The second factor of $\mathrm{III}_1$ is dominated by $([Z,Z]^j_T) \le S_T[Z]$. Its first factor tends to zero if $\gamma_0$ is refined so that $Z_{-S_i} - Z_{S_{-1}} \to 0$. This can be done since $Z$ has no oscillatory discontinuities (2.5), e.g., by insisting that if $S_{i-1} \in \gamma_0$ then $S_i \le \inf\{t \ge S_{i-1};\ |Z_t - Z_{S_{-1}}| < \delta\}$ and letting $\delta \downarrow 0$. The proof is finished. It is worth noting that III $\to 0$, which reads after polarization:

5.6 COROLLARY. $\lim_\gamma \sum_i (Y_{-S_i} - Y_{S_{i-1}})(Z_{S_i} - Z_{-S_i}) = 0$ *in* $L^0$.

5.7 COROLLARY. *For any as finite time T,*

(5.7.1) $[Y, Z]^T = [Y, Z^T], \{Y, Z\}^T = \{Y, Z^T\}$, *and* $\langle Y; Z\rangle^T = \langle Y, Z^T\rangle$;

*and for any bounded previsible process X,*

(5.7.2) $[Y, X * Z] = X * [Y, Z], \{Y, X * Z\} = X * \{Y, Z\}$, *and* $\langle Y, X * Z\rangle = X * \langle Y, Z\rangle$.

PROOF. (5.7.1) follows from 5.1 and is identical[1] with (5.7.2) when $X$ is the stochastic interval $[0, T]$. Linearity and the usual monotone class argment yield (5.7.2) in general.

5.8 THEOREM. *(Itô's formula). Let $D \subset \mathbf{R}^d$ be open and convex, $F$: $D \to \mathbf{R}$ of class $C^2$, and $\vec{Z} = (Z', \dots, Z^d)$ a vector of $L^0$-integrators with values in D such that the paths of Z and $Z_-$ stay in D at all times. Then $F(Z)$ is an $L^0$-integrator, and for any* a.s. *finite time T*

$$F(Z_T) = F(Z_0) + \int_0^T F'_a(\vec{Z}_-)\, dZ^a + J_T^F + \tfrac{1}{2}\int_0^T F''_{ab}(\vec{Z}_-)\, d\{Z^a, Z^b\},$$

*where* $J_T^F = J_T^F[Z] = \sum_{0\le s\le T} (\Delta_S F(\vec{Z}) - F'_a(\vec{Z}_{-s})\Delta_s Z^a)$, *the sum converging* a.s. *absolutely.*

PROOF. Let $r_n \downarrow 0$, $B_n$ the ball of radius $r_n > 0$ and center zero, and let $D_n$ be an increasing sequence of relatively compact open convex sets with union $D$ such that $D_n + B_n \subset D_{n+1}$. On the compact set $\bar{D}_n$, $F$ is the difference of two convex functions of class $C^2$ (merely subtract $c\cdot\sum x_i^2$ with $c > 0$ suitably large). The idea is to establish the theorem with $D$ and $Z$ replaced by $D_n$ and $Z^n = Z\cdot[0, T_n)$, respectively, where $T_n = \inf\{t: Z_t \notin D_n\}$. Noting that $Z^n = Z$ up to and excluding time $T_n$, and that $T_n \uparrow \infty$ by assumption, the formula will follow for all times $T$ (4.2). In other words, we may assume that $F$ is $C^2$ and convex in a neighborhood $D + B_\epsilon$ of $D$, $D$ bounded. Now, for any partition $\gamma$: $0 \le S_0 \le S_1 \le \cdots$

$$\begin{aligned}
&\textstyle\sum_i (F(\vec{Z}_{S_i}) - F(\vec{Z}_{S_{i-1}}) - F'_a(\vec{Z}_{S_{i-1}})(Z^a_{S_i} - Z^a_{S_{i-1}})) \\
&\quad - \textstyle\sum_i (F(\vec{Z}_{S_i}) - F(\vec{Z}_{-S_i}) - F'_a(\vec{Z}_{-S_i})(Z^a_{S_i} - Z^a_{-S_i})) \\
&= \textstyle\sum_i (F'_a(Z_{-S_i}) - F'_a(Z_{S_{i-1}}))(Z^a_{S_i} - Z^a_{-S_i}) \\
&\quad + \textstyle\sum_i (F(Z_{-S_i}) - F(Z_{S_{i-1}}) - F'_a(Z_{S_{i-1}})(Z^a_{-S_i} - Z^a_{S_{i-1}})).
\end{aligned}$$

The left-hand side converges to $C_T^F[Z]$ as $\gamma$ is refined. Assume for the moment that $F$ is of class $C^3$. Then $F'_a \in C^2$ and $F'_a(\vec{Z})$ is an $L^0$-integrator (5.1). The first sum on the right then converges to zero in $L^0$ (5.6). The second sum on the right can be written as

$$\tfrac{1}{2}\textstyle\sum_i F''_{ab}(\vec{Z}_{S_{i-1}})(Z^a_{-S_i} - Z^a_{S_{i-1}})(Z^b_{-S_i} - Z^b_{S_{i-1}}) + \sum_i o(Z_{-S_i} - Z_{S_{i-1}})^2$$

and converges to $\tfrac{1}{2}\int_0^T F''_{ab}(\vec{Z}_-)d\{Z^a, Z^b\}$ by (5.5.2). To get rid of the assumption that $F$ belong to the class $C^3$, note that every convex $F \in C^2(D + B_\epsilon)$ can be approximated uniformly on the compact set $\bar{D}$, together with its first and second derivatives, by smooth convex functions. Merely convolve $F$ with a positive smooth approximate identity having support in $B_\epsilon$. Itô's formula persists in the limit by the DCT.

5.9. The non-classical term $\tfrac{1}{2}\int F''d\{,\}$ in Itô's formula deserves some attention. Suppose $Z$ is increasing. Then it is the sum $Z = Z^c + Z^j$ of a continuous r.c. process and a process constant between jumps. By 5.5, $\{Z, Z\} = [Z^c, Z^c] = 0$, so that $[Z, Z]_t = [Z^j, Z^j]_t = \sum \{(\Delta_s Z)^2: 0 \le s \le t\}$. Taking differences shows that the same observation applies to processes $Z$ of finite variation. We shall see later that every $L^0$-integrator $Z$ is the sum $Z = M + V$ of a local martingale $M$ and a process $V$ of finite variation, so that $\{Z, Z\} = \{M, M\}$. Furthermore, $M$ has a decomposition $M = M^c + M^j$ into a continuous local martingale $M^c$ and a r.c. martingale $M^j$ that has the same jumps as $M$ and $\{M^j, M^j\} = 0$. So $\{Z, Z\} = \{M^c, M^c\}$ owes its existence to the 'continuous martingale part of $Z$'.

5.10. It is worth noting that the very definition of $[Z, V]$ contains the classical formula for integration by parts: When $V$ is of finite variation then $\{Z, V\} = 0$ by 5.4 and $[Z, V] = [Z, V]^j = \Delta Z * V$. Hence

$$Z \cdot V = Z_0 V_0 + Z * V + V_- * Z.$$

**6. Local time.** The results of this section will not be used later on, except for the information that the $L^p$-integrators form a vector lattice. Of course, this fact can be derived as well from 2.6 with a small effort. The section is based on the little smoothing trick in the proof of Itô's formula 5.8. It turns out that, in dimension $d = 1$[5], it can be used to more general effect. We follow Meyer [M5].

Consider a convex function $F$ on the line. It is necessarily continuous, differentiable at all but countably many points, and is the indefinite integral of a unique *left continuous* function $f$.[6] Let $e_n$ be a positive smooth approximate identity with support in $[0, 1/n]$ and $\int e_n(x)\,dx = 1$, and let $F_n$, $f_n = F_n'$ be the convolution of $F$, $f$ with $e_n$: $F_n(x) = \int F(x-y)e_n(y)\,dy$ etc. Note that $f_n \uparrow f$ pointwise, with $\sup_n |f_n(x)|$ bounded on each compact interval.

Consider the equation

$$F_n(Z_T) - F_n(Z_0) = \int_0^T f_n(Z_-)\,dZ + A_T^{F_n}$$

of (5.1), $T$ a.s. finite. As $n \to \infty$, $\int_0^T f_n(Z_-)\,dZ$ converges to $\int_0^T f(Z_-)\,dZ$ by the DCT. Hence

$$F(Z_T) - F(Z_0) = \int_0^T f(Z_-)\,dZ + A_T^F,$$

where $A^F = A^F[Z]$ is an increasing process, as the limit of such (cf. 5.1). Note now that $A^F$ depends only on the measure $\mu = F''$: it will not change if an affine function is added to $F$. It is therefore justified to rename $A^F$ by $A^\mu$. Taking differences of convex functions and identifying jumps as in 5.1 yields:

6.1 PROPOSITION. *Let $Z$ be an $L^0$-integrator, and let $\mu$ be a signed measure of finite—but not necessarily totally finite—variation on the line. There exists a* r.c.l.l. *process $A^\mu = A^\mu[Z]$ of finite variation such that*

$$F(Z_T) - F(Z_0) = \int_0^T f(Z_-)\,dZ + A_T^\mu$$

*for any left-continuous distribution function $f$ of $\mu$, any indefinite integral $F$ of $f$, and any* a.s. *finite time $T$. $A^\mu$ is the sum of a* r.c.l.l. *pure jump process $J^\mu = J^\mu[Z]$ of finite variation given by*

$$J_t^\mu = \textstyle\sum_{0 \le s \le t} \{\Delta_s F(Z) - f(Z_{-s})\Delta_s Z\},$$

*the sum converging* a.s. *absolutely, and a continuous process $C^\mu = C^\mu[Z]$ of finite variation, which is termed the local time of $Z$ at $\mu$. When $\mu$ is positive, $A$, $J$, and $C$ are increasing. We have $A_0 = J_0$, $C_0 = 0$, and the maps $\mu \to J^\mu$, $\mu \to C^\mu$ are linear and*

[5] This restriction is due to the lack of the author's knowledge about the Choquet theory of the convex cone of convex functions on $\mathbf{R}^d$, $d > 1$. It is probably also the wrong cone to consider.

[6] The convention to consider left-continuous cumulative distribution functions is in accordance with [IM]. cf. also Yor [Y3].

*isotone. For any* a.s. *finite time T,*

$$A^\mu[Z^T] = (A^\mu[Z])^T \quad \text{and} \quad C^\mu[Z^T] = (C^\mu[Z])^T.$$

*If $\mu$ is Lebesgue measure $\lambda$ then*

$$A^\lambda[Z] = [Z, Z], \qquad J_t^\lambda[Z] = \textstyle\sum_{0\le s\le t}(\Delta_s Z)^2, \qquad C^\lambda = \{Z, Z\}.$$

*Suppose $\mu_n$ is a sequence of measures such that $\mu_n(g) \to \mu(g)$ for every bounded right-continuous function of compact support. Then*

$$A^{\mu_n} \to A^\mu, \qquad J^{\mu_n} \to J^\mu, \qquad C^{\mu_n} \to C^\mu$$

*uniformly on every compact interval,* a.s.

PROOF. Only the very last statement is not obvious. Choose the l.c. distribution functions $f$, $f_n$ of $\mu$, $\mu_n$ and their integrals $F$, $F_n$ to vanish at zero. Then $f_n \to f$ pointwise and $F_n \to F$ uniformly on every compact interval. By the uniform boundedness principle $\sup_n |f_n(z)| \leqq \sup \int_{-z}^{z} |d\mu_n| = g(z) < \infty$, so that $\sup|f_n(Z_-)|_T^* < \infty$ a.s. for any a.s. finite time $T$ (2.5). By 4.4, 4.1, and the DCT, $A^{\mu_n} \to A^\mu$ a.s. uniformly on $[0, T]$. From this, the convergence of $J^{\mu_n}$ and $C^{\mu_n}$ follows immediately.

6.2 PROPOSITION. *For any locally bounded Borel function g let $g\mu$ denote the measure with density g and base $\mu$. Then*

$$C^{g\mu}[Z] = g(Z_-) * C^\mu[Z].$$

PROOF. When $g$ is continuous and $\mu = \lambda$ then this is just Itô's formula. Thus

$$G_T^{gh\lambda} = \int_0^T g(Z_-)\, dC^{h\lambda} \qquad \text{a.s.}$$

for $g$, $h$ continuous and $T$ a.s. finite. Now there is a sequence $h_n$ of continuous functions such that $h_n\lambda \to \mu$ in the sense of proposition 6.1; for example if $\mu \ge 0$ set $h_n = d(\mu * e_n)/d\lambda$. Thus

$$(*) \qquad C_T^{g\mu} = \int_0^T g(Z_-)\, dC^\mu \qquad \text{a.s.}$$

if $g$ is continuous. A monotone class argument finishes the proof: if $g_n$ satisfies $(*)$ and, say, increases to $g$ then $g_n\mu \to g\mu$ in the sense of 6.1, and the DCT shows that $g$ satisfies $(*)$. Taking [1] $g = B$ and observing that $dC$ does not charge the countable set $[Z \ne Z_-]$ yields

6.3 COROLLARY. *Suppose $\mu$ is carried by the Borel set B. Then the random measure $dC^\mu[Z]$ is carried by the set $[Z \in B] \cap [Z_- \in B]$,* a.s.

