The Annals of Probability
1981, Vol. 9, No. 6, 909–936

LIMITING POINT PROCESSES FOR RESCALINGS OF COALESCING AND ANNIHILATING RANDOM WALKS ON Z^d

BY RICHARD ARRATIA

University of Southern California

Let $p(x, y)$ be an arbitrary random walk on Z^d. Let ξ_t be the system of coalescing random walks based on p, starting with all sites occupied, and let η_t be the corresponding system of annihilating random walks. The spatial rescalings $P(0 \in \xi_t)^{1/d}\xi_t$ for $t \geq 0$ form a tight family of point processes on R^d. Any limiting point process as $t \to \infty$ has Lesbesgue measure as its intensity, and has no multiple points. When p is simple random walk on Z^d these rescalings converge in distribution, to the simple Poisson point process for $d \geq 2$, and to a non-Poisson limit for $d = 1$. For a large class of p, we prove that $P(0 \in \eta_t)/P(0 \in \xi_t) \to \frac{1}{2}$ as $t \to \infty$. A generalization of this result, proved for nearest neighbor random walks on Z^1, and for all multidimensional p, implies that the limiting point process for rescalings $P(0 \in \xi_t)^{1/d}\eta_t$ of the system of annihilating random walks is the *one half thinning* of the limiting point process for the corresponding coalescing system.

1. Introduction. We consider two interacting particle systems on the d-dimensional integer lattice Z^d: coalescing random walks ξ_t, and annihilating random walks η_t. Each process consists of identical particles, one starting from each site $x \in Z^d$. Each particle undergoes a continuous time random walk on Z^d, with mean one exponential holding times between jumps, based on some fixed transition kernel p. These random walks are independent, except that whenever a particle jumps to a site which is already occupied by another, there is interference. In the coalescing system ξ_t, the two particles coalesce into one (one particle vanishes); in the annihilating system η_t both particles in a collision vanish. The state space for each system is $\mathscr{S} = \{\text{all subsets of } Z^d\}$, where $x \in \xi_t$ or $x \in \eta_t$ if there is a particle present at site x at time t. The basic ergodic theory of these particle systems is easy; the configuration ϕ is a trap, and starting from Z^d or any other initial configuration

$$\xi_t \to_d \delta_\phi, \qquad \eta_t \to_d \delta_\phi.$$

Here, the convergence in distribution of ξ_t to δ_ϕ, the probability measure on $\mathscr{S}$ which is concentrated on the single configuration ϕ, means that for any finite $K \subset Z^d$,

$$P(\xi_t \cap K \neq \phi) \to 0 \quad \text{as } t \to \infty.$$

See Griffeath (1979) for an exposition of this and other basic results about interacting particle systems.

Since $\xi_t \to_d \delta_\phi$ it is natural to consider spatial rescalings

$$\nu_t \equiv \alpha_t \xi_t,$$

choosing α_t so that the density of particles per unit volume in R^d is always one. Rescalings of infinite particle systems are also considered in Holley and Stroock (1979) and Bramson and Griffeath (1979). For $\alpha > 0$, $x = (x_1, \cdots, x_d) \in R^d$, and $A \subset R^d$, write $\alpha x = (\alpha x_1, \cdots, \alpha x_d)$, $\alpha A = \{\alpha x \colon x \in A\} \subset R^d$. The following notation, which depends on the underlying random walk p, will be used throughout this paper. Let

$$(1) \qquad p_t = P(0 \in \xi_t), \qquad \alpha_t = p_t^{1/d},$$

and for any $B \subset R^d$, $t \geq 0$ let

$$B_t = \{x \in Z^d \colon \alpha_t x \in B\} = (p_t^{-1/d} B) \cap Z^d.$$

Received August 7, 1980.
AMS 1970 *subject classification.* Primary 60K35.
Key words and phrases. Interacting Particle System.

Thus, for any compact convex $B \subset R^d$ having Lebesgue measure $m(B) > 0$,

(2) $$E\,|\,\alpha_t\xi_t \cap B\,| = E\,|\,\xi_t \cap B_t\,| = p_t\,|\,B_t\,| \to m(B) \quad \text{as } t \to \infty.$$

Consider $\nu_t = \alpha_t\xi_t$ as the random measure

(3) $$\nu_t = \sum_{x\in\xi_t} \delta_{\alpha_t x}$$

on R^d having an atom of mass one at the rescaled location of each particle. Statement (2) shows that the family $\{\nu_t,\ t \ge 0\}$ is tight, so every sequence ν_{t_i} with $t_i \to \infty$ must have a subsequence which converges to a limiting point process μ on R^d. In general, for arbitrary p, we can show that any limit μ is a *simple* point process (i.e. μ has no multiple points) having Lebesgue measure m as its intensity. What limits μ are possible?

This question can be answered completely when p is *simple* random walk on Z^d: for each $d = 1, 2, \cdots$, there is a point process μ_d on R^d such that

$$\alpha_t\xi_t \to_d \mu_d \quad \text{as } t \to \infty.$$

Theorem 1 states that for $d \ge 2$, the limit above exists and is the basic Poisson point process. For $d = 1$, the limiting point process μ_1 on the line is *not* Poisson (Arratia 1979); formula (20) given in Section 2 specifies μ_1 in terms of its zero function. The proof of Theorem 1 depends upon knowing the asymptotic behavior of $p_t = P(0 \in \xi_t)$, obtained for simple random walks on Z^d, $d \ge 2$, in Bramson and Griffeath (1980b):

(4) $$\begin{aligned} p_t &\approx (\pi t)^{-1/2} && d = 1 \\ &\approx (\pi t/\log t)^{-1} && d = 2 \\ &\approx (\gamma_d t)^{-1} && d \ge 3, \end{aligned}$$

where γ_d is the probability that a d-dimensional simple random walk never returns to its origin. The other key ingredient for Theorem 1 is a negative correlation result, that

$$P(x, y \in \xi_t) \le P(x \in \xi_t)P(y \in \xi_t) = p_t^2,$$

which we prove for arbitrary random walk p, using Harris's correlation inequality (Harris 1977).

What relation is there between the systems ξ_t, coalescing random walks, and η_t, annihilating random walks? There is a coupling such that, for all $t \ge 0$, for every ω,

(5) $$\eta_t \subset \xi_t.$$

Recall that we start with all sites occupied: $\eta_0 = \xi_0 = Z^d$. The coupling is easy to construct directly: in each system, particles undergo the same random walks, and when two particles collide, one of them disappears in ξ_t, while both of them disappear in η_t. It may be brash to suggest that since twice as many particles vanish per collision in η_t compared with ξ_t, then the ratio of the density of particles in the two systems should go to one half as t goes to infinity:

(6) $$P(0 \in \eta_t)/P(0 \in \xi_t) = P(0 \in \eta_t\,|\,0 \in \xi_t) \to \tfrac{1}{2} \quad \text{as } t \to \infty.$$

Indeed, by the standard duality of coalescing and annihilating random walks with the finite voter model ζ_t^x starting with a single individual at x, (6) is equivalent to

(7) $$P(0 \in \eta_t\,|\,0 \in \xi_t) = P(|\,\zeta_t^0\,| \text{ is odd}\,|\,|\,\zeta_t^0\,| > 0) \to \tfrac{1}{2} \quad \text{as } t \to \infty.$$

Since $|\,\zeta_t^0\,|$ is a time change of simple (symmetric) random walk on the line, starting at one, with absorption at zero, (7) is highly plausible for any p. The only case for which (7) was previously known is that of p being a nearest neighbor walk on the integers; in this case the holding times for $|\,\zeta_t^0\,|$ before absorption are all exponential with mean ½. The reader is invited to try to prove (7) on his own for a special case such as simple random walk on Z^2. Theorem 3 establishes this one half density relation for any genuinely multidimensional p, and for random walks on the integers having $\sum p(0, x)\,|\,x\,| = \infty$. In the remaining cases,

p a non-nearest neighbor walk on the integers with finite expectation, relation (7) remains unproved.

For a large class of random walks p, Theorem 3 reduces the problem of finding the asymptotic density of particles in the annihilating system η_t to a problem about the survival probability for the finite voter model ζ_t^0. Partial results describing $P(\zeta_t^0 \neq \phi)$ appear in Sudbury (1976), Kelly (1977), and Sawyer (1979). For *simple* random walk on Z^d, $d \geq 2$, asymptotics for $p_t = P(0 \in \xi_t) = P(\zeta_t^0 \neq \phi)$ were finally established by Bramson and Griffeath (1980b). Thus, for $\eta_t = \eta_t^{Z^d}$, the system of annihilating *simple* random walks starting from all sites occupied,

$$\begin{aligned} P(0 \in \eta_t) &\approx 1/(2\sqrt{\pi t}) & d &= 1 \\ &\approx \log t/(2\pi t) & d &= 2 \\ &\approx 1/(2\gamma_d t) & d &\geq 3. \end{aligned}$$

The case with $d = 1$ above is in Griffeath (1979); the cases with $d \geq 2$ are an immediate consequence of Theorem 3 and the asymptotics (4) found by Bramson and Griffeath.

The one half density relation suggests that for large t, η_t may be approximately a "one half thinning" of ξ_t, i.e. a subset of ξ_t obtained by tossing a fair coin for each particle in ξ_t to decide whether to retain or delete that particle. Theorem 4 gives a precise version of this: if p is a multidimensional random walk on Z^d, or a nearest neighbor random walk on Z^1, then for any compact $K \subset R^d$, with K_t given by (1),

$$\textstyle\sum_{A \subset K_t} (P(\xi_t \cap K_t = A) \sum_{B \subset A} |2^{-|A|} - P(\eta_t \cap K_t = B \mid \xi_t \cap K_t = A)|) \to 0 \quad \text{as } t \to \infty.$$

A more palatable restatement of this appears as Corollary 3: there exist versions of $\Theta(\alpha_t \xi_t)$, the one half thinnings of the rescaled point processes $\alpha_t \xi_t$, such that for any compact $K \subset R^d$,

$$P((\alpha_t \eta_t)|_K \neq (\Theta \alpha_t \xi_t)|_K) \to 0 \quad \text{as } t \to \infty,$$

where $|_K$ denotes the restriction of a measure to K. This implies a weaker result, Corollary 4: for multidimensional random walk, or nearest neighbor random walk on the line,

$$\text{if} \quad \alpha_t \xi_t \to_d \mu \text{ (along some sequence)} \quad \text{then} \quad \alpha_t \eta_t \to \Theta\mu$$

along the same sequence, where $\Theta\mu$ is the one half thinning of the point process μ on R^d.

Combining the one half thinning result with the convergence results for rescalings of coalescing *simple* random walks on Z^d (a Poisson limit for $d \geq 2$, a non-Poisson limit when $d = 1$) and the asymptotic formulas (4) for $p_t = \alpha_t^d$, we get:

For the system η_t of annihilating *simple* random walks on Z^d starting with particles everywhere,

$d = 1$: $(\pi t)^{-1/2} \eta_t \to_d \Theta\mu_1$

$d = 2$: $(\pi t / \log t)^{-1/2} \eta_t \to_d \Theta\mu_2 =$ Poisson, intensity ½

$d \geq 3$: $(\gamma_d t)^{-1/d} \eta_t \to_d \Theta\mu_d =$ Poisson, intensity ½.

Here $\Theta\mu_1$ is the one half thinning of the point process μ_1 specified by formula (20). For $d \geq 2$, the one half thinning $\Theta\mu_d$ of the intensity one Poisson process μ_d on R^d is the Poisson point process with intensity one half.

The one half thinnings $\Theta\mu_d$ that are the rescaled limits for rescaled annihilating simple random walks are examples of a "compound point process." In general, if $\beta, \beta_1, \beta_2, \cdots$ are i.i.d., R^+-valued random variables which are independent of a point process $\mu = \sum \delta_{x_i}$, then the β-compound of μ is the random measure $\sum \beta_i \delta_{x_i}$. In the one half thinning example above, β is the fair coin variable with $P(\beta = 1) = P(\beta = 0) = ½$. For $d \geq 2$, another example of a β-compound of the Poisson point process μ_d on R^d, with β exponentially distributed, arises as the limit of rescalings of a system γ_t of coalesing simple random walks on Z^d in which mass is preserved. Start with a particle of mass 1 at each site $x \in Z^d$. Whenever two particles in γ_t collide, they coalesce into a single particle whose mass is the sum of the colliding masses. All particles, regardless of their masses, still undergo identical random walks, independent apart from the coalescing interference. The state space now is $(Z^+)^{Z^d}$, where $\gamma_t(x) = n$ means that there is a particle of mass n at site $x \in Z^d$, $n = 0$ (representing no particle present), 1, 2, $\cdots$. This system arises naturally in studying coalescing and annihilating random walks; the coupling (5) may be achieved by taking

$$\xi_t = \{x: \gamma_t(x) > 0\}, \qquad \eta_t = \{x: \gamma_t(x) \text{ is odd}\}.$$

The appropriate rescaling of γ_t is the random measure μ_t on R^d defined by

$$\mu_t = \textstyle\sum_{x \in \xi_t} p_t \gamma_t(x) \delta_{\alpha_t x}, \tag{8}$$

having atoms of mass p_t, $2p_t$, $\cdots$, carried on the lattice $\alpha_t Z^d$. Extending a moment calculation by Sawyer (1979), Bramson and Griffeath (1980b) show for *simple* random walk on Z^d, $d \geq 2$, that for any $a \geq 0$,

$$\lim_{t\to\infty} P\left(\gamma_t(x) > \frac{a}{p_t} \,\middle|\, \gamma_t(x) > o\right) = e^{-a}. \tag{9}$$

Theorem 2 says that in a rescaled limit, these exponential masses are independent of the particle locations and each other. More precisely, for simple random walk on Z^d with $d \geq 2$, μ_t converges in distribution to the β-compound of the simple, intensity one Poisson point process, where β is exponential with mean one. In terms of Laplace transforms, Theorems 1 and 2 are equivalent to, and are proved by showing: for any continuous non-negative f having compact support in R^d,

$$L_{\nu_t}(f) \equiv Ee^{-\int f d\nu_t} \to L_{\mu_d}(f) = e^{-\int (1-e^{-f})\, dm},$$

$$L_{\mu_t}(f) \equiv Ee^{-\int f d\mu_t} \to L_{\mu_d}(-\log L_\beta \circ f) = e^{-\int f/(1+f)\, dm}$$

(where ν_t and μ_t are specified in (3) and (8).)

