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A STABILITY CRITERION FOR ATTRACTIVE NEAREST NEIGHBOR SPIN SYSTEMS ON $\mathbb{Z}$

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We study attractive nearest neighbor spin systems on the integers having "all 0's" and "all 1's" as traps. A necessary and sufficient condition for stability under small one-sided perturbations of the flip rates is established.

1. Introduction. This paper deals with the stability of certain stochastic spin systems under perturbations. The reader is referred to a concise survey by Liggett [10] for references to the extensive literature on spin systems, and for the framework and basic notation adopted here. Specifically, the processes we discuss are the attractive nearest neighbor spin systems on the integers, a class studied in Liggett's paper [11]. Intuitively, the $\{0, 1\}^{\mathbb{Z}}$-valued process (ξ_t) consists of interacting two state continuous time Markov chains $\xi_t(x)$, one at each site $x \in \mathbb{Z}$, where the Q-matrix at x at time t is given by

$$Q_t(x) = \begin{pmatrix} -\beta_{ij} & \beta_{ij} \\ \delta_{ij} & -\delta_{ij} \end{pmatrix},$$

with $i = \xi_t(x-1)$, $j = \xi_t(x+1)$. Thus the *birth rate* β and *death rate* δ at the site x are functions of the states of the chains at the two nearest neighboring sites $x - 1$ and $x + 1$. The dynamics are evidently translation invariant. The *attractiveness* assumption is:

$$\begin{aligned} &\beta_{ij} \quad \text{is nondecreasing as } i + j \text{ increases,} \\ &\delta_{ij} \quad \text{is nonincreasing as } i + j \text{ increases.} \end{aligned} \tag{1}$$

Intuitively, the more 1's on the neighbor set of x, the greater is the tendency for a 1 to occupy x. The eight values β_{ij}, δ_{ij} $(i, j = 0$ or $1)$, called the *flip rates*, uniquely determine the transition mechanism of (ξ_t).

The ergodic theory of spin systems is concerned with identification of invariant measures and their domains of attraction. Let $\mathscr{I}$ denote the class of invariant measures for a given system, $\mathscr{I}_e$ its extreme points. The main result of [11] asserts that if (ξ_t) is one of the spin systems on $\mathbb{Z}$ described above, and if the positivity assumption

$$\beta_{ij} + \delta_{ij} > 0 \qquad (i, j = 0 \quad \text{or } 1) \tag{2}$$

is satisfied, then

$$|\mathscr{I}_e| = 1 \quad \text{or } 2.$$

(Condition (2) rules out the possibility of degenerate invariant measures other than μ_0 and μ_1, which are the two measures concentrated at the states $\mathbf{0}$ = "all 0's" and $\mathbf{1}$ = "all 1's" respectively.) When $|\mathscr{I}_e| = 1$, the system is called *ergodic*; if ν is the unique invariant measure, then

$$P_\xi(\xi_t \in \cdot) \Rightarrow \nu \qquad \text{as} \quad t \to \infty$$

for any initial state ξ. If $|\mathscr{I}_e| = 2$, the system is *nonergodic*. It has been widely conjectured, but not proved, that if

$$\min\{\beta_{00}, \delta_{11}\} > 0, \tag{3}$$

(i.e., all the flip rates are positive), then the system is always ergodic. If (3) does not hold,

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say if $\beta_{00} = 0$, then nonergodicity can arise. The most widely studied example of this phenomenon is the basic one-dimensional contact process of Harris [5], which (up to a change in time scale) has flip rates of the form

$$\beta_{ij} = i + j$$
(4)
$$\delta_{ij}^{\varepsilon} \equiv \varepsilon.$$

Here $\varepsilon > 0$ is a parameter. For this model it is known that $|\mathscr{I}_e| = 1$ if $\varepsilon \geq 1$ (see [5]), whereas $|\mathscr{I}_e| = 2$ when $\varepsilon \leq 1/2$ (see [7]). One extreme invariant measure is clearly μ_0. For large ε, μ_0 is the only invariant measure, but for small ε there is a nonatomic extreme steady state ν_1 such that

(5) $$P_1(\xi_t \in \cdot) \Rightarrow \nu_1 \neq \mu_0 \quad \text{as} \quad t \to \infty.$$

Monotonicity arguments establish a critical value ε_c such that

$$|\mathscr{I}_e| = 1 \quad \text{for } \varepsilon > \varepsilon_c, \qquad |\mathscr{I}_e| = 2 \quad \text{for } \varepsilon < \varepsilon_c.$$

The value of ε_c and the behavior at $\varepsilon = \varepsilon_c$ are not known.