6.4 COROLLARY. *When $\mu$ is Dirac measure $\epsilon_x$ we write $C^x$ for $C^{\epsilon_x}$; $dC^x$ is carried by $[Z = Z_- = x]$ and does not charge the interior of that set,* a.s.

PROOF. Let $S$ be the $n$th time that $Z$ equals $x$, $T = \inf\{t > S, Z_t \ne x\}$. Then $Z = x$ on $(S, T)$, and due to the local nature of the stochastic integral (4.2), $f(Z_-) * Z$ is constant on that interval. So is clearly $F(Z)$, hence the difference $A^\mu$ and its continuous part $C^\mu$.

The process $L^x[Z] = 2C^x[Z]$ is called the *local time of Z at x*. As $2(Z - x) \vee 0$ and $|Z - x|$ differ by an affine function, $L^x$ is given by

$$|Z_t - x| = |Z_t - 0| + 2\int_0^t [Z_- > x]\, dZ + 2J_t^x[Z] + L_t^x.$$

6.5 PROPOSITION. *For any* a.s. *finite time T, the maps* $x \to (A_T^x)^\cdot$, $x \to (J_T^x)^\cdot$, $x \to$

$(C_T^x)^{\cdot}$ *from* $\mathbf{R}$ *to* $L^0$ *are Borel measurable, and*

$$A_T^\mu \in \int (A_T^x)^{\cdot}\mu\,(dx), \qquad J_T^\mu \in \int (J_T^x)^{\cdot}\mu\,(dx), \qquad C_T^\mu \in \int (C_T^x)^{\cdot}\mu\,(dx).$$

PROOF. By linearity, it suffices to assume that $\mu$ is positive and carried by a bounded interval $(a, b)$. For any such $\nu$ let $f^\nu$, $F^\nu$ denote that l.c. distribution function and its indefinite integral which vanish at $a - 1$. Then

$$f^\nu(z) = \int f^{\epsilon_x}(z)\nu\,(dx) \quad \text{and} \quad F^\nu(z) = \int F^{\epsilon_x}\nu\,(dx).$$

The theorem of Fubini applied to $\mu$ and counting measure yields

$$\begin{aligned} J_T^\mu &= \textstyle\sum_{0\le s\le T} \{F^\mu(Z_s) - F^\mu(Z_{-s}) - f^\mu(Z_{-s})(Z_s - Z_{-s})\} \\ &= \int \textstyle\sum_{0\le s\le T} \{F^{\epsilon_x}(Z_s) - \cdots\}\,\mu\,(dx) = \int J_T^x\mu\,(dx) \qquad \text{a.s.} \end{aligned}$$

It suffices therefore to prove the integral decomposition for $A_T^\mu$. Consider the equation

$$(*) \qquad F^{\epsilon_x * e_n}(Z_T) - F^{\epsilon_x * e_n}(Z_0) - \int_0^T f^{\epsilon_x * e_n}(Z_-)\,dZ = A_T^{\epsilon_x * e_n}.$$

Since $f^{\epsilon_x * e_n} \to f^{\epsilon_y * e_n}$ and $F^{\epsilon_x * e_n} \to F^{\epsilon_y * e_n}$ uniformly as $x \to y$, $x \to A_T^{\epsilon_x * e_n}$ is a continuous map from $[a, b]$ to $L^0$. Now

$$\int\int_0^T f^{\epsilon_x * e_n}(Z_-)\,dZ\,\mu(dx) = \int_0^T f^{\mu * e_n}(Z_-)\,dZ.$$

This is evident if $\mu$ is discrete and follows in general by approximating $f^{\mu * e_n}$ uniformly by $f^{\nu * e_n}$, $\nu$ discrete. Integrating $(*)$ now yields

$$\dot A_T^{\mu * e_n} = \int \dot A^{\epsilon_x * e_n}\mu\,(dx),$$

and taking the limit as in 6.1 results in

$$\dot A_T^\mu = \int \dot A_T^x\,\mu\,(dx).$$

6.6 Proposition. *The $L^p$-integrators form a vector lattice under pointwise operations.*

PROOF. The formulae $Y \vee Z = Y + 0 \vee (Z - Y)$ and $Y \wedge Z = Y + Z - Y \vee Z$ show that it suffices to prove that $0 \vee Z$ is an $L^p$-integrator when $Z$ is. Now $0 \vee Z = 0 \vee Z_0 + [Z_- > 0] * Z + A^{\epsilon_0}$. Clearly $[Z_- > 0] * Z$ is an $L^p$-integrator. Thus$\|A_t^{\epsilon_0}\|_{L^p} < \infty$ for all $t$, and the increasing process is an $L^p$-integrator (3.1). This exhibits $0 \vee Z$ as the sum of three $L^p$-integrators.

**7. Structure of $\mathbf{L^p}$-integrators.** We shall now show, as promised earlier, that an $L^0$-integrator $Z$ is a semimartingale $M + V$. The decomposition $Z = M + V$ is, of course, not unique. The question arises whether the local martingale $M$ and the process of finite variation $V$ can be chosen to be $L^p$-integrators when $Z$ is one, $p > 0$. This and the problem of how to characterize the $L^p$-integrators, $p > 0$, amongst the $L^0$-integrators go hand-in-hand with the problem of finding a manageable replacement for the unwieldy upper gauge $G_p^Z$. For $p \ge 1$ there are some easy, well known [Y1, 4] answers:

7.1 THEOREM. *Let $Z$ be an $L^p$-integrator, $1 \le p < \infty$, and $Z = \tilde Z + \langle Z\rangle$ its Doob decomposition. Then $\tilde Z$ and $\langle Z\rangle$ are $L^p$-integrators; in fact, there exist constants $C_p$ depending on $p$ alone such that, for any stopping time $T$,*

$$(7.1) \qquad \gamma_T^p[\tilde Z] \le C_p\gamma_T^p[Z] \quad \text{and} \quad \gamma_T^p[\langle Z\rangle] \le C_p\gamma_T^p[Z].$$

PROOF. Note first that the measure $m_Z = EdZ$ is σ-additive and of finite variation, so that $Z$ has, indeed, a Doob decomposition (3.4 or 3.9.4). The jump of $Z$ at zero is in $\mathscr{L}^p$, so we may subtract it and assume that $Z_0 = 0$. For $p = 1$, there is nothing to prove (3.4). Assume then that $p > 1$, and let $M$ be a bounded r.c. martingale with $\| M_T \|_{L^{p'}} \leq 1$. Suppose for the moment that $Z$ is bounded on $[0, T)$. The first two processes on the right of

$$Z^T M^T = Z_- * M^T + M_- * \tilde{Z}^T + M_- * \langle Z \rangle^T + [Z, M]^T$$

are then uniformly integrable martingales vanishing at zero, and the penultimate process has expectation at $T$ $E(M_T \langle Z \rangle_T)$, since $\langle Z \rangle$ is previsible (3.3.1). Hence

$$\begin{aligned} E(M_T \langle Z \rangle_T) &\leq E\,|[M, Z]_T| + E\,|M_T Z_T| \leq \| S_T[M] \|_{L^{p'}} \, \| S_T[Z] \|_{L^p} \\ &\quad + \| M_T \|_{L^{p'}} \, \| Z_T \|_{L^p} \\ &\leq (C_{p'}^{(5.3)} \gamma_T^{p'}[M] C_p^{(5.3)} + 1) \gamma_T^p[Z] \\ &\leq (C_{p'}^{(5.3)} C_{p'}^{(3.8)} C_p^{(5.3)} + 1) \gamma_T^p[Z] := C_p \cdot \gamma_T^p[Z]. \end{aligned}$$

Hence $\| \langle Z \rangle_T \|_{L^p} \leq C_p \gamma_T^p[Z]$. The restriction on $T$ can now be lifted by observing that both sides increase with $T$. Now let $X$ be a previsible process of absolute value one such that $|d\langle Z \rangle| = X d\langle Z \rangle$ (3.3.2). Then $X*Z = X*\tilde{Z} + X*\langle Z \rangle$, and the same estimate applies and yields

$$\gamma_T^p[\langle Z \rangle] = \gamma_T^p[X * \langle Z \rangle] = \| \langle X*Z \rangle_T \|_{L^p} \leq C_p \gamma_T^p[X*Z] = C_p \gamma_T^p[Z].$$

The other inequality follows by subtraction.

Let $\mathscr{T}^p$ denote the vector lattice of $L^p$-integrators, equipped with the distance functions $\mathrm{dist}(Y, Z) = \gamma_T^p[Y - Z]$, and $\mathscr{M}^p$ the linear subspace consisting of the local martingales in $\mathscr{T}^p$. The proposition says that $\mathscr{M}^p$ is a complemented subspace of $\mathscr{T}^p$, if $1 \leq p < \infty$. For $0 \leq p < 1$ it is not even clear whether $\mathscr{M}^p$ is closed. Let us also introduce the spaces $\mathscr{T}_\infty^p$, $\mathscr{M}_\infty^p$ of processes, resp local martingales, that are global $L^p$-integrators: $\gamma_\infty^p[Z] < \infty$, and the spaces $\mathscr{T}_{\mathrm{loc}}^p$, $\mathscr{M}_{\mathrm{loc}}^p$ of local $L^p$-integrators, and local martingales in $\mathscr{T}_{\mathrm{loc}}^p$, respectively. Again, $\mathscr{M}_\infty^p \subset \mathscr{T}_\infty^p$ and $\mathscr{M}_{\mathrm{loc}}^p \subset \mathscr{T}_{\mathrm{loc}}^p$ are complemented subspaces, if $1 \leq p < \infty$.

When $Z$ has a Doob decomposition $Z = \tilde{Z} + \langle Z \rangle$ into two $L^p$-integrators, the task of finding a manageable replacement for $G_p^Z$ is reduced to finding one for $G_p^{\tilde{Z}}$, since $G_p^{\langle Z \rangle}(X) = \| \int |X| \, |d\langle Z \rangle| \, \|_{L^p}$ is reasonable enough (3.3). Again, the case $p \geqq 1$ is simple and known [G2, M6, Y1, 4 and 5]:

7.2 THEOREM. *Let $M$ be a local martingale. In order that $M$ be an $L^p$-integrator it is necessary and sufficient that $S[M]$ be one when $1 \leq p < \infty$; when $0 < p \leq 2$ a sufficient condition is that $M$ be a local $L^2$-integrator and $s[M]$ an $L^p$-integrator. In fact*[7], *there are constants $C_p$, $c_p$ depending on $p$ alone such that for any stopping time $T$*

(7.2a) $$C_p^{-1} \| S_T[M] \|_{L^p} \leq \gamma_T^p[M] \leq C_p \| S_T[M] \|_{L^p}, \qquad 1 \leq p < \infty,$$

(7.2b) $$C_p^{-1} \| S_T[M] \|_{L^p} \leq \gamma_T^p[M] \leq c_p \| s_T[M] \|_{L^p}, \qquad 0 < p \leq 2.$$

*Consequently* (2.7), *for $X$ previsible*

(7.2ā) $$C_p^{-1} \bar{G}_p^M(X) \leq G_p^M(X) \leq C_p \bar{G}_p^M(X) \quad \text{with} \quad \bar{G}_p^M(X) = \left\| \left( \int X^2 d[M, M] \right)^{1/2} \right\|_{L^p},$$

$$1 \leq p < \infty$$

[7] These inequalities are well known, albeit in different forms. For $p = 2$ they make Meyer's integration theory work [M5]. For $1 < p < \infty$, they are the reason that Meyer's $\| \cdot \|_{H_p}$-norms [M6] are relevant and useful. cf. also [Y1].

$$(7.2\bar{b})\qquad G_p^M(X) \le \left(c_p \left\| \left(\int X^2 d\langle M, M\rangle\right)^{1/2} \right\|_{L^p}\right)^{p\wedge 1} \quad \textit{if} \quad 0 < p \le 2.$$

PROOF. The barred inequalities follow from the unbarred ones and (2.7). The inequalities on the left are but a repetition of 5.3. The second inequality of (a) is easy if $p > 1$: Let $X \in \mathscr{R}_T$ with $|X| \le 1$ and let $N$ be a bounded martingale with $\| N_T \|_{L^{p'}} \le 1$. Then, by 5.4,

$$E((X*M)_T \cdot N_T) = E[X * M, N]_T \le E \int_0^T |d[M, N]|$$
$$\le \| S_T[M] \|_{L^p} \| S_T[N] \|_{L^{p'}} \le \| S_T[M] \|_{L^p} C_{p'}^{(5.3)} C_{p'}^{(3.8)} \| N_T \|_{L^{p'}}$$
$$\le C_p \| S_T[M] \|_{L^p}.$$

For $p = 1$, the inequality in question follows similarly from Fefferman's inequality

$$(7.2c)\qquad E \int_0^T |d[M, N]| \le C \| S_T[M] \|_{L^1} \cdot \| N^T \|_{BMO}$$

where $\| N \|_{BMO} = \inf\{c: E((N_\infty - N_{-S})^2 | \mathscr{F}_S) < c^2 \text{ for all stopping times } S\} \le 2 \| N_\infty \|_{L^\infty}$. For completeness' sake we furnish a proof of (c), following Meyer [M5]. Write $S = S[M]$, so that $S^2 = [M, M]$. By Kunita-Watanabe (5.4),

$$\int_0^T |d[M, N]| \le 2^{1/2} \int_0^T (S_- + S)^{-1/2} S^{1/2} |d[M, N]|$$
$$\le 2^{1/2} \left(\int_0^T (S_- + S)^{-1} d(S^2)\right)^{1/2} \cdot \left(\int_0^T S d[N, N]\right)^{1/2}.$$

Now $S^2 = 2S_- * S + [S, S] = (S_- + S) * S$, or $(S_- + S)^{-1} d(S^2) = dS$. Concerning the second factor, integration by parts (5.10) gives $\int_0^T [N, N]_T \, dS = [N, N]_T S_T = \int_0^T [N, N]_- \, dS + \int_0^T S d[N, N]$, so that $\int_0^T S d[N, N] = \int_0^T ([N, N]_T - [N, N]_-) \, dS$. Let $T_t = \inf\{s: S_s > t\}$. Using the classical formula

$$\int_0^T X \, dS = \int_0^{S_T} X_{T_t} \, dt,$$

which one checks for intervals $X = [0, U]$ first and extends to all $X$ by a monotone class argument, and using Hölder's inequality we obtain

$$E \int_0^T |d[M, N]| \le 2^{1/2} (ES_T)^{1/2} \left(\int \int [0 \le t \le S_T]([N, N]_T - [N, N]_-)_{T_t} \, dP \, dt\right)^{1/2}.$$

Now $[0 \le t \le S_T] \in \mathscr{F}_{T_t}$, and taking conditional expectations with respect to $\mathscr{F}_{T_t}$, we see that the double integral on the right is majorized by $\| N^T \|_{BMO}^2 \cdot \int \int [0 \le t \le S_T] \, dt dP = \| N^T \|_{BMO} \cdot ES_T$. Hence (c) follows, with $C = \sqrt{2}$.

