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ON THE WILLIAMS-BJERKNES TUMOUR GROWTH MODEL I

BY MAURY BRAMSON AND DAVID GRIFFEATH

University of Minnesota and University of Wisconsin

Williams and Bjerknes introduced in 1972 a stochastic model for the spread of cancer cells; independently, this model has since surfaced within the field of interacting particle systems as the biased voter model. Cells, normal and abnormal (cancerous), are situated on a planar lattice. With each cellular division, one daughter stays put, while the other usurps the position of a neighbor; abnormal cells reproduce at a faster rate than normal cells. We treat here the long-term behavior of this system. In particular, we show that, provided it lives forever, the tumour will eventually contain a ball of linearly expanding radius. This also demonstrates the ergodicity of the interacting particle system, the coalescing random walk with nearest neighbor births. Our techniques include the use of dual processes, and of different numerical computations involving the use of imbedded processes.

1. Introduction. In a 1972 paper, Williams and Bjerknes [20] proposed a simple stochastic model for the spread of cancerous cells. Based on biological considerations, they restricted attention to the basal layer of an epithelium, thereby obtaining a two-dimensional setting. The cells, normal and abnormal, are situated on a planar lattice. With each cellular division, one daughter cell stays put while the other usurps the position of a neighbor. Splitting of each normal cell is assumed to occur at exponential rate 1, whereas, due to "carcinogenic advantage," each cancerous cell splits at rate $\kappa > 1$.

Assuming the lattice to be Z^2, these axioms give rise to a simple continuous time Markov chain on the state space $S_0 = \{\text{finite subsets of } Z^2\}$. (Actually, the hexagonal lattice was preferred in [20], though the square and triangular lattices were considered as well. Our results apply to all three lattices; Z^2 is chosen largely for notational convenience.) If we let ξ_t^A denote the set of sites occupied by cancer cells at time t, given that the original cancerous population occupies $A \in S_0$, then the processes $(\xi_t^a)_{t\ge 0}$ are Markov. Their jump rates are given by

$$(1)\qquad \begin{aligned} &A \to A \cup \{x\}\ (x \notin A) &&\text{at rate}\quad \kappa\,|\{y \in A: \|y - x\| = 1\}|,\\ &A \to A - \{x\}\ (x \in A) &&\text{at rate}\quad |\{y \in A^c: \|y - x\| = 1\}|, \end{aligned}$$

where $|\Lambda|$ is the cardinality of $\Lambda \in S_0$, and $\|x\|$ is the distance from x to 0 ($\|\ \|$ = Euclidean norm). For simplicity, Williams and Bjerknes restricted attention to the process (ξ_t^0) starting with a single abnormal cell at the origin. They noted that if τ_n is the time of the nth jump of (ξ_t^0), and if $S_n = |\xi_{\tau_n}^0|$, then (S_n) is a simple random walk with positive drift $(\kappa - 1)/(\kappa + 1)$ on $\{1, 2, \cdots\}$ and absorption at 0. Hence, letting $\tau_\varnothing^0$ be the hitting time of (ξ_t^0) for the trap $\varnothing$ (= total recovery), the gambler's ruin formula yields

$$(2)\qquad P(\tau_\varnothing^0 < \infty) = P(\mathbf{S}_n = 0 \quad \text{for some } n) = \kappa^{-1} < 1.$$

Thus $P(\tau_\varnothing^0 = \infty) > 0$, and on $\{\tau_\varnothing^0 = \infty\}$ the question arises: how fast does (ξ_t^0) grow, and what is the geometric nature of ξ_t^0 for large t? Unable to obtain rigorous results beyond (2), Williams and Bjerknes resorted to computer simulations. Some tantalizing Monte Carlo graphics, a few of which are included in [20], led the authors to formulate surprising conjectures concerning the growth rate of the tumour and the dimensionality of its

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boundary. Their simulations showed that the abnormal region was more or less a single "blob," the average radius of which grew faster than linearly, and the boundary of which seemed to have asymptotic dimension greater than one.

Shortly thereafter, Mollison [14] gave a rigorous proof that the Williams-Bjerknes conjectures were wrong, that $|\xi_t^0|^{1/2}$ grows at most linearly in t, and hence that the boundary of ξ_t^0 remains one-dimensional in the limit as t tends to ∞. Mollison's proof is based on comparison of (ξ_t^0) with a process $(\bar{\xi}_t^0)$ for which recovery is impossible, i.e., where transitions $A \to A \cup \{x\}$ are as in (1), but no transitions $A \to A - \{x\}$ take place. By a change of time scale, $(\bar{\xi}_t^0)$ may be thought of as the Williams-Bjerknes model with $\kappa = \infty$. Hammersley [7] had shown that the maximal radius of this process without recovery grows at most linearly, and since there is a joint realization of (ξ_t^0) and $(\bar{\xi}_t^0)$ such that $P(\xi_t^0 \subset \bar{\xi}_t^0 \ \forall t) = 1$, the same result must also hold for (ξ_t^0). More precisely, it can be shown that there is a constant $C < \infty$ such that

$$P(\exists t^* < \infty\colon \ \xi_t^0 \subset D_{Ct} \ \forall \ t \geq t^*) = 1, \tag{3}$$

where $D_R \overset{\text{def}}{=} \{x \in Z^2\colon \|x\| \leq R\}$.

To date, equations (2) and (3) seem to be the only known rigorous results for (ξ_t^0). Several authors have studied this process, but have either relied on Monte Carlo simulations ([4], [17]), or considered the limiting cases $\kappa = \infty$ [16] and $\kappa = 1$([3], [11]). Among these papers, the work of Richardson [16] is particularly relevant for our purposes. He showed that if $\kappa = \infty$, then there is a norm $\| \quad \|'$, defined implicitly in terms of the process, such that

$$\forall \, \epsilon > 0 \ \exists \ t^* < \infty\colon \ P(D'_{(1-\epsilon)t} \subset \bar{\xi}_t^0 \subset D'_{(1+\epsilon)t}) \geq 1 - \epsilon \qquad \forall \ t > t^*. \tag{4}$$

Here, $D'_R = \{x \in Z^2\colon \|x\|' \leq R\}$. Thus, $\bar{\xi}_t^0$ has an asymptotic shape, the "radius" of which grows linearly in time. An almost sure version of (4) was subsequently proved by Kesten [12]. Richardson's methodology relies heavily on the theory of subadditive processes, pioneered by Hammersley [7] and developed further by Kingman [13]. The impossibility of extinction is needed to exploit subadditivity directly; this explains the difficulty in proving a result analogous to (4) for $(\xi_t^0 \mid \tau_\varnothing^0 = \infty)$.