If $\beta_{00} = \delta_{11} = 0$ and (2) holds then $|\mathscr{I}_e| = 2$ since μ_0 and μ_1 are both extreme invariant. One way of viewing the contact model (4) is as a one-sided ε-perturbation of the system with flip rates:

(6) $$\beta_{ij} = i + j \qquad \delta_{ij} \equiv 0.$$

Note that (4) is derived from (6) by adding ε to all the death rates. If (5) holds, we say that the state **1** is *stable* with respect to such a perturbation. In fact, if P^{ε} governs the contact system (ξ_t^{ε}) with parameter ε, one can show that

(7) $$\lim_{\varepsilon \to 0} \sup_{t \geq 0} P_1^{\varepsilon}(\xi_t^{\varepsilon}(0) = 0) = 0.$$

By analogy, given any rates β_{ij}, δ_{ij} with $\beta_{00} = \delta_{11} = 0$, one can consider the one-sided perturbation:

$$\beta_{ij}^{\varepsilon} = \beta_{ij}$$
(8)
$$\delta_{ij}^{\varepsilon} = \delta_{ij} + \varepsilon,$$

and ask the same question:

(9) Is **1** stable, i.e. does (7) hold?

If (7) does hold, then it follows from attractiveness and Liggett's Theorem that for sufficiently small positive ε, $\mathscr{I}_e^{\varepsilon} = \{\mu_0, \nu_1^{\varepsilon}\}$, where $\nu_1^{\varepsilon} = \lim_{t \to \infty} P_1^{\varepsilon}(\xi_t^{\varepsilon} \in \cdot) \neq \mu_0$. Since attractiveness implies the existence of a critical constant ε_c such that

$$|\mathscr{I}_e^{\varepsilon}| = 1 \quad \text{for} \quad \varepsilon > \varepsilon_c, \qquad |\mathscr{I}_e^{\varepsilon}| = 2 \quad \text{for} \quad \varepsilon < \varepsilon_c,$$

(we allow $\varepsilon_c = 0$, in which case the second alternative is vacuous), (9) is equivalent to:

$$\text{Is} \quad \varepsilon_c > 0?$$

Our object in this paper is to settle (9) for any attractive nearest neighbor system with $\beta_{00} = \delta_{11} = 0$. We will consider rates of the form (8), which we call ε-perturbations. It is not hard to identify a *necessary* condition for stability, namely

(10) $$\beta_{01} + \beta_{10} > \delta_{01} + \delta_{10}.$$

If (10) fails, then in the ε-perturbation, a block of 0's of size 2 or more decreases by one at rate $\beta_{01} + \beta_{10}$ and increases by at least one at rate $\delta_{01} + \delta_{10} + 2\varepsilon$. Thus blocks of 0's tend to grow. Using translation invariance it is not hard to show that for every $\varepsilon > 0$,

(11) $$\lim_{t\to\infty} P_1^{\varepsilon}(\xi_t^{\varepsilon}(0) = 0) = 1,$$

so that $\mathscr{I}^{\varepsilon} = \{\mu_0\}$. In particular, (7) fails. The reader should have no difficulty supplying a rigorous proof that (10) is necessary for stability, based on the above heuristics.

