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REFLECTED BROWNIAN MOTION ON AN ORTHANT[1]

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We consider a K-dimensional diffusion process Z whose state space is the nonnegative orthant. On the interior of the orthant, Z behaves like a K-dimensional Brownian motion with arbitrary covariance matrix and drift vector. At each of the (K-1) dimensional hyperplanes that form the boundary of the orthant, Z reflects instantaneously in a direction that is constant over that hyperplane. There is no extant theory of multidimensional diffusion that applies to this process, because the boundary of its state space is not smooth. We adopt an approach that requires a restriction on the directions of reflection, but Reiman has shown that this restriction is met by all diffusions Z arising as heavy traffic limits in open K-station queuing networks.

Our process Z is defined as a path-to-path mapping of K-dimensional Brownian motion. From this construction it follows that Z is a continuous Markov process and a semimartingale. Using the latter property, we obtain a change of variable formula from which one can develop a complete analytical theory for the process Z.

1. **Introduction.** Let K be a positive integer, $A = (a_{ij})$ a $K \times K$ covariance matrix (symmetric and nonnegative definite), and $b = (b_j)$ a K-vector. (All vectors should be envisioned as row vectors. The indices i and j will always take values $1, \ldots, K$.) Let $Q = (q_{ij})$ be a nonnegative $K \times K$ matrix with zeros on the diagonal and spectral radius strictly less than unity.

Let S be the nonnegative orthant R_+^K. Our objective in this paper is to construct and characterize a K-dimensional stochastic process $Z = \{Z(t);\, t \geq 0\}$ which has the following properties. First, Z is a Markov process with stationary transition probabilities, continuous sample paths, and state space S. Second, Z behaves on the interior of S like a K-dimensional Brownian Motion with covariance matrix A and drift vector b. Third, Z reflects instantaneously at the boundary of S. Finally, the direction, of reflection everywhere on the boundary surface $Z_i = 0$ is the ith row of the *reflection matrix* I-Q. In assuming that the reflection matrix has ones on the diagonal, we are simply adopting a convenient normalization, since these diagonal elements must be positive. (The direction of reflection at each point on the boundary must have a positive inward normal.) In assuming that Q is nonnegative, we restrict attention to the case where reflection is downward (toward the origin) at every point on the boundary. With Q assumed nonnegative, the requirement that it have spectral radius less than unity is in a sense necessary, as we shall explain later.

Following Skorokhod's (1961) approach to construction of one-dimensional diffusions with reflection, we cast our problem in precise mathematical terms as follows. We seek a pair of K-dimensional processes $Z = \{Z(t);\, t \geq 0\}$ and $Y = \{Y(t);\, t \geq 0\}$ which jointly satisfy the following conditions:

$$Z(t) = X(t) + Y(t)(I - Q), \qquad t \geq 0, \tag{1}$$

where $X = \{X(t);\, t \geq 0\}$ is a K-dimensional Brownian motion with covariance matrix A,

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drift vector b, and $X(0) \in S$;

(2) $$Z(t) \in S, \qquad t \geq 0;$$

(3) $$Y_j(\cdot) \text{ is continuous and nondecreasing with } Y_j(0) = 0 \qquad j = 1, \dots, K;$$

and

(4) $$Y_j(\cdot) \text{ increases only at those times } t \text{ where } Z_j(t) = 0 \qquad j = 1, \dots, K.$$

It will be shown in Section 2 that for any given Brownian motion X there exists a unique pair of processes Y and Z satisfying conditions (1)–(4). Each component of Z is the sum of a continuous martingale and a continuous process of bounded variation. Using that sample path structure, we derive in Section 3 a change of variable formula (a variation of Itô's lemma) from which one can develop a complete analytical theory of the process Z. The usefulness of this formula is illustrated in Section 3, but we do not attempt a comprehensive treatment. Section 4 contains some miscellaneous concluding remarks.