Quite independently, the Williams-Bjerknes model has surfaced within the field of interacting particle systems, where it is known as the biased voter model. Interacting particle systems are Markov processes on the state space $S = \{\text{all subsets of } Z^d\}$, for some $d \geq 1$. The state $A \in S$ is interpreted as the (typically infinite) set of sites occupied by particles, and the process (ξ_t^A) represents the evolution in time starting from A. The biased voter model, first considered by Schwartz [18], is a "spin system" with "flip rates" of the form

$$\begin{aligned} c_x(A) &= \kappa \, |\{y \in A\colon \|y - x\| = 1\}| && x \notin A, \\ &= |\{y \in A^c\colon \|y - x\| = 1\}| && x \in A, \end{aligned}$$

for some $\kappa > 1$. (Intuitively, $c_x(A)\,dt = P(\xi_{dt}^A = A \triangle \{x\})$.) The voting interpretation is as follows: ξ_t^A is the set of sites occupied by voters in favor of some proposition, while voters on $Z^d - \xi_t^A$ are against the proposition. Each individual changes opinion at a rate determined by neighboring voters, the rate being proportional to the number of neighbors with the opposite opinion. But, "for" voters have more influence than "against" voters by a factor of $\kappa > 1$. When A is *finite* (and $d = 2$), (ξ_t^A) is simply a Williams-Bjerknes process. For *infinite* A, one expects convergence of ξ_t^A to Z^d as $t \to \infty$, in some appropriate sense. Partial results along these lines were obtained by Schwartz [18]. She proved that $\delta_\varnothing$ and δ_{Z^d}, the distributions concentrated at $\varnothing$ and Z^d, are the only invariant measures for the system $\{(\xi_t^A); A \in S\}$, and showed that

$$P(\xi_t^A \in \cdot\,) \Rightarrow \delta_{Z^d} \qquad \text{as} \quad t \to \infty \tag{5}$$

($\Rightarrow$ means weak convergence) for certain infinite initial configurations A. It seems intuitively most plausible that almost sure convergence to Z^d should occur starting from arbitrary infinite A; but even the weaker question as to whether (5) holds for all A, i.e., whether

$$\lim_{t\to\infty} P(0 \in \xi_t^A) = 1, \qquad A \text{ infinite}, \tag{6}$$

has remained unanswered.

Our purpose in this paper and its sequel is to resolve some of the outstanding problems concerning the Williams-Bjerknes model and its infinite counterpart, the biased voter model. The main result of the present paper is a companion to (3), stating that (ξ_t^0) eventually contains a ball of linearly expanding radius provided the tumour survives forever. Our methods apply equally well in any dimension, so our theorems will be stated and proved for general $d \geq 1$. We remark, however, that the one-dimensional results are quite elementary; they are discussed in [6].

THEOREM 1. *Let (ξ_t^0) be the d-dimensional Williams-Bjerknes model with carcinogenic advantage $\kappa > 1$. There is a constant $c = c(d, \kappa) > 0$ such that*

$$P(\exists\, t^* < \infty\colon \ D_{ct} \subset \xi_t^0 \ \forall\, t \geq t^* \mid \tau_\varnothing^0 = \infty) = 1. \tag{7}$$

As an easy consequence of Theorem 1, we will obtain an almost sure result for the biased voter model.

THEOREM 2. *Let $\{(\xi_t^A);\, A \in S\}$ be the d-dimensional biased voter model with bias $\kappa > 1$. For any $A \in S$, $A \neq \varnothing$, (7) holds with 0 replaced by A. For any infinite $A \in S$, $P(\tau_\varnothing^A = \infty) = 1$. Therefore,*

$$P(\lim_{t\to\infty} \xi_t^A = Z^d) = 1. \tag{8}$$

(Any finite subset of Z^d will eventually be permanently occupied.)

The proof of Theorem 1 relies on the *additive* nature of $\{(\xi_t^A)\}$, and the resultant duality theory (cf. [8], [9] or [10]). In particular, the dual interacting system $\{(\hat\xi_t^A)\}$, comprised of *coalescing random walks with nearest neighbor births*, provides the key to the analysis. Each particle in $\hat\xi_t^A$ attempts to jump to any one of its $2d$ neighboring sites at rate $1/2d$, and also attempts to give birth to a new particle at any one of its neighboring sites at rate $(\kappa - 1)/2d$. Whenever a particle attempts to occupy a site which is already occupied, the two particles coalesce. We also make use of certain Markov chains $\{(X_t^x);\, x \in Z^d\}$, which are imbedded in $\{(\hat\xi_t^A)\}$, to obtain estimates for certain occupational probabilities which are relevant to the demonstration of Theorem 1. The processes (X_t^x) were previously exploited in [18]. The system $\{(\hat\xi_t^A)\}$ is not without interest in its own right: its countervailing effects of coalescence and nearest neighbor birth suggest a stable ergodic theory. Using Theorem 2, we can obtain a "complete convergence" result for $\{(\hat\xi_t^A)\}$:

THEOREM 3. *Let $\{(\hat\xi_t^A)\}$ be the system of coalescing random walks on Z^d with nearest neighbor births, of rate κ. There is a unique invariant measure ν for the system $\{(\hat\xi_t^A);\, A \in S - \{\varnothing\}\}$; ν satisfies*

$$\hat P(\hat\xi_t^A \in \cdot) \Rightarrow \nu \quad as \quad t \to \infty \ \forall\, A \in S - \{\varnothing\}. \tag{9}$$

Section 2 contains the main steps in the derivation of Theorem 1. Proofs of several of the necessary lemmas are relegated to Section 3. Finally, in Section 4 we prove Theorems 2 and 3, and make some concluding remarks.

In a subsequent paper we will show how to use Theorem 1 and a modified version of Richardson's technique to prove that $(\xi_t^0 \mid \tau_\varnothing^0 = \infty)$ has an asymptotic shape. As in [16], the shape is defined by means of an implicit norm $\| \|^*$. The Williams-Bjerknes process, conditioned on nonextinction, thus grows like $a \| \|^*$-ball with linearly increasing radius.