The main result of this paper asserts that condition (10) is also sufficient for stability:

THEOREM. *Let (ξ_t) be an attractive nearest neighbor spin system on $\mathbb{Z}$ with $\beta_{00} = \delta_{11} = 0$. Let (ξ_t^{ε}) be the one-sided perturbation with flip rates* (8). *If*

$$\beta_{01} + \beta_{10} > \delta_{01} + \delta_{10},$$

then **1** *is stable, i.e.* (7) *holds.*

The proof of the Theorem is quite involved; it is based on the so-called "contour method" of percolation theory. Roughly speaking, we think of the space-time set

$$\{(x, t) \in \mathbb{Z} \times \mathbb{R}^+ : \xi_t(x) = 1\}$$

as representing the flow of liquid in $\mathbb{Z} \times \mathbb{R}^+$. In order to show that $\delta_t(0) = 1$ with large probability, one observes that if the origin is not wet at time t, then the flow must be blocked by a barrier, or *contour.* Thus it suffices to prove that such a contour can only exist with small probability. As far as we are aware, this methodology has not been applied previously to continuous time systems. For certain discrete time deterministic operators under one-sided perturbations, a similar approach has been exploited extensively by Toom [13], [14], [15]. In the oriented percolation setting, this technique is due to Hammersley [4]. At least in spirit, one is also reminded of the Peierls argument for phase transition in the two-dimensional Ising model [12], [3]. Our work here is most directly inspired by Toom's beautiful theory of "eroders" [15]. For his models, **1** is stable if the unperturbed deterministic process reaches **1** from any state that has only finitely many 0's. Quite remarkably, his analysis yields necessary and sufficient conditions for stability in any dimension d. We note in passing that the notion of stability of deterministic operators subject to random noise goes back to Von Neumann [16]. An outline of von Neumann's theory may be found in [1]. For us, matters are not so nice: major complications arise from the fact that the unperturbed system (ξ_t) is already random. Clearly (10) is the correct "eroder condition" in the one-dimensional nearest neighbor case, but even here the problem is barely tractable. For non-nearest neighbor or higher dimensional systems it seems unlikely that a manageable theory will emerge.

In order to give structure to the proof of the Theorem, we have divided it into four parts. Section 2 contains relatively concrete representations of the systems under discussion. To facilitate computations, we define (ξ_t), as far as possible, with the aid of a *substructure* of Poisson processes. Our construction may be viewed as a generalization of Harris' graphical representation of additive systems (cf. [6] or [2]) to cases where the processes with different initial configurations do not fit together quite so well. In Section 3 we define the *contour* Γ_T *surrounding* $(0, T)$ in the space-time diagram of (ξ_t). Using the construction of Section 2, we are able, in Lemma 8, to obtain a function which is an upper bound for the probability density for contours of a certain description. At this point, for the sake of clarity, the proof of the Theorem is divided into two cases. It turns out that if

(12) $$\beta_{11} \geq \beta_{01} + \beta_{10},$$

then a relatively straightforward calculation, based on Lemma 8 and some combinatorics, yields the stability property (7). In this case one can actually determine a numerical lower bound for ε_c. Harris' basic contact process (4) satisfies (12); for that model we obtain the estimate

(13) $$\varepsilon_c \geq 14 - 8\sqrt{3} \approx .14.$$

This is not as good as the remarkable Holley-Liggett bound $\varepsilon_c \geq .5$, but percolation methods are notoriously crude. Section 4 contains the proof of stability assuming (12). We elimimate the hypothesis (12) in Section 5, but in order to do so we must resort to unpleasant surgical operations on the contour which obfuscate the essential idea. When (12) fails, our proof is too complicated to yield any calculable positive lower bound for ε_c.

2. A representation of (ξ_t). In this section we show how to use Poisson point locations on the space-time graph $\mathbb{Z} \times [0, \infty)$ to construct any translation invariant nearest neighbor spin system with given attractive flip rates β_{ij} and δ_{ij}. Recall that a *Poisson point location* (P.p.ℓ.) on R^+ with density $\rho > 0$ is a random set of points $\mathscr{B} = \{X_1, X_2, \cdots\} \subset \mathbb{R}^+$ such that $X_1, X_2 - X_1, \cdots$ are i.i.d. exponential random variables with mean ρ^{-1}. If $\rho = 0$, then $\mathscr{B} = \varnothing$. To begin the construction, let β_k and δ_k, $0 \leq k \leq 3$, be the β_{ij} and δ_{ij} arranged in increasing order. Define