2. Rescaling Coalescing Random Walks

2.1 Arbitrary Random Walks. We continue with the notation $\nu_t = \alpha_t \xi_t$, where $p_t = P(0 \in \xi_t)$, $\alpha_t = (p_t)^{1/d}$, and $\xi_t = \xi_t^{Z^d}$ is a system of coalescing random walks on Z^d, starting with all sites occupied, based on an arbitrary random walk p. This rescaling is appropriate in the sense that the family $\{\nu_t, t \geq 0\}$ of point processes is tight, with respect to the vague topology.

[Here are the details: Let B^ϵ denote the ϵ-neighborhood of B in the Euclidean metric on R^d. We have, for any $t \geq 0$, and B bounded

$$E\nu_t(B) = |\alpha_t Z^d \cap B| p_t \leq m(B^{\alpha_t\sqrt{d}}) p_t = m(B^{\sqrt{d}}) < \infty. \tag{10}$$

Using Chebyshev's inequality, this implies, for compact B, that

$$\lim_{a\to\infty} \sup_{t\geq 0} P(\nu_t B > a) = 0,$$

which is equivalent to tightness for the family $\{\nu_t\}$. See Kallenberg (1975) for a general reference on random measures.]

Consider a limiting point process μ on R^d, that is, suppose that $\nu_{t_i} \to_d \mu$ for some sequence $t_i \to \infty$. It is easily seen that for any compact, convex $B \subset R^d$,

$$E\mu(B) \leq \limsup E\nu_t(B) = m(B).$$

If we are merely given some translation invariant spatially ergodic $\xi_t \subset Z^d$ having $p_t \to 0$, without knowing that the ξ_t are obtained from coalescing random walks, then a limiting point process μ might have $E\mu(B) < m(B)$. This can happen iff the random variables

$\nu_t(B)$ are not uniformly integrable, which indicates clustering in ξ_t. Another way to detect clustering in ξ_t, relative to the scale of distance $\alpha_t = P(0 \in \xi_t)^{1/d}$, would be through the existence of multiple points—atoms of mass 2, 3, $\cdots$, in a limiting point process μ. Lemma 1 will be the key to showing that no such clustering occurs—the full strength of Lemma 1 is not needed here; an estimate such as: for $x \neq y \in Z^d$

$$P(x, y \in \xi_t) \leq c p_t^2, \tag{11}$$

for some finite constant c, would be enough to establish Corollaries 1 and 2. It is necessary, in our proof of Theorem 1, to have (11) with $c = 1$ in order to conclude that for simple random walk on Z^d, $d \geq 2$, the limiting point process μ is a *simple* Poisson process, rather than some *mixture* of Poisson point processes (i.e. a Cox process).

LEMMA 1. *For any $A \subset Z^d$, and for an arbitrary random walk p on Z^d, let ξ_t^A be the system of coalescing random walks starting with a particle at each site $x \in A$. For any $x \neq y \in Z^d$,*

$$P(x, y \in \xi_t^A) \leq P(x \in \xi_t^A) P(y \in \xi_t^A). \tag{12}$$

In the special case $A = Z^d$, this becomes

$$P(x, y \in \xi_t) \leq p_t^2.$$

PROOF. Let $(\zeta_t^B, B \subset Z^d)$ be the family of voter models based on p, all constructed together via a single random substructure $\mathscr{P}$. ($\mathscr{P}$ is a collection of Poisson flows $T(x, y)$ on $[0, \infty)$ having rate $p(x, y) = p(0, y - x)$; the event times of $T(x, y)$ tell the voter at x when to discard his opinion and adopt the opinion held by the voter at y.) Thus for any $B, C \subset Z^d$, for each ω, $\zeta_t^{B \cup C} = \zeta_t^B \cup \zeta_t^C$. By the usual duality between coalescing random walks and the voter model, (12) is equivalent to

$$P(\zeta_t^x \cap A \neq \phi, \zeta_t^y \cap A \neq \phi) \leq P(\zeta_t^x \cap A \neq \phi) P(\zeta_t^y \cap A \neq \phi). \tag{13}$$

We apply Harris's (1977) elegant theorem on positive correlations: for a monotone Markov process on a finite partially ordered state space E, a necessary and sufficient condition for the set of measures on E having positive correlations to be preserved by the semigroup is that the process can only jump up or down. Ignore momentarily the requirement that the state space E be finite. Take

$$E = \{(B, C) : B \subset C \subset Z^d\}$$

with the ordering $(B, C) \leq (B', C')$ iff $B \subset B'$ and $C \subset C'$. Define the process starting at $(B, C) \in E$ to be

$$X_t^{(B,C)} = (\zeta_t^B, \zeta_t^C);$$

this process is monotone and has only jumps up and down. For an initial distribution having positive correlations, take the deterministic configuration $(\{x\}, Z^d - \{y\})$ which has zero correlations; thus we are considering the process

$$X_t = (\zeta_t^x, \zeta_t^{Z^d - \{y\}}) = (\zeta_t^x, Z^d - \zeta_t^y).$$

The conclusion, that for every $t \geq 0$ the distribution of X_t has positive correlations, says that for any increasing functions f, g on E,

$$E(f(X_t) g(X_t)) \geq Ef(X_t) Eg(X_t).$$

Take

$$f((B, C)) = 1_{B \cap A \neq \phi}, \qquad g((B, C)) = 1_{A \subset C};$$

these are increasing functions on E. The previous inequality becomes

$$P(\zeta_t^x \cap A \neq \phi, \zeta_t^y \cap A = \phi) \geq P(\zeta_t^x \cap A \neq \phi) P(\zeta_t^y \cap A = \phi). \tag{14}$$

Replacing the event $(\zeta_t^y \cap A = \phi)$ by its complement yields formula (13).

To comply with the requirement in Harris's theorem that the state space E be finite, an approximation is needed. For any $r > \max(|x|, |y|)$ let $S_r = \{z \in Z^d : |z| \le r\}$. For each ω modify the substructure $\mathscr{P}$ to create a new substructure $\mathscr{P}_r$, by deleting all event times of the clocks $T_{(z,z')}$ having $|z| > r$ or $|z'| > r$. Construct a family of voter models $({}_r\zeta_t^B, B \subset Z^d)$ based on $\mathscr{P}_r$; in this family no site outside S_r can give or receive influence. Let $E_r = \{(B, C) : B \subset C \subset S_r\} \subset E$, and for $(B, C) \in E_r$ define the process ${}_rX_t^{(B,C)} = ({}_r\zeta_t^B, {}_r\zeta_t^C)$; this is a monotone Markov process on the finite state space E_r. Take unit mass on $(\{x\}, S_r - \{y\})$ as the initial distribution. Define increasing functions f_r and g_r on E_r by

$$(15) \qquad f_r(B, C) = 1_{B \cap A \ne \phi}, \qquad g_r(B, C) = 1_{A \cap S_r \subset C}.$$

The theorem on positive correlations yields

$$(16) \qquad P({}_r\zeta_t^x \cap A \ne \phi, {}_r\zeta_t^y \cap A = \phi) \ge P({}_r\zeta_t^x \cap A \ne \phi) P({}_r\zeta_t^y \cap A = \phi).$$

Define $\tau_r = \inf\{t : \zeta_t^x \ne {}_r\zeta_t^x \text{ or } \zeta_t^y \ne {}_r\zeta_t^y\}$. Almost surely, $\tau_r \to \infty$ as $r \to \infty$; this is equivalent to the claim that the substructure $\mathscr{P}$ for the voter model has no influence from ∞. Thus, taking limits as $r \to \infty$ in (16) yields formula (14). □

A slight generalization of this is needed for Theorem 2; Lemma 1 is exactly the special case $m = n = 1$ of the following lemma.

LEMMA 2. *For an arbitrary random walk p on Z^d, $A \subset Z^d$, let γ_t^A be the system of coalescing random walks with mass conserved, starting with a particle of mass one at each site $x \in A$. Then $\forall\ t, m, n \ge 0$, $x \ne y \in Z^d$,*

$$(17) \qquad P(\gamma_t^A(x) \ge m, \gamma_t^A(y) \ge n) \le P(\gamma_t^A(x) \ge m) P(\gamma_t^A(y) \ge n).$$

PROOF. In terms of the usual coupling with the family of voter models, (17) is equivalent to

$$P(|\zeta_t^x \cap A| \ge m, |\zeta_t^y \cap A| \ge n) \le P(|\zeta_t^x \cap A| \ge m) P(|\zeta_t^y \cap A| \ge n).$$

The proof of this is exactly the proof given for Lemma 1, with the increasing functions f_r and g_r of (15) replaced by

$$f_{r,m}(B, C) = 1_{|B \cap A| \ge m}, \qquad g_{r,n}(B, C) = 1_{|(A \cap S_r) - C| < n}. \qquad \square$$

COROLLARY 1. *Let ν_t be the random measure on R^d defined by* (3), *for an arbitrary random walk p on Z^d. For any bounded $B \subset R^d$,*

$$(18) \qquad \operatorname{var}(\nu_t B) \le E(\nu_t B).$$

If, along some sequence $t_i \to \infty$, $\nu_{t_i} \to_d \mu$, then

$$E\mu(B) = m(B),$$

i.e. the intensity of any limiting point process μ is Lebesgue measure m on R^d.

PROOF. With B_t given by (1), $\nu_t(B) = |\xi_t \cap B_t|$. Identify ξ_t with its indicator function, i.e. write $\xi_t(x) = 1$ if $x \in \xi_t$; $\xi_t(x) = 0$ otherwise. Lemma 1 says that $E(\xi_t(x)\xi_t(y)) \le p_t^2$ if $x \ne y$. Thus

$$E(\nu_t B)^2 = E(\textstyle\sum_{x \in B_t} \xi_t(x))^2 = \sum_{x \in B_t} E(\xi_t(x)) + \sum_{x \ne y \in B_t} E(\xi_t(x)\xi_t(y))$$

$$\le |B_t| p_t + |B_t|^2 p_t^2 = E(\nu_t B) + (E(\nu_t B))^2,$$

which shows (18). Let $B \subset R^d$ be bounded and convex. A calculation like (10) shows that $E\nu_t B \to m(B)$ as $t \to \infty$. It follows that $E(\mu B) \le \lim_i E(\nu_{t_i} B) = m(B)$; the intensity of the limiting point process is absolutely continuous with respect to Lebesgue measure. Since

$m(\partial B) = 0$ implies $E\mu(\partial B) = 0$ and hence $\mu(\partial B) = 0$ a.s., B is a μ-continuity set. Thus as random variables, $\nu_{t_i}B \to_d \mu B$, and formula (18) shows that the $\nu_{t_i}B$ are uniformly integrable, so that $E\mu B = \lim E\nu_{t_i}B = m(B)$. □

COROLLARY 2. *With the same hypotheses as Corollary* 1, *any limiting point process* μ *is simple* (*or orderly*), *i.e.*

$$P(\mu(\{x\}) > 1 \text{ for some } x \in S_1) = 0.$$

PROOF. It follows easily from Lemma 1 that for any bounded $B \subset R^d$,

$$P(\nu_t B > 1) = P(\xi_t \cap B_t > 1) \le \sum_{x \ne y \in B_t} P(x, y \in \xi_t) \le |B_t|^2 p_t^2 = (E\nu_t B)^2.$$

Write $S_r = \{x \in R^d : |x| < r\}$. There is a constant c depending only on d, such that for any $r \in (0, 1)$, S_1 can be covered by $n_r \le c/m(S_r)$ translates $U_1, U_2, \cdots, U_{n_r}$ of S_r. Now

$$\begin{aligned} P(\mu(\{x\}) > 1 \text{ for some } x \in S_1) &\le P(\mu(U_j) > 1 \text{ for some } j,\ 1 \le j \le n_r) \\ &\le \sum_{j=1}^{n_r} P(\mu(U_j) > 1) = \sum_j \lim_i P(\nu_{t_i}(U_j) > 1) \\ &\le \sum_j \lim_i (E\nu_{t_i} U_j)^2 \\ &= \sum m(U_j)^2 \le (c/m(S_r))(m(S_r))^2 = c \cdot m(S_r). \end{aligned}$$

Taking a limit as $r \to 0$,

$$P(\mu(\{x\}) > 1 \text{ for some } x \in S_1) = 0.$$ □

2.2 Simple Random Walks. For the remainder of this section we will consider only the case where p is simple random walk on Z^d, i.e. $p(x, y) = 1/(2d)$ if $|x - y| = 1$, $p(x, y) = 0$ otherwise. Thus, when a particle moves, it chooses any one of the $2d$ neighboring sites in the lattice Z^d with equal probability. For the system of coalescing *simple* random walks, the asymptotic behavior of $p_t = P(0 \in \xi_t)$ is known in all cases ($d = 1$ in Bramson and Griffeath 1980, $d \ge 2$ in Bramson and Griffeath 1980b). The d^{th} root of these asymptotics is:

$$\text{(19)} \qquad \begin{aligned} &\alpha_t \approx (\pi t)^{-1/2} && d = 1 \\ &\alpha_t \approx (\pi t/\log t)^{-1/2} && d = 2 \\ &\alpha_t \approx (\gamma_d t)^{-1/d} && d \ge 3. \end{aligned}$$

Since $t^{-1/2}$ is the appropriate normalizing factor for a simple random walk on Z^d for any d, we see that $\alpha_t/t^{-1/2} \to \infty$ as $t \to \infty$, when $d \ge 2$. Thus it is quite plausible that the limiting point process should be Poisson when $d \ge 2$, and not Poisson when $d = 1$. The case $d = 1$, coalescing simple random walk on the line, is analyzed in Arratia (1979). The result is that $\nu_t \to_d \mu_1$, where the limiting point process μ_1 on R can be realized as the state at time $1/\pi$ of a system of coalescing standard Brownian motions on the line, starting with a particle at each $x \in R$. A self-duality relation for the system of coalescing Brownian motions leads to a formula expressing μ_1 in terms of its zero function:

$$\text{(20)} \qquad P(\mu_1(B) = 0) = P(\tilde{\eta}_{1/\pi}^{\partial B} = \phi).$$

Here $B \subset R$ is a finite disjoint union of intervals, and $\tilde{\eta}_t^{\partial B}$ is a finite system of annihilating standard Brownian motions on R, starting with a particle at each site in ∂B. For contrast, the zero function for the limiting point process μ_d for $d \ge 2$ is

$$P(\mu_d(B) = 0) = e^{-m(B)}$$

for any Borel set $B \subset R^d$.