2. The proof of Theorem 1: main steps. An explicit graphical representation of $\{(\xi_t^A); A \in S\}$ will be exploited throughout the remainder of the paper. This approach is due to Harris [9]; the reader is referred to his paper or [6] for more details concerning the construction. Start with the "space-time diagram" $Z^d \times T$. For each site $x \in Z^d$ and each neighbor y of x, draw an infinite sequence of arrows with δ's on the head: $\delta_x \leftarrow_y$, from $(y, \tau_{x,y}^1)$ to $(x, \tau_{x,y}^1)$, from $(y, \tau_{x,y}^2)$ to $(x, \tau_{x,y}^2)$, etc. The values $\tau_{x,y}^1, \tau_{x,y}^2 - \tau_{x,y}^1, \cdots$ are taken to be independent exponential variables with mean $1/2d$. Similarly, for each x and neighbor y of x, put arrows without δ's on the head: ${}_x\!\leftarrow_y$ from $(y, \bar\tau_{x,y}^1)$ to $(x, \bar\tau_{x,y}^1)$, from $(y, \bar\tau_{x,y}^2)$ to $(x, \bar\tau_{x,y}^2)$, etc., where the $\bar\tau_{x,y}^n$ occur at rate $(\kappa - 1)/2d$. Say *there is a path up from* (y, s) *to* (x, t), $x, y \in Z^d$, $0 \le s \le t < \infty$, if there is a chain of "upward vertical" (i.e., increasing in time with fixed spatial coordinates) and directed "horizontal" (fixed time) edges in the resulting diagram which leads from (y, s) to (x, t) without passing "vertically" through a δ. The δ's may be thought of as obstructions to the flow (or "percolation") of liquid. Some pictures of graphical representations for similar systems may be found in [6]. Now, define

$$\xi_t^A = \{x\text{: there is a path up from } (y, 0) \text{ to } (x, t) \text{ for some } y \in A\}.$$

A little thought reveals that (ξ_t^A) is the biased voter process starting from A, with bias κ. In particular, if $A = \{0\}$, then (ξ_t^0) is the basic Williams-Bjerknes process. This construction has the useful feature that the entire system $\{(\xi_t^A)\}$ is defined on a single probability space in such a way that *additivity* holds, i.e.,

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B \qquad \forall A, B \in S, \qquad t \ge 0. \tag{10}$$

Hence, there is *monotonicity*:

$$\xi_t^A \subset \xi_t^B \qquad \forall A \subset B, \qquad t \ge 0, \tag{11}$$

a property which will be used repeatedly.

Let us now proceed to the proof of Theorem 1. $B \in S_0$ is a *box centered at* 0 if $B = B_R = \{(x_1, \cdots, x_d): |x_i| \le R \ \forall \ i\}$ for some $R \ge 1$, and is a *box* if $B = B_{x,R} = B_R + x$ (B_R translated by x) for some $R \ge 1$, $x \in Z^d$. The basic strategy will be to establish two claims:

(a) the process $(\xi_t^0 \mid \tau_\varnothing^0 = \infty)$ covers a box $B_{x,R}$ for some random $x \in Z^d$ at some random time $\sigma_R < \infty$ P – a.s., for arbitrary $R > 0$; and

(b) If B is a very large box, then (ξ_t^B) grows (at least) linearly for all large times with overwhelming probability. Let us formalize (a) and (b), and show how they yield (7). We state (a) as a lemma, the proof of which will be deferred to Section 3. (The proof is not difficult.)

LEMMA 1. *Let* $R > 0$, $\sigma_R = \min\{t\text{: } \xi_t^0 \supset B_{x,R}$ *for some* $x \in Z^d\}$ ($= \infty$ if no such x, t exist). Then,

$$P(\sigma_R < \infty \mid \tau_\varnothing^0 = \infty) = 1. \tag{12}$$

Linear growth is expressed by means of the events

$$E_{x,c}^\Lambda = \{\exists\, t^* < \infty\text{: } \ B_{x,c't} \subset \xi_t^\Lambda \ \forall\, t \ge t^*,\, c' < c\} \qquad x \in Z^d,\, c > 0,\, \Lambda \in S_0.$$

Our main task will be to prove the following version of (b):

PROPOSITION 1. *There is a* $c > 0$ *such that for any* $\epsilon > 0$,

$$\exists\, R < \infty\text{: } \ P(E_{0,c}^{B_R}) \ge 1 - \epsilon. \tag{13}$$

The proof of Proposition 1 is rather involved, requiring a number of preliminary steps.

Assuming Lemma 1 and Proposition 1, however, Theorem 1 follows quite easily.

PROOF OF THEOREM 1 ASSUMING (12) AND (13). Note that
(i) $E^{\Lambda}_{x,c} \subset \{\tau^{\Lambda}_{\varnothing} = \infty\}$,
(ii) $E^{\Lambda}_{x,c}$ is a tail event, and
(iii) $E^{A}_{x,c} \subset E^{B}_{x,c}$ if $A \subset B$ (by (11)).
Also, $E^{\Lambda}_{x,c} = E^{\Lambda}_{0,c}$ for any x, and by translation invariance of $\{(\xi^A_t)\}$, $P(E^{\Lambda}_{x,c}) = P(E^{\Lambda-x}_{o,c})$ ($\Lambda - x$ = the translate of Λ by $-x$). Thus,
(iv) $P(E^{B_{x,R}}_{0,c}) = P(E^{B_R}_{0,c})\ \forall\, x \in Z^d$.
Assume Lemma 1 and Proposition 1. Given $\epsilon > 0$, choose R so that (13) holds. By (12), (13) and (i)–(iv), we have

$$\begin{aligned} P(E^{\{0\}}_{0,c}, \tau^0_{\varnothing} = \infty) &= \textstyle\sum_\Lambda \displaystyle\int_s P(\sigma_R \in ds,\quad \xi^0_{\sigma_R} = \Lambda) P(E^{\Lambda}_{0,c}) \\ &\geq \textstyle\sum_\Lambda \displaystyle\int_s P(\sigma_R \in ds,\quad \xi^0_{\sigma_R} = \Lambda) P(E^{B_R}_{0,c}) \\ &= P(\sigma_R < \infty) P(E^{B_R}_{0,c}) \\ &\geq P(\tau_{\varnothing} = \infty)\cdot(1-\epsilon). \end{aligned}$$

Since ϵ is arbitrary, $P(E^{\{0\}}_{0,c} \mid \tau^0_{\varnothing} = \infty) = 1$. Finally, since $D_R \subset B_R$ for any R, (7) follows (with a slightly smaller c). □

The remainder of this section is devoted to the proof of Proposition 1. We will estimate the probability that $\xi^{B_R}_t$ contains the box $B_{R(t)}$ for all times $t \geq t_1$, where $R(0) = R$ and $R(t)$ grows approximately linearly in t. By choosing t_1 and $R(t)$ appropriately, and by letting $R \to \infty$, we will show that this probability tends to one, thereby demonstrating (13).