$$m_{ij} = k \quad \text{if} \quad \beta_{ij} = \beta_k$$

$$n_{ij} = k \quad \text{if} \quad \delta_{ij} = \delta_k.$$

For each $x \in \mathbb{Z}$, and $0 \leq k \leq 3$, let $\mathscr{B}_k(x)$ and $\mathscr{D}_k(x)$ be P.p.ℓ.'s on $\{x\} \times \mathbb{R}^+$ with densities $\beta_k - \beta_{k-1}$ and $\delta_k - \delta_{k-1}$ respectively, where $\beta_{-1} = \delta_{-1} = 0$. Take all the P.p.$\ell$.'s to be independent on a common probability space $(\Omega, \mathscr{F}, P_1)$. Next, introduce

$$\mathscr{B}_{ij}(x) = \cup_{0 \leq k \leq m_{ij}} \mathscr{B}_k(x), \qquad \mathscr{D}_{ij}(x) = \cup_{0 \leq k \leq n_{ij}} \mathscr{D}_k(x),$$

$$\mathscr{J}(x) = \cup_k (\mathscr{B}_k(x) \cup \mathscr{D}_k(x)).$$

Note that $\mathscr{B}_{ij}(x)$, $\mathscr{D}_{ij}(x)$ and $\mathscr{J}(x)$ are P.p.ℓ.'s on $\{x\} \times \mathbb{R}^+$ with densities β_{ij}, δ_{ij} and $\beta_{11} + \delta_{00}$ respectively.

We will now define the desired spin system (ξ_t), starting from $\mathbf{1}$ = "all 1's", on $(\Omega, \mathscr{F}, P_1)$. First, we need some finite approximations. For each *finite* $A \subset \mathbb{Z}$, inductively define

$$\tau_{n+1}^A = \inf\{t > \tau_n^A : (x, t) \in \mathscr{J}(x) \quad \text{for some} \quad x \in A\}$$

$(\tau_0^A \equiv 0)$. Introduce the process (ξ_t^A) given by

$$\xi_t^A(x) = \xi^{n,A}(x) \qquad t \in [\tau_n^A, \tau_{n+1}^A), \tag{14}$$

where the right side of (14) is identically 1 for $x \notin A$, and for $x \in A$ is defined inductively by:

$$\xi^{0,A}(x) = 1,$$

and when $\xi^{n,A}(x-1) = i$ and $\xi^{n,A}(x+1) = j$,

$$\begin{aligned} \xi^{n+1,A}(x) &= 1 && \text{if} \quad \xi^{n,A}(x) = 0,\ (x, \tau_{n+1}^A) \in \mathscr{B}_{ij}(x); \\ &= 0 && \text{if} \quad \xi^{n,A}(x) = 1,\ (x, \tau_{n+1}^A) \in \mathscr{D}_{ij}(x); \\ &= \xi^{n,A}(x) && \text{otherwise.} \end{aligned}$$

It is easily verified that (ξ_t^A) is the spin system which starts in configuration $\mathbf{1}$ and has flip rates at x equal to $\beta_{ij} I_A(x)$ and $\delta_{ij} I_A(x)$. Thus (ξ_t^A) can be identified with a finite Markov chain on $\{0, 1\}^A$. It is also easy to see that for each finite A and B, $(\xi_t^A, \xi_t^B)_{t \geq 0}$ is the *basic coupling* of (ξ_t^A) and (ξ_t^B) (cf. section 2.1 of [10]). Since the flip rates are attractive, it follows that whenever $B \subset A$,

$$\xi_t^B(x) \geq \xi_t^A(x) \qquad \text{for all} \quad x, t \tag{15}$$

with probability one. To avoid null sets as much as possible, it is convenient at this point to redefine $(\Omega, \mathscr{F}, P_1)$ by removing the following sets of measure zero:

$$\{\omega : |\mathscr{J}(x) \cap (\{x\} \times [0, T])| = \infty \quad \text{for some} \quad x \quad \text{and} \quad T\},$$

$$\{\omega : (x, t) \in \mathscr{J}(x) \quad \text{and} \quad (y, t) \in \mathscr{J}(x) \quad \text{for some} \quad x \neq y \quad \text{and} \quad t\},$$

$$\{\omega : \text{(15) does not hold for some } B \subset A\}.$$

We now define

$$(16)\qquad \xi_t(x) = \lim_{A\uparrow Z} \xi_t^A(x) \qquad (\text{for } all\ \omega).$$

It is not hard to see that (ξ_t) is a process with the desired flip rates.

LEMMA 1. *For any* $x \in \mathbb{Z}$, $T > 0$,

$$(17)\qquad \lim_{A\uparrow Z} \sup_{t\in[0,T]} |\xi_t(x) - \xi_t^A(x)| = 0 \qquad (\textit{for all } \omega),$$

and hence the process defined by (16) *is a spin system with flip rates* β_{ij} *and* δ_{ij} *and with initial state* 1.

(We omit the easy proof. A convergence theorem of Holley and Stroock (Theorem (2.3) of [8]) can be applied once (17) is established.)

The construction just completed allows us to make use of the underlying P.p.$\mathscr{L}$'s in our analysis of (ξ_t). The lemma which follows shows the relationship between points in the P.p.$\mathscr{L}$'s and the times of various types of transitions of the system (ξ_t). It will be exploited in the next section.

LEMMA 2. *Given* $x \in \mathbb{Z}$, $u > 0$, *let* $\xi_u(x-1) = i$, $\xi_u(x+1) = j$.

(*i*) *Suppose that* $\lim_{t\nearrow u} \xi_t(x) \neq \xi_u(x)$. *Then*

$$(x, u) \in \mathscr{B}_{ij}(x) \quad \text{if} \quad \xi_u(x) = 1;$$
$$(x, u) \in \mathscr{D}_{ij}(x) \quad \text{if} \quad \xi_u(x) = 0.$$

(ii) *Suppose that* $\lim_{t\nearrow u} \xi_t(x) = \xi_u(x)$. *Then*

$$(x, u) \notin \mathscr{D}_{ij} \quad \text{if} \quad \xi_u(x) = 1;$$
$$(x, u) \notin \mathscr{B}_{ij}(x) \quad \text{if} \quad \xi_u(x) = 0.$$

PROOF. By construction, the lemma holds if (ξ_t) is replaced by (ξ_t^A) for any finite $A \subset \mathbb{Z}$. Property (17) guarantees that the claims (i) and (ii) hold in the limit as $A \nearrow \mathbb{Z}$. □

3. The contour estimate. In this section, let (ξ_t) be an attractive spin system with flip rates β_{ij} and δ_{ij}, such that

$$\beta_{00} = 0,$$

constructed as in Section 2. Our first objective in this section is to define the contour Γ_T surrounding $(0, T)$ in the space-time diagram for (ξ_t), and to establish P_1-a.s. equality of the events $\{\xi_T(0) = 0\}$ and $\{\Gamma_T \neq \varnothing\}$. Embed $\mathbb{Z} \times R^+$ in $R \times R^+$ in the natural way. Let

$$\mathscr{O} = \text{interior of } \{(r, t) \in R \times R^+ : \xi_t(x) = 0 \quad \text{and} \quad |x - r| \leq 1/2 \qquad \text{for some} \quad x \in \mathbb{Z}\},$$

and, for $T > 0$, let $\mathscr{O}_T = \varnothing$ if $(0, T), \notin \mathscr{O}$ and if $(0, T) \in \mathscr{O}$ let $\mathscr{O}_T$ be the component of $\mathscr{O} \cap (R \times [0, T])$ whose boundary contains $(0, T)$. The *contour* Γ_T *surrounding* $(0, T)$ is defined by

$$\Gamma_T = \text{the boundary of } \mathscr{O}_T \quad \text{if} \quad (0, T) \in \mathscr{O};$$
$$= \varnothing \quad \text{if} \quad (0, T) \notin \mathscr{O}.$$

LEMMA 3. $\{\xi_T(0) = 0\} = \{\Gamma_T \neq \varnothing\}$ $\quad P_1$-a.s. *for each* $T > 0$.