THEOREM 1. *For $d \geq 2$, as $t \to \infty$,*

$$\nu_t \equiv \alpha_t \xi_t \to_d \mu_d,$$

where $\mu_d = P_m$ is the simple Poisson point process on R^d whose intensity is Lebesgue measure m. Here, ξ_t is the system of coalescing simple random walks on Z^d, starting with all lattice sites occupied; α_t is given asymptotically by (19).

PROOF. To show that $\nu_t = \alpha_t \xi_t$ is approximately Poisson, we will run the system for a while with the collision mechanism suspended—particles will follow independent random walks with no interference, and lattice sites may be multiply occupied. For any $t \geq e$ $(= 2.7 \cdots)$, set

$$\Delta t = \Delta t(t) = t/\sqrt{\log t} \qquad d = 2$$
$$= t^{1/2+1/d} \qquad d \geq 3$$

so that $\alpha_t \sqrt{\Delta t} \to \infty$, and $\Delta t = o(t)$, hence $p_{t-\Delta t}/p_t \to 1$, as $t \to \infty$. Set

$$s = s(t) = t - \Delta t, \tag{21}$$

and let $\bar{\xi}_t$ be the system of coalescing random walks with the collision mechanism suspended from time s to time $t = s + \Delta t$. Write $\bar{\xi}_t(x) = 0, 1, 2, \cdots$ for the number of particles at $x \in Z^d$; for each t there is a coupling such that $\forall\, \omega$, $\forall\, x \in Z^d$,

$$\xi_t(x) \leq \bar{\xi}_t(x).$$

Let

$$\bar{\nu}_t \equiv \alpha_t \bar{\xi}_t \equiv \textstyle\sum_{x \in Z^d} \bar{\xi}_t(x) \delta_{\alpha_t x}.$$

We claim that for any compact $B \subset R^d$,

$$P(\nu_t|_B \neq \bar{\nu}_t|_B) \to 0 \quad \text{as } t \to \infty, \tag{22}$$

where $|_B$ denotes the restriction of a measure to B. [Proof: Write $p_t(x, y)$ for the transition density of continuous time simple random walk on Z^d, run for time t. The density of particles in $\bar{\xi}_t$ is

$$E\bar{\xi}_t(y) = E(\textstyle\sum_x \xi_s(x) p_{\Delta_t}(x, y)) = \sum_x p_s p_{\Delta_t}(x, y) = p_s.$$

Thus

$$P(\xi_t(x) \neq \bar{\xi}_t(x)) \leq E(\bar{\xi}_t(x) - \xi_t(x)) = p_s - p_t,$$

so

$$P(\nu_t|_B \neq \bar{\nu}_t|_B) \leq \textstyle\sum_{x \in B_t} P(\xi_t(x) \neq \bar{\xi}_t(x))$$
$$\leq |B_t|(p_s - p_t) = (E\nu_t B)(p_s - p_t)/p_t \to 0 \quad \text{as } t \to \infty.$$

This establishes the claim (22).]

For each t define a transition kernel $q_t(x, dy)$ on R^d which describes the motion of each particle of $\nu_s = \alpha_s \xi_s$ to its location in $\bar{\nu}_t = \alpha_t \bar{\xi}_t$:

$$q_t(x, B) = p_{\Delta_t}(\alpha_s^{-1} x, \alpha_t^{-1} B).$$

(These do not form a semigroup.) Since $\alpha_t \sqrt{\Delta t} \to \infty$,

$$\sup_{x \in R^d} q_t(x, B) \to 0 \quad \text{as } t \to \infty \tag{23}$$

for any compact B. By the independence of the particle motions going from ν_s to $\bar{\nu}_t$,

$$E\left(\exp\left[-\int f(y) \bar{\nu}_t(dy)\right] \Big| \nu_s\right) = \exp\left[\int \log\left(\int e^{-f(y)} q_t(x, dy)\right) \nu_s(dx)\right] \tag{24}$$

for any Borel function f on R^d. For a similar analysis of a completely independent particle system, see Liggett (1978).

For each t define a random measure M_t on R^d by

$$M_t(B) = \int q_t(x, B)\nu_s(dx) = \textstyle\sum_{x\in\xi_s} p_{\Delta_t}(x, B_t) \tag{25}$$

so that

$$E(\bar{\nu}_t(B) \mid \xi_s) = M_t(B).$$

(To check that this is really a random measure, having $M_t(B) < \infty$ a.s. for compact convex B, compute

$$EM_t(B) = EE(\bar{\nu}_t(B) \mid \xi_s) = E\bar{\nu}_t(B) = p_s \mid B_t \mid \to m(B) \quad \text{as } t \to \infty.$$

This also shows that the family $\{M_t\}$ is tight.)

Start with any convergent sequence of $\{\nu_t\}$, say $\nu_t \to_d \nu$ as $t \to \infty$. Since $P(\nu_t|_B \neq \bar{\nu}_t|_B) \to 0$, $\bar{\nu}_t \to_d \nu$ also, as $t \to \infty$ along this sequence. Take a subsequence along which $M_t \to_d M$. The Laplace transform of ν can be computed as the limit of the Laplace transform of $\bar{\nu}_t$, along the subsequence for which $\bar{\nu}_t \to_d \nu$ and $M_t \to_d M$. Let $f \in C_c^+(R^d)$, the set of continuous non-negative functions on R^d having compact support. Write $g_t(x) = -\log \int \exp(-f(y))q_t(x, dy)$. By (23), $\sup_x | \int exp(-f(y))q_t(x, dy) - 1 | \to 0$ as $t \to \infty$; thus

$$\frac{g_t(x)}{\int (1 - \exp(-f(y)))q_t(x, dy)} \to 1 \quad \text{uniformly in } x. \tag{26}$$

Now along our subsequence,

$$L_\nu f \equiv E \exp\left(-\int f(x)\nu(dx)\right)$$

$$= \lim L_{\bar{\nu}_t}(f) = \lim EE\left(\exp\left[-\int f(x)\bar{\nu}_t(dx)\right] \middle| \nu_s\right)$$

(by (24)) $$= \lim E(e^{-\int g_t(x)\nu_s(dx)})$$

(by (26)) $$= \lim E(e^{-\int [\int (1-e^{-f(y)})q_t(x,dy)]\nu_s(dx)})$$

(by (25)) $$= \lim E(e^{-\int (1-e^{-f(y)})M_t(dy)})$$

$$= \lim L_{M_t}(1 - e^{-f}) = L_M(1 - e^{-f}).$$

Thus $\nu = P_M$, the mixture of Poisson processes directed by the random measure M.

We want to show that ν is the simple Poisson process with intensity m, i.e. that $M = m$ (a.s.). Take B to be compact and convex, and consider the random variables $\nu(B)$ and $\Lambda \equiv M(B)$. The distribution of $\nu(B)$ is a mixture of ordinary Poisson distributions with parameter λ directed by Λ, so that $E\nu(B)^2 = E(\Lambda^2) + E\Lambda$. We know that $E\Lambda = E\nu(B) = m(B)$; the bound (18) of Corollary 1, together with $\nu_t(B) \to_d \nu(B)$ yields

$$E\nu(B)^2 \le \lim\sup E(\nu_t B)^2 \le \lim\sup[(E\nu_t B)^2 + E\, \nu_t B] = m(B)^2 + m(B).$$

Thus $E(\Lambda^2) \le (E\Lambda)^2$, so $\Lambda = m(B)$, (a.s.). We have $M(B) = m(B)$ a.s., for each compact convex B, so $M = m$ (a.s.). Thus $\nu = P_m$, the simple Poisson point process on R^d with intensity one. Since this limit is obtained along a subsequence of an arbitrary convergent sequence (with $t_i \to \infty$) from the tight family $\{\nu_t, t \ge 0\}$, the theorem is proved. □

Let γ_t be the system of coalescing random walks on Z^d with mass conserved, starting with a particle of mass one at each lattice site. Write $\gamma_t(x) = n$ to indicate that there is a particle of mass $n = 0, 1, 2, \cdots$ at $x \in Z^d$; interpret $\gamma_t(x) = 0$ as "no particle present." In terms of the usual coupling of the family $(\xi_t^A, A \subset Z^d)$ of coalescing random walks, which has the additive property: for all $A, B \subset Z^d$,

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B,$$

the system γ_t may be defined by

$$\gamma_t(x) = |\{y \in Z^d : \xi_t^{\{y\}} = \{x\}\}|.$$

For each site x, the mass $\gamma_t(x)$ at x has mean

$$E\gamma_t(x) = \sum_{y \in Z^d} p_t(y, x) = 1.$$

The particle locations for γ_t are the same as those for ξ_t: for each ω,

$$x \in \xi_t \quad \text{iff} \quad \gamma_t(x) > 0.$$

The expected mass of a given particle in γ_t is

$$E(\gamma_t(x) \mid \gamma_t(x) > 0) = E\gamma_t(x)/P(\gamma_t(x) > 0) = 1/p_t.$$

Thus we rescale both the spatial locations and the masses of the particles; let

$$\mu_t \equiv \sum_{x \in Z^d} p_t \gamma_t(x) \delta_{\alpha_t x} = \sum_{x \in \xi_t} p_t \gamma_t(x) \delta_{\alpha_t x}.$$

THEOREM 2. *Let p be simple random walk on Z^d, $d \geq 2$. As $t \to \infty$, μ_t converges in distribution to the β-compound of P_m. Here, β is the exponential distribution with mean one on R^+ and P_m is the simple Poisson point process on R^d whose intensity is Lebesgue measure m. In terms of Laplace transforms,*

$$L_{\mu_t}(f) \to L_{P_m}(-\log L_\beta \circ f) = e^{-\int f/(1+f)\, dm}$$

for every $f \in C_c^+(R)$.

PROOF. Define a particle system ξ^* on $Z^d \times Z^+$ by: for $x \in Z^d$, $n = 1, 2, \cdots$,

$$(x, n) \in \xi_t^* \quad \text{iff} \quad \gamma_t(x) = n.$$

Consider a spatial rescaling ν_t^* of ξ_t^* as a point process on $R^d \times R^+$:

$$\nu_t^* \equiv \sum_{(x,n) \in \xi_t^*} \delta_{(\alpha_t x, p_t n)}.$$

We will repeat the proof of Theorem 1 to show that

$$\nu_t^* \to_d P_{m \times \beta}, \tag{27}$$

where $P_{m \times \beta}$ is the Poisson point process on $R^d \times R^+$ with intensity $m \times \beta$.

To see that this establishes Theorem 2, compute Laplace transforms. Given $f \in C_c^+(R^d)$, define g on $R^d \times R^+$ by

$$g(x, a) = af(x).$$

Now

$$\begin{aligned} L_{\mu_t} f &= E \exp(-\textstyle\sum_{z \in Z^d} p_t \gamma_t(z) f(\alpha_t z)) = E \exp\left(-\int g\, d\nu_t^*\right) \\ &= L_{\nu_t^*}(g) \to L_{P_{m \times \beta}}(g) = \exp\left(-\int (1 - e^{-g})\, d(m \times \beta)\right) \\ &= \exp\left(-\int (1 - e^{-af(x)}) \beta(da) m(dx)\right) \\ &= \exp\left(-\int (1 - L_\beta(f(x)) m(dx)\right) \\ &= L_{P_m}(-\log L_\beta \circ f). \end{aligned} \tag{28}$$

This shows that as random measures on R^d,

$$\mu_t \to_d \text{the } \beta\text{-compound of } P_m.$$

The expression (28) for the Laplace transform of the limiting random measure can be simplified; since $L_\beta(t) = (1+t)^{-1}$,

$$L_{P_m}(-\log L_\beta \circ f) = \exp\left(-\int (1 - L_\beta(f(x))m(dx)\right) = e^{-\int f(x)/(1+f(x))m(dx)}.$$

To prove the Poisson convergence (27), take s and t as given by (21). Let $\bar{\xi}_t^*$ be the system ξ_t^* run with the collision mechanism suspended from time s to t, writing $\bar{\xi}_t^*(x, n) = 0, 1, 2, \cdots$ for the number of particles at (x, n), $x \in Z^d$, $n = 1, 2, \cdots$. The corresponding point process on $R^d \times R^+$ is

$$\bar{\nu}_t^* = \sum \bar{\xi}_t^*(x, n)\delta_{(\alpha_t x, p_t n)}.$$

We need to show that there are couplings of ξ_t^* and $\bar{\xi}_t^*$ such that

$$P(\nu_t^* |_{B\times R^+} \neq \bar{\nu}_t^* |_{B\times R^+}) \to 0 \quad \text{as } t \to \infty \tag{29}$$

for every compact $B \subset R^d$. In Theorem 1, there was a coupling such that $\nu_t \le \bar{\nu}_t$, but such a relation for the $*$ system is not possible. In ξ^*, a particular at (x, n) may jump to (y, n) if x and y are nearest neighbors in the lattice Z^d; if there already is a particle at (y, m) (for $m = 1, 2, \cdots$, possibly even $m = n$) then both particles vanish in collision and are replaced by one particle at $(y, m+n)$. Consider a joint construction of the systems ξ_t, ξ_t^*, $\bar{\xi}_t$, and $\bar{\xi}_t^*$ in which for each $r \in [s, t]$,

$$x \in \xi_r \quad \text{iff} \quad (x, n) \in \xi_r^* \quad \text{for some } n = 1, 2, \cdots;$$

$$\bar{\xi}_r(x) = \textstyle\sum_{n\ge 1} \bar{\xi}_r^*(x, n).$$

Consider $(\gamma_u^{\xi_s}, 0 \le u \le \Delta t)$, a coalescing system in which mass is conserved, starting with a particle of mass one at each $x \in \xi_s$. Thus with the natural coupling, $\gamma_u^{\xi_s}(x) = n$ indicates that n particles from ξ_s have coalesced together to form a single particle in ξ_{s+u} at site x, so for each $u \in [0, \Delta t]$,

$$x \in \xi_{s+u} \quad \text{iff} \quad \gamma_u^{\xi_s}(x) \ge 1.$$

With this coupling, for any $x \in Z^d$,

$$\{\omega: \xi_t^*(x, n) \neq \bar{\xi}_t^*(x, n) \text{ for some } n\} = \{\xi_t(x) \neq \bar{\xi}_t(x)\} \cup \{\gamma_{\Delta t}^{\xi_s}(x) \ge 2\}. \tag{30}$$

Compute

$$E\gamma_u^{\xi_s}(x) = \textstyle\sum_{y\in Z^d} p_u(y, x) P(y \in \xi_s) = \sum_y p_u(y, x)p_s = p_s.$$

Now

$$P(\gamma_{\Delta t}^{\xi_s}(x) \ge 1) = P(x \in \xi_t) = p_t,$$

so

$$P(\gamma_{\Delta t}^{\xi_s}(x) \ge 2) \le p_s - p_t.$$

Therefore the event in (30) has probability $\le 2(p_s - p_t)$, so

$$P(\bar{\nu}^* |_{B\times R^+} \neq \nu^* |_{B\times R^+}) \le \textstyle\sum_{x\in B_t} P(\bar{\xi}_t^*(x, n) \neq \xi_t^*(x, n) \text{ for some } n)$$

$$\le |B_t|\, 2(p_s - p_t) \to 0 \quad \text{as } t \to \infty.$$

This shows that (29) holds.