Now, let

$$R = \beta + \gamma,$$

where γ is a positive constant, depending only on κ and d, which will be identified in Section 3. We will let β approach ∞, so that R will also. For $k = 0, 1, \cdots$, put

$$\beta_k = 2^k\beta, \qquad s_k = 2\sqrt{d}\,\beta_k/\mu,$$

where $\mu > 0$ is another constant, depending only on κ and d, which will be prescribed in Section 3. Introduce a sequence of increasing times

$$t_k = \textstyle\sum_{j=1}^k s_j \qquad\qquad (k = 0, 1, \cdots;\quad t_0 = 0,$$

and a corresponding sequence of boxes $B_{R(k)}$, where

$$R(k) = \beta_k + \gamma.$$

We consider the probability that for all $k \geq 1$, $\xi^{B_R}_t$ covers $R(k)$ if $t \in [t_k, t_{k+1}]$, an event which is represented by the shaded region of the space-time "cross-section" in Figure i. Let

$$\mathbf{k}_R = \min\{k \geq 1 \colon B_{R(k)} \not\subset \xi^{B_R}_t \text{ for some } t \in [t_k, t_{k+1}]\} \qquad (= \infty \text{ if no such } t \text{ exists}).$$

Note that $\{\mathbf{k}_R = \infty\} \subset E^{B_R}_{0,c}$, where $c = \beta/t_1 = \mu/4\sqrt{d}$ is the slope of the right dotted line in Figure i (as a function of t). To prove Proposition 1, it therefore suffices to show that $P(\mathbf{k}_R < \infty) \to 0$ as $R \to \infty$. For $k \geq 1$, write

$$\begin{aligned} F_k &= \{\exists\, t \in [t_k, t_{k+1}] \colon B_{R(k)} \not\subset \xi^{B_R}_t\}, \\ G_k &= \{B_{R(k)} \subset \xi^{B_R}_{t_k}\}. \end{aligned}$$

Note that

$$P(\mathbf{k}_R = k) \leq P(F_k \cap G_{k-1}) \leq P(F_k \mid G_{k-1}),$$

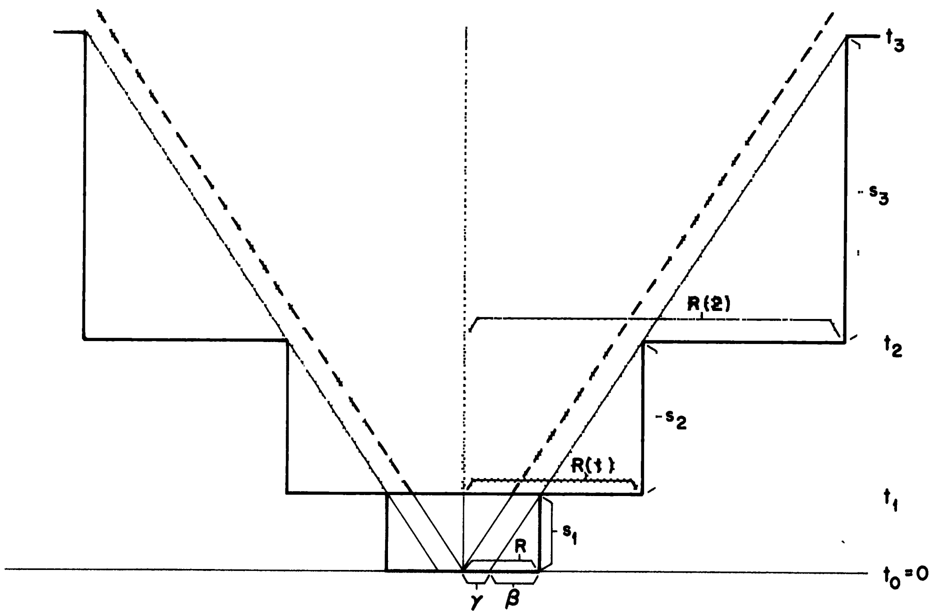

figure i

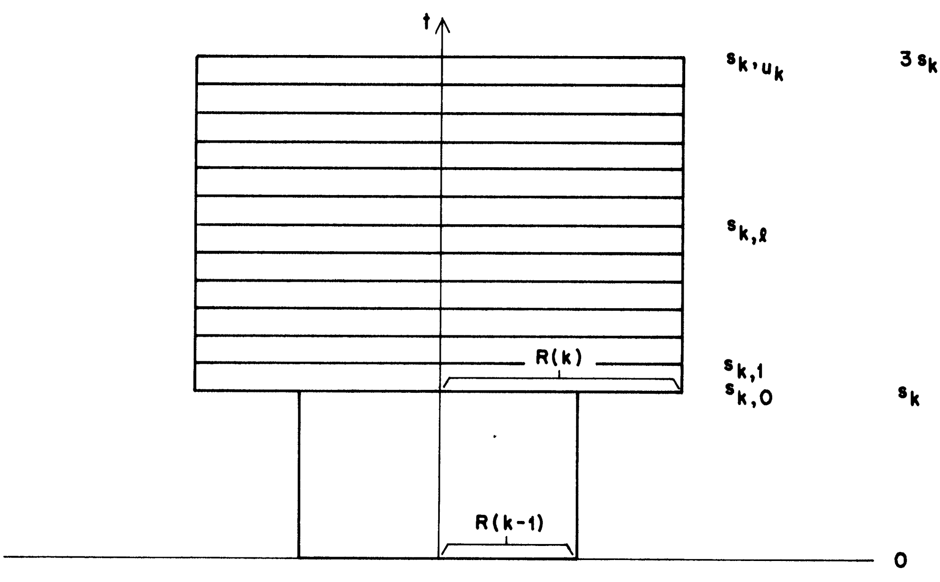

figure ii

and therefore that

$$P(\mathbf{k}_R < \infty) \le \sum_{k=1}^{\infty} P(F_k \mid G_{k-1}).$$

Applying the Markov property at time t_{k-1}, and using (11), we get

$$\begin{aligned} P(F_k \mid G_{k-1}) &= \sum_{\Lambda \supset B_{R(k-1)}} P(\xi_{t_{k-1}}^{B_R} = \Lambda \mid G_{k-1}) P(\exists t \in [s_k, 3s_k]\colon B_{R(k)} \not\subset \xi_t^{\Lambda}) \\ &\le \sum_{\Lambda \supset B_{R(k-1)}} P(\xi_{t_{k-1}}^{B_R} = \Lambda \mid G_{k-1}) P(\exists t \in [s_k, 3s_k]\colon B_{R(k)} \not\subset \xi_t^{B_{R(k-1)}}) \\ &= P(\exists t \in [s_k, 3s_k]\colon B_{R(k)} \not\subset \xi_t^{B_{R(k-1)}}) \overset{\text{def}}{=} p_R(k). \end{aligned}$$