We turn to the one remaining inequality, the one on the right in (b). Note first that $s[M]$ exists, since $M$ is assumed to be a local $L^2$-integrator (5.4). Suppose then that $0 < p \le 2$ and $\sigma = \| s_T[M] \|_{L^p} < \infty$. We may assume that $s_T[M]$ is bounded and that both $[M, M]^T - \langle M, M\rangle^T$ and $M^T$ are global $L^1$-integrators. This restriction can be removed later by the observation that arbitrarily large[4] such times exist (3.9.3), and that both sides of the inequality depend isotonically on $T$. Now set $s = s[M^T]$ and $\bar{M} = s^{(p-2)/2} * M^T$. Then $\langle \bar{M}, \bar{M}\rangle = \langle M, M\rangle^{(p-2)/2} * \langle M; M\rangle^T$ (5.7.2). Integration by parts gives $\langle M, M\rangle^{p/2} = \langle M, M\rangle^{(p-2)/2} \cdot \langle MM\rangle = \langle M, M\rangle^{(p-2)/2} * \langle M, M\rangle + \langle M, M\rangle_- * \langle M, M\rangle^{(p-2)/2} \ge \langle M, M\rangle^{(p-2)/2} * \langle M, M\rangle = \langle M, \bar{M}\rangle$. Thus $\langle \bar{M}, \bar{M}\rangle_T \le \langle M, M\rangle_T^{p/2}$ and $E[\bar{M}, \bar{M}]_T \le \sigma^p$, and so $\gamma_T^2[\bar{M}] \le$

$(C_2^{(7.2a)})^{1/2}\sigma^{p/2}$. Now $M_T = (s^{(2-p)/2}*\bar{M})_T = s_T^{(2-p)/2}\cdot\bar{M}_T - (\bar{M}_-*s^{(2-p)/2})_T \leq 2\bar{M}_T^* s_T^{(2-p)/2}$. With Hölder's inequality and Doob's maximal inequality this results in $E|\bar{M}_T|^p \leq 4^p E\bar{M}_T^{*p} s_T^{p(2-p)/2} \leq 4^p \|\bar{M}_T^*\|_{L^2}^p\ \sigma^{p(2-p)/2} \leq 8^p \|\bar{M}_T\|_{L^2}^p \sigma^{p(2-p)/2} \leq 8^p(\gamma^2{}_T[\bar{M}])^p \sigma^{p(2-p)/2} \leq 8^p (C_2^{(7.2a)})^{p/2}\cdot\sigma^p = c\cdot\sigma^p$. The same estimate obtains for $X*M$, whenever $X \in \mathscr{R}_T$ with $|X| \leq 1$, and so, finally, $\gamma_T^p[M] \leq c_p\cdot\sigma$, as desired. The argument is adapted from Garsia [G2].

Almost every result on the structure of $\mathscr{T}^p$, $\mathscr{M}^p$ and the existence of solutions to stochastic differential equations, topics that concern us during the remainder of the paper, is a consequence of the following deep and powerful lemma.

7.3 Main Lemma. *Let $\vec{Z} = (Z^1, \ldots, Z^d)$ be a vector of right continuous adapted processes with left limits, and let $T$ be a stopping time such that $dZ^a$: $\mathscr{R}_T \to L^p(P)$ is bounded for $1 \leq a \leq d$. For $0 < p$ this reads*

$$\gamma_T^p[Z^a; P] < \infty, \qquad 1 \leq a \leq d.$$

*If $0 \leq p < q \leq 2$ there is a measure $P'$ equivalent to $P$ and with bounded derivative $g = dP'/dP$ such that the stopped processes $Z^{aT}$ are global $L^q(P')$-integrators*:

$$\gamma_T^q[Z^a; P'] < \infty, \qquad 1 \leq a \leq d.$$

*In the range $0 < p < q \leq 2$, there exist constants $C_{pqd}$ depending only on $p$, $q$, $d$ such that, for some choice of $P'$, the following inequalities hold*:

$$\text{(7.3a)} \qquad \gamma_T^q[Z^a; P'] \leq C_{pqd}\cdot\gamma_T^p[Z^a; P], \qquad 1 \leq a \leq d$$

$$\text{(7.3b)} \qquad 0 \leq g \leq C_{pqd}; \quad \textit{and with} \quad g' = g^{-1} = dP/dP'$$

$$\text{(7.3c)} \qquad \|g'\|_{L^{r/q}(P)} \leq C_{pqd}; \qquad 1/r + 1/q = 1/p,$$

$$\text{(7.3d)} \qquad \|g'\|_{L^{r/p}(P')} \leq C_{pqd}, \qquad r/p = q/(q-p),$$

$$\text{(7.3e)} \qquad \|f\|_{L^p(P)} \leq C_{pqd}\|f\|_{L^q(P')}, \qquad f \textit{ measurable.}$$

Proof. There is a deep factorization theorem by Maurey [M1] and Rosenthal [R1] stating the following. Let $\nu$ be a measure, $0 \leq p < q \leq 2$, and $U$: $L^\infty(\nu) \to L^p(P)$ a continuous linear map; then there is a function $\Phi \geq 0$ in $L^r(P)$, $1/r + 1/q = 1/p$, and a continuous linear map $V$: $L^\infty(\nu) \to L^q(P)$ such that $U$ factorizes as

$$U(X) = \Phi\cdot V(X), \qquad X \in \text{dom}(U) = L^\infty(\nu).$$

Moreover, if $p > 0$ there are constants $m_{pq}$ depending on $p$ and $q$ alone such that, for some choice of $\Phi$, $V$,

$$\|\Phi\|_{L^r(P)}\cdot\|V\|_q \leq m_{pq}\|U\|_p.$$

Here $\|U\|_p$ is the modulus of continuity of $U$, $\|U\|_p = \sup\{\|U(X)\|_{L^p(P)}: \|X\|_{\text{dom}(U)} \leq 1\}$, and $\|V\|_q$ similar.

A close look at the proof of this theorem reveals that it uses only the structure of the finite-dimensional subspaces of the domain of $U$, $L^\infty(\nu)$ in the quote above, and that it actually applies as well when the domain of $U$ is a space $C(K)$, $K$ compact. To apply it in the situation at hand, let $K$ denote the closure of the stochastic interval $[0, T]$ in the compactification $\mathscr{B}$ of $B$ (2.4). The vector lattice $\hat{\mathscr{R}}_T$ of functions on $K$ that is generated by the extensions via uniform continuity of the constants and the functions in $\mathscr{R}_T$ is dense in $C(K)$, due to the theorem of Stone-Weierstrass. The continuous linear map $\hat{X} \to \int XdZ^a$ from $\hat{\mathscr{R}}_T$ to $L^p(P)$ can be extended to a continuous linear map $U^a$: $C(K) \to L^p(P)$, and to this we apply the theorem. We restrict the map $V^a$: $C(K) \to L^q(P)$ provided by the theorem to $\mathscr{R}_T$ and arrive at the following situation: There are continuous linear maps $V^a$: $\mathscr{R}_T \to L^q(P)$ and functions $0 \leq \Phi^a \in L^r(P)$, $1 \leq a \leq d$, such that

$$\|\Phi^a\|_{L^r(P)} \leq 1 \quad \text{and} \quad dZ^a(X) = \Phi^a\cdot V^a(X), \qquad X \in \mathscr{R}_T;$$

and

$$\| V^a \|_q \le m_{pq} \gamma_T^p[Z^a; P] \qquad \text{if} \quad p > 0.$$

Now set $g = c(1 + \Phi^1 + \ldots + \Phi^d)^{-q}$, where the constant $c$ is chosen so that $P' = g \cdot P$ is a probability measure. Then if $X \in \mathscr{R}_T$ with $|X| \le 1$

$$(*) \qquad \left| \int dZ^a(X) \right|^q dP' \le c \int |(\Phi^a)^{-1} \cdot dZ^a(X)|^q dP$$
$$= c \| V^a(X) \|^q_{L^q(P)} \le c m^q_{pq} (\gamma^p_T[Z^a; P])^q$$

and so $\gamma^q_T[Z^a; P'] < \infty$. The case $p = 0$ is established, and we may turn to the inequalities (7.3) for $0 < p < q \le 2$. The first order of business is to estimate the constant $c$. Set $G = (1 + \Phi^1 + \ldots + \Phi^d)^{-q}$, so that

$$c^{-1} = \int G dP \le 1.$$

Let $s > 0$ be such that $P[G \ge s] \ge ½$ and $P[G \le s] \ge ½$. Then

$$c^{-1} \ge \int_{[G \ge s]} G dP \ge s/2,$$

and

$$s^{-r/q}/2 \le \int_{[G \le s]} G^{-r/q} dP \le \int (1 + \Phi^1 + \ldots + \Phi^d)^r dP \le (d + 1)2^d, \; s \ge ((d + 1)2^{d+1})^{-q/r}.$$

Hence $c$ is majorized by a universal constant depending only on $p$, $q$, $d$. This proves (b). Inequality (a) follows from this and $(*)$. As $g' = g^{-1} \le G^{-1}$, $\int g'^{r/q} dP \le (d + 1)2^d$ and (c) follows, and that in turn implies (d):

$$\int g'^{r/p} \, dP' = \int g'^{r/p} g'^{-1} \, dP = \int g'^{r/q} \, dP \quad \text{since} \quad r/p - 1 = r/q.$$

Finally, (e) follows from this and Hölder's inequality with conjugate exponents $q/p$ and $r/p$:

$$\int f^p \, dP = \int f^p g' \, dP' \le \left( \int f^q \, dP' \right)^{p/q} \cdot \left( \int g'^{r/p} dP' \right)^{p/r}.$$

Helpful as this lemma is, in reducing the $L^p$-theory to the $L^2$-theory—which is in general much easier to handle—, it leaves room for improvement. For one thing, one would like to find a measure $P'$ that works for all times $T$. Also, one might expect estimates similar to (7.3a–e), of $P'$ and $\gamma^q_T[Z; P']$ in terms of the size of $Z$ as an $L^o(P)$-integrator.

No numerical expression of this size has been given so far. We correct this omission now. First, for any $f \in L^o(P)$ and $0 < \lambda < 1$ set

$$j_\lambda[f] := \inf\{c : P[|f| > c] \leqq \lambda\}.$$

This is nothing but the nonincreasing rearrangement $f^*(\lambda)$ of $f$. Next set

$$\gamma^{o,\lambda}_T[Z] := \sup\left\{ j_\lambda\left[ \int XdZ \right] : X \in \mathscr{R}_T, \, |X| \leqq 1 \right\}, \qquad 0 < \lambda < 1.$$

The answer to the second question raised above is in the next statement. Its proof follows the lines of proof of (7.3a–e), making use of the quantitative version of the Maurey-Rosenthal factorization for $p = 0$ and $0 < q \leqq 2$.

7.3.1 COMPLEMENT. Let $Z$ be an $L^0(P)$-integrator and $0 < q \leqq 2$. For any stopping time $T$ there exists a measure $P'$ equivalent with $P$ such that $dP'/dP \in L^\infty$,

$$\gamma_T^q[\dot{Z}; P'] \leqq \inf_\lambda\{(1-\lambda)^{-1/q}(1 + B_\lambda \gamma_T^{o,\lambda^2}[Z; P])\},$$

and

$$j_\lambda[dP/dP'] \leqq (1 + B_\lambda \gamma_T^{o,\lambda^2}[Z; P])^q \qquad \text{for } 0 < \lambda < 1.$$

The constants $B_\lambda$ depend on $\lambda$ only. A similar statement holds for vectors of $L^0$-integrators.