The transition kernel of $R^d \times R^+$ which describes the motion of each particle in ν_s^* to its new location in $\bar{\nu}_t^*$ is

$$q_t^*((x, a), B \times I) = p_{\Delta t}(\alpha_s^{-1}x, \alpha_t^{-1}B)1_I(p_t a/p_s);$$

for any compact $B \subset R^d$

$$\sup_{(x,a)} q_t^*((x, a), B \times R^+) \to 0 \quad \text{as } t \to \infty.$$

Define random measures on $R^d \times R^+$ by

$$M_t^*(B \times I) = \sum_{(z,n)\in\xi_s^*} p_{\Delta t}(z, B_t)1_I(p_t n) = \int q_t^*((x, a), B \times I)\, d\nu_s^*(x, a).$$

so that

$$E(\bar{\nu}_t^*(B \times I) \mid \xi_s^*) = M_t^*(B \times I).$$

For $I \subset R^+$ of the form $[a, \infty)$, $x \in R^d$

$$E(\textstyle\sum_{(z,n)\in\xi_s^*} p_{\Delta t}(z, x)1_I(p_t n)) = \sum_{z\in Z^d, n\ge a/p_t} P((z, n) \in \xi_s^*)p_{\Delta t}(z, x)$$

$$= \textstyle\sum_{n\ge a/p_t} P((0, n) \in \xi_s^*) = P(\gamma_s(0) > a/p_t)$$

$$\approx e^{-a}p_s \text{ (by (9) and } p_s/p_t \to 1.)$$

Thus, for $I = [a, \infty)$ and B compact convex $\subset R^d$,

$$E\bar{\nu}_t^*(B \times I) = EM_t^*(B \times I) = E \textstyle\sum_{x\in B_t} \sum_{(z,n)\in\xi_s^*} p_{\Delta t}(z, x)1_I(p_t n)$$

$$\approx |B_t|\, e^{-a}p_s \approx m(B)e^{-a}$$

$$= (m \times \beta)(B \times I).$$

Note that $E(\bar{\nu}_t^*(B \times R^+)) = E(\bar{\nu}_t(B)) \le c < \infty$ for compact B, so that the familes $\{\bar{\nu}_t^*, t \ge 0\}$ and $\{M_t^*, t \ge 0\}$ of random measures on $R^d \times R^+$ are tight. Starting with any convergent sequence of ν_t^*, say $\nu_t^* \to_d \nu^*$, take a subsequence along which M_t^* converges, say $M_t^* \to_d M^*$. A Laplace transform calculation similar to (26ff.) shows that ν^* is the mixture of Poisson processes on $R^d \times R^+$ directed by the random measure M^*. Lemma 2 is now used to compare first and second moments of the random variable $\nu^*(B \times [a, \infty))$ and show that $M^* = m \times \beta$ almost surely. This establishes (27) and concludes the proof of Theorem 2. □

3. Annihilating Random Walks. The one half thinning relationship between the system η_t of annihilating random walks and the system ξ_t of coalescing random walks is natural when viewed in terms of the dual system, the family of voter models $(\zeta_t^A, A \subset Z^d)$. (See Griffeath (1979) for an exposition of all the material in this paragraph.) The voter model ζ_t is the spin flip system on Z^d in which the voter at any site x changes opinion at a rate equal to the proportion of his neighbors (weighted by p) who hold the opposite opinion; equivalently, the flip rates are

$$c(x, \zeta) = \textstyle\sum_{y\in Z^d: \zeta(x)\ne\zeta(y)} p(x, y).$$

Identify the state space $\{0, 1\}^{Z^d}$ for this spin system with $\mathscr{S} = \{\text{all subsets of } Z^d\}$, and write ζ_t^A for the voter model starting with opinion 1 held by the voters at sites $x \in A$, opinion 0 held everywhere else. When $A = \{x\}$, write ζ_t^x for the voter model $\zeta_t^{\{x\}}$ starting with a lone dissenting opinion at x. Thus $\{\omega: \zeta_t^x \ne \phi\}$ is the event that this dissenting opinion survives until time t, and $|\zeta_t^x|$ is the size at time t of the dynasty of converts to that dissenting opinion. For each $t \ge 0$, there is a coupling, based on a random substructure $\mathscr{P}$ of event times, of ξ_t, the system of coalescing random walks starting with all sites occupied, η_t, the corresponding system of annihilating random walks, and $(\zeta_t^A, A \subset Z^d)$, the family of voter models started at each possible initial configuration, such that for every ω,

$$(31) \qquad \xi_t = \{x: \zeta_t^x \ne \phi\},\ \eta_t = \{x: |\zeta_t^x| \text{ is odd}\}.$$

Thus the one half density result of Theorem 3 is equivalent to a statement about the parity of the finite voter model:

(32) $$P(0 \in \eta_t)/P(0 \in \xi_t) = P(|\zeta_t^0| \text{ is odd} \mid \zeta_t^0 \neq \phi) \to \tfrac{1}{2} \quad \text{as } t \to \infty.$$

When the voter model is in state A, for some finite $A \subset Z^d$, it grows in size by one (i.e. $A \to A \cup \{x\}$ for some $x \notin A$) at rate $\sum_{x \notin A, y \in A} p(x, y)$, and decreases in size by one at rate $\sum_{x \in A, y \notin A} p(x, y)$. By the translation invariance of p, these two rates are equal. We write

(33) $$r(A) = \textstyle\sum_{x \notin A, y \in A} p(x, y) + \sum_{x \in A, y \notin A} p(x, y) = 2 \sum_{x \in A, y \notin A} p(x, y)$$

for the total jump rate out of state A. The configuration ϕ is a trap for the voter model, i.e. $r(\phi) = 0$; but for any finite $A \neq \phi$, $r(A) \geq 2(1 - p(0, 0)) > 0$. Thus $|\zeta_t^0|$ is a time change of S_t, a *simple* random walk on Z^+, started at one, with absorption at zero, and with mean one exponential holding times between jumps. In the special case where p is a nearest neighbor random walk on Z, the state of the voter model ζ_t^0 before absorption is a block of consecutive integers, with $r(\zeta_t^0) = 2$. Using the reflection principle and the local central limit theorem, taking X_t to be simple (unstopped) random walk on Z started at zero,

$$\begin{aligned} P(|\zeta_t^0| \text{ is odd} \mid \zeta_t^0 \neq \phi) &= P(S_{2t} \text{ is odd} \mid S_{2t} > 0) \\ &= P(X_{2t} = 0)/P(X_{2t} = 0 \text{ or } 1) \\ &\to \tfrac{1}{2} \quad \text{as } t \to \infty. \end{aligned}$$

This is the only case of p for which the one half density relation (32) had been established. Theorem 3 extends this to arbitrary multidimensional random walks p, and to p on the integers Z having $\sum |x| p(0, x) = \infty$.

For *any* random walk p, the size of the finite voter model, conditional on survival, tends in probability to infinity: for any m,

(34) $$P(|\zeta_t^0| \geq m \mid \zeta_t^0 \neq \phi) \to 1 \quad \text{as } t \to \infty.$$

This is stated and proved as Lemma 3. If p is multidimensional (Lemma 4) or p has $\sum_{x \in Z} p(0, x) |x| = \infty$ (Lemma 5), then

$$\lim_{m \to \infty} \inf_{A \subset Z^d, m \leq A < \infty} r(A) = \infty.$$

Combining this with (34) yields, for these random walks, that the border size of the voter model ζ^0, conditional on survival, tends in probability to infinity: for any r_0,

(35) $$P(r(\zeta_t^0) \geq r_0 \mid \zeta_t^0 \neq \phi) \to 1 \quad \text{as } t \to \infty.$$

Contrast this with the nearest neighbor case on the line, where for all t

(36) $$P(r(\zeta_t^0) = 2 \mid \zeta_t^0 \neq \phi) = 1.$$

Here is a quick proof of Theorem 3, that the one half density relation (32) holds for any random walk which satisfies (35). This argument cannot be extended to derive the one half thinning relation, Theorem 4. A second proof of (32), involving a randomization of the substructure $\mathscr{P}$ of event times for the voter model, will be given as our formal proof of Theorem 3. This second proof of Theorem 3 is really a special case of the proof of Theorem 4, but this latter proof is sufficiently complicated that it is useful to write out the full argument in the special case.

FIRST PROOF OF THEOREM 3. We want to show that (32) holds. Given $\epsilon \in (0, 1)$, set

(37) $$r_0 = 8\epsilon^{-3}\log(1/\epsilon)$$

and using (35), take t_0 so large that $t > t_0$ implies

(38) $$P(r(\zeta_t^0) > r_0 \mid \zeta_t^0 \neq \phi) > 1 - \epsilon.$$

At any jump of ζ^0, $r(\zeta_t^0)$ changes by at most 2. Let

(39) $$\Delta t = \epsilon^3/16.$$

For any $c \geq 1$,

$$P(|\, r(\zeta_s^0) - c\,| > \epsilon c \text{ for some } s \in [t, t+\Delta t] \mid r(\zeta_t^0) = c)$$

(40) $$\leq P(\zeta^0 \text{ has at least } \epsilon c/2 \text{ jumps during } [t, t+\Delta t] \mid r(\zeta_t^0) = c)$$

$$\leq P(X \geq \epsilon c/2), \text{ where } X \text{ is a mean } \Delta t(1+\epsilon)c \text{ Poisson random variable}$$

$$\leq \frac{E(X^2)}{(\epsilon c/2)^2} = \frac{[\Delta t(1+\epsilon)c][\Delta t(1+\epsilon)c+1]}{(\epsilon c/2)^2} \leq \frac{16\Delta t(\Delta t + 1/(2c)}{\epsilon^2} < \frac{16\Delta t}{\epsilon^2} = \epsilon.$$

For a continuous time Markov chain Y whose states can be partitioned into "odd" and "even" states which alternate with each other, and with all jump rates lying in the interval $[(1-\epsilon)c, (1+\epsilon)c]$ it is easily proved that

(41) $$P(Y_{\Delta t} \text{ is odd}) \geq \frac{1-\epsilon}{2} - \frac{1}{2}e^{-2c\Delta t}.$$

[Here is a proof of (41). Let $f(t) = P(Y_t \text{ is odd})$, and write $[a, b]$ for the interval of rates. Then

$$f'(t) \geq a(1-f(t)) - bf(t) = -(a+b)(f(t) - a/(a+b)).$$

Gronwall's inequality yields

$$(f(t) - a/(a+b)) \geq e^{-(a+b)t}(f(0) - a/(a+b)) \geq -a/(a+b)e^{-(a+b)t},$$

$$f(t) \geq \frac{a}{a+b}(1 - e^{-(a+b)t}),$$

from which (41) follows.] Using r_0 and Δt as specified by (37) and (39), we get, for any $c \geq r_0$,

(42) $$P(Y_{\Delta t} \text{ is odd}) \geq \frac{1-\epsilon}{2} - \frac{1}{2}e^{-2r_0\Delta t} = \frac{1}{2} - \epsilon.$$

Given that $\zeta_t^0 = A$ with $r(A) = c \geq r_0$, (40) shows that ζ^A can be coupled to a comparison Markov chain Y as described above, so that

$$P(\zeta_{\Delta t}^A \neq Y_{\Delta t}) < \epsilon,$$

and hence, by (42)

$$P(|\zeta_{\Delta t}^A| \text{ is odd}) > ½ - 2\epsilon.$$

Thus

$$P(|\zeta_{t+\Delta t}^0| \text{ is odd} \mid r(\zeta_t^0) \geq r_0) > ½ - 2\epsilon,$$

and this together with (38) yields, for any $t > t_0$,

$$P(|\zeta_{t+\Delta t}^0| \text{ is odd}) \geq (½ - 2\epsilon)(1-\epsilon)P(\zeta_t^0 \neq \phi) > (½ - 3\epsilon)P(\zeta_{t+\Delta t}^0 \neq \phi).$$

The same argument shows that for $t > t_0$,

$$P(|\zeta_{t+\Delta t}^0| \text{ is even}, \zeta_{t+\Delta t}^0 \neq \phi) > (½ - 3\epsilon)P(\zeta_{t+\Delta t}^0 \neq \phi).$$

Thus, for $t > t_0$,

$$|\, ½ - P(|\zeta_{t+\Delta t}^0| \text{ is odd} \mid \zeta_{t+\Delta t}^0 \neq \phi)| < 3\epsilon.$$

This establishes (32) and concludes the first proof of Theorem 3. The second proof also starts from statement (35); the heart of this second proof is presented as Lemma 6.