Hence, Proposition 1 will be proved once we have shown that

$$\lim_{R\to\infty} \sum_{k=1}^{\infty} p_R(k) = 0. \tag{14}$$

Next, define

$$u_k = 4^{kd'+\theta\beta},$$

where $d' = d + 1$ and θ is a small positive number which will be chosen later. To obtain (14), we subdivide $[s_k, 3s_k]$ into intervals $I_{k,l}$ of equal length:

$$I_{k,l} = [s_{k,l}, s_{k,l+1}] \qquad 0 \le l \le u_k - 1,$$

where $s_{k,l} = s_k + (l/u_k)s_{k+1}$, $0 \le l \le u_k$. Such a partition is illustrated in Figure ii. Also, introduce the events

$$\begin{aligned} H_{k,l} &= \{B_{R(k)} \not\subset \xi_{s_{k,l}}^{B_{R(k-1)}}\} \\ \bar{H}_{k,l} &= \{\exists t \in I_{k,l}\colon B_{R(k)} \not\subset \xi_t^{B_{R(k-1)}}\} \cap \{B_{R(k)} \subset \xi_{s_{k,l}}^{B_{R(k-1)}} \cap \xi_{s_{k,l+1}}^{B_{R(k-1)}}\}. \end{aligned}$$

Observe that

$$\begin{aligned} p_R(k) &= P([\cup_l H_{k,l}] \cup [\cup_l \bar{H}_{k,l}]) \\ &\le u_k[\max_l P(H_{k,l}) + \max_l P(\bar{H}_{k,l})]. \end{aligned} \tag{15}$$

The verification of (14) is accomplished with the aid of the following two estimates.

LEMMA 2.

$$\max_l P(\bar{H}_{k,l}) \le (2R(k))^d \cdot \mu_1 \beta^2 4^{-2(kd'+\theta\beta)+k}, \tag{16}$$

where μ_1 is a constant depending on only κ and d.

PROPOSITION 2. *There are constants $C_\lambda < \infty$ and $\lambda > 0$, depending only on κ and d, such that for $\beta \ge \gamma$,*

$$\max_l P(H_{k,l}) \le (2R(k))^d \cdot 2C_\lambda e^{-\lambda\beta_{k-1}}. \tag{17}$$

The arguments for (16) and (17) will be presented in the next section.

PROOF OF PROPOSITION 1 ASSUMING (16) AND (17). By (15)–(17), if $\beta \ge \gamma$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} p_R(k) &\le \sum_{k=1}^{\infty} u_k (2R(k))^d [\mu_1 \beta^2 4^{-2(kd'+\theta\beta)+k} + 2C_\lambda e^{-\lambda\beta_{k-1}}] \\ &= 2^d \mu_1 \beta^2 4^{-\theta\beta} \sum_{k=1}^{\infty} (2^k\beta + \gamma)^d 4^{-kd} + C_\lambda 2^{d+1} 4^{\theta\beta} \sum_{k=1}^{\infty} (2^k\beta + \gamma)^d 4^{kd'} e^{-\lambda 2^{k-1}\beta}. \end{aligned}$$

Both sums are convergent, and of the same order as their $k = 1$ terms, uniformly in β for β bounded below by a positive constant. Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} p_R(k) &= O\left(\frac{\beta^d}{4^{\theta\beta}}\right) + O\left(\beta^d \left(\frac{4^{\theta}}{e^{\lambda}}\right)^{\beta}\right) \\ &\to 0 \quad \text{as} \quad \beta \to \infty, \end{aligned}$$

provided $\theta > 0$ is chosen small enough that $4^\theta < e^\lambda$. □

In the next section we will complete the derivation of Theorem 1 by presenting the proofs of Lemmas 1 and 2 and Proposition 2.

3. The proof of Theorem 1: details.

PROOF OF LEMMA 1. Fix $R > 0$. Set

$$\sigma_R^\Lambda = \min\{t\colon \xi_t^\Lambda \supset B_{x,R} \text{ for some } x \in Z^d\}$$
$$(= \infty \text{ if no such } x, t \text{ exist});$$

$\sigma_R = \sigma_R^{\{0\}}$. By translation invariance, $\sigma_R^{\{y\}}$ has the same distribution as σ_R for each $y \in Z^d$. Let $P(\sigma_R^{\{y\}} \le 1) \equiv p$. Clearly $p > 0$, since each $\{y\} \in Z^d$ communicates with any $B_{x,R}$. Using the Markov property and (11), we have for any $n \ge 0$,

$$\begin{aligned} P(n < \sigma_R \le n+1) &= \textstyle\sum_{\Lambda\ne\varnothing} P(\xi_n^0 = \Lambda,\ \sigma_R > n) P(\sigma_R^\Lambda \le 1) \\ &\ge \textstyle\sum_{\Lambda\ne\varnothing} P(\xi_n^0 = \Lambda,\ \sigma_R > n) P(\sigma_R \le 1) \\ &= pP(\sigma_R > n,\ \tau_\varnothing^0 > n). \end{aligned}$$

Thus,

$$\begin{aligned} P(\sigma_R > n+1,\ \tau_\varnothing^0 > n+1) &\le P(\sigma_R > n+1,\ \tau_\varnothing^0 > n) \\ &\le (1-p) P(\sigma_R > n,\ \tau_\varnothing^0 > n), \end{aligned}$$

and hence by induction,

$$P(\sigma_R > n,\ \tau_\varnothing^0 > n) \le (1-p)^n.$$

Let $n \to \infty$ to get $P(\sigma_R = \infty,\ \tau_\varnothing^0 = \infty) = 0$, or equivalently, $P(\sigma_R < \infty \mid \tau_\varnothing^0 = \infty) = 1$. □