Concerning the first question raised above, there is a positive answer as well. It is due to Dellacherie (oral communication; see also [D4]).

7.3.2 COMPLEMENT. Let $Z$ be an $L^0(P)$-integrator and $0 < q \leqq 2$. There is a measure $P'$ equivalent with $P$ such that $Z$ is an $L^q(P')$-integrator.

PROOF. For each integer instant $n$ let $P_n = g_n P$ be such that $\gamma_n^q[Z; P_n] < \infty$. Next find $a_n > 0$ such that $\sum P[g_n^{-1} > a_n] < \infty$ and then $c_n > 0$ so that $\sum a_n c_n < \infty$. By the Borel-Cantelli lemma, $\sum c_n g_n^{-1} < \infty$ a.s. Set $g = c(\sum c_n g_n^{-1})^{-1}$, where $c$ is chosen so that $P' = gP$ is a probability. Evidently, $P'$ meets the description.

Note that this argument destroys the control over the size of $Z$ as an $L^q(P')$-integrator offered by 7.3 and 7.3.1. Here is a first application of the lemma showing how it can be used to reduce the $L^p$-theory to the $L^2$-theory. Again, only the case $0 < p < 1$ is new (cf. e.g., [E1]).

7.4 THEOREM. *Let $Z$ be an $L^p$-integrator, $0 \le p < \infty$. Then so is its maximal process $Z^*$. In fact, there are constants $C_p$ depending on $p$ alone such that*

$$\|Z_T^*\|_{L^p(P)} \le C_p \cdot \gamma_T^p[Z] \tag{7.4}$$

*for any time $T$ and any $p \in (0, \infty)$.*

(For $p \geqq 1$ this is due to Meyer [M6].)

PROOF. For $p = 0$ this follows from 2.4, and 3.1. For $p > 1$, it is still easy: In the Doob decomposition $Z = \tilde{Z} + \langle Z \rangle$, both components are $L^p$-integrators (7.1). From Doob's maximal inequality,

$$\|\tilde{Z}_T^*\|_{L^p} \le p/(p-1)\,\|\tilde{Z}_T\|_{L^p} \le p/(p-1)\gamma_T^p[\tilde{Z}] \le p/(p-1)\cdot C_p^{(7.1)}\gamma_T^p[Z].$$

Also, by 3.3, $\|\langle Z\rangle_T^*\|_{L^p} \le \|\int_0^T |d\langle Z\rangle|\,\|_{L^p} = \gamma_T^p[\int |d\langle Z\rangle|] = \gamma_T^p[\langle Z\rangle] \le C_p^{(7.1)}\gamma_T^p[Z]$.
Now consider the remaining case $0 < p \le 1$. We choose $q = 2$ and $d = 1$, and let $P'$ be the measure provided by Lemma 7.3. Then

$$\begin{aligned}\|Z_T^*\|_{L^p(P)} &\le C_{p21}^{(7.3e)}\|Z_T^*\|_{L^2(P')} \\ &\le C\cdot C_2^{(7.4)}\gamma_T^2[Z; P'] \le C\cdot C_{p21}^{(7.3a)}\gamma_T^p[Z; P].\end{aligned}$$

7.5. Note that this together with previous results yields the martingale inequalities of Burkholder-Davis-Gundy [BDG] and Davis [D1]:

$$\|M_T^*\|_{L^p} \le C_p\|s_T[M]\|_{L^p}, \qquad 0 < p \le 2, \tag{7.5a}$$

with $C_p = C_p^{(7.4)}C_p^{(7.2b)}$. And with $C_p = C_p^{(7.4)}C_p^{(7.2a)}$

(7.5b) $$\| M_T^* \|_{L^p} \le C_p \| S_T[M] \|_{L^p}, \qquad 1 \le p < \infty.$$

The inequality (b) has the converse

(7.5c) $$\| S_T[M] \|_{L^p} \le c_p \| M_T^* \|_{L^p}, \qquad 1 \le p < \infty.$$

For $p > 1$, $c_p = C_p^{(5.3)} C_p^{(3.8)}$ will do. The case $p = 1$ is one of Davis' inequalities [D1]. A proof of (7.5c) at $p = 1$ can be found in [M5, page 105]. See also Garsia [G2].

The next application of Lemma 7.3 shows the long-promised identity between $L^0$-integrators and semimartingales. For a direct proof see [M5].

7.6 THEOREM. *A process $Z$ is an $L^0$-integrator if and only if it is a semimartingale; in fact, then there exists a decomposition $Z = M + V$ where $V$ is right continuous of finite variation with $V_0 = 0$ and the local martingale $M$ a local $L^2$-integrator.*

PROOF. Since we know that a semimartingale is an $L^0$-integrator (3.9.1), only the converse needs to be shown. Let $T$ be any a.s. finite stopping time, choose $q = 2$, and let $P' = gP$ be the measure provided by Lemma 7.3 (cf. 3.9.2), let $Z^T = M' + V'$ be the Doob decomposition of the $L^2(P')$-integrator $Z^T$ with respect to $P'$, and denote by $G$ the r.c. bounded $P$-martingale $E(g \mid \mathscr{F})$. Clearly

$$M' G = G_- * M' + M'_- * G + [M'; G]$$

is a square-integrable $P$-martingale. Now once $G_-$ reaches zero it stays there a.s. As $G_\infty = g > 0$ a.s., $G_-^{-1*}$ is finite at all finite times a.s. and $G_-^{-1} * M'G$ exists by 4.4. Hence

$$M' = G_-^{-1} * (G_- * M') = \{G_-^{-1} * (M'G) - (G_-^{-1} \cdot M') * G\} - G_-^{-1} * [M', G].$$

This exhibits $M'$ as the sum of a local $L^2(P)$-integrator martingale and the finite variation process $-G_-^{-1} * [M', G]$. Hence $Z^T = M' + V'$ is of the same description. Repeating the process with $Z$ replaced by $Z - Z^T$, etc., finishes the proof.

7.7 PROBLEM. Suppose $Z$ is an $L^p$-integrator, $0 < p < 1$. Can the decomposition $Z = M + V$ be chosen so that $M$ and $V$ are $L^p$-integrators?

A r.c. process $V$ of finite variation has a unique decomposition $V = V^c + V^{jpr} + V^{jti}$ into a continuous process $V^c$, a process $V^{jpr}$ all of whose jumps occur across the graphs of predictable stopping times, and a process $V^{jti}$ all of whose jumps occur at totally inaccessible stopping times. All three processes are of finite variation, the two pure jump-processes are right continuous and constant between consecutive jumps. The question arises whether there is a similar decomposition for the complementary class of $L^0$-integrators, the local martingales. And since the decomposition $Z = M + V$ of an $L^0$-integrator is not unique, the same question can be asked about the $L^0$-integrators themselves. Let us begin with the local martingales. If one wishes to end up with a unique decomposition, one evidently has to define what is to be meant by a pure jump martingale with predictable or totally inaccessible jumps. Since a martingale with a jump at a totally inaccessible time $T$ cannot be constant before reaching $T$, one cannot define a jump martingale as one constant between jumps. The right definition is this

7.8 DEFINITION. A local martingale $M$ is a *pure jump local martingale* if $\{M, M\} = 0$, equivalently $[M, M] = [M, M]^j$. It is said to have purely predictable or totally inaccessible jump times, respectively, if $[M, M]$ does.

7.9 THEOREM. *Let $M$ be a local martingale. There exists a unique decomposition $M = M^c + M^{jpr} + M^{jti}$ into the sum of three local martingales: $M^c$ is continuous with $M_0^c = 0$, $M^{jpr}$ is a local pure jump martingale with previsible jump times, and $M^{jti}$ is a local pure jump martingale with totally inaccessible jump times. Furthermore, $[M^c, M^c] =$*

*$\{M, M\}$, $[M^{jpr}, M^{jpr}] = [M, M]^{jpr}$ and $M^{jpr}$ is constant between jumps, and $[M^{jti}, M^{jti}] = [M, M]^{jti}$. There exist constants $C_p$, $0 < p < \infty$, depending on p alone such that for every stopping time T*

$$\gamma_T^p[M^c] \leq C_p \gamma_T^p[M];\ \gamma_T^p[M^{jpr}] \leq C_p \gamma_T^p[M];\ \gamma_T^p[M^{jti}] \leq C_p \gamma_T^p[M]. \tag{7.9}$$

PROOF. *Step* 1, *uniqueness.* Suppose $M^c + M^{jpr} + M^{jti} = \bar{M}^c + \bar{M}^{jpr} + \bar{M}^{jti}$ are two such decompositions. Then $\bar{M}^c - M^c = (M^{jpr} - \bar{M}^{jpr}) + (M^{jti} - \bar{M}^{jti}) = L^{jpr} - L^{jti}$. The square function of the left-hand side is continuous, while the one of the right-hand side is constant between jumps. By 7.2, both sides vanish. Then $L^{jpr}$ cannot jump across any predictable time, since $L^{jti}$ does not, and so is continuous, has square function zero, and vanishes as well.

*Step* 2, *taking care of* $M^{jpr}$. Let $S_1, S_2, \ldots$ be the predictable times at which $M$ jumps, and let $X \in \mathscr{P}$ be the union of their graphs. Then $M^{jpr} = X * M$ has square bracket $[M, M]^{jpr}$ and is a local pure jump martingale with predictable jumps. Incidentally, the middle inequality in (7.9) is satisfied with $C_p = 1$. We need to worry only about the local martingale $M - M^{jpr}$, whose only jumps can occur at totally inaccessible times. In other words, we may henceforth assume that the jumps of $M$ occur at the totally inaccessible times $T_1, T_2, \ldots$, whose graphs do not intersect.

*Step* 3, *the case* $p \geq 1$. Let $T$ be a stopping time with $\gamma_T^p[M] < \infty$, and let $J_n$ denote the process $\Delta_{T_n} M^T \cdot [T \wedge T_n, \infty)$. It is a process of finite total variation $|\Delta_{T \wedge T_n} M|$ and an $L^p$-integrator with $\gamma_T^p[J_n] \leq \|\Delta_{T \wedge T_n} M\|_{L^p}$. Let $J_n = L_n + V_n$ be its Doob decomposition. Clearly, $V_n$ is continuous (3.3). Hence $[L_n, L_m] = [J_n, J_m] = \delta_{nm} J_n^2$, and

$$[\textstyle\sum_N^\infty L_n, \sum_N^\infty L_n]_T = \sum_N^\infty (\Delta_{T_n} M^T)^2 \leq [M, M]_T^j.$$

If $N \to \infty$, $S_T[\sum_N^\infty L_n] \to 0$ in $L^p$ (5.3), thus $\sum_{n=1}^\infty L_n$ exists in $\mathscr{T}^p$. Actually, it is a martingale $M^{jti}$ in $\mathscr{M}^p$ with $[M^{jti}, M^{jti}] = [M, M]^{jT}$ and $\gamma_T^p[M^{jti}] \leq C_p^{(7.2a)} C_p^{(5.3)} \gamma_T^p[M]$. Since $\sum_{n=1}^\infty L_n$ converges uniformly (7.4), the difference $M^c = M - M^{jti}$ has continuous paths, and the statement follows, including the inequalities, at least up to time $T$. The global decomposition can be obtained using the uniqueness and the fact that $T$ can be chosen arbitrarily large.

*Step* 4, *the case* $0 < p < 1$. Let $T$, $T_n$ be as above. Choosing $q = 2$ and $d = 1$, we let $P' = gP$ be the measure provided by Lemma 7.3, and denote by $G$ the r.c. $P$-martingale $E(g\,|\,\mathscr{F})$. Let $M^T = M' + V'$ be the Doob decomposition of $M^T$ with respect to $P'$. Then $M' = M'^c + M'^{jti}$, and as in the proof of 7.6 we may write

$$M^T = \{G_-^{-1} * (M'^c G) - (G_-^{-1} M'^c_-) * G\} - G_-^{-1} * [M'^c, G] + M'^{jti} + V'.$$

The second term is continuous, so is the sum $M'^c$ of the first and second term, and then so is the first. It is a local $P$-martingale $M^c$ with $[M^c, M^c] = \{M', M'\} = \{M, M\}^T \leq [M, M]^T$. Hence $\gamma_T^p[M^c] \leq C_p^{(7.2b)} \| s_T[M^c] \|_{L^p(P)} \leq C \cdot C_{p21}^{(7.3e)} \| S_T[M] \|_{L^2(P')} \leq C C_p^{(5.3)} \gamma_T^2[M;\, P'] \leq C \cdot C_{p21}^{(7.3a)} \gamma_T^p[M]$. We define $M^{jti}$ as $M - M^c$, check the desired properties, and let $T \to \infty$. The case $p = 0$ is simpler and is left to the reader.

PROBLEM. For $1 \leq p < \infty$, $M^{jti}$ is the sum *in* $\mathscr{M}^p$ of martingales that are continuous except for one jump. Is the same true for all $p > 0$? For $1 \leq p < \infty$, the decomposition

$$Z = M + V = M^c + M^{jpr} + M^{jti} + V^c + V^{jpr} + V^{jti}$$

of an $L^p$-integrator can be made unique by requiring that the finite variation process $V$ be previsible with $V_0 = 0$; and then one has control over the size of the six constituents: $\gamma_T^p[M^c] \leq C_p \gamma_T^p[Z]$, etc. This is not so anymore if $p < 1$. However, some parts are unique: $M^c$ is the same for all decompositions, as Step 1 in the previous proof shows, and there is control. The sum $M^{jpr} + V^{jpr} = Z^{jpr}$ is unique, with $\gamma_T^p[Z^{jpr}] \leq C_p \gamma_T^p[Z]$, because this process equals $X * Z$, where $X$ is the union of the graphs of the predictable times where $Z$ jumps. And then $V^c + M^{jti} + V^{jti}$ is unique, controlled in size by $Z$.