A natural generalization of the one half density relation (32) between annihilating and coalescing random walks is that such a relation should hold independently at each of n sites. Let $A = \{x_1, x_2, \cdots, x_n\} \subset Z^d$, $|A| = n$; it should be the case that for each $B \subset A$,

$$P(\eta_t \cap A = B \mid A \subset \xi_t) \to 2^{-n} \quad \text{as } t \to \infty.$$

Expressed in terms of duality (31) with the family of voter models, this is: for any $q_1, q_2, \cdots, q_n \in \{0, 1\}$,

$$P(|\zeta_t^{x_i}| \equiv q_i \bmod 2,\ i = 1 \text{ to } n \mid \zeta_t^{x_i} \neq \phi,\ i = 1 \text{ to } n) \to 2^{-n} \quad \text{as } t \to \infty.$$

Although this is plausible, it is intractable; there is no random walk p for which we can prove the required analog of (34), namely that for each finite $A \subset Z^d$, for every m

$$P(|\zeta_t^x| \geq m\ \forall\ x \in A \mid \zeta_t^x \neq \phi\ \forall\ x \in A) \to 1 \text{ as } \quad t \to \infty.$$

Intuitively, the worst case is when the set A is a cluster; a proof along the same lines as the proof of Lemma 3 requires a lower bound on the probability of the conditioning event of the form: for every $a > 0$,

$$e^{at}P(\zeta_t^x \neq \phi\ \forall\ x \in A) = e^{at}P(A \subset \xi_t) \to \infty \quad \text{as } t \to \infty.$$

A different way of generalizing the one half density result (32) involves the notion of a one half thinning $\Theta\xi_t$ of ξ_t: independently toss a fair coin for each particle in ξ_t to decide whether or not to retain that point. Equivalently, if $\nu_{1/2}$ is a random subset of Z^d whose distribution is product measure with density one half, and $\nu_{1/2}$ independent of ξ_t, then

$$\Theta\xi_t = \xi_t \cap \nu_{1/2},$$

or any random set with this distribution, will be called a one half thinning of ξ_t. With the usual notion of convergence in distribution for random elements of $\{0, 1\}^{Z^d}$, the statement that η_t gets close to a one half thinning of ξ_t as t gets large is: for any finite $K \subset Z^d$,

$$0 = \lim_{t\to\infty} \textstyle\sum_{A\subset K} P(\xi_t \cap K = A)(\sum_{B\subset A} |P(\eta_t \cap K = B \mid \xi_t \cap K = A) - 2^{-|A|}|).$$

This result, for each fixed K, holds almost trivially since

$$P(\xi_t \cap K = A) \to 1 \quad \text{if} \quad A = \phi \tag{43}$$

$$\to 0 \quad \text{if} \quad A = \phi,$$

i.e. since ξ_t converges in distribution to ϕ.

We cure this, and arrive at the statement of Theorem 4, by replacing K above with sets $K_t \subset Z^d$ such that $E|\xi_t \cap K| \to c > 0$. The uniform integrability estimate (Corollary 1) on the random variables $|\xi_t \cap K_t|$ now ensures that lim sup $P(\xi_t \cap K_t = \phi) < 1$, so that objection (43) has been overcome.

Corollary 3 gives an almost literal translation of Theorem 4 in terms of the rescaled point processes discussed in Section 2: for each t there exists a version $\Theta\xi_t$ of the one half thinning of ξ_t such that, for every compact $K \subset R^d$,

$$P((\alpha_t\eta_t)|_K \neq (\alpha_t\Theta\xi_t)|_K) \to 0 \quad \text{as } t \to \infty. \tag{44}$$

Using Laplace transforms, Corollary 4 follows immediately:

$$\text{if } \alpha_t\xi_t \to_d \mu, \quad \text{then} \quad \alpha_t\eta_t \to_d \Theta\mu, \tag{45}$$

the limit in distribution for rescaled annihilating random walks (along some sequence $t_i \to \infty$) exists and is the one half thinning of the limiting point process for the coalescing random walks.

Since the notion of convergence of point processes requires only that corresponding atoms get *close* in position, statement (45) is not as strong as statement (44). As an example, we will produce $\{0, 1\}^{Z^d}$ valued processes $\eta_t' \subset \xi_t'$ for which (45) holds, with a limiting Poisson point process, while (44) fails. Let the distribution of ξ_t' be ν_{p_t}, product measure with density $p_t \to 0$. Let

$$\eta'_t = \xi'_t \cap \{x \in Z^d : x_1 + x_2 + \cdots + x_d \text{ is even}\}.$$

Then $\alpha_t\xi'_t$ and $\alpha_t\eta'_t$ converge to the simple Poisson point processes on R^d having intensities respectively 1 and ½, but for any version of $\Theta\xi'_t$, for any compact convex $K \subset R^d$ having Lebesgue measure $m(K) > 0$,

$$P(\alpha_t\eta'_t|_K \neq \alpha_t\Theta\xi'_t|_K) \geq P(\Theta\xi'_t \cap \{x \in K_t : x_1 + \cdots + x_d \text{ is odd}\} \neq \phi) \to 1 - e^{-m(K)/4}.$$

LEMMA 3. *For the voter model based on an arbitrary nontrivial random walk p, for every m*

$$P(|\zeta_t^0| \geq m \mid \zeta_t^0 \neq \phi) \to 1 \quad \text{as } t \to \infty.$$

PROOF. The jump rate for ζ_t^0, conditional on $\zeta_t^0 \neq \phi$, is at least $c = 2(1 - p(0, 0)) > 0$. By comparison to a process which always jumps at rate c, then exists Δt such that

$$P(\zeta^A \text{ has at least } m - 1 \text{ jumps during } [0, \Delta t], \text{ or } \zeta_{\Delta t}^A = \phi) > ½,$$

for all finite $A \subset Z^d$. With probability $2^{-(m-1)}$ the first $m - 1$ jumps are down, so

$$P(\zeta_{\Delta t}^A = \phi) \geq 2^{-m} \tag{46}$$

for all $A \subset Z^d$ having $|A| < m$. An upper bound on the rate of extinction of the voter model ζ_t^0 is available from duality with coalescing random walks ξ_t, and Lemma 1 (recall that $p_t = P(0 \in \xi_t) = P(\zeta_t^0 \neq \phi)$):

$$-p'_t = -\frac{d}{dt}P(0 \in \xi_t) = \textstyle\sum_{0 \neq x \in Z^d} P(0, x \in \xi_t)p(0, x) \leq \sum_{x \neq 0} p_t^2 p(0, x) \leq p_t^2.$$

By (46),

$$p_{t+\Delta t} \leq p_t - 2^{-m}P(0 < |\zeta_t^0| < m)$$

so that

$$P(0 < |\zeta_t^0| < m) \leq 2^m(p_t - p_{t+\Delta t}) \leq 2^m p_t^2 \Delta t.$$

Thus

$$P(|\zeta_t^0| < m \mid \zeta_t^0 \neq \phi) \leq 2^m p_t \Delta t \to 0 \quad \text{as } t \to \infty. \qquad \square$$

For any $A \subset Z^d$, define

$$\partial A = \{(x, y) : x \neq y \in Z^d, p(x, y) > 0, \quad \text{and } 1 = |\{x, y\} \cap A|\}. \tag{47}$$

For any $C \subset (Z^d)^2$, define

$$r_p(C) = \textstyle\sum_{(x,y) \in C} p(x, y). \tag{48}$$

With this notation, formula (33) for the jump rates of the voter model is

$$r(A) = r_p(\partial A) = 2 \textstyle\sum_{x \in A, y \notin A} p(x, y).$$

LEMMA 4. *For any genuinely multidimensional random walk p on Z^d,*

$$\lim_{n \to \infty} \inf_{A \subset Z^d : n \leq |A| < \infty} r_p(\partial A) = \infty. \tag{49}$$

PROOF. Fix two linearly independent $x_1, x_2 \in Z^d$ with

$$\lambda \equiv \min(p(0, x_1), p(0, x_2)) > 0.$$

Consider a subset of the support of p:

$$S = \{(x, y) \in (Z^d)^2 : y - x = x_1 \quad \text{or} \quad y - x = x_2\}.$$

For $i = 1, 2$ define maps

$$B_i : A \to S \cap \partial A$$

$$x \to (x + kx_i, x + (k+1)x_i),$$

where k is the maximal interger such that $x + kx_i \in A$. The map $x \to (B_1(x), B_2(x))$ from A into $(S \cap \partial A)^2$ is $1 - 1$ since each of the pairs B_i determines a line in R^d, and these two lines intersect exactly at x. Thus

$$|A| \le |S \cap \partial A|^2, \quad \text{so} \quad r_p(\partial A) \ge r_p(S \cap \partial A) \ge \lambda \sqrt{|A|}. \qquad \square$$

LEMMA 5. *For any random walk on Z such that* $\Sigma |x| p(0, x) = \infty$, *the conclusion* (49) *of the previous lemma holds.*

PROOF. We may assume without loss of generality that p is symmetric and irreducible. Thus there is a set $K = \{x_1, x_2, \cdots, x_k\}$ whose elements are positive integers with greatest common divisor 1, such that

$$\lambda \equiv \min_{1 \le i \le k} p(0, x_i) > 0.$$

Let $\ell = \max(x_1, \cdots, x_k)$. Since $1 = a_1 x_1 + \cdots + a_k x_k$ for some integer coefficients a_i, it follows for any integers $a < b$ that there is a path

$$a = s_0, s_1, s_2, \cdots, s_j = b \tag{50}$$

from a to b, with displacements $|s_i - s_{i-1}| \in K$, which stays inside the interval $(a - \ell, b + \ell)$. Let $r > 0$ be given; take m so large that

$$\textstyle\sum_{x=\ell}^{m} xp(0, x) \ge r.$$

For integers $0 \le i < x$ write

$$A_{ix} = (i + xZ) \cap A.$$

If $A_{i,x} \ne \phi$ define

$$a_{i,x} = \min(A_{i,x})$$

so that $a_{i,x} \in A$, and $a_{i,x} - x \notin A$. If for each i, x with $\ell \le x \le m$ and $0 \le i < x$ we have $A_{i,x} \ne \phi$, then those pairs $(a_{i,x}, a_{i,x} - x)$ are sufficient:

$$r_p(\partial A) \ge \textstyle\sum_{x=\ell}^{m} \sum_{i=0}^{x-1} p(a_{i,x}, a_{i,x} - x) = \sum_{x=\ell}^{m} xp(0, x) \ge r.$$

If, on the other hand, $A_{i,x} = \phi$ for some i, x with $\ell \le x \le m$, $0 \le i < x$, then choose any integer $j \in [0, 3x)$ such that $|A_{j,3x}| \ge |A|/(3x)$. The assumption $A_{i,x} = \phi$ guarantees, for every n such that $a = 3nx + j \in A$, that $b = 3nx + i \notin A$. A path of the form (50) leads to an edge $e_n = (c_n, d_n) \in \partial A$ having $p(c_n, d_n) \ge \lambda$, and

$$(3n - 1)x \le c_n < d_n < (3n + 2)x.$$

This last relation is needed in order to conclude that for different n, the edges e_n are really distinct. Now if $|A| > 3xr/\lambda$, then

$$r_p(\partial A) \ge \textstyle\sum_{n:3nx+j \in A} p(c_n, d_n) \ge (|A|/3x)\lambda > r.$$

LEMMA 6. *For any* $\epsilon \in (0, 1)$, *with*

$$\Delta t = \epsilon/8, \text{ and } r = (4/\epsilon)\log(1/\epsilon),$$

for any random walk p on Z^d, *for any finite* $A \subset Z^d$, $r_p(\partial A) \ge r$ *implies*

$$|\tfrac{1}{2} - P(|\zeta_{\Delta t}^A| \text{ is odd})| < \epsilon.$$

Here, ζ^A *is the voter model on* Z^d *based on p, starting from A.*

PROOF. We use a randomization—take a flow of jump times having double the usual rate, and then select with a fair coin which times will be used to run the process. Such a device appears in Griffeath (1979) to prove that "cancellative systems having pure births are exponentially ergodic." This randomization can be viewed as a modification of the usual coupling proof (run two copies of the process independently, starting one of them in equilibrium, until they reach the diagonal, then run both together along the diagonal) that a finite state irreducible Markov process converges to its equilibrium distribution. In the context here, our proof is a modification of the coupling proof that a two state (even, odd) Markov chain which flips back and forth between even and odd at rate r each way, starting from a pure state δ_{even} or δ_{odd} can be coupled to the chain in equilibrium $(\frac{1}{2})(\delta_{\text{even}} + \delta_{\text{odd}})$ by time Δt with probability exactly $1 - (\frac{1}{2})\exp(-2r\Delta t)$.

Construct the voter models ζ_t^B for every $B \subset Z^d$ simultaneously from a substructure $\mathscr{P}$ consisting of independent exponential alarm clocks $T_{(x,y)}$ which ring at rates $p(x, y)$. When the clock for the pair (x, y) rings, the voter at site x is influenced by the voter at site y: the configuration either flips at x or else remains unchanged, so that its components at x and y agree.

Write $\alpha = (x, y)$ for a pair of sites in Z^d, and let $T_\alpha = T_{(x,y)}$ be the random set of times at which the clock for (x, y) rings, with

$$T_\alpha = \{t_\alpha(1) < t_\alpha(2) < \cdots\}.$$

To say that T_α is a Poisson flow or Poisson point process on $[0, \infty)$ with rate or intensity $p(x, y)$ is to say that $t_\alpha(1), t_\alpha(2) - t_\alpha(1), t_\alpha(3) - t_\alpha(2), \cdots$ are independent and exponentially distributed with mean $1/p(x, y)$. Thus, the substructure $\mathscr{P}$ is a family of independent random flows T_α of event times, indexed by pairs $\alpha = (x, y) \in (Z^d)^2$.