Multidimensional diffusions with reflecting boundaries have been studied by Stroock and Varadhan (1971) and by Watanabe (1971). They allow the covariance matrix and the drift vector to be functions of state and time, but they require that the boundary of the state space be smooth and that the direction of reflection vary continuously on the boundary. To obtain the boundary behavior described above, we must use entirely different methods, but our process has sample path properties and analytical characteristics very similar to those obtained by the aforementioned authors.

Diffusion processes of the type discussed here arise in the heavy traffic theory for networks of queues. Harrison (1978) has shown that one very special two-dimensional case occurs as a heavy traffic limit for tandem queues. Reiman (1977) has developed the heavy traffic theory for general K-station open networks, obtaining as limits precisely the class of K-dimensional processes Z studied here. For a survey of these and other results, see Lemonie (1978).

2. Definition of the process. Let C be the space of continuous functions x: $[0, \infty) \to R^K$, endowed with the topology of uniform convergence on compact intervals. Component functions will be denoted $x_j(t)$ for $t \geq 0$ and $j = 1, \dots, K$. Let C_S be the set of $x \in C$ such that $x(0) \in S$.

THEOREM 1. *For each $x \in C_S$ there exists a unique pair of functions $y \in C$ and $z \in C$ satisfying*

(5) $$z_j(t) = x_j(t) + y_j(t) - \textstyle\sum_{i=1}^K q_{ij} y_i(t), \qquad t \geq 0,$$

(6) $$z_j(t) \geq 0, \qquad t \geq 0,$$

(7) $$y_j(\cdot) \quad \textit{is nondecreasing with} \quad y_j(0) = 0,$$

and

(8) $$y_j(\cdot) \quad \textit{increases only at those times } t \textit{ where} \quad z_j(t) = 0$$

for all $j = 1, \dots, K$. Moreover, setting $y = \psi(x)$ and $z + \phi(x)$, we have the following.

(9) *The restrictions of y and z to $[0, T]$ depend only on the restriction of x to $[0, t]$.*

(10) *Both ψ and ϕ are continuous mappings $C_S \to C$.*

Fix $x \in C_S$ and $T > 0$. Let $y = \psi(x)$ and $z = \phi(x)$ as above.

(11) *Define $x^*(t) = z(T) + x(T + t) - x(T)$, $y^*(t) = y(T + t) - y(T)$, and $z^*(t) = z(T + t)$. Then $y^* = \psi(x^*)$ and $z^* = \phi(x^*)$.*

PROOF. For a nonnegative $K \times K$ matrix P we denote by $\| P \|$ the maximal row sum. Recall that Q is nonnegative and has spectral radius less than unity. Thus there exists a diagonal matrix Λ, having positive diagonal elements, such that the nonnegative matrix $Q^* = \Lambda^{-1} Q \Lambda$ satisfies $\| Q^* \| < 1$ (cf., Veinott (1969), Lemma 3). Observe that two functions $y \in C$ and $z \in C$ satisfy (5)–(8) for Q and $x \in C_S$ if and only if $y\Lambda$ and $z\Lambda$ satisfy (5)–(8) for Q^* and $x\Lambda$. Thus, for the remainder of the proof we can assume without loss of generality that $\| Q \| = \alpha < 1$.

Let $x \in C_S$ be fixed until further notice. Let C_0 be the set of $y \in C$ that are nondecreasing with $y(0) = 0$. (Here we make no notational distinction between the vector and scalar zeros. When we say that y is nondecreasing, we mean that each component function is.) Define a mapping $\pi: C_0 \to C_0$ by setting

$$\pi_j(y)(t) = \left[-\inf_{0 \le s \le t} \{ x_j(s) - \textstyle\sum_{i=1}^K q_{ij} y_i(s) \} \right]^+ \tag{12}$$
$$= \sup_{0 \le s \le t} \left[\textstyle\sum_{i=1}^K q_{ij} y_i(s) - x_j(s) \right]^+$$

for $t \ge 0$ and $j = 1, \ldots, K$. Observe that $\pi_j(y)$ does not depend on y_j because $q_{jj} = 0$. We can write (12) more compactly as

$$\pi(y)(t) = \sup_{0 \le s \le t} [y(s) Q - x(s)]^+,$$

with the understanding that the positive part and the supremum are to be computed component-wise. The key observation is that (5)–(8) are completely equivalent to