7.10 THEOREM. *Let $\mathscr{D}$ denote the family of Doob processes $Z = \tilde{Z} + \langle Z \rangle$ such that both $\tilde{Z}$ and $\int |d\langle Z \rangle|$ are bounded. Then $\mathscr{D}$ is dense in $\mathscr{I}^p_\infty$ for $0 \le p < \infty$ and $\mathscr{D} \cap \mathscr{M}^p_\infty$ is dense in $\mathscr{M}^p_\infty$ for $1 \le p < \infty$.*

PROOF. *Case* 1, $p > 1$. To approximate $Z = \tilde{Z} + \langle Z \rangle \in \mathscr{I}^p_\infty$, let $S$ announce $T = \inf\{t: \int_0^t |d\langle Z \rangle| > n\}$. As $n \to \infty$, $S \to \infty$, and $\gamma^p_\infty[\langle Z \rangle - \langle Z \rangle^S] = G_p^{\langle Z \rangle}([S, \infty)) \to 0$. That is, the component $\langle Z \rangle$ can be approximated arbitrarily closely by $\langle Z \rangle^S \in \mathscr{D}$. To approximate $\tilde{Z}$, we find a sequence $y_n$ of bounded functions converging in $L^p$ to $\tilde{Z}_\infty$. The bounded r.c. martingales $Y_n = E(y_n | \mathscr{F})$ converge to $\tilde{Z}$ in $\mathscr{I}^p_\infty$ by 3.8.

*Case* 2, $0 \le p < 2$. Choose $T = \infty$, $d = 1$, $q = 2$, and let $P' = gP$ be the measure provided by Lemma 7.3, and $g' = g^{-1} = dP/dP'$. Given $\epsilon > 0$, let $Y \in \mathscr{D}[P']$ such that $\gamma^2_\infty[Z - Y; P'] < \epsilon$. Then $\gamma^p_\infty[Z - Y; P] \le C_{p21}^{(7.3e)} \cdot \epsilon$. Now $Y$ is a global $L^q(P)$-integrator for any $q < \infty$, because for $X \in \mathscr{R}$, $|X| \le 1$

$$\int \left| \int X dY \right|^q dP = \int |\ldots|^q g'\, dP' \le \left( \int \left| \int X\, dY \right|^{2q/p} dP' \right)^{p/2} \cdot \left( \int g'^{2/(2-p)}\, dP' \right)^{2-p/2}$$

$$\le C_{p21}^{(7.3d)} \cdot (\gamma_\infty^{2q/p}[Y])^q < \infty.$$

By Step 1, $Y$ can be approximated in the stronger norm of $\mathscr{I}^2_\infty[P]$ by an element of $\mathscr{D}[P]$. The triangle inequality yields the claim.

*Case* 3, *approximating martingales for* $1 \le p$. If $Y = \tilde{Y} + \langle Y \rangle \in \mathscr{D}$ approximates $M \in \mathscr{M}^p_\infty$ as close as $\epsilon > 0$, $1 \le p < \infty$, then $\gamma^p_\infty[M - \tilde{Y}] \le C_p^{(7.1)} \epsilon$.

PROBLEM. Is $\mathscr{D} \cap \mathscr{M}^p_\infty$ dense in $\mathscr{M}^p_\infty$ for $0 \le p < 1$?

The next proposition extends results established previously for $p, q \geqq 1$ [M6, E1, Y2].

7.11 PROPOSITION. *Let $Z$ be an $L^p$-integrator, $p, q > 0$ and $1/r = 1/p + 1/q$. Then $Z$ is an $L^r$-integrator, and every left continuous process $X$ with $\| |X|^*_\infty \|_{L^q} < \infty$ is $dZ$-$r$-integrable. In fact, there are constants $C_{pq}$ depending on p, q alone such that, for any stopping time T*

$$\gamma^r_T[X*Z] \le C_{pq} \cdot \|X^*_T\|_{L^q} \cdot \gamma^p_T[Z]. \tag{7.11}$$

PROOF. Since $\gamma^r_T[X*Z] = \{G^Z_r(|X|\cdot[0, T])\}^{1\vee 1/r} \le \{G^Z_r(X^*\cdot[0, T])\}^{1\vee 1/r} = \gamma^r_T[X^* * Z]$, we may assume that $X$ is increasing. The proof will be broken into several steps.

*Step* 1, *the case* $r \ge 1$. Let $Z = \tilde{Z} + \langle Z \rangle$ be the Doob decomposition of $Z$. Then

$$\gamma^r_T[X*\tilde{Z}] \le C_r^{(7.2\bar{a})} \| (\textstyle\int_0^T X^2 d[\tilde{Z}, \tilde{Z}])^{1/2} \|_{L^r} \le C \| X_T \cdot S_T[\tilde{Z}] \|_{L^r} \le C \| X_T \|_{L^q} \cdot C_p^{(5.3)} \cdot C_p^{(7.1)} \cdot \gamma^p_T[Z].$$

Also, $\gamma^r_T[X*\langle Z \rangle] \le \| X_T \cdot \int_0^T |d\langle Z \rangle| \|_{L^r} \le \|X_T\|_{L^q} \cdot C_p^{(7.1)} \cdot \gamma^p_T[Z]$. The two inequalities result in (7.11).

*Step* 2, *the case* $r \le 2$, $p \ge 2$. The second estimate above stays unchanged. The first is replaced by

$$\gamma^r_T[X * \tilde{Z}] = (G_r^{\tilde{Z}}(X \cdot [0, T]))^{1/r} \le C_r^{(7.2\bar{b})} \cdot \left\| \left( \int_0^T X^2\, d\langle \tilde{Z}, \tilde{Z} \rangle \right)^{1/2} \right\|_{L^r}$$

$$\le c \| X_T \cdot \langle \tilde{Z}; \tilde{Z} \rangle_T^{1/2} \|_{L^r} \le c \|X_T\|_{L^q} \cdot \left( \int \langle [\tilde{Z}; \tilde{Z}] \rangle_T^{p/2}\, dP \right)^{1/p}$$

$$\le c \|X_T\|_{L^q} C_{p/2}^{(7.1)} \| S_T[\tilde{Z}] \|_{L^p} \le c \|X_T\|_{L^q} \cdot C \cdot C_p^{(5.3)} C_p^{(7.1)} \gamma^p_T[Z].$$

*Step* 3, *the case* $p \le 2$. Note that the r.c. version $Y$ of $(X^T)^{q/p}$ is an $L^p$-integrator with $\gamma^p_T[Y] = \|X_T\|^{q/p}_{L^q}$. Let $P' = gP$ be the measure provided by Lemma 7.3, so that both $Y^T$ and $Z^T$ are $L^2(P')$-integrators, and let $Z^T = M + V$ be the Doob decomposition of $Z^T$ with respect to $P'$. For $F$ previsible with $|F| \le 1$, $g' = g^{-1}$,

$$\left\| \int_0^T FX\,dV \right\|_{L^r} \le \left( \int \left( \left( \int_0^T X|dV| \right) g'\,dP' \right)^r \right)^{1/r} \le \left( \int \left( X_T \cdot \int_0^T |dV| \right)^r \cdot g'\,dP' \right)^{1/r}$$

$$\le \left( \int X_T^{2r/p} \cdot \left( \int_0^T |dV| \right)^{2r/p} dP' \right)^{p/2r} \cdot \left( \int g'^{2/(2-p)}\,dP' \right)^{(2-p)/2r}$$

$$\le \left( \int X_T^{2r/(p-r)}\,dP' \right)^{(p-r)/2r} \cdot \left( \int \left( \int_0^T |dV| \right)^2 dP' \right)^{1/2} \cdot (C_{p22}^{(7.3d)})^{1/r}$$

$$= C\left( \int (X_T^{q/p})^2\,dP' \right)^{p/2q} \gamma_T^2[V; P'] \le C \cdot (\gamma_T^2[Y; P'])^{p/q}$$

$$\times C_2^{(7.1)} \gamma_T^2[Z; P'] \le C \cdot (C_{p22}^{(7.3a)})^{1+p/q} (\gamma_T^p[Y, P])^{p/q} \gamma_T^p[Z, P]$$

$$\le C \| X_T \|_{L^q(P)} \cdot \gamma_T^p[Z; P].$$

In a similar way,

$$\left\| \int_0^T FX\,dM \right\|_{L^r(P)} \le \left( \int \left| \int_0^T FX\,dM \right|^{2r/p} dP' \right)^{p/2r} \| g' \|_{L^{2/(2-q)}(P')}^{1/r}$$

$$\le (C_{p22}^{(7.3d)})^{1/r} \left\| \int_0^T FX\,dM \right\|_{L^{2r/p}(P')}$$

Now if $2r/p \ge 1$, equivalently $q \ge p$, we estimate the last expression as in Step 1 by

$$C_{2r/p}^{(7.2\bar{a})} \left\| \left( \int_0^T X^2\,d[M; M] \right)^{1/2} \right\|_{L^{2r/p}(P')}$$

$$\le C \cdot \| X_T \cdot S_T[M] \|_{L^{2r/p}(P')} \le C \cdot \| X_T^{q/p} \|_{L^2(P')}^{p/q} \cdot \| S_T[M] \|_{L^2(P')}$$

$$\le C \cdot (C_{p22}^{(7.3a)})^{1+p/q} \cdot \gamma_T^p[Y; P]^{p/q} \cdot \gamma_T^p[Z, P] \le C \cdot \| X_T \|_{L^q} \gamma_T^p[Z, P].$$

If, on the other hand, $2r/p < 1$ then we use the estimate of Step 2: Since $(2r/p)^{-1} = 2^{-1} + (2q/p)^{-1}$,

$$\left\| \int_0^T FX\,dM \right\|_{L^{2r/p}(P')} \le \left( \int (X_T \cdot \langle M \cdot M \rangle_T^{1/2})^{2r/p}\,dP' \right)^{p/2r}$$

$$\le \left( \int (X_T^{q/p})^2\,dP' \right)^{p/2q} \cdot \left( \int \langle MM \rangle_T\,dP' \right)^{1/2}$$

$$\le (\gamma_T^2[Y, P'])^{p/q} \cdot C_2^{(7.2a)} \cdot \gamma_T^2[M; P']$$

$$\le C(C_{p22}^{(7.3a)})^{1+p/q} (\gamma_T^p[Y; P])^{p/q} C_2^{(7.1)} \gamma_T^p[Z; P]$$

$$\le C \cdot \| X_T \|_{L^q(P)} \cdot \gamma_T^p[Z].$$

Adding these estimates results again in (7.11).

PROBLEM. If $1/p + 1/q = 1/r$, is every $Z \in \mathcal{T}_\infty^r$ the product of a $Z_1 \in \mathcal{T}_\infty^p$ with a $Z_2 \in \mathcal{T}_\infty^q$? (The answer is affirmative if $Z$ is continuous.) Is $\mathcal{T}_\infty^q$ the exact class of multipliers from $\mathcal{T}_\infty^p$ to $\mathcal{T}_\infty^r$? That every $Y \in \mathcal{T}_\infty^q$ is an $(\mathcal{T}_\infty^p, \mathcal{T}_\infty^r)$-multiplier follows from the next result. (For $\mathcal{T}_\infty^p$, $\mathcal{T}_\infty^p$)-multipliers see [Y1], for $p, q > 1$ [Y2].)

7.12 COROLLARY. *If $Y \in \mathcal{T}^q$ and $Z \in \mathcal{T}^p$ then $Y \cdot Z \in \mathcal{T}^r$, $1/r = 1/p + 1/q$; there are constants $C_{pq}$ such that for any stopping time $T$,*

$$\gamma_T^r[YZ] \le C_{pq} \cdot \gamma_T^p[Z] \cdot \gamma_T^q[Y]. \tag{7.12}$$

PROOF. Write $Y \cdot Z = Y_0 Z_0 + Y_- * Z + Z_- * Y + [Y; Z]$. Then $\| Y_0; Z_0 \|_{L^r} \le \gamma_T^p[Z] \cdot \gamma_T^q[Y]$. The second and third terms are controlled in a similar fashion by the last result and 7.4. The last term is controlled by Corollary 5.4 and 5.3.

7.13 COROLLARY. *If $Z$ is an $L^p$-integrator and $q > 1$ then $Z^q$ is an $L^{p/q}$-integrator, and there are constants $C_{pq}$ such that for any time $T$*

$$\gamma_T^{p/q}[Z^q] \le C_{pq} \gamma_T^p[Z]. \tag{7.13}$$

PROOF. Write $Z^q = Z_0^q + qZ_-^{q-1} * Z$ + increasing (5.1). The first term is an $L^{p/q}$-intregator. As $(Z_-^{q-1})_T^* \in L^{p/(q-1)}$ (7.4) and $p^{-1} + (p/(q-1))^{-1} = (p/q)^{-1}$, the second term belongs to $\mathcal{I}^{p/q}$ by 7.11. To estimate the increasing term, simply integrate.