Construct $\mathscr{P}$ as the one half thinning of another substructure $\mathscr{P}'$ having double the usual rates. That is, let $\mathscr{P}' = (T'_{(x,y)}, x, y \in Z^d)$, where $T'(x, y) = \{t'_\alpha(1) < t'_\alpha(2) < \cdots\}$ is a Poisson flow with intensity $2p(x, y)$, let $\mathscr{C} = (c_{(x,y)}(i);\ x, y \in Z^d,\ i = 1, 2, \cdots)$ be a collection of fair coin tossing variable $(P(c_\alpha(i) = 0) = P(c_\alpha(i) = 1) = \frac{1}{2})$ independent of $\mathscr{P}$, and set

$$T_\alpha(\omega) = \{t'_\alpha(i, \omega) : c_\alpha(i, \omega) = 1,\ i = 1, 2, \cdots\}.$$

When $c_\alpha(i) = 0$ so that $t'_\alpha(i)$ is not an event time in the substructure $\mathscr{P}$, we say "there is a phantom effect at time $t'_\alpha(i)$," and when $c_\alpha(i) = 1$ we call the effect at time $t'_\alpha(i)$ "real." For any $C \subseteq (Z^d)^2$ having $r_p(C) \equiv \sum_{(x,y)\in C} p(x, y) < \infty$, write

$$T'_C = \cup_{\alpha\in C} T'_\alpha$$

for the set of all event times of $\mathscr{P}'$ for edges in C; T'_C is a Poisson flow with intensity $2r_p(C)$. Let

$$\tau(\omega) = \inf T'_{\partial A}, \text{ and}$$

$$\alpha_\tau(\omega) = (x_\tau, y_\tau) = \text{the pair } \alpha \text{ such that } \tau = t'_\alpha(1)$$

be the time and place of the first effect in $\mathscr{P}'$ on some edge in ∂A. Let

$$G = \{\omega : c_{\alpha_\tau}(1) = 0\}$$

be the event, having probability $\frac{1}{2}$, that the effect at time τ in $\mathscr{P}'$ is a phantom effect. Thus, for all ω,

$$|\zeta_\tau^A \cap \{x_\tau, y_\tau\}| \equiv 1_G \pmod 2.$$

Define H to be the event

$$H = \{\omega : |\zeta_{\Delta t}^A - \{x_\tau, y_\tau\}| \equiv 1 \pmod 2\}.$$

Although G and H are not independent, there is an event F having $P(F) > 1 - 2\epsilon$, conditional on which G and H are independent. F is a $\mathscr{P}'$-measurable event which guarantees that $\tau \leq \Delta t$ and each of the sites x_τ, y_τ can neither give nor receive influence at any time before Δt other than τ. To make this precise, for any $x \in Z^d$ let

$$N_x \equiv \partial(\{x\}) = \{(x, y) : p(x, y) > 0, y \neq x\} \cup \{(y, x) : p(x, y) > 0, y \neq x\},$$

and for any pair $\alpha = (x, y) \in (Z^d)^2$, let

$$N_\alpha = N_x \cup N_y.$$

Note that $r_p(N_x) \leq 2$ and $r_p(N_\alpha) \leq 4$. For each $\alpha \in \partial A$, define the event

$$F_\alpha = \{\omega : \alpha_\tau = \alpha, \tau \leq \Delta t, [0, \Delta t] \cap (T'_{N_\alpha} - \{\tau\}) = \phi\}.$$

Define the event F by

$$F = \cup_{\alpha \in \partial A} F_\alpha.$$

It can be seen that

$$\text{for } \omega \in F, |\zeta^A_{\Delta t}| \equiv 1_G + 1_H \pmod 2.$$

Here is the argument that $P(F) > 1 - 2\epsilon$. For any $\alpha \in \partial A$ and $t \in [0, \Delta t]$,

$$\begin{aligned} P(F_\alpha \mid \tau = t, \alpha_t = \alpha) \\ &= P(T'_{N_\alpha \cap \partial A} \cap (t, \Delta t] = \phi, T'_{N_\alpha - \partial A} \cap [0, \Delta t] = \phi \mid \tau = t, \alpha_\tau = \alpha) \\ &= P(T'_{N_\alpha \cap \partial A} \cap (t, \Delta t] = \phi, T'_{N_\alpha - \partial A} \cap [0, \Delta t] = \phi) \\ &\geq e^{-2r_p(N_\alpha)\Delta t} \geq e^{-8\Delta t} = e^{-\epsilon} > 1 - \epsilon. \end{aligned}$$

Integrate over $t \in [0, \Delta t]$ to get

$$P(F_\alpha) \geq (1 - \epsilon) P(\tau \leq \Delta t, \alpha_\tau = \alpha)$$

and sum over $\alpha \in \partial A$ to get

$$P(F) \geq (1 - \epsilon) P(\tau \leq \Delta t).$$

Since $\mathscr{P}'$ has double the usual rates, and $r_p(\partial A) \geq r \equiv (4/\epsilon)\log(1/\epsilon)$, and $\Delta t = \epsilon/8$,

$$P(\tau > \Delta t) = e^{-2r_p(\partial A)\Delta t} \leq e^{-2r\Delta t} = \epsilon.$$

Thus $P(F) \geq (1 - \epsilon)^2 > 1 - 2\epsilon$.

Next we show that G and H are independent, conditional on F, and that G is independent of F. Notice that for any $\alpha = (x, y) \in \partial A$, the event $F_\alpha \cap \{|\zeta^A_{\Delta t} - \{x, y\}| \equiv 1 \bmod 2\}$ is measurable with respect to the σ-field generated by $\mathscr{P}'$, $(c_\alpha(i) : i \geq 2)$, and $(c_\beta(i), \beta \in (Z^d)^2 - \{\alpha\}, i \geq 1)$, i.e. with respect to the information in $\mathscr{P}'$ and $\mathscr{C}$ excluding the variable $c_\alpha(1)$. Thus we can compute

$$\begin{aligned} P(F \cap G \cap H) &= \textstyle\sum_{\alpha=(x, y)\in\partial A} P(F_\alpha, c_\alpha(1) = 0, |\zeta^A_{\Delta t} - \{x, y\}| \equiv 1) \\ &= \textstyle\sum \frac{1}{2} P(F_\alpha, |\zeta^A_{\Delta t} - \{x, y\}| \equiv 1) \\ &= \tfrac{1}{2} P(F \cap H). \end{aligned}$$

The same argument using the complements of G or H establishes the claims of independence.

Finally,

$$\begin{aligned} P(|\zeta^A_t| \text{ is odd}) &\geq P(F \cap G \cap H^c) + P(F \cap G^c \cap H) \\ &= \tfrac{1}{2} P(F \cap H^c) + \tfrac{1}{2} P(F \cap H) \\ &= \tfrac{1}{2} P(F) > \tfrac{1}{2}(1 - 2\epsilon) = \tfrac{1}{2} - \epsilon \end{aligned}$$

and similarly

$$P(|\zeta_{\Delta t}^{A}| \text{ is even}) > \tfrac{1}{2} - \epsilon. \qquad \square$$

THEOREM 3. *Let p be any random walk on Z^d for which statement* (35) *about the voter model holds. In particular, p may be any genuinely multidimensional random walk, or a random walk on Z having $\Sigma |x| p(0, x) = \infty$. Then*

$$P(0 \in \eta_t)/P(0 \in \xi_t) \to \tfrac{1}{2} \quad \text{as } t \to \infty. \tag{51}$$

Here, η_t is the system of annihilating random walks, starting from all sites occupied, and ξ_t is the corresponding system of coalescing random walks.

PROOF. Given $\epsilon \in (0, 1)$, set $r = (4/\epsilon)\log(1/\epsilon)$ and $\Delta t = \epsilon/8$. Lemma 6 guarantees that for finite $A \subset Z^d$, $r_p(\partial A) \ge r$ implies

$$|\tfrac{1}{2} - P(|\zeta_{\Delta t}^{A}| \text{ is odd})| < \epsilon.$$

Since the flip rates for the voter model at any site are at most 1,

$$P(\zeta_{\Delta t}^{A} \ne \phi) \ge e^{-\Delta t} > 1 - \epsilon, \text{ for any } A \ne \phi.$$

Thus $r_p(\partial A) \ge r$ also implies

$$|\tfrac{1}{2} - P(|\zeta_{\Delta t}^{A}| \text{ is even}, \zeta_{\Delta t}^{A} \ne \phi)| < 2\epsilon.$$

By relation (35), (which follows easily for those particular cases of p mentioned by using Lemmas 3, 4, and 5,) there exists t_0 such that $t > t_0$ implies

$$P(r_p(\partial(\zeta_t^0)) \ge r \mid \zeta_t^0 \ne \phi) > 1 - \epsilon.$$

Thus for $t > t_0$,

$$P(|\zeta_{t+\Delta t}^0| \text{ is odd}) \ge (1 - \epsilon)(\tfrac{1}{2} - 2\epsilon)P(\zeta_t^0 \ne \phi) > (\tfrac{1}{2} - 3\epsilon)P(\zeta_{t+\Delta t}^0 \ne \phi).$$

The same inequality holds for $P(|\zeta_{t+\Delta t}^0| \text{ is even}, \zeta_{t+\Delta t}^0 \ne \phi)$, so that

$$|\tfrac{1}{2} - P(|\zeta_{t+\Delta t}^0| \text{ is odd} \mid \zeta_{t+\Delta t}^0 \ne \phi)| < 3\epsilon.$$

Thus $P(|\zeta_t^0| \text{ is odd} \mid \zeta_t^0 \ne \phi) \to \tfrac{1}{2}$ as $t \to \infty$, which is equivalent to (51) by duality. $\square$

Now we embark on a proof of Theorem 4. The main part of the argument lies in Lemma 8, which generalizes the technique of Lemma 6.

LEMMA 7. *Let p be a genuinely multidimensional random walk on Z^d; let $r > 0$ and $n > 0$ be given. There exists m such that for any disjoint finite sets $A_1, A_2, \cdots, A_n \subset Z^d$, $|A_i| \ge m$ for $i = 1$ to n implies the existence of $C_1, C_2, \cdots, C_n \subset (Z^d)^2$ satisfying, for $1 \le i, j \le n$,*

$$C_i \subset \partial A_i, \text{ and } r_p(C_i) \ge r; \tag{52a}$$

(52*b*) there is a basis $\{d_1, d_2, \cdots, d_n\}$ for $(Z/2)^n$ such that for any $(x, y) \in C_i$, $|\{x, y\} \cap A_j| = 1$ iff the jth component of d_i is 1;

(52c) if $i \ne j$, $(x, y) \in C_i$, and $(u, v) \in C_j$, then $\{x, y\} \cap \{u, v\} = \phi$.

Informally, condition (52b) *says that for the voter models ζ^{A_i}, if the first jump is caused by an alarm clock for a pair $(x, y) \in C_i$, then d_i will be the parity change in $(|\zeta^{A_i}| \bmod 2, i = 1 \text{ to } n) \in (Z/2)^n$. Condition* (52c) *may be restated: if $i \ne j$, $\alpha \in C_i$, then $C_j \cap N_\alpha \ne \phi$.*

PROOF. Continue with the notations x_1, x_2, λ, S from the proof of Lemma 4. Let

$$m_1 = \ulcorner r_0/ \urcorner,$$

(m_1 is the number of pairs from S needed in each C_i to yield a rate $\geq r_0$), let

$$m_2 = 6m_1 n,$$

(m_2 is the number of pairs from S needed for sets $\bar{C}_i$ so that subsets C_i of size exactly m_1 may be chosen to satisfy the "no interference" condition (52c)), and take

$$m = (m_2 n^2)^2$$

for the value m whose existence this lemma asserts. Assume now that we are given disjoint sets $A_i \subset Z^d$, satisfying $m \leq |A_i| < \infty$ for $i = 1$ to n. Let

$$A_0 = Z^d - \cup_{i=1} A_i,$$

$$N = \{0, 1, \cdots, n\}.$$

Define a subset of the border between A_i and A_j

$$B_{ij} = (\partial A_i) \cap (\partial A_j) \cap S.$$

Define an undirected graph G on N by

$$G = \{(i, j) \in N^2 : |B_{ij}| \geq m_2\}.$$

We prove that G is a connected graph by showing that any nonempty connected component $I \subset N$ must contain 0. Suppose to the contrary that $0 \notin I$. Let $A = \cup_{i \in I} A_i$, so that $m \leq |A| < \infty$ and by Lemma 4,

$$|\partial A \cap S| \geq \sqrt{m} = m_2 n^2. \tag{53}$$

In the partition

$$\partial A \cap S = \cup_{i \in I} \cup_{j \in N-I} B_{ij},$$

each B_{ij} has $|B_{ij}| < m_2$, since the choice of I as a connected component means $(i, j) \notin G$. This partition involves at most n^2 sets B_{ij}, giving $|\partial A \cap S| < n^2 m_2$ in contradiction to (53). Thus, 0 is an element of every connected component of G.

Since G is connected, there is a subgraph $T \subset G$ which is a connected tree on N. Fix such a T and define a map

$$s : N - \{0\} \to N$$

$$s(i) = \text{the unique vertex: } (i, s(i)) \text{ is the first edge of a path in } T \text{ from } i \text{ to } 0.$$

Let $e_1, e_2, \cdots, e_n$ be the standard basis for $(Z/2)^n$ and let $e_0 = 0$. For $i = 1$ to n, define

$$d_i = e_i + e_{s(i)}.$$

To see that $\{d_1, d_2, \cdots, d_n\}$ is a basis for $(Z/2)^n$, check that each e_i is in its linear span. Just follow the path in T from i to 0, i.e., with $h(i) = \min\{k : s^k(i) = 0\}$,

$$e_i = \sum_{k=0}^{h(i)-1} d_{s^k(i)}.$$

For $i = 1$ to n, let

$$\bar{C}_i = B_{is(i)},$$

so that the $\bar{C}_i$ are disjoint subsets of S, and the $\bar{C}_i$ in place of the C_i satisfy (52b), and $|\bar{C}_i| \geq m_2 = 6m_1 n$. Observe that for any pair $\alpha \in S$,

$$|(N_\alpha - \{\alpha\}) \cap S| = 6.$$

Choose any subset $C_1 \subset \bar{C}_1$ having $|C_1| = m_1$. Proceed recursively, at the i^{th} stage, $i = 2$ to n, choosing

$$C_i \subset \bar{C}_i - (\cup_{1 \leq j < i} \cup_{\alpha \in C_j} (N_\alpha \cap \bar{C}_i))$$

to have $|C_i| = m_1$. This is possible since $|\bar{C}_i| \geq m_2 = 6m_1n$; the union is taken over exactly $(i-1)m_1$ values of α, and each $N_\alpha \cap \bar{C}_i$ has at most 6 elements. This produces sets $C_1, C_2, \cdots, C_n$ such that $i<j$, $\alpha \in C_i$, and $\beta \in C_j$ imply $\beta \notin N_\alpha$. This establishes (52c). Notice that $r_p(C_i) \geq \lambda |C_i| = \lambda(\ulcorner r/\lambda \urcorner) \geq r$. □

LEMMA 8. *For any genuinely multidimensional random walk p, $\epsilon \in (0, 1)$, and positive interger n_0 there exists m with the following property*:
Given disjoint finite subsets $A_1, A_2, \cdots, A_n \subset Z^d$, $n \leq n_0$, with each $|A_i| \geq m$, define the random joint parity vector in $(Z/2)^n$:

$$Q_t \equiv (|\zeta_t^{A_i}| \bmod 2,\ i = 1 \text{ to } n).$$

For $\Delta t = \epsilon/8n_0$,

$$\sum_{q\in(Z/2)^n} |2^{-n} - P(Q_{\Delta t} = q)| < 4\epsilon. \tag{54}$$

PROOF. Set $r = (4n_0/\epsilon)\log(n_0/\epsilon)$ and take m as determined by Lemma 7 (depending on p, r and n_0; this m will also work for $n = 1, 2, \cdots, n_0$ in place of n_0.) Let $A_1, A_2, \cdots, A_n$ be any disjoint finite subsets of Z^d having $|A_i| \geq m$, with $n \leq n_0$. The unqualified indices i, j will always be taken to range over $1, 2, \cdots, n$. By Lemma 7, there are sets of pairs $C_i \subset \partial A_i$ having $r_p(C_i) \geq r$, satisfying also (52b) and (52c). Condition (52b) says that if the first effect in $\mathscr{P}$ to change any of the $\zeta_t^{A_i}$ is at $\alpha \in C_i$, then the change in Q_t produced by that jump will be d_i.