PROOF. Easy. Recall that our ultimate goal is to prove (7) assuming (10). According to Lemma 3, it suffices to show that

$$(18)\qquad \lim_{\epsilon\to 0} \sup_{T>0} P_1^\epsilon(\Gamma_T \neq \varnothing) = 0,$$

where $\Gamma_T = \Gamma_T^\epsilon$ is the contour surrounding $(0, T)$ for (ξ_t^ϵ). We will verify (18) by analyzing probabilistically the possible shapes for nonempty contours. Some preliminary observations concerning Γ_T are summarized in the next lemma. The terms *right, left, up, down, horizontal* and *vertical* will have their usual meanings throughout this paper, where we think of $R \times R^+$ as embedded in R^2 with the standard orientation. Thus, for example, *up* means increasing the R^+ (i.e. time) coordinate with the R (i.e. space) coordinate fixed.

LEMMA 4. *Let $\Omega_T = \{\omega: \mathcal{O}_T$ is bounded$\}$. Then $P_1(\Omega_T) = 1$ for each $T > 0$. Moreover, for any $\omega \in \Omega_T$, either $\Gamma_T = \varnothing$ or the following four properties hold:*

(i) *Γ_T consists only of vertical and horizontal line segments.*
(ii) *Any horizontal segment in $\Gamma_T \cap (R \times (0, T))$ has unit length and a midpoint (x, t) with $x \in Z$ and $(x, t) \in \mathcal{J}(x)$.*
(iii) *Γ_T is a simple closed curve.*
(iv) *$\Gamma_T \cap (R \times \{T\})$ is a connected segment of positive integer length containing $(0, T)$.*

PROOF. By the definition of $\mathcal{O}_T$ and Lemma 2,

$$\Omega_T \supset \{\exists\, x \in \mathbb{Z}^+, y \in \mathbb{Z}^-: \xi_t(x) = \xi_t(y) = 1 \quad \text{for all} \quad t \in [0, T]\}$$
$$\supset \{\exists\, x \in \mathbb{Z}^+, y \in \mathbb{Z}^-: \mathcal{J}(x) \cap (\{x\} \times [0, T]) = \mathcal{J}(y) \cap (\{y\} \times [0, T]) = \varnothing\}.$$

The last event has probability one by Borel-Cantelli, so $P_1(\Omega_T) = 1$. Assuming $\Gamma_T \neq \varnothing$, we now verify (i)–(iv) for $\omega \in \Omega_T$. Property (i) follows from the definition of Γ_T, (ii) from the definition of Γ_T and Lemma 2(i).

To prove (iii), recall from the definition of $\mathcal{O}$ that $\mathcal{O}$ is open and connected, so (iii) holds unless $\mathcal{O}$ is not simply connected. It is not hard to see that if $\mathcal{O}$ were not simply connected, then there would be some site x where a birth occurred between two 0's. This is impossible by Lemma 2(i) and the fact that $\mathcal{B}_{00}(x) = \varnothing$. Similar reasoning also implies (iv). □

Our next objective is to formulate the manner in which the shape of Γ_T gives information about the underlying P.p.ℓ's. In order to facilitate analysis of the contour, we orient it so that $\Gamma_T \cap (R \times \{T\})$ is directed to the right. Thus, if you walked around the contour following this direction, you would always see a 0 on your right and a 1 on your left. Let

$$\mathcal{G}_T = \{\text{oriented } \gamma \subset R \times R^+ : \Gamma_T(\omega) = \gamma \quad \text{for some} \quad \omega \in \Omega_T\},$$
$$\mathcal{G} = \cup_{T>0}\, \mathcal{G}_T.$$

At this point we need a good deal of descriptive notation to encode the oriented shape of a contour. Given $\gamma \in \mathcal{G}_T$, define

$$H_0 = \gamma \cap (R \times \{T\}),$$
$$p_0 = \text{the right endpoint of } H_0,$$
$$N = \text{the number of horizontal segments in } \gamma - H_0.$$

For $1 \le i \le N + 1$, set

V_i = the ith vertical segment in γ, starting with the vertical segment which contains p_0, and indexing around γ according to the orientation.