PROOF OF LEMMA 2. For $\bar{H}_{k,l}$ to take place, some site $x \in B_{R(k)}$ must be infected at times $s_{k,l}$ and $s_{k,l+1}$ but healthy at some intermediate time. Thus,

$$P(\bar{H}_{k,l}) \le (2R(k))^d \max_{x\in B_{R(k)}} P(\exists t \in I_{k,l}\colon x \notin \xi_t^{B_{R(k-1)}}, x \in \xi_{s_{k,l}}^{B_{R(k-1)}} \cap \xi_{s_{k,l+1}}^{B_{R(k-1)}}). \tag{18}$$

Consider the graphical representation of $\{(\xi_t^A)\}$. The event on the right side of (18) can only occur if an arrow of type $\delta \leftarrow$ arrives at (x, t) for some $t \in I_{k,l}$ and then an arrow of either type arrives at (x, t') for some later $t' \in I_{k,l}$. The probability of this is

$$\begin{aligned} \iint_{0<t<t'<u_k^{-1}s_{k+1}} (e^{-t}\,dt)(\kappa e^{-\kappa t'}\,dt') &\le \left[\int_{0<t<u_k^{-1}s_{k+1}} e^{-t}\,dt\right]\left[\int_{0<t'<u_k^{-1}s_{k+1}} \kappa e^{-\kappa t'}\,dt'\right] \\ &= (1 - e^{-u_k^{-1}s_{k+1}})(1 - e^{-\kappa u_k^{-1}s_{k+1}}) \\ &\le \kappa u_k^{-2} s_{k+1}^2 = \mu_1 \beta^2 4^{-2(kd'+\theta\beta)+k}. \end{aligned}$$

where $\mu_1 = 16\, kd/\mu^2$. Using this estimate in (18), we obtain Lemma 2. □

PROOF OF PROPOSITION 2. Since

$$P(H_{k,l}) \le (2R(k))^d \max_{x\in R(k)} P(x \notin \xi_{s_{k,l}}^{B_{R(k-1)}}),$$

it suffices to show that for $\beta \ge \gamma$,

$$P(x \notin \xi_{s_{k,l}}^{B_{R(k-1)}}) \le 2C_\lambda e^{-\lambda\beta_{k-1}} \qquad \forall\ x \in B_{R(k)}, \qquad k \ge 1. \tag{19}$$

We analyse the left side of (19) with the aid of a duality equation, which asserts that

$$P(\xi_t^B \cap A = \varnothing) = \hat{P}(\hat{\xi}_t^A \cap B = \varnothing) \qquad A, B \in S, \tag{20}$$

where $\{(\hat{\xi}_t^A)\}$ is the system of coalescing random walks with nearest neighbor births described in Section 1. There are several different approaches to (20); we refer the reader to [6], [8], [9] or [10] for various proofs. Here we simply mention that there is a graphical representation of $\{(\hat{\xi}_t^A)\}$ precisely like the one for $\{(\xi_t^A)\}$ described in Section 2, except that the obstructions δ occur at the tails of the second type of arrow rather than at the heads. One proof is based on this fact. Taking $A = \{x\}$ in (20), one has

(21) $$P(x \notin \xi_t^B) = \hat{P}(\hat{\xi}_t^x \cap B = \varnothing).$$

Next, we introduce a certain Markov chain (X_t^x) which is imbedded in $(\hat{\xi}_t^x)$, i.e., which satisfies

(22) $$X_t^x \in \hat{\xi}_t^x \qquad \forall\, t \geq 0.$$

In the graphical representation of $\{(\hat{\xi}_t^A)\}$, the process (X_t^x) starts at x and displaces according to any random walk arrow $\delta \rightarrow$ it encounters, but only according to certain selected branching arrows $\rightarrow$. Namely, the chain follows any branching arrow which carries it closer to the origin; if some coordinate equals 0, it also follows a branching arrow leading to the neighbor with the same coordinate equal to 1 (this last condition is simply a technicality). Clearly (22) holds, since $((X_t^x, t))$ is a path up in the space-time diagram. The jump rates for (X_t^x) are:

(23)
$$\begin{aligned} &y \rightarrow z \text{ at rate } \kappa/2d \text{ if } \|z - y\| = 1 \text{ and } \textit{either } \|z\| < \|y\| \\ &\qquad \textit{or } y = (y_1, \ldots, y_d) \text{ and } z = (z_1, \ldots, z_d) \text{ satisfy} \\ &\qquad y_i = 0,\ z_i = 1 \text{ for some } i\colon \leq i \leq d. \\ &y \rightarrow z \text{ at rate } 1/2d \text{ if } \|z - y\| = 1 \text{ and the other above} \\ &\qquad \text{condition does not hold.} \end{aligned}$$

(No other transitions take place.) The key observation for our purposes is that (X_t^x) has uniformly positive drift toward the origin off of some ball D_γ. It is not difficult to show that for large values of γ, the drift of $\|X_t^x\|$ toward 0 is minimized over states in $Z^d - D_\gamma$ at sites located on the axes, and that this minimal drift is asymptotically $(\kappa - 1)/2d$ as $\gamma \rightarrow \infty$. Since exactly half of the neighbors z of any site y satisfy the first condition in (23), the total rate at which (X_t^x) leaves any state is $(\kappa + 1)/2$. Therefore, the minimal expected displacement toward 0 when a jump occurs, for states in $Z^d - D_\gamma$, is asymptotically $(1/d$ $(\kappa - 1)/(\kappa + 1)$.