7.14. As another application of the factorization Lemma 7.3 we show that the integral $X * Z$ can be evaluated *pathwise* when the integrand $X$ is left continuous and has no oscillatory discontinuities. This fact has important applications to stochastic differential equations and Markov processes.

THEOREM. *Let $Z$ be an $L^0$-integrator and $X$ a* r.c.l.l. *process with* $|X|_t^* < \infty$ a.s. *for all $t > 0$. Then for almost every $\omega \in \Omega$, $(X_- * Z)_s(\omega)$ is, on every bounded interval $0 \le s \le t$, the uniform limit of the* r.c.l.l. *processes*

$$Y_s^n(\omega) = X_{-0}(\omega) Z_0(\omega) + \textstyle\sum_{i=1}^{\infty} X_{s\wedge T_i^n}(\omega)(Z_{s\wedge T_{i+1}^n}(\omega) - Z_{s\wedge T_i^n}(\omega))$$

*where $T_0^n = 0$, $T_{i+1}^n = \inf\{t > T_i^n : |X_t - X_{T_i^n}| \ge 2^{-n}\}$. In fact*

$$\textstyle\sum_{n\ge1} | Y^{n+1} - Y^n |_t^* < \infty \qquad \text{a.s. } \textit{for all} \quad t > 0,$$

*and if $Z$ is an $L^p$-integrator, $0 < p < \infty$, then*

$$\textstyle\sum_{n\ge1} \gamma_t^p[Y^{n+1} - Y^n] < \infty \qquad \textit{and} \qquad \sum_{n\ge1} \| | Y^{n+1} - Y^n |_t^* \|_{L^p} < \infty \qquad \textit{for all} \quad t.$$

PROOF. Fix $t \ge 0$. As $X$ has no oscillatory discontinuities and $|X|_t^* < \infty$ a.s., $T_i^n \to \infty$ a.s. as $i \to \infty$. Let[1]

$$X^n = X_{-0} \cdot [0] + \textstyle\sum_i X_{T_i^n} \cdot (T_i^n, T_{i+1}^n].$$

Then $|X_- - X^n| \le 2^{-n}$ uniformly on $B = \Omega \times [0, \infty)$ and $Y^n = X^n * Z$. Now let $P'$ be a measure equivalent to $P$ and such that $Z^t$ is a global $L^1(P')$-integrator. Since

$$\textstyle\sum_n G_1^{Z;P'}((X^{n+1} - X^n) \cdot [0, t]) \le \sum_n G_1^{Z;P'}(2^{-n+2} \cdot [0, t]) = 4 \sum_n 2^{-n} \gamma_t^1[Z, P'] < \infty,$$

(2.7) and (7.4) apply and show that $|X_- * Z - Y^n|_t^* = |(X_- - X^n) * Z^t|_\infty^* \to 0$ a.s. In fact

$$\| \textstyle\sum | Y^{n+1} - Y^n |_t^* \|_{L^1(P')} < \infty.$$

The second statement follows from 7.3.e.

**8. Stochastic differential equations.** Let $Z = (Z^a)^{a=1,\cdots,d}$ be a $d$-vector of $L^0$-integrators with $Z_0 = 0$ and $(F_a^b)_{a=1,\cdots,d}^{b=1,\cdots,e}$ an $e \times d$-matrix of functions on $\mathcal{R}^e$. The problem is to find an $e$-vector $X = (X^b)^{b=1,\cdots,e}$ of $L^0$-integrators satisfying the initial conditions $X_0^b = C^b \in \mathcal{L}^0(F_0; P)$, and $dX^b = \sum_{a=1}^d F_a^b(X_-^1, \cdots, X_-^e)\, dZ^a$. In matrix notation this is

$$X = C + FX_- * Z. \tag{8.1a}$$

There is a vast literature concerning (8.1a) in the classical case that some of the components of the driving force $Z$ are sure and theothers Wiener processes, e.g., [A1, F1-2, GS2, I1, M3], to name but a very few. The existence and uniqueness of solutions of (8.1a) with general semimartingales driving was essentially settled by Doleans-Dade [D4], after work in this direction by Kazamaki [K1]. See also the concurrent work by Protter [P4, 5, 7]. Emery [E1] and Protter [P7] have investigated the dependence of the solution on $C$, $F$, and $Z$, and their Markoff character [P6]. Meyer [M6] and Yor [Y1, 2, 4, 5] have added to the scope of the problem and developed the proper topologies to be used in this context. See [DM] for a survey.

To describe these topologies, we paraphrase two recent results. (1) Suppose $Z$ is an $L^0$-integrator and $|C_m = C|^*_- \to 0$ locally in $L^p$. Then the solutions $X_m$, $X$ corresponding to these initial conditions do the same: $|X_m - X|^*_- \to 0$ locally in $p$-mean. (This means, of course, that $|X_m - X|^*_{-T} \to 0$ in $L^p$ for arbitrarily large[4] stopping times $T$; it has been termed weak-local convergence by Protter [P7]. It is natural to consider this mode of convergence as $Z$ and then the $X_m$ might have unexpected huge jumps at any time $T$.) A similar statement governs the dependence on $F$ ($1 \le p$; see [E1]). (2) If $Z_m \to Z$ weak-locally in $\mathscr{I}^p$ then $X_m \to X$ in the previous sense ($1 \le p$; see [P2]). We shall not try to generalize the last quoted result to $0 < p < 1$, since the topology of $\mathscr{I}^p$ seems too strong.

The purpose of this section is to review the general theory with emphasis on, and generalizing it to, the case $0 \le p \le 2$, and to add the following facts. If $C_m \to C$ in the previous sense then $X_m \to X$ weak-locally in $\mathscr{I}^p$; and similarly for $F$. There are a priori estimates on the size of the solution and the rate of convergence of the iterative method. The latter estimate is good enough to show that the solution can actually be evaluated *pathwise*. This fact has an interesting application in the theory of Markoff processes.

All the results quoted have been obtained by Picard's iterative method. For this to work, $F$ has to be Lipschitz. Meyer [E1] has observed, though, that the correspondence $X \to FX$ need not be of functional type; all that is needed is that

(8.1b) For every $e$-vector $X$ of r.c.l.l. adapted processes $FX$ is an $e \times d$-matrix of r.c.l.l. adapted processes;

(8.1c) The left-continuous version $FX_-$ of $FX$ depends progressively on $X_-$; i.e., $X^T_- = Y^T_-$ implies $(FX_-)^T = (FY_-)^T$ a.s., for all stopping times $T$;

(8.1d) $$|FX - FY|^* \le K \cdot |X - Y|^* \text{ for some constant } K.$$

Here as henceforth $|X|$, $|FX|$, etc. stand for $\sum_b |X^b|$, $\sum_{a,b} |(FX)^b_a|$, etc. Similarly, $\gamma^p_T[X] = \sum \gamma^p_T[X^a]$ and $[Z; Z] = \sum [Z^a; Z^a]$, etc. When the system is written in the form (8.1a) it makes sense even when the vector $C$ is not a constant process. We agree on

(8.1e) $Z$ is a $d$-vector of $L^p$-integrators and the components of $C$ and $F0$ are $L^q$-integrators, $0 \le p, q < \infty$.

As usual, the system (8.1) will be solved by starting with an $L^q$-integrator $X^0$—arbitrary, e.g., produced by a good guess—and showing that the *iterates*

$$X^{n+1} := C + FX^n_- * Z$$

converge to a solution $X$. To have a quantitative expression for the mode of convergence it is convenient to use a notion of Emery's [E1]: for any $L^0$-integrator $Y$ and stopping time $S$ set

$$Y^{-S} := [0, S)Y + Y_{-S}[S, \infty) \qquad \text{and} \qquad \gamma^q_{-S}[Y] := \gamma^q[Y^{-S}].$$

The r.c.l.l. process $Y^{-S}$ is an $L^q$-integrator when $Y$ *is:*

(8.1-) $$\gamma^q_{-S}[Y] \leqq C_q \gamma^q_S[Y]$$

for universal constants $C_q$. In fact, $\gamma^q_{-S}[Y]^{1\wedge q} \leqq \gamma^q_S[Y]^{1\wedge q} + \|\Delta_S Y\|^{1\wedge q}_{L^q} \le (1 + (2C^{(7.4)}_q)^{1\wedge q})\gamma^q_S[Y]^{1\wedge q}$.

8.2. THEOREM. *Suppose $p = q = 0$. There is a unique solution of* (8.1) *in $\mathscr{I}^0$. It is* a.s. *the uniform limit on compact intervals of the iterates $X^n$. In fact, for any* a.s. *finite stopping time $T$*

(8.2$a$) $$\textstyle\sum_{n=0}^{\infty} |X^{n+1} - X^n|_T^* < \infty \qquad \text{a.s.}$$

*Moreover, when C and F0 are local $L^q$-integrators, $0 < q < \infty$, then for arbitrarily large stopping times T both $L := \| |\Delta Z|_{-T}^* \|_{L^\infty} < \infty$ and*

(8.2$b$) $$\gamma_{-T}^q[X - X^1] \leqq \left(\textstyle\sum_{n=1}^{\infty} \gamma_{-T}^q[X^{n+1} - X^n]\right)^{1 \vee 1/q} \leqq e^{c \cdot c_T} \cdot \| |X^1 - X^0|_{-T}^* \|_{L^q} < \infty, \qquad q > 0.$$

*The constant c is of the form $\alpha + \beta KL$ with $\alpha$ and $\beta$ depending only on q, d, e; while $c_T$ depends on q, d, e, and the product KZ.*

The point of the estimate (8.2b) is this. If after running the iterative scheme for a while the change from $X^n$ to $X^{n+1}$ is small when measured by $\| |X^{n+1} - X^n|_{-T}^* \|_{L^q}$, then $X^{n+1}$ is close to the solution, with the deviation in $\mathscr{I}^q$, $\gamma_{-T}^q[X - X^{n+1}]$, bounded by $e^{c \cdot c_T} \| |X^{n+1} - X^n|_{-T}^* \|_{L^q}$. Below, we shall give a bound for $c_T$ in terms of the size of $Z$ in $\mathscr{I}^p$.

PROOF. We assume $d = e = 1$. In higher dimension the proof is practically identical, a bit more cumbersome without offering any new insight.

(8.2a) follows easily from (8.2b): $C$ and $F0$ are $L^1(P')$-integrators for some equivalent measure $P'$ (7.3.2). By (8.2b), $E' \sum |X^{n+1} - X^n|_{-T}^* < \infty$ for arbitrarily large times $T$. This clearly implies (8.2a). By 4.4 and the DCT, $X := X^0 + \sum_{n=0}^{\infty} (X^{n+1} - X^n)$ solves (8.1) and is an $L^0$-integrator. We defer the uniqueness.

Let us prove (8.2b) next in the case $q \geqq 1$. There are arbitrarily large times $U$ so that $C^U$ and $F0^U$ are global $L^q$-integrators and the jumps of $Z$ are bounded on $[0, U)$, $|\Delta Z|_{-U}^* \leqq L = \text{const} < \infty$, say. It will suffice to produce stopping times $T \leqq U$ arbitrarily close to $U$ and satisfying (8.2b). It is thus permissible to assume that $\Delta_U Z = 0$, replacing $Z$ with $Z^{-U}$, if necessary; this will not alter the iterates on $[0, U)$ (4.3).