For $1 \le i \le N$, put

$$H_i = \text{the horizontal segment of } \gamma \text{ connecting } V_i \text{ to } V_{i+1},$$
$$(x_i, t_i) = \text{the midpoint of } H_i.$$

Each segment of γ inherits an orientation; use the letters ℓ, r, u, d for the orientations left, right, up and down respectively. For $1 \le i \le N$, write

Δ_i = the ordered triple of orientations corresponding to (V_i, H_i, V_{i+1}),
$\mathbf{\Delta} = (\Delta_1, \cdots, \Delta_N)$,

$\rho_i(\mathbf{\Delta})$ =	if Δ_i =
δ_{01}	(d, ℓ, d)
β_{01}	(d, r, d)
δ_{10}	(u, ℓ, u)
β_{10}	(u, r, u)
δ_{00}	(u, ℓ, d)
δ_{11}	(d, ℓ, u)

With the aid of Lemma 2, we are now ready to show the connection between the Δ_i and the substructure of P.p.ℓ.'s.

LEMMA 5. *If $\Gamma_T = \gamma$, and $1 \leq i \leq N$, then*
(i) $\rho_i(\mathbf{\Delta}) = \delta_{jk}$ *implies* $(x_i, t_i) \in D_{jk}(x_i)$
(ii) $\rho_i(\mathbf{\Delta}) = \beta_{jk}$ *implies* $(x_i, t_i) \in B_{jk}(x_i)$, $j \neq k$
(iii) Δ_i *cannot equal* (d, r, u)
(iv) Δ_i *cannot equal* (u, r, d).

PROOF. (i)-(iii) follow from Lemma 2(i), (iii) because $\mathscr{B}_{00}(x) = \varnothing$. Since $(0, T) \in \Gamma_T$, if $\Delta = (u, r, d)$, then $(\mathscr{O}_T \cup \Gamma_T) \cap (R \times \{t\})$ must contain two disjoint horizontal line segments. By translation invariance we can assume $x = 0$, in which case Γ_t violates Lemma 4 (iv). Hence (u, r, d) cannot occur, so (iv) holds. □

LEMMA 6. *Suppose $\Gamma_T = \gamma$. Let $V = V_i$ − {the endpoints of V_i} for some $i: 1 \leq i \leq N + 1$, and let $o \in \{u, d, r, \ell\}$ be the orientation of V_i.*
(i) *If $(x + 1/2, t) \in V$ and $o = d$, then $(x, t) \notin \mathscr{B}_{01}(x)$.*
(ii) *If $(x + 1/2, t) \in V$ and $o = u$, then $(x, t) \notin \mathscr{D}_{10}(x)$.*
(iii) *If $(x - 1/2, t) \in V$ and $o = d$, then $(x, t) \notin \mathscr{D}_{01}(x)$.*
(iv) *If $(x - 1/2, t) \in V$ and $o = u$, then $(x, t) \notin \mathscr{B}_{10}(x)$.*

PROOF. (i) By the definition of Γ_T and Lemma 2(ii), $(x, t) \notin \mathscr{B}_{ij}(x)$, where $i = \xi_t(x - 1)$ and $j = \xi_t(x + 1) = 1$. Since $\mathscr{B}_{11}(x) \supset \mathscr{B}_{01}(x)$, the claim follows. The arguments for (ii)-(iv) are analogous. □

LEMMA 7. *Let γ, V and o be as in Lemma 6, and let $V' = V_j$ − {the endpoints of V_j} for some $j: 1 \leq j \leq N + 1$.*
(i) *If $(x - 1/2, t) \in V$, $(x + 1/2, t) \in V'$ and $o = d$, then $x \notin \mathscr{D}_{00}(x)$.*
(ii) *If $(x - 1/2, t) \in V$, $(x + 1/2, t) \in V'$ and $o = u$, then $x \notin \mathscr{B}_{11}(x)$.*