We now define a family of continuous time processes on $[-\gamma, \infty) \subset R^1$ by

(24) $$Z_t^{(x,\alpha)} = \|X_{2t/(\kappa+1)}^x\| - \gamma,$$

where $\alpha = \|x\| - \gamma$. Each $(Z_t^{(x,\alpha)})$ makes jumps at total rate 1, and is, in a sense, a projection of $(X_{2t/(\kappa+1)}^x)$. For γ chosen large enough, we may select $\mu > 0$, $\epsilon > 0$ so that $(Z_t^{(x,\alpha)})$ has minimal drift toward 0 of $\mu + \epsilon > 0$ on $[0, \infty)$; in this case μ and ϵ will clearly satisfy

$$0 < \mu < \mu + \epsilon < \frac{1}{d}\frac{\kappa - 1}{\kappa + 1}.$$

The constants γ and μ so chosen are the ones which appeared in the previous section; note that they depend only on κ and d. Using (21), (22), (24) and the inclusions $D_R \subset B_{R'}$ $B_{R'} \subset D_{\sqrt{d}R'}$, we have for any $x \in B_{R'}$,

$$\begin{aligned} P(x \notin \xi_t^{B_R}) &= \hat{P}(\hat{\xi}_t^x \cap B_R = \varnothing) \\ &\leqq \hat{P}(X_t^x \notin B_R) \\ &= P(Z_t^{(x,\alpha)} \geq R - \gamma) \\ &\leq \max_{\alpha \leq \sqrt{d}R'} P(Z_t^{(x,\alpha)} \geq R - \gamma). \end{aligned}$$

Therefore, for $\beta \geq \gamma$,

(25) $$\max_{x \in B_{R(k)}} P(x \notin \xi_{s_{k,l}}^{B_{R(k-1)}}) \leq \max_{\alpha \leq 2\sqrt{d}\beta_k} P(Z_{s_{k,l}}^{(x,\alpha)} \geq \beta_{k-1}).$$

$(Z_t^{(x,\alpha)})$ has a positive minimal drift toward 0 on $[0, \infty)$. A large deviations estimate is needed to assert that the right side is of order $e^{-\lambda\beta_{k-1}}$. This will establish (19), and complete the proof of Theorem 1. A general result which suits our needs is the following.

PROPOSITION 3. *Let (Z_t^α) be a continuous time pure jump process, with $Z_0^\alpha = \alpha$. (Z_t^α) is assumed to have independent mean-1 exponential holding times. Let N_t denote the number of jumps made by (Z_t^α) up to time t, and let (Y_n^α) be the imbedded discrete time process,* i.e., *Y_n^α is the value of Z_t^α after exactly n jumps. Assume that N_t and Y_n^α are independent, that*

(26) $$|Y_{n+1}^\alpha - Y_n^\alpha| \leq M < \infty \qquad \forall n,$$

and that

(27) $$E[Y_{n+1}^\alpha - Y_n^\alpha \mid \mathscr{F}_n] \leq -\mu - \epsilon \qquad \text{a.s. whenever } Y_n^\alpha \geq 0,$$

where $\mu, \epsilon > 0$. ($\mathscr{F}_n$ is the σ-algebra generated by $Y_0^\alpha, Y_1^\alpha, \ldots Y_n^\alpha$.) Then, there is a $\lambda > 0$ and a $C_\lambda < \infty$ such that

(28) $$P(Z_t^\alpha \geq \beta) \leq C_\lambda e^{-\lambda\beta}(1 + e^{-\lambda(\mu t - \alpha)}) \qquad \forall \alpha, \beta, t.$$

The proof of Proposition 3 is complicated by the fact that the bound on the drift in (27) is only assumed on $[0, \infty)$. So before proceeding to the proof, we show that (19) follows immediately from (25) and (28).

PROOF OF PROPOSITION 2 ASSUMING PROPOSITION 3. For any choice of x, $Z_t^\alpha = Z_t^{(x,\alpha)}$ satisfies the hypotheses of Proposition 3, so

$$\max_{\alpha \leq 2\sqrt{d}\beta_k} P(Z_{s_{k,l}}^{(x,\alpha)} \geq \beta_{k-1}) \leq C_\lambda e^{-\lambda\beta_{k-1}}(1 + e^{-\lambda(\mu s_{k,l} - 2\sqrt{d}\beta_k)})$$
$$\leq C_\lambda e^{-\lambda\beta_{k-1}}(1 + e^{-\lambda(\mu s_k - 2\sqrt{d}\beta_k)}),$$

which, by definition of s_k, equals

$$2C_\lambda e^{-\lambda\beta_{k-1}}.$$

Using (25), we obtain (19) as desired. □

Let us now state and prove the preliminary large deviation type results which we will need to obtain Proposition 3. We now demonstrate Proposition 3, thereby completing the proof of Theorem 1.

PROOF OF PROPOSITION 3. Inequality (28) follows from

(29) $$E[e^{\lambda Z_t^\alpha}] \leq C_\lambda(1 + e^{-\lambda(\mu t - \alpha)})$$

by means of a Chebyshev estimate. Inequality (29) is in turn a consequence of the differential inequality

(30) $$\frac{d}{dt} E[e^{\lambda Z_t^\alpha}] \leq -\lambda\mu\, E[e^{\lambda Z_t^\alpha}] + \lambda\mu + e^{\lambda M},$$

which we now proceed to prove. Making use of the independence of N_t and Y_n^α, we obtain

(31) $$\frac{d}{dt} E[e^{\lambda Z_t^\alpha}] = \frac{d}{dt} \sum_{n=0}^\infty e^{-t} \frac{t^n}{n!} E[e^{\lambda Y_n^\alpha}]$$
$$= \sum_{n=0}^\infty e^{-t} \frac{t^n}{n!} E[e^{\lambda Y_{n+1}^\alpha} - e^{\lambda Y_n^\alpha}].$$

(The reversal of limits is easy to justify.) Now,

$$(32)\qquad E[e^{\lambda Y^{\alpha}_{n+1}} - e^{\lambda Y^{\alpha}_n}] = E[e^{\lambda Y^{\alpha}_n}(e^{\lambda(Y^{\alpha}_{n+1}-Y^{\alpha}_n)} - 1)] = E[e^{\lambda Y^{\alpha}_n}E[e^{\lambda(Y^{\alpha}_{n+1}-Y^{\alpha}_n)} - 1 \mid \mathscr{F}_n]].$$

Linearization of the exponential function at 0, together with (26) and (27), shows that for $\lambda > 0$ chosen small enough,

$$(33)\qquad E[e^{\lambda(Y^{\alpha}_{n+1}-Y^{\alpha}_n)} - 1 \mid \mathscr{F}_n] \le -\lambda\mu$$

for $Y^{\alpha}_n \ge 0$. On the other hand, for $Y^{\alpha}_n < 0$,

$$(34)\qquad E[e^{\lambda(Y^{\alpha}_{n+1}-Y^{\alpha}_n)} - 1] \le e^{\lambda M}.$$