$$y \in C_0 \tag{13}$$

$$y = \pi(y), \tag{14}$$

and

$$z = x + y(I - Q). \tag{15}$$

Conditions (13) and (15) are just restatements of (7) and (5) respectively, and it is easy to verify that (13)–(15) imply (6) and (8) as well. To complete the proof that (5)–(8) are equivalent to (13)–(15), it remains to show that (5)–(8) imply (14). Suppose y and z satisfy (5)–(8), and let $v = \pi(y)$. We must show that $y_j(t) = v_j(t)$ for all j and t. From (5)–(7) it is immediate that $y_j(t) \ge v_j(t)$. If $y_j(t) > v_j(t)$ for some t, then it must be that $y_j(t_0) > v_j(t_0)$ for some t_0 that is a point of increase for $y_j(\cdot)$. But (5) and $y_j(t_0) > v_j(t_0)$ together imply $z_j(t_0) > 0$, which contradicts (8). Thus $y_j(t) = v_j(t)$ for all j and t, which establishes the desired equivalence.

We now use a contraction argument to prove that (13)–(14) has a unique solution y. Let $C_0[0, T]$ and $C_S[0, T]$ be defined in the obvious way, and define the norm

$$\| y \| = \max_{1 \le j \le K} \sup_{0 \le t \le T} | y_j(t) | \quad \text{for} \quad y \in C[0, T].$$

Observe that $C_0[0, T]$ is a complete metric space with this norm. Viewing π as a mapping $C_0[0, T] \to C_0[0, T]$ for the moment, it is easy to verify that

$$\| \pi(y) - \pi(y') \| \le \alpha \| y - y' \| \quad \text{for} \quad y, y' \in C_0[0, T].$$

Thus π is a contraction, implying that it has a unique fixed point $y \in C_0[0, T]$. Furthermore, that fixed point y is given by the following construction. Let

$$y^0(t) \equiv 0 \tag{16}$$

$$y^1(t) = \pi(y^0)(t) = \sup_{0 \le s \le t} [-x(s)]^+, \tag{17}$$

and

$$y^{n+1}(t) = \pi(y^n)(t) = \sup_{0 \le s \le t} [y^n(s) Q - x(s)]^+ \tag{18}$$

for general $n = 1, 2, \ldots$. Then the contraction property gives us

$$\|y^n - y\| \to 0 \quad \text{as} \quad n \to \infty. \tag{19}$$

Finally, since our choice of T was arbitrary, we have that $y^n \to y$ uniformly on compact intervals, where $y^0(t), y^1(t), \ldots$ are defined by (16)–(18) for all $t \geq 0$ and y is the unique solution of (13)–(14). Then z is defined in terms of y by (15).

The first part of the theorem has now been proved, and property (9) is immediate from (16)–(19). To prove (11), given the uniqueness statement of the theorem, one need only verify that y^* and z^* satisfy (5)–(8) when x^* is substituted for x. It remains to prove the continuity property (10). For this purpose, suppose $x, x' \in C_S[0, T]$ and define $y$yn$(x)(t)$ and $y^n(x')(t)$ as in (16)–(18) for $n = 0, 1, \ldots$ and $0 \leq t \leq T$. It is easy to show from (18) that

$$\|y^{n+1}(x) - y^{n+1}(x')\| \leq \|x - x'\| + \alpha\|y^n(x) - y^n(x')\|,$$

and obviously $\|y^0(x) - y^0(x')\| = 0$, so we have $\|y(x) - y(x')\| \leq \|x - x'\|/(1 - \alpha)$ from (19) and induction. Thus the mapping ψ, which carries x into y, is continuous in the topology of uniform convergence on compact intervals, and the continuity of ϕ is then immediate from (15). This completes the proof of the theorem.