We continue with notation from the proof of Lemma 6: there is a double rate substructure $\mathscr{P}'$, an independent collection of fair coins $\mathscr{C}$ used to thin $\mathscr{P}'$, and the resulting ordinary substructure $\mathscr{P}$ is used to construct the family of voter model $(\zeta^A, A \subset Z^d)$. Define

$$\tau_i = T'_{C_i}$$

to be the time of the first effect in the double rate substructure $\mathscr{P}'$ to involve a pair in C_i. Define the random pair

$$\beta_i = (x_i, y_i) \in C_i$$

to be the pair α such that $\tau_i = \tau'_\alpha(1)$, i.e., the place at which the effect at τ_i occurs. Define the 0, 1-valued variable

$$c_i = c_{\beta_i}(1)$$

to be the value of the coin in $\mathscr{C}$ which determines whether the effect at time τ_i is real ($c_i = 1$) or phantom ($c_i = 0$). The vector of these c_i is a random element of $(Z/2)^n$

$$c(\omega) = (c_1, c_2, \cdots, c_n)$$

for which each possible value has probability 2^{-n}.

Consider the set of sites involved at the random times τ_i:

$$B(\omega) = \cup_{i=1}^n \{x_i, y_i\}.$$

Note that $|B| = 2n$ since $i \neq j$ implies $\beta_i \notin N_{\beta_j}$. Define a modified parity vector Q_t^*, similar to Q_t, but which "resets" the opinions on B to what they were initially, just before counting up the $\zeta_t^{A_i}$ (mod 2):

$$Q_t^*(\omega) = (|\zeta_t^{A_i} - B| + |A_i \cap B| \bmod 2,\ i = 1 \text{ to } n).$$

For any $\alpha = (x, y) \in C_i$ define the event

$$F_\alpha = \{\beta_i = \alpha,\ \tau_i \leq \Delta t,\ (T'_{N_\alpha} - \{\tau_i\}) \cap [0, \Delta t] = \phi\},$$

which says that $\tau_i \leq \Delta t$ is caused by an effect in $\mathscr{P}'$ at α, and that sites x and y are not involved in any other effects in $\mathscr{P}'$ during $[0, \Delta t]$. Thus, on F_α there cannot be any interaction before Δt between what happens at time τ_i (a real or phantom effect, indicated by $c_i(\omega) = 1$ or 0) and the rest of the evolution of the family of voter models. Define F, representing successful coupling, with no interference possible, by

$$F = \cup_{\alpha_1 \in C_1} \cdots \cup_{\alpha_n \in C_n} (\cap_{i=1}^n F_{\alpha_i}).$$

A little reflection shows that for $\omega \in F$,

$$Q_{\Delta t}(\omega) = Q_{\Delta t}^*(\omega) + \sum_{i=1}^n c_i(\omega)\, d_i.$$

Taking D to be the $n \times n$ matrix over $Z/2$ whose rows are $d_1, d_2, \cdots, d_n$, this may be rewritten as

$$Q_{\Delta t}(\omega) = Q_{\Delta t}^*(\omega) + c(\omega)D, \text{ for } \omega \in F.$$

By arguing that conditional on F, $Q_{\Delta t}^*$ and c are independent, and that c is independent of F, it can be seen that for all $q_1, q_2 \in (Z/2)^n$

$$P(Q_{\Delta t}^* = q_1, c = q_2, F) = 2^{-n}P(Q_{\Delta t}^* = q_1, F). \tag{55}$$

Thus, for any $q \in (Z/2)^n$,

$$\begin{aligned} P(Q_{\Delta t}(\omega) = q) &\geq P(Q_{\Delta t}(\omega) = q, F) \\ &= \textstyle\sum_{q_1 \in (Z/2)^n} P(Q_{\Delta t}^* = q_1, c = (q - q_1)D^{-1}, F) \\ &= \textstyle\sum 2^{-n} P(Q_{\Delta t}^* = q_1, F) \\ &= 2^{-n}P(F). \end{aligned}$$

Once it is shown that

$$P(F) > 1 - 2\epsilon,$$

we are done since

$$\textstyle\sum_{q \in (Z/2)^n} |\, 2^{-n} - P(Q_{\Delta t}(\omega) = q)| \leq 2P(F^c) < 4\epsilon.$$

Thus, all that remains is to demonstrate the independent claim (55), and to show that $P(F) > 1 - 2\epsilon$.

Here is the proof that $P(F) > 1 - 2\epsilon$. For any choices of $\alpha_i \in C_i$ and $t_i \in [0, \Delta t]$, using $i \neq j$ implies $C_j \cap N_{\alpha_i} = \phi$,

$$\begin{aligned} &P(\cap_{i=1}^n F_{\alpha_i} \mid \tau_i = t_i, \alpha_{\tau_i} = \alpha_i \text{ for } i = 1 \text{ to } n) \\ &\qquad = P(T'_{N_{\alpha_i} \cap C_i} \cap (t_i, \Delta t] = \phi, T'_{N_{\alpha_i} - C_i} \cap [0, \Delta t] = \phi, \\ &\qquad\qquad \text{for } i = 1 \text{ to } n \mid \tau_i = t_i, \alpha_{\tau_i} = \alpha_i \text{ for } i = 1 \text{ to } n) \\ &\qquad = P(T'_{N_{\alpha_i} \cap C_i} \cap (t_i, \Delta t] = \phi, T'_{N_{\alpha_i} - C_i} \cap [0, \Delta t] = \phi \text{ for } i = 1 \text{ to } n) \\ &\qquad \geq \Pi \exp(-2r_p(N_{\alpha_i})\Delta t) \\ &\qquad \geq \exp(-8n\Delta t) > 1 - \epsilon. \end{aligned}$$

Integrating over $t_i \in [0, \Delta t]$, $i = 1$ to n yields

$$P(\cap_{i=1}^n F_{\alpha_i}) \geq (1 - \epsilon)P(\tau_i \leq \Delta t, \alpha_{\tau_i} = \alpha_i \text{ for } i = 1 \text{ to } n).$$

Summing this over $C_1 \times C_2 \times \cdots \times C_n$ yields

$$P(F) \geq (1 - \epsilon)P(\tau_i \leq \Delta t \text{ for } i = 1 \text{ to } n).$$

For each i,

$$P(\tau_i > \Delta t) = \exp(-2r_p(C_i)\Delta t) \leq \exp(-2r\Delta t) = \exp\left(-2\left(\frac{4n_0}{\epsilon}\log\frac{n_0}{\epsilon}\right)\frac{\epsilon}{8n_0}\right) = \epsilon/n_0.$$

Since $n \le n_0$,

$$P(\tau_i \le \Delta t \text{ for } i = 1 \text{ to } n) > 1 - \epsilon,$$

so that

$$P(F) > (1 - \epsilon)^2 > 1 - 2\epsilon.$$

To show the independence result (55), start with any choices of $\alpha_i \in C_i$, and $q_1, q_2 \in (Z/2)^n$. On the event $\cap_{i=1}^n F_{\alpha_i}$, c is determined by coin tossing variables $c_{\alpha_i}(1)$, $i = 1$ to n. The event $\cap F_{\alpha_i}$ is measurable with respect to $\mathscr{P}'$ (without using any of the information in $\mathscr{C}$, saying which of the event times in $\mathscr{P}'$ are real and which are phantom). On the event $\cap F_{\alpha_i}$, $Q^*_{\Delta t}$ is measurable with respect to $\mathscr{P}'$ and that part of $\mathscr{C}$ which excludes the first coin $c_{\alpha_i}(1)$ for each pair of α_i. Thus, conditional on $\cap F_{;\, g\alpha_i}$, $Q^*_{\Delta t}$ and c are independent:

$$\begin{aligned} P(c = q_2, \cap F_{\alpha_i}, Q^*_{\Delta t} = q_1) &= P((c_{\alpha_i}(1), i = 1 \text{ to } n) = q_2, Q^*_{\Delta t} = q_1 \mid \cap F_{\alpha_i})P(\cap F_{\alpha_i}) \\ &= P((c_{\alpha_i}(1), i = 1 \text{ to } n) = q_2 \mid \cap F_{\alpha_i})P(Q^*_{\Delta t} = q_1 \mid \cap F_{\alpha_i})P(\cap F_{\alpha_i}) \\ &= 2^{-n}P(Q^*_{\Delta t} = q_1, \cap F_{\alpha_i}). \end{aligned}$$

Summing the outer equality over all choices of the $\alpha_i \in C_i$ yields (55) and completes the proof. □

The counterpart to Lemma 8, for a nearest neighbor random walk p on the line, is easy because the jump rate (36) for this voter model is constant. In contrast to Lemma 8, where $\Delta t = \epsilon/8n_0$, in Lemma 8′ below Δt must be taken very large to accommodate either ϵ small or n_0 large.

LEMMA 8′. *Let p be a nearest neighbor random walk on Z^1, and let $\epsilon > 0$ and n_0 be given. There exist m and $\Delta t > 0$ such that, for any $n \le n_0$ and $a_0 < a_1 < \cdots < a_n$ having $a_i - a_{i-1} \ge m$ for $i = 1$ to n, the joint voter model parity*

$$Q_t \equiv (|\zeta_t^{[a_{i-1}, a_i)}| \bmod 2, \qquad i = 1 \text{ to } n)$$

satisfies

$$\textstyle\sum_{q \in (Z/2)^n} |2^{-n} - P(Q_{\Delta t} = q)| < \epsilon.$$

PROOF. Let R_t be the random walk on $(Z/2)^n$, starting at Q_0, and having jumps, expressed in terms of the standard basis $\{e_1, e_2, \cdots, e_n\}$,

$$\begin{aligned} &q \to q + e_i &&\text{at rate 1, for } i = 1 \text{ or } n \\ &q \to q + e_i + e_{i+1} &&\text{at rate 2, for } i = 1 \text{ to } n - 1. \end{aligned}$$

Up until time

$$\tau = \min_{1 \le i \le n} \inf\{t : \zeta_t^{[a_{i-1}, a_i)} = \phi\},$$

Q and R can be coupled so that $Q_t = R_t$. Take m so large that $P(\zeta_{\Delta t}^{[0,m)} = \phi) < \epsilon/(4n_0)$, and thus $P(\tau \le \Delta t) < \epsilon/4$. The distribution PR_t^{-1} of R_t converges to π_n, having mass 2^{-n} at each point of $(Z/2)^n$. Take Δt so large that the total variation distance $\| PR_{\Delta t}^{-1} - \pi_n \|$, which is the same regardless of which pure state R is started in, is less than $\epsilon/2$. Now

$$\| \pi_n - PQ_{\Delta t}^{-1} \| \le \| \pi_n - PR_{\Delta t}^{-1} \| + \| PR_{\Delta t}^{-1} - PQ_{\Delta t}^{-1} \| < \epsilon/2 + \epsilon/2 = \epsilon. \qquad \square$$

THEOREM 4. *Let p be a genuinely multidimensional random walk on Z^d or a nearest neighbor walk on Z^1. Let $K \subset R^d$ be compact and convex, and define sets $K_t \subset Z^d$ by* (1), *so that*

$$\sup_{t \geq 0} E(\xi_t \cap K_t) \leq c < \infty. \tag{56}$$

The system η_t of annihilating random walks, starting with all sites occupied, is asymptotically the one half thinning of the corresponding system ξ_t of coalescing random walks, i.e.

$$\lim_{t\to\infty} \textstyle\sum_{A\subset K_t} P(\xi_t \cap K_t = A)(\sum_{B\subset A} |2^{-|A|} - P(\eta_t \cap K_t = B \mid \xi_t \cap K_t = A)|) = 0. \tag{57}$$

PROOF. We show that (57) holds for each choice of $c < \infty$, uniformly in arbitrary choices $K_t \subset Z^d$ satisfying (56); the geometric structure of the K_t does not enter into the argument, except in the case $d = 1$ where it is necessary to assume that each K_t is an interval.