Therefore, by (33) and (34), (32) is at most

$$E[e^{\lambda Y^{\alpha}_n}(-\lambda\mu\chi_{[0,\infty)}(Y^{\alpha}_n) + e^{\lambda M}\chi_{(-\infty,\,0)}(Y^{\alpha}_n))],$$

where $\chi.$ denotes the indicator function. By simple algebra this is at most

$$-\lambda\mu E[e^{\lambda Y^{\alpha}_n}] + \lambda\mu + e^{\lambda M},$$

which shows that

$$E[e^{\lambda Y^{\alpha}_{n+1}} - e^{\lambda Y^{\alpha}_n}] \le -\lambda\mu E[e^{\lambda Y^{\alpha}_n}] + \lambda\mu + e^{\lambda M}.$$

Together with (31), this last estimate proves (30), and hence the proposition. □

4. Additional results and remarks. In this last section, we give the proofs of Theorems 2 and 3, and make a few remarks.

PROOF OF THEOREM 2. Let $\tau^A_{\varnothing} = \min\{t: \xi^A_t = \varnothing\}$ ($= \infty$ if no such t exists). With $E^{\Lambda}_{x,c}$ defined as in Section 2, and c appropriately chosen, we have shown that $E^{\{0\}}_{0,c} \approx \{\tau^0_{\varnothing} = \infty\}$ ($\approx$ means equality up to a P-null set). By translation invariance, $E^{\{x\}}_{0,c} \approx \{\tau^x_{\varnothing} = \infty\}$ for each $x \in Z^d$. Therefore, making use of (10) and (11), we have

$$P(\tau^A_{\varnothing} = \infty) = P(\cup_{x\in A}\{\tau^x_{\varnothing} = \infty\}) = P(\cup_{x\in A}E^{\{x\}}_{0,c} \le P(E^A_{0,c}).$$

Since $E^A_{0,c} \subset \{\tau^A_{\varnothing} = \infty\}$, (7) holds when $\{0\}$ is replaced by any $A \in S$. If A is infinite, then by (10),

$$P(\tau^A_{\varnothing} < \infty) \le P(\cap^{\infty}_{n=1}\{\tau^{A\cap B_n}_{\varnothing} < \infty\}) = \lim_{n\to\infty}P(\tau^{A\cap B_n}_{\varnothing} < \infty) = \lim_{n\to\infty}\kappa^{-|A\cap B_n|} = 0,$$

where we have made use of the gambler's ruin observation mentioned in the introduction. For A infinite, (8) now follows as a weaker variant of (7). □

PROOF OF THEOREM 3. Theorem 2 implies the weaker result:

$$(35)\qquad P(\xi^A_t \in \cdot) \Rightarrow P(\tau^A_{\varnothing} < \infty)\,\delta_{\varnothing} + P(\tau^A_{\varnothing} = \infty)\,\delta_{Z^d} \quad \text{as} \quad t \to \infty,$$

for any $A \in S$. It was noted in [5] that if the "complete convergence" (35) holds, and if $\{(\hat{\xi}^A_t)\}$ is the dual system related to $\{(\xi^A_t)\}$ by (20), then there is an invariant measure ν such that

$$\hat{P}(\hat{\xi}^A_t \in \cdot) \Rightarrow \hat{P}(\tau^A_{\varnothing} < \infty)\,\delta_{\varnothing} + \hat{P}(\tau^A_{\varnothing} = \infty)\nu \quad \text{as} \quad t \to \infty,$$

i.e., complete convergence holds for the dual system as well. ν is defined by

$$\nu = \lim_{t\to\infty} \hat{P}(\xi_t^{Z^d} \in \cdot).$$

(A monotonicity argument shows that the limit exists.) In our case, complete convergence therefore holds for the coalescing random walks with nearest neighbor births, the dual system for the biased voter model. Since our dual system satisfies

$$\hat{P}(\tau_\varnothing^A < \infty) = 1 \qquad A = \varnothing$$
$$= 0 \qquad A \neq \varnothing,$$

we obtain (9). Uniqueness of the invariant measure ν on $S - \{\varnothing\}$ follows easily from ergodicity on $S - \{\varnothing\}$. □

REMARKS. (i) By keeping track of constants, and by being more careful at various points in the argument, it is possible to show that one can take c to be any number less than $(\kappa - 1)/2d$ in Theorem 1. In particular, this involves the use of balls D_R rather than boxes B_R throughout the proof, a somewhat more awkward procedure. We note that $(\kappa - 1)/2d$ is the correct growth constant when $d = 1$. (For $d > 1$, the correct constant is greater than $(\kappa - 1)/2d$.)

(ii) The key to our analysis lies in the fact that the dual processes (ξ_t^x) cannot die out. This ensures that the imbedded chain lives for arbitrarily long stretches of time. A model which is not susceptible to our techniques is the d-dimensional basic *contact system* $\{(\xi_t^A)\}$, such that for some $\lambda > 0$,

$$A \to A \cup \{x\} (x \notin A) \quad \text{at rate} \quad \lambda \,|\{y \in A\colon \|y - x\| = 1\}|,$$
$$A \to A - \{x\} (x \in A) \quad \text{at rate} \quad 1.$$

It is known that for λ sufficiently large, $P(\tau_\phi^0 = \infty) > 0$, so that on $\{\tau_\phi^0 = \infty\}$ one can ask about the asymptotic "shape" of the set ξ_t^0. Now, since recovery always occurs at rate 1, ξ_t^0 is not a solid blob. Nevertheless, its frontier should expand linearly in radius, while its interior should settle down to equilibrium. A result along these lines can be proved in one dimension; see [6] for example. In higher dimensions, it is not even known whether $|\xi_t^0|$ grows like t^d. The best result, due to Harris [9], states that

$$P\left(\liminf_{t\to\infty} \frac{|\xi_t^0|}{t} > 0 \,\Big|\, \tau_\phi^0 = \infty\right) = 1$$

for λ sufficiently large. Since a basic contact system is its own dual, one cannot find a chain imbedded in the dual system. Thus, a new technique is needed to treat these more delicate systems.

(iii) The limiting case $\kappa = 1$ is critical, so $P(\tau_\varnothing^0 < \infty) = 1$. Therefore, the behavior of $\xi_t^0 \,|\, \tau_\phi^0 = \infty$ for large t is much more difficult to determine. To date, the asymptotics for

$$E(|\xi_t^0| \,|\, \tau_\phi^0 = \infty)$$

are not known. Partial results may be found in [1] and [11].

ADDED IN PROOF. See the supplementary references [21], [22] for recent work concerning Remarks (ii) and (iii).

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MATHEMATICS DEPARTMENT
VINCENT HALL
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455

MATHEMATICS DEPARTMENT
VAN VLECK HALL
480 LINCOLN DR.
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706