To start with, let $Z = N + A$ be a decomposition into a local $L^2$-integrator martingale $N$ and a process $A$ of finite variation (7.6). With $\lambda = [4KC_q^{(7.4)}(C_q^{(7.2a)} + 1)]^{-1}$ set

$$J_t = \textstyle\sum \{\Delta_s N \colon s \leqq t, |\Delta_s N| > \lambda/2\} \leqq 2^{-1}[N, N]_t.$$

$J$ is a local $L^1$-integrator (5.3), and so $N - J$ has a Doob decomposition $N - J = M + B$ (3.9.4). As $B$ is previsible and $|\Delta(N - J)| \leqq \lambda/2$, we have $|\Delta M| \leqq \lambda$. We write $V = A + J + B$, so that $Z = M + V$, and we abbreviate: $W = \int |dV|$. Next, we define a partition of $[0, U)$ by $T_0 = 0$ and

$$T_{k+1} = U \wedge \inf\{t > T_k : [M, M]_t > \lambda^2 + [M, M]_{T_k} \quad \text{or} \quad W_t > \lambda + W_{T_k}\}.$$

Clearly $T_k \uparrow U$ a.s. The fact that $|\Delta M| \leqq \lambda$ permits an estimate of the effect of the map $Y \to Y' := Y_- * Z$ across any one of the open intervals $(R, S) = (T_{k-1}, T_k)$: since $Y'^{-S} = Y_- * Z^{-S}$ (4.3),

$$\begin{aligned}
\gamma_{-S}^q[Y'] &\leqq \gamma_R^q[Y'] + G_q^{Z^{-S}}((R, S]Y_-) \\
&\leqq \gamma_R^q[Y'] + G_q^{M^S}((R, S]Y_-) + G_q^{M^S - M^{-S}}((R, S]Y_-) + G_q^{V^{-S}}((R, S]Y_-) \\
&\leqq \gamma_R^q[Y'] + C_q^{(7.2a)} \left\| \left( \int_{(R}^{S]} Y_-^2 \, d[M, M] \right)^{1/2} \right\|_{L^q} \qquad (1) \\
&\quad + \lambda \| Y_{-S} \|_{L^q} + \left\| \int_{(R}^{S)} |Y_-| \, dW \right\|_{L^q} \\
&\leqq \gamma_R^q[Y'] + C^{(1)} \| |Y|_{-S}^* \|_{L^q} \leq \gamma_R^q[Y'] + \gamma_{-S}^q[Y]/2K,
\end{aligned}$$

with $C^{(1)} = 2\lambda(C_q^{(7.2a)} + 1)$. Thus, with $C^{(2)} = C^{(1)} + L$,

(2) $$\gamma_S^q[Y'] \leqq \gamma_{-S}^q[Y'] + \| \Delta_S Y' \|_{L^q} \leqq \gamma_R^q[Y'] + C^{(2)} \| |Y|_{-S}^* \|_{L^q},$$

and

(3) $$\gamma^q_{T_k}[Y'] \leqq kC^{(2)} \| |Y|^*_{-T_k} \|_{L^q}.$$

This shows in particular that the stopped iterates $(X^n)^{T_k}$ are global $L^q$-integrators, provided the initial one $X^0$ is. For instance,

(4) $$\gamma^q_{T_k}[X^1] \leqq \gamma^q_{T_k}[C] + kKC^{(2)} \| |X^0|^*_{-T_k} \|_{L^q},$$

and this is finite: since $\Delta^0 = (C - X^0) + (FX^0 - F0)_- * Z + F0_- * Z$, (3) yields

(5) $$\| |\Delta^0|^*_{-T_k} \|_{L^q} \leqq \| |C - X^0|^*_{-T_k} \|_{L^q} + kKC^{(2)} \| |X^0|^*_{-T_k} \|_{L^q} + kC^{(2)} \| |F0|^*_{-T_k} \|_{L^q} < \infty.$$

With these inequalities, the stage is set for the proof proper of (8.2b). Applying (1) to $\Delta^n := X^{n+1} - X^n = (FX^n - FX^{n-1})_- * Z$, we obtain

$$\gamma^q_{-S}[\Delta^n] \leqq \gamma^q_R[\Delta^n] + 1/2\gamma^q_{-S}[\Delta^{n-1}], \qquad \text{for} \quad n \geqq 1.$$

An iteration stopping at $n = 1$ with the use of (1), and a summation yield, in turn,

$$\gamma^q_{-S}[\Delta^n] \leqq \textstyle\sum_{i=0}^{n-1} 2^{-i}\gamma^q_R[\Delta^{n-1}] + 2^{-n}KC^{(1)} \| |\Delta^0|^*_{-S} \|_{L^q}$$

and

(6) $$\textstyle\sum_{n=1}^{\infty} \gamma^q_{-S}[\Delta^n] \leqq 2 \sum_{n=1}^{\infty} \gamma^q_R[\Delta^n] + KC^{(1)} \| |\Delta^0|^*_{-S} \|_{L^q}.$$

Now

$$\gamma^q_S[\Delta^n] \leqq \gamma^q_{-S}[\Delta^n] + \| \Delta_S \Delta^n \|_{L^q} \leqq \gamma^q_{-S}[\Delta^n] + KL \| |\Delta^{n-1}|^*_{-S} \|_{L^q},$$

and therefore, with $c_1 = 2KL + KC^{(1)}$ and $c_2 = 2(1 + KLC_q^{(7.4)}) \geqq c_1$,

$$\textstyle\sum_{n=1}^{\infty} \gamma^q_S[\Delta^n] \leqq c_2 \sum_{n=1}^{\infty} \gamma^q_R[\Delta^n] + c_1 \| |\Delta^0|^*_{-S} \|_{L^q}.$$

Recalling that $R = T_{k-1}$ and $S = T_k$ we see by induction in $k$ that

$$\gamma^q_{T_k}[X - X^1] \leqq \textstyle\sum_{n=1}^{\infty} \gamma^q_{T_k}[\Delta^n] \leqq c_1 \cdot e^{c_2 \cdot k} \cdot \| |\Delta^0|^*_{-T_k} \|_{L^q} \leqq e^{\bar{c}k} \cdot \| |\Delta^0|^*_{-T_k} \|_{L^q},$$

where $\bar{c} = c_2 + 1$. With $c = \bar{c} \cdot \ln C_q^{(8.1\text{-})}$, we arrive finally at

(7) $$\gamma^q_{-T_k}[X - X^1] \leqq \textstyle\sum_{n=1}^{\infty} \gamma^q_{-T_k}[X^{n+1} - X^n] \leqq e^{ck} \| |\Delta^0|^*_{-T_k} \|_{L^q},$$

which is finite by (5). For $q \geqq 1$, (8.2b) is established with $T = T_k$ and $c_T = k$. If $0 < q < 1$, one notes that $M$ is a local $L^2$-integrator and replaces $[M, M]$ by $\langle M, M \rangle$ in the definition of the $T_k$, using (7.2b) instead of (7.2a) in the estimates. The details are left to the reader.

Only the uniqueness assertion of the theorem remains to be established. For this and later use we put together (4), (5), (7), and (8.1-), choosing $X^0 = C$ and shifting $\gamma^q_{-T_k}[X^1]$ to the right-hand side. We get, with $c' = c + 1$,

(8.2c) $$\gamma^q_{-T_k}[X] \leqq e^{c'k} \{\gamma^q_{-T_k}[C] + K^{-1} \| |F0|^*_{-T_k} \|_{L^q}\}, \qquad q > 0.$$

To establish the uniqueness from this, we employ a trick of Emery's [E1]. Let $(\bar{C}, \bar{F}, Z)$ be another system (8.1), driven by the same $L^0$-integrator $Z$, and let $\bar{X}$ be any solution of it. Then $Y = X - \bar{X}$ solves the equation $Y = D + GY_- * Z$, where $D = C - \bar{C} + (FX - \bar{F}X)_- * Z$ and $GY = \bar{F}X - \bar{F}(X - Y)$. Since $G0 = 0$ and $Y$ equals its iterates, an application of (8.2c) and (3) gives

(8.2d) $$\gamma^q_{-T_k}[X - \bar{X}] \leqq e^{c''k} \{\gamma^q_{-T_k}[C - \bar{C}] + \| |FX - \bar{F}X|^*_{-T_k} \|_{L^q}\},$$

where $c'' = (c' + 1)(1 \vee 2^{1/q-1})$. In particular, when $C = \bar{C}$ and $F = \bar{F}$ then $X - \bar{X}$ is evanescent on each of the intervals $[0, T_k)$; the solution is unique. Even when $C$ and $F0$ are merely $L^0$-integrators this argument works. We see to it that they become $L^1$-integrators for some new measure and conclude that $|X - \bar{X}|^*_{-U} = 0$ a.s. for arbitrarily large times $U$.

8.3. Information about the size of $Z$ as an $L^p$-integrator, $0 < p$, can be turned into an estimate of the constant $c_T$:

COROLLARY. *For any* $\epsilon > 0$ *and instant* $u$ *there exists a stopping time* $T \leqq u$ *with* $P[T < u] < \epsilon$ *such that*

$$\gamma^q_{-T}[X - X^1] \leqq \| |X^1 - X^0|^*_{-T}\|_{L^q} \cdot E_{pq}, \tag{8.3b}$$

$$\gamma^q_{-T}[X] \leqq \{\gamma^q_{-T}[C] + K^{-1}\| |F0|^*_{-T}\|_{L^q}\} \cdot E_{pq}, \tag{8.3c}$$

*and*

$$\gamma^q_{-T}[X - \bar{X}] \leqq \{\gamma^q_{-T}[C - \bar{C}] + \| |FX - \bar{F}X|^*_{-T}\|_{L^q}\} \cdot E_{pq} \tag{8.3d}$$

*whenever* $\bar{X}$ *is the solution of another system* $Y = \bar{C} + \bar{F}Y_- * Z$ *driven by* $Z$. *The* $E_{pq}$ *are functions of the form*

$$E_{pq}(KZ; u; \epsilon) = \exp[(\alpha + \beta K\gamma^p_u[Z])^3 \cdot \epsilon^{-3/p}]$$

*with* $\alpha$ *and* $\beta$ *depending only on* $p$, $q$, $d$, $e$.

PROOF. Again we assume $d = e = 1$, and we start with the case $p \geqq 1$. Set $L = (\epsilon/2)^{-1/p}C_p^{(7.4)}\gamma^p_u[Z]$, and take for the stopping time $U$ of the previous proof the time $U = u \wedge \inf\{t: |Z_t| > L\}$. Then $P[U < u] \leqq \epsilon/2$.

For the decomposition $Z = N + A$ we choose the Doob decomposition. This gives us control over $\gamma^p_T[N]$ and $\gamma^p_T[A]$ in terms of $\gamma^p_T[Z]$, for any stopping time $T$ (7.1). Since $[M, M] \leqq 2[N, N]$, we get control of $\gamma^p_T[M]$ (7.2a) and then of $\gamma^p_T[V] = \| W_T\|_{L^p}$. After all is said and done, we end up with an inequality

$$\| [M, M]^{1/2}_T\|_{L^p} \vee \| W_T\|_{L^p} \leqq \alpha'\gamma^p_T[Z],$$

where $\alpha'$ is a constant depending only on $p$ through the constants of 3.5, 7.1, and 7.2. The time $T$ sought will be one of the $T_k$. Now, whatever $k$, we have $[M, M]_{T_k} \geqq \lambda^2k/2$ or $W_{T_k} \geqq \lambda k/2$ at any point of $[T_k < U]$, so that

$$P[T_k < U]^{1/p} \leqq \{(\lambda\sqrt{k/2})^{-1} + (\lambda k/2)^{-1}\}\alpha'\gamma^p_u[Z] \leqq 4\lambda^{-1}k^{-1/2}\alpha'\gamma^p_u[Z].$$

Choosing $T = T_k$ with $k = (4\lambda^{-1}\alpha'\gamma^p_u[Z]2^{1/p}\epsilon^{-1/p})^2$ we have $P[T < U] \leqq \epsilon/2$ and thus $P[T < u] \leqq \epsilon$. (Note that we can always decrease $\epsilon$ a bit, thereby improving the estimate and turning the expression for $k$ into an integer.) Now note that $k$ and the constants $c$, $c'$, $c''$ of 8.2 are of the form $(\alpha + \beta K\gamma^p_u[Z])\epsilon^{-1/p}$, and insert $k$ and $L$ as specified above into (8.2b–d). The inequalities (8.3b–d) result.

The case $0 < p < 1$ is reduced to the previous one by a change of measure (7.3), familiar by now.

8.4. REMARKS. (1) Suppose the correspondence $X \to FX$ is *pathwise*; that is to say, a.s. $X_.(\omega) = Y_.(\omega)$ implies $FX_.(\omega) = FY_.(\omega)$. This happens, in particular, in the frequent case that $F$ is of *functional type*: $FX = f \circ X$ for some Lipschitz function $f: \mathbf{R}^e \to \mathbf{R}^d \times \mathbf{R}^e$. Then each of the iterates $X^n$ can be evaluated pathwise a.s. by 7.14, and then so can the solution $X$, by (8.2a). Let us go into this in detail. We start with $X^0 = C$. Whenever the algorithm provided by 7.14 and used to compute $C + FX^n_- * Z$ converges uniformly on each bounded interval, we set $X^{n+1}$ equal to its limit. Else we set $X^{n+1} = 0$. Wherever $C + \sum(X^{n+1} - X^n)$ converges uniformly on compacta, we set $X$ equal to that limit; else we set $X = 0$. Evidently, we obtain every path $X_.(\omega)$ of the solution via progressively measurable operations from the paths $C_.(\omega)$ and $Z_.(\omega)$. The exceptional set, where convergence does not take place, is negligible for every measure with respect to which $C$ and $Z$ are $L^0$-integrators. We shall give an application of this to Markoff processes in 8.7 below.

(2) Suppose $FX = f \circ X$ is of functional type[8] with $f$ merely locally Lipschitz. For every

[8] A restriction for convenience's sake; it can be replaced by adopting a proper notion of a local Lipschitz correspondence $X \to FX$.

initial value $C$ there is then a stopping time $\zeta = \zeta_C$ and a solution $X = X_C$ of $X = C + f(X)_- * Z$ on $[0, \zeta)$ such that $\lim\{\|X_t\| : t \uparrow \zeta\} = \infty$ a.s. on $[\zeta < \infty]$. $\zeta$ is called the life time or time of explosion of $X$. Both $X$ and $\zeta$ are unique.

It will suffice to give a sketch of the proof. For each $n$ let $f^n$ be a global Lipschitz function that coincides with $f$ on the ball $B_n$ in $\mathbf{R}^e$ that has radius $n$ and is centered at the origin, and let $X^n$ be the solution of $X = C + f^n(X_-) * Z$. By uniqueness, $X^m = X^n$ up to the first time $T_m$ at which $X^m$ leaves $B_m$, for $m < n$. Then $\zeta = \sup T_m$ and $X = \lim X_m$ satisfy the description of the statement. A problem arises: How do $\zeta$ and $X$ depend on $C$ and $f$?