For the family of voter models $(\zeta_t^A, A \subset Z^d)$ with the standard additive coupling induced by a substructure $\mathscr{P}$:

$$\zeta_t^A = \cap_{x\in A} \zeta_t^x$$

introduce abbreviation for the set of individuals whose initial opinions survive until time t:

$$S_t = \{x \in Z^d : \zeta_t^x \neq \phi\}$$

and for the set of those whose dynasty of followers at time t has odd cardinality:

$$S_t^{\text{odd}} = \{x \in Z^d : |\zeta_t^z| \text{ is odd}\}. \tag{58}$$

The usual coupling on $[0, t]$ is coalescing random walks ξ_t, annihilating random walks η_t, (both starting from Z^d, all sites occupied) and the family of voter models yields, for all ω,

$$\xi_t = S_t, \qquad \eta_t = S_t^{\text{odd}}. \tag{59}$$

Fix $c < \infty$ and sets $K_t \subset Z^d$ satisfying (56); in $d = 1$ also require that the K_t be intervals. Introduce the abbreviation, for finite $A \subset Z^d$, $t \geq 0$

$$g_t(A) = \textstyle\sum_{B\subset A} |2^{-|A|} - P(S_t^{\text{odd}} \cap K_t = B \mid S_t \cap K_t = A)|.$$

Since each $g_t(A)$ is the total variation distance between two probability measures on the collection of all subsets of A,

$$0 \leq g_t(A) \leq 2.$$

Using the coupling (59), our goal (57) becomes

$$\lim_{t\to\infty} \textstyle\sum_{A\subset K_t} P(S_t \cap K_t = A) g_t(A) = 0. \tag{60}$$

Let $\epsilon \in (0, 1)$ be given; we will find t_0, depending on ϵ, p, and c, such that the sum in (60) is less than 10ϵ when $t > t_0$.

Set $n_0 = c/\epsilon$, so that by Chebyshev's inequality and (56),

$$P(|K_t \cap S_t| > n_0) < \epsilon. \tag{61}$$

In the multidimensional case, let m, depending on n_0, p, and ϵ, be determined by Lemma 8, and set $\Delta t = \epsilon/(8n_0)$. In the nearest neighbor, $d = 1$ case, let m and Δt, depending on n_0 and ϵ, be determined by Lemma 8′. Fix any $t > \Delta t$ and set

$$s = t - \Delta t.$$

For $n \leq n_0$ and any disjoint finite $A_1, A_2, \cdots, A_n \subset Z^d$ satisfying $|A_i| \geq m$ [require also, in the $d = 1$ case, that the A_i as well as $\cup A_i$ be intervals] Lemma 8 or Lemma 8′ says

$$\textstyle\sum_{q\in(Z/2)^n} |2^{-n} - P((\zeta_{\Delta t}^{A_i}, \quad i = 1 \text{ to } n) = q \bmod 2)| < 4\epsilon.$$

Since the part of the substructure $\mathscr{P}$ up to time s is independent of the part of $\mathscr{P}$ after s, for any $A \subset K_t$ with $|A| = n$, $A = \{x_1, \cdots, x_n\}$,

$$\textstyle\sum_q |2^{-n} - P((|\zeta_t^{x_i}| \; i = 1 \text{ to } n) \equiv q \,|\, \zeta_s^{x_i} = A_i \text{ for } i = 1 \text{ to } n, K_t \cap S_s = A)| < 4\epsilon.$$

Average this over all allowable choices for the $A_1, \cdots, A_n$ to get:

$$\textstyle\sum_q |2^{-n} - P((|\zeta_t^{x_i}|, i = 1 \text{ to } n) \equiv q \,||\, \zeta_t^{x_i}| \ge m \text{ for } i = 1 \text{ to } n, K_t \cap S_s = A)| < 4\epsilon.$$

[In the $d = 1$ case each $\zeta_s^{x_i}$ is an interval, so $K_t \cap S_s = A$, together with K_t being an interval, implies that $\cup_{i=1}^n \zeta_s^{x_i}$ is an interval.] Using the notation (58), this can be rewritten

$$\text{(62)} \qquad \textstyle\sum_{B\subset A} |2^{-|A|} - P(S_t^{\text{odd}} \cap K_t = B \,|\, S_s \cap K_t = A, |\zeta_s^x| \ge m \; \forall x \in A)| < 4\epsilon.$$

Consider the events

$$E_t = \{|\zeta_s^x| \ge m \; \forall x \in K_t \cap S_s, \quad K_t \cap S_s = K_t \cap S_t\}.$$

Our final goal will be to show that

$$\text{(63)} \qquad P(E_t) \to 1 \quad \text{as } t \to \infty.$$

To show how (60) follows from (62) and (63), define

$$\mathscr{G}_t = \{A \subset K_t\colon P(E_t \,|\, K_t \cap S_t = A) > 1 - \epsilon\}.$$

Choose t_0 so that $t \ge t_0$ implies $P(E_t^c) < \epsilon^2$. Since

$$\epsilon^2 > P(E_t^c) = \textstyle\sum_{A\subset K_t} P(S_t \cap K_t = A) P(E_t^c \,|\, S_t \cap K_t = A)$$

$$\ge \textstyle\sum_{A\subset K_t : A \notin \mathscr{G}_t} P(S_t \cap K_t = A) \cdot \epsilon = \epsilon P(S_t \cap K_t \notin \mathscr{G}_t),$$

$t > t_0$ implies

$$\text{(64)} \qquad P(S_t \cap K_t \in \mathscr{G}_t) > 1 - \epsilon.$$

Now for $A \in \mathscr{G}_t$ with $|A| = n \le n_0$,

$$P(K_t \cap S_s = A, \quad |\zeta_s^x| \ge m \; \forall x \in A \,|\, K_t \cap S_t = A) > 1 - \epsilon$$

and (62) together imply

$$\text{(65)} \qquad \textstyle\sum_{B\subset A} |2^{-|A|} - P(S_t^{\text{odd}} \cap K_t = B \,|\, K_t \cap S_t = A)| < 2\epsilon + 4\epsilon.$$

[This argument has the form: if a measure $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ is a mixture of two probability measures, then for the total variation distance from a probability measure ν,

$$\|\nu - \mu\| \le \lambda \|\nu - \mu_1\| + (1 - \lambda)\|\nu - \mu_2\| \le \lambda(2) + 1\,\|\nu - \mu_2\|,$$

with $\lambda < \epsilon$ and $\|\nu - \mu_2\| < 4\epsilon$.] Formula (65) may be rewritten: for $A \in \mathscr{G}_t$ with $|A| = n \le n_0$,

$$g_t(A) < 6\epsilon,$$

so the sum in (60) may be estimated, for $t > t_0$:

$$\textstyle\sum_{A\subset K_t} P(S_t \cap K_t = A) g_t(A) \le \sum_{A\in \mathscr{G}_t : |A| \le n_0} P(S_t \cap K_t = A) \cdot 6\epsilon$$

$$+ \textstyle\sum_{A\subset K_t : A \notin \mathscr{G}_t \text{or} |A| > n_0} P(S_t \cap K_t = A) \cdot 2$$

(using (64) and (61)) $\le 1 \cdot 6\epsilon + 2\epsilon \cdot 2 = 10\epsilon$.

Thus all that remains is to verify (63). Recall that $s = t - \Delta t$; Δt is fixed as t varies; we write $p_t = P(0 \in \xi_t) = P(\zeta_t^x \ne \phi)$ so that condition (56) is: $|K_t| p_t \le c$. Define events, for $k = 1, 2, \cdots$

$$E_{t,k} = \{|\zeta_s^x| \ge k \; \forall x \in K_t \cap S_s, \quad K_t \cap S_s = K_t \cap S_t\},$$

so that $E_t \equiv E_{t,m} \supset E_{t,k}$ if $k \geq m$. To get $P(E_t^c) \to 0$, write

$$E_{t,k}^c = \cup_{x \in K_t} (\{0 < |\zeta_s^x| < k\} \cup \{|\zeta_s^x| \geq k, \quad \zeta_t^x = \phi\}),$$

so that by translation invariance and (56),

$$(66) \quad P(E_t^c) \leq \inf_{k:k\geq m} P(E_{t,k}^c) \leq (c/p_t) \inf_{k:k\geq m}[P(0 < |\zeta_s^0| < k) + P(|\zeta_s^0| \geq k, \quad \zeta_t^0 = \phi)].$$

Since the flip rate at any site in the voter model is at most 1,

$$p_t/p_s = P(\zeta_t^x \neq \phi \mid \zeta_s^x \neq \phi) \geq e^{-1\Delta t}.$$

By conditioning on ζ_s^0,

$$P(|\zeta_s^0| \geq k, \zeta_t^0 = \phi) \leq P(|\zeta_s^0| \geq k) \sup_{A:|A|\geq k} P(\zeta_{\Delta t}^A = \phi) \leq p_s \sup_{A:|A|\geq k} P(\zeta_{\Delta t}^A = \phi).$$

Thus, continuing with (66),

$$P(E_t^c) \leq ce^{\Delta t} \inf_{k:k\geq m}[P(|\zeta_s^0| < k \mid \zeta_s^0 \neq \phi) + \sup_{A:|A|\geq k} P(\zeta_{\Delta t}^A = \phi)].$$

Lemma 3 says that the first term above goes to zero as $t \to \infty$, for any fixed k. To see that the second term can be made arbitrarily small by choosing k large enough, note that the embedded Markov chain for ζ^A must pass through a sequence $A_k, A_{k-1}, \cdots, A_1$ of configurations having $|A_i| = i$ before reaching ϕ. The jump rate out of A_i is $r_p(\partial A_i) \leq |A_i| = i$. For a pure death process which starts at k and jumps from i to $i - 1$ at rate i, the time until extinction has mean $\sum_{i=1}^k (1/i)$ and variance $\sum 1/i^2$; the probability that this process, starting at k, is extinct by time Δt tends to 0 as k increases. A comparison of ζ^A with this process shows that

$$\sup_{A:|A|\geq k} P(\zeta_{\Delta t}^A = \phi) \to 0 \quad \text{as } k \to \infty.$$

Thus $P(E_t^c) \to 0$ as $t \to \infty$, concluding this proof. □

COROLLARY 3. *Let p be any genuinely multidimensional random walk on Z^d, $d \geq 2$, or a nearest neighbor random walk on Z^d, $d = 1$. For each $t \geq 0$ there exists versions of η_t, and of $\Theta\xi_t$, a one half thinning of ξ_t, such that for any compact $K \subset R^d$,*

$$P(\eta_t \cap K_t \neq \Theta\xi_t \cap K_t) \to 0.$$

PROOF. When the sum in (57) is ϵ, there exist versions of η_t and $\Theta\xi_t$ such that

$$P(\eta_t \cap K_t \neq \Theta\xi_t \cap K_t) = \epsilon.$$

Let S^n be the sphere of radius n centered at the origin in R^d, and fix an increasing sequence $t_n \to \infty$ such that for $t \geq t_n$, with $K = S_n$, the sum in (57) is less than $1/n$. For $t \in [t_n, t_{n+1})$ select versions of η_t and $\Theta\xi_t$ such that

$$P(\eta_t \cap S_t^n \neq \Theta\xi_t \cap S_t^n) < \frac{1}{n}.$$

These are the required versions. □

COROLLARY 4. *Let p be a genuinely multidimensional random walk, or a nearest neighbor random walk on the line. If for some sequence $t_i \to \infty$,*

$$\alpha_{t_i}\xi_{t_i} \to_d \mu,$$

then

$$\alpha_{t_i}\eta_{t_i} \to_d \Theta\mu,$$

where the point process $\Theta\mu$ is the one half thinning of the point process μ on R^d.

PROOF. Write $C_c^+(R^d)$ for the class of all nonnegative continuous functions on R^d having compact support. The Laplace transform L_μ a random measure μ on R^d is defined by $L_\mu f = E(\exp(-\int f\,d\mu))$ for $f \in C_c^+$. The one half thinning $\Theta\mu$ of a random measure is characterized by

$$L_{(\Theta\mu)} f = L_\mu(-\log(1 - \tfrac{1}{2}(1 - e^{-f}))).$$

Suppose $f \in C_c^+$ with support contained in a compact set K. Let $g = -\log(1 - \tfrac{1}{2}(1 - e^{-f}))$, so that $L_{\Theta\mu} f = L_\mu g$. Take versions of η_t and $\Theta\xi_t$ as in Corollary 3. We have

$$|L_{\alpha_t\eta_t} f - L_{\Theta\mu} f| \le |L_{\alpha_t\eta_t} f - L_{\Theta\alpha_t\xi_t} f| + |L_{\alpha_t\xi_t} g - L_\mu g|.$$

The first term above is dominated by $P(\eta_t \cap K_t \neq \Theta\xi_t \cap K_t)$ which goes to zero as t approaches infinity; since $g \in C_c^+$ the second term goes to zero along the subsequence t_i by the hypothesis $\alpha_{t_i}\xi_{t_i} \to_d \mu$. Thus $L_{\alpha_{t_i}\eta_{t_i}} f \to L_{\Theta\mu} f$ for every $f \in C_c^+$, so $\alpha_{t_i}\eta_{t_i} \to_d \Theta\mu$. □

REFERENCES

[1] ARRATIA, R. (1979). Coalescing Brownian Motions on the Line. Ph.D. thesis, University of Wisconsin at Madison.

[2] BRAMSON, M. and GRIFFEATH, D. (1979). Renormalizing the 3-dimensional voter model. *Ann. Probability* **7** 418–432.

[3] BRAMSON, M. and GRIFFEATH, D. (1980). Clustering and dispersion rates for some interacting particle systems on Z^1. *Ann. Probability* **8** 183–213.

[4] BRAMSON, M. and GRIFFEATH, D. (1980b). Asymptotics for interacting particle systems on Z^d. *Z. Warsch. verw. Gebiete* **53** 183–196.

[5] GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724.** Springer, New York.

[6] HARRIS, T. (1977). A correlation inequality for Markov processes in partially ordered state spaces. *Ann Probability* **5** 451–454.

[7] HOLLEY, R. and STROOCK, D. (1979). Central limit phenomena of various interacting systems. *Ann. of Math.* **110** 333–393.

[8] KALLENBERG, O. (1975). *Random Measures.* Akademie-Verlag, Berlin.

[9] KELLY, F. P. (1977). The asymptotic behavior of an invasion process. *J. Appl. Probability* **14** 584–590.

[10] LIGGETT, T. (1978). Random invariant measures for Markov chains, and independent particle systems. *Z. Wahrsch. verw. Gebiete* **45** 297–313.

[11] SAWYER, S. (1979). A limit theorem for patch sizes in a selectively-neutral migration model. *J. Appl. Probability* **16** 482–495.

[12] SUDBURY, A. (1976). The size of the region occupied by one type in an invasion process. *J. Appl. Probability* **13** 355–356.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90007