Now to construct our process Z we begin with any probability space $(\Omega, \mathscr{F}, P)$ on which is defined a K-dimensional Brownian motion $X = \{X(t); t \geq 0]$ with covariance matrix A, drift vector b, and $X(0) \in S$ almost surely. (In saying that X is a Brownian motion, we include the requirement that $X(0)$ be independent of $X(t) - X(0)$ for $t > 0$.) Let $\mathscr{F}_t = \mathscr{F}(X(s); 0 \leq s \leq t)$ for $t \geq 0$. Let $Y = \psi(X)$ and $Z = \phi(X)$ on that subset of Ω where $X \in C_S$. Let $Y(t) = Z(t) = 0$ for all $t \geq 0$ on the exceptional set.

COROLLARY 1. (a) *Both $Y(t)$ and $Z(t)$ are measurable with respect to $\mathscr{F}_t$ for each $t \geq 0$.* (b) *The processes Y and Z uniquely satisfy* (1)–(4) *except perhaps on a set of measure zero.* (c) *Z is a Markov process with stationary transition probabilities.*

PROOF. Part (a) is immediate from (9) and (10), and part (b) is immediate from the first statement of Theorem 1, and (c) follows directly from (11) and the stationary, independent increments of X.

To conclude this section, we shall show that if Q is nonnegative with spectral radius $\rho \geq 1$, and if the covariance matrix A is nonsingular, then there can be no pair of processes Y and Z satisfying (1)–(4). (The assumption that A is nonsingular is not needed except for this argument.) From the Frobenius theory of nonnegative matrices, we know that there is a nonnegative right eigenvector w, having at least one positive element (say w_i), such that $Qw = \rho w$. Let N be the set of $u \in R^K$ with $u_i < 0$ and $u_j \leq 0$ for all j. If $u \in N$ and $v \in S$, then we have

$$[u + v(I - Q)]w = uw - (\rho - 1)vw < 0.$$

Thus $[u + v(I - Q)] \notin S$. If X is a Brownian motion as described above, then the event $X(t) \in N$ has positive probability for any $t > 0$, regardless of the initial state $X(0)$. If Y and Z are to satisfy (1)–(4), then we must have $Y(t) \in S$ and $X(t) + Y(t)(I - Q) \in S$ for all t with probability one. Substituting $X(t)$ for u and $Y(t)$ for v in the argument above, we see that this is impossible when $\rho \geq 1$.

3. A Change of Variable Formula. It will be convenient to represent X in the form $X(t) = \xi(t) + bt$, where $\xi = \{\xi(t); t \geq 0\}$ is a Brownian motion with covariance matrix A, zero drift, and $\xi(0) = X(0) = Z(0)$. Then we have

$$Z_j(t) = \xi_j(t) + B_j(t), \qquad t \geq 0, \tag{20}$$

where

$$B_j(t) = b_j t + Y_j(t) - \textstyle\sum_{i=1}^K q_{ij} Y_i(t), \qquad t \geq 0, \tag{21}$$

for $j = 1, \ldots, K$. Observe that ξ_j is a martingale over the σ-fields $\{\mathcal{F}_t\}$ and that B_j is a continuous adapted process of bounded variation. Thus each Z_j is a continuous semimartingale, and one can develop the analytical theory of the vector process Z from the following version of Itô's formula. For twice differentiable functions on $\mathbb{R}^K$ we define the (constant coefficient) differential operators

$$L = \frac{1}{2} \sum_{i=1}^K \sum_{j=1}^K a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^K b_j \frac{\partial}{\partial x_j}$$

and

$$D_i = \frac{\partial}{\partial x_i} - \sum_{j=1}^K q_{ij} \frac{\partial}{\partial x_j} \qquad \text{for } i = 1, \ldots, K.$$

Observe that D_i is the directional derivative in the direction of reflection at boundary $Z_i = 0$. Let $\mathcal{S}$ (for smooth) be the set of functions $f(t, x)$ that are continuously differentiable in $t \geq 0$ and twice continuously differentiable in $x \in R^K$.