(3) Inequality (8.3d) has the version

$$(8.4\text{d}*) \qquad \| |X - \bar{X}|^*_{-T} \|_{L^q} \leqq \{ \| |C - \bar{C}|^*_{-T} \|_{L^q} + \| |FX - \bar{F}X|^*_{-T} \|_{L^q} \} E_{pq}.$$

This does not permit us to conclude that the solutions $X^m$ of $Y = C^m + FY_- * Z$ converge almost surely pathwise to the solution $X$ of $Y = C + FY_- * Z$ when the paths of $C^m$ converge uniformly to the paths of $C$, $\| |C - C^m|^*_\infty \|_{L^\infty} \to 0$, not even when the $C^m$ are constant on $B = \Omega \times \mathbf{R}_+$. Is it true, nevertheless? Clearly, if $\sum |C^{m+1} - C^m|^*_t < \infty$ a.s. for all $t$ then $\sum |X^{m+1} - X^m|^*_t < \infty$ a.s. for all $t$ as well: with a suitable change of measure we arrive at $\sum \| |C^{m+1} - C^m|^*_{-T} \|_{L^1(P')} < \infty$ and apply (8.4d*).

(4) Assume again that $FX = f(X)$ is of functional type, $f$ Lipschitz and of class $C^{k+1}$. Suppose $C$ depends $k$-times differentiably on a parameter $u \in \mathbf{R}^n$, in the sense of Frechet and with respect to the metrics $d_{-T}(X, Y) = \| |X - Y|^*_{-T} \|_{L^q}$, $T \in \mathcal{T}$. Then the solutions $X^u = C^u + FX^u_- * Z$ depend $k$-times differentiably on $u$, with respect to the metrics $d_{-T}$, $T$ arbitrarily large. In fact, the $k$th derivative $D^k X^u$ can be computed pathwise from $(C^u, \ldots, D^k C^u; Z)$ as the solution of a differential equation driven by $Z$. As in (3), this does *not* imply that the paths $X^u_.(\omega)$ depend differentiably on $C^u_.(\omega)$ in some suitable space of functions on $\mathbf{R}_+$. Problem: Do they ever?

NOTE ADDED IN PROOF. P.-A. Meyer has recently given the answer: always. The proof consists in the application of an old lemma of Kolmogoroff's to (8.4d*). The details will appear in an article on stochastic flows on manifolds by P. A. Meyer, forthcoming Séminaire de Probabilités de l'Université de Strasbourg, *Lecture Notes in Mathematics*. Springer, Berlin.

8.5. THE EULER-PEANO METHOD. The fact that the solution $X$ of (8.1) can be evaluated pathwise from $C$ and $Z$ is of evident interest in principle, e.g., in linear filtering theory. For computational purposes its demonstration in 8.4(1) is not too helpful, though, since two consecutive limits are involved; this makes error analysis very hard, if not impossible.

The Euler-Peano method of little straight steps offers better prospects in this regard, involving as it does only one limit. Moreover, because of its progressive nature it seems more appropriate to, e.g., the filtering problem in the first place. Emery [E1] has proved that it does, indeed, converge to the solution, in probability. We shall now show that this method, with a suitable and natural choice of the subdivision, actually furnishes an algorithm for the pathwise computation of the solution, complete with error estimates.

We resume the assumption that $F$ is a pathwise correspondence. The natural and most economical choice of subdivisions for the Euler-Peano scheme is this: Given a $\delta > 0$, set $S_0 = 0$, $Y_0 = C_0$, and

$$S_{n+1} = \inf\{t > S_n : |C_t - C_{S_n} + (F(Y^{S_n})_- * (Z - Z^{S_n}))_t| > \delta\},$$

$$Y = Y_{S_n} + C - C_{S_n} + F(Y^{S_n})_- * (Z - Z^{S_n}) \qquad \text{on} \quad (S_n, S_{n+1}].$$

In other words, the prescription is not to proceed with a linear approximation until a certain time has elapsed or the driving term $Z$ has changed a given amount, but to wait until its *effect* $C_t - C_{S_n} + \int_{S_n}^t (FY^{S_n})_- \, dZ$ is large enough to warrant a new computation. Then reassess and start from there. It is clear how to program an electronic device so that it will compute the $\delta$-approximate $Y = Y^\delta$ above progressively and pathwise as it receives the signals $C_t(\omega)$ and $Z_t(\omega)$.

THEOREM. *Assume $p = q = 0$ in* (8.1). *Then the $2^{-n}$-approximates $Y^{2^{-n}}$ converge* a.s. *uniformly on bounded intervals to the solution $X$ of* (8.1). *In fact, for any* a.s. *finite stopping time $T$*

(8.5a) $$\textstyle\sum_n |X - Y^{2^{-n}}|_T^* < \infty \qquad \text{a.s.}$$

*Now assume $p \neq 0 \neq q$. Then for every instant $u$ and $\epsilon > 0$ there exists a stopping time $T \leqq u$ with $P[T < u] < \epsilon$ so that*

(8.5b) $$\gamma_{-T}^q[X - Y^\delta] \leqq \delta \cdot E_{pq}(KZ; u, \epsilon)$$

*for every Euler-Peano approximate $Y^\delta$, $\delta > 0$.*

PROOF. We start by showing that $S_\infty := \sup S_n = \infty$ a.s. To this end, set

$${}^\delta H := \textstyle\sum H_{S_n} \cdot [S_n, S_{n+1}) + H_{S_\infty} \cdot [S_\infty, \infty)$$

for any r.c.l.l. process $H$, and denote by $\bar{Y}$ the solution of $\bar{Y} = C + F({}^\delta\bar{Y})_- * Z$. Clearly $Y = \bar{Y}$ on $[0, S_\infty)$. Let $t > 0$, $M > 0$, and split the interval $[-M, M]$ into finitely many subintervals of length less than $\delta/3$. If $\omega \in [|\bar{Y}|_t^* \leqq M]$ then a.s. the path of the $L^0$-integrator $\bar{Y}$ crosses each of these intervals at most finitely many times before $t$ (2.5), and so $S_\infty(\omega) > t$. Hence $S_\infty = \infty$ a.s., and $Y = \bar{Y}$. We start now Picard's iterative scheme with $X^0 = {}^\delta Y$, and observe that $X^1 = Y$ differs from $X^0$ uniformly by less than $\delta$. The second statement is now immediate from (8.3b), and the first is established from this routinely along the lines leading from (8.2b) to (8.2a).

8.6. EXAMPLE. Let $Z$ be a single $L^0$-integrator. There is then a unique solution $E = E[Z]$ of the equation $X = 1 + X_- * Z$, called the exponential of $Z$. It has an explicit representation as

$$E_t = e^{Z_t - 1/2\{Z,Z\}_t} \cdot \textstyle\prod_{0 \leqq s \leqq t} (1 + \Delta_s Z) e^{-\Delta_s Z},$$

the product converging a.s. uniformly. Indeed, using $dE = E_-\, dZ$ and $d\{E, E\} = E_-^2\, d\{Z, Z\}$, Itô's formula yields

$$\begin{aligned} \ln E_t &= \int_0^t E_-^{-1}\, dE + \textstyle\sum_{0 \leqq s \leqq t} \{\Delta_s \ln E - E_-^{-1}\Delta_s E\} - 1/2 \int_0^t E_-^{-2}\, d\{E, E\} \\ &= Z_t - 1/2\{Z, Z\}_t + \textstyle\sum_{0 \leqq s \leqq t} \{\ln(E_s/E_{-s}) - \Delta_s Z], \end{aligned}$$

the sum converging absolutely. Now $E_s/E_{-s} = 1 + \Delta_s E/E_{-s} = 1 + \Delta_s Z$, and the formula follows by exponentiating.

8.7. ON THE MARKOV CHARACTER OF THE SOLUTIONS. We shall use the language of Blumenthal-Getoor [BG] but leave the finer details such as augmentation at infinity to the reader.

(1) Suppose the $d$-vector $Z$ of $L^0$-integrators is also a (temporally homogeneous) Markov process with translation operators $\theta_s: \Omega \to \Omega$, $s \in [0, \infty)$, $Z_t \circ \theta_s = Z_{t+s}$, and field of probability measures $\{P^z: z \in \mathbf{R}^d\}$. Let $f: \mathbf{R}^e \to \mathbf{R}^{d \times e}$ be locally Lipschitz, and denote by $X_t(\omega, x, s)$ the special version produced by 8.4(1, 2), of the solution to $X = x + f(X_-) * (Z - Z^s)$. Set $X_t(\omega, x) = X_t(\omega, x, 0)$. We want to show that the process $Y_t = Y_t(\omega, x) = (Z_t(\omega), X_t(\omega, x)) \in \mathbf{R}^d \times \dot{\mathbf{R}}^e$ is a Markoff process on the enlarged probability space $\tilde{\Omega} = \Omega \times \dot{\mathbf{R}}^e$ with $\sigma$-fields $\tilde{\mathscr{F}}_t = \mathscr{F}_t \times \text{Borel}(\dot{\mathbf{R}}^e)$, translation operators

$$\tilde{\theta}_s(\omega, x) = (\theta_s(\omega), X_s(\omega, x))$$

and field of probability measures

$$P^y = P^z \times \epsilon_x, \qquad y = (z, x) \in \mathbf{R}^d \times \dot{\mathbf{R}}^e.$$

The explicit construction of $X_{t+s}(\omega, x, s) = X_{t+s}((Z_u(\omega): s \le u \le t+s), x, s) = X_t((Z_u \circ \theta_s(\omega): 0 \le u \le t), x, 0)$ shows that $X_{t+s}(\omega, x, s) = X_t(\theta_s(\omega), x)$, thus

$$X_{t+s}(\omega, x) = X_{t+s}(\omega, X_s(\omega, x), s) = X_t(\theta_s(\omega), X_s(\omega, x)).$$

Consequently $\tilde{\theta}_s$ has the required property

$$Y_{t+s} = Y_t \cdot \tilde{\theta}_s.$$

It is left to be shown that the Markov property holds, to wit

$$E^y(\varphi \circ Y_{t+s} \mid \tilde{\mathscr{F}}_s) = E^{Y_s}(\varphi \circ Y_t)$$

a.s. for any continuous bounded function $\varphi$ on $\mathbf{R}^d \times \dot{\mathbf{R}}^e$ and $y = (z, x) \in \mathbf{R}^d \times \dot{\mathbf{R}}^e$. Now

$$\varphi(Y_{t+s}(\omega, x)) = \varphi(Z_t(\theta_s(\omega)), X_t(\theta_s(\omega), X_s(\omega, x)))$$

depends measurably on the two arguments $\theta_s(\omega)$ and $X_s(\omega, x)$. If it were a product of the form

$$f(\theta_s(\omega), X_s(\omega, x)) = f_1(\theta_s(\omega)) \cdot f_2(X_s(\omega, x))$$

then its conditional expectation with respect to $\tilde{\mathscr{F}}_s$ would a.s. have the value

$$f_2(X_s(\omega, x)) \cdot E^{Z_s(\omega)}(f_1(\cdot)) = E^{Y_s(\omega, x)}(f(\cdot, \cdot))$$

at any $(\omega, x) \in \tilde{\Omega}$. The same applies to sums of such products, and the usual monotone class argument results in

$$E^y(\varphi \circ Y_{t+s} \mid \tilde{\mathscr{F}}_s)(\omega, x) = E^{Y_s(\omega, x)}(\varphi(Z_t(\cdot), X_t(\cdot, \cdot))),$$

as desired.

(2) The instant $s$ in the arguments can be replaced by an $\tilde{\mathscr{F}}$-stopping time $S$, showing that if $Z$ is strong Markov so is $Y$. Also, when $Z$ is a Hunt process, $Y$ will be as well. If the lifetime $\eta$ of $Z$ is not infinite, an obvious stopping argument will result in the corresponding statements. These matters have been dealt with in detail by Protter [P6]. Note here that the identification of semimartingales with $L^0$-integrators renders superfluous some labor concerning the uniqueness of decompositions $Z = M + V$ with respect to the various measures $P^z$.

(3) *Problem.* Express the semigroup of transition probabilities of $Y$,

$$Q_t(\varphi, y) = E^y(\varphi(Y_t)), \varphi \in C(\mathbf{R}^d \times \dot{\mathbf{R}}^e)$$

and its generator in terms of $F$ and the corresponding objects for $Z$.

(4) Given a Markov process $Z$ with values in $\mathbf{R}^d$, the problem arises when its components are $L^0$-integrators for each of the $P^z$. Work on this is in progress by Protter and Sharpe [PS]. See also [J1] for $Z$ with stationary independent increments.

(5) Now suppose in addition that $Z$ has independent increments. Then $E^z f$ is independent of $z \in \mathbf{R}^d$ if $f$ is a function of the process $Z - Z_0$ alone. For $\varphi \in C(\dot{\mathbf{R}}^e)$, $\varphi(X_t(\cdot, x))$ is such a function, and so, with $P^x = P^0 \times \epsilon_x$,

$$E^x(\varphi(X_{t+s}) \mid \tilde{\mathscr{F}}_s) = E^{X_s}(\varphi(X_t)):$$

$(\tilde{\Omega}, \tilde{\mathscr{F}}_t, \tilde{\theta}_t, E^x : x \in \dot{\mathbf{R}}^e, X_t)$ is a Markov process in its own right. Again, if $Z$ is strong Markov or a Hunt process then so is $X$.

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