THEOREM 2. *If $f \in \mathcal{S}$ then*

$$\begin{aligned} f(t, Z(t)) - f(0, Z(0)) = & \int_0^t \left[\frac{\partial}{\partial u} f(u, Z(u)) + Lf(u, Z(u)) \right] du \\ & + \sum_{j=1}^K \int_0^t \frac{\partial}{\partial x_j} f(u, Z(u))\, d\xi_j(u) \\ & + \sum_{j=1}^K \int_0^t D_j f(u, Z(u))\, dY_j(u) \end{aligned} \tag{22}$$

for all $t \geq 0$. Here the integrals involving $d\xi_j(u)$ are of the Itô type, and those involving $dY_j(u)$ are defined path-by-path as ordinary Riemann-Stieltjes integrals.

PROOF. We first claim that

$$\begin{aligned} f(t, Z(t)) = & f(0, Z(0)) + \int_0^t \frac{\partial}{\partial u} f(u, Z(u))\, du \\ & + \sum_{j=1}^K \int_0^t \frac{\partial}{\partial x_j} f(u, Z(u))\, dZ_j(u) \\ & + \frac{1}{2} \sum_{i=1}^K \sum_{j=1}^K \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(u, Z(u))\, dZ_i(u)\, dZ_j(u), \end{aligned} \tag{23}$$

where the differentials $dZ_j(u)$ are computed from (20) and (21) in the obvious way and

$$dZ_i(t) dZ_j(t) = a_{ij}\, dt \tag{24}$$

by convention. If the process $B(t)$ were absolutely continuous, then (23)–(24) would be a direct application of Itô's, formula (cf., McKean (1969), page 44), and it is well known that this formula extends to the case where $B(t)$ is continuous and of bounded variation (cf., Kunita and Watanabe (1967), Theorem 2.2). The Kunita-Watanabe change of variable formula does not directly imply (23)–(24), because they take f to be a function of $x \in R^K$ only, but with minor changes their proof will cover this slightly more general situation as well. From (23)–(24) we obtain (22) by simply collecting similar terms.

The following (more or less standard) calculation illustrates the usefulness of Theorem 2. Suppose that $g(t,x)$ is an element of $\mathcal{S}$ having bounded first-order partials with respect to x and further satisfying

$$\frac{\partial}{\partial t} g(t,x) = Lg(t,x), \tag{25}$$

(26) $$D_j g(t, x) = 0 \qquad \text{if } x_j = 0 \ (j = 1, \ldots, K),$$

(27) $$g(0, x) = h(x)$$

where $h: S \to \mathbb{R}$ is bounded and continuous. Fix $t > 0$ and apply Theorem 2 to the function

(28) $$f(u, x) = g(t - u, x), \qquad 0 \le u \le t.$$

The first integral on the right side of (22) vanishes because of (25) and (28). Similarly, each of the integrals in the final sum on the right side of (22) vanishes because of (26), (28) and the fact that $Y_j(\cdot)$ increases only when $Z_j(\cdot) = 0$. Thus, substituting (27) and (28) into the remaining terms of (22), we have

(29) $$h(Z(t)) = g(t, Z(0)) + \sum_{j=1}^{K} \int_0^t \frac{\partial}{\partial x_j} g(t - u, Z(u))\, d\xi_j(u).$$

Since the first-order partials of g with respect to x are bounded by assumption, the Itô integrals on the right side of (29) are martingales, and upon taking expectations we have

(30) $$E[g(t, Z(0))] = E[h(Z(t))].$$

Denoting by $P_x(\cdot)$ the distribution on the path space of Z corresponding to initial state $Z(0) = x$, and by $E_x(\cdot)$ the associated expectation operator, (30) can be restated as

(31) $$g(t, x) = E_x[h(Z(t))].$$

See Harrison and Reiman (1980) for more on the analytical properties of the process Z.

4. Concluding remarks. In the construction of Section 2, one could replace the Brownian Motion X by any K-dimensional process having stationary, independent increments and no negative jumps. (The continuity of the boundary process Y depends only on the absence of negative jumps, and the Markov property of Z depends only on the stationary, independent increments.) This would yield a class of multidimensional reflected processes, generalizing the class of one-dimensional reflected processes that have been studied (with great effect) in storage theory, cf. Harrison (1977).

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