PROOF. (i) Since $o = d$, it is not hard to see that V' must be oriented up. Hence, by definition of Γ_T and Lemma 2(ii), $\xi_t(x - 1) = \xi_t(x + 1) = 0$ and $(x, t) \notin \mathscr{D}_{00}(x)$. The proof of (ii) is similar. □

Given $\Gamma_T = \gamma$, let

$$\mathbf{\lambda} = (\lambda_1, \cdots, \lambda_N), \qquad \text{where } \lambda_i = \text{the length of } V_i, 1 \leq i \leq N.$$

In order to economize on notation, we will use the symbols N, $\mathbf{\Delta}$, and $\mathbf{\lambda}$ to denote both the functions of γ defined above, and also to denote canonical values of these functions. Which meaning we have in mind will be clear from the context. For each possible pair of N-tuples $\mathbf{\Delta}$ and $\mathbf{\lambda}$, there is a whole class of curves γ with direction vector $\mathbf{\Delta}$ and length vector $\mathbf{\lambda}$; these curves are all (space-time) translates of one another. Therefore, quantities which depend only on the shape of γ may be thought of as functions of $\mathbf{\Delta}$ and $\boldsymbol{\lambda}$. For N a positive integer, write

$$\mathscr{D}_N = \{\mathbf{\Delta} : \mathbf{\Delta}(\gamma) = \mathbf{\Delta} \text{ for some } \gamma \in \mathscr{G} \text{ such that } N(\gamma) = N\},$$

and for $\mathbf{\Delta} \in \mathscr{D}_N$, let

$$\mathscr{L}(\mathbf{\Delta}) = \{\boldsymbol{\lambda} \in (R^+)^N : \text{for some } T > 0 \text{ and } \gamma \in \mathscr{G}, \mathbf{\Delta}(\gamma) = \mathbf{\Delta} \text{ and } \boldsymbol{\lambda}(\gamma) = \boldsymbol{\lambda}\}.$$

Given $T > 0$ and $\mathbf{\Delta} \in \mathscr{D}_N$ for some $N > 0$, define a finite Borel measure $\mu_T^{\mathbf{\Delta}}$ on $(R^+)^N$ by

$$\mu_T^{\mathbf{\Delta}}(\cdot) = P_1(p_0(\Gamma_T) = (\tfrac{1}{2}, T), \mathbf{\Delta}(\Gamma_T) = \mathbf{\Delta}, \boldsymbol{\lambda}(\Gamma_T) \in \cdot). \tag{19}$$

The main result of this section is an upper estimate for $\mu_T^{\mathbf{\Delta}}$. To state it, we need to introduce

$$\rho(\mathbf{\Delta}) = \textstyle\prod_{i=1}^N \rho_i(\mathbf{\Delta}). \tag{20}$$

Also, write

$$I_u(\mathbf{\Delta}) = \{i: \text{the first coordinate of } \Delta_i = u\},$$

$$I_d(\mathbf{\Delta}) = \{i: \text{the first coordinate of } \Delta_i = d\}.$$

Finally, if γ is any of the translates such that $\mathbf{\Delta}(\gamma) = \mathbf{\Delta}$ and $\boldsymbol{\lambda}(\gamma) = \boldsymbol{\lambda}$, introduce

$$b(\mathbf{\Delta}, \boldsymbol{\lambda}) = \textstyle\sum_{x \in \mathbb{Z}} \nu\{t : \exists\, i \in I_u(\mathbf{\Delta}), j \in I_d(\mathbf{\Delta}) \quad \text{such that} \quad (x - \tfrac{1}{2}, t) \in V_i^0(\gamma) \text{ and } (x + \tfrac{1}{2}, t) \in V_j^0(\gamma)\}. \tag{21}$$

Here ν is Lebesgue measure on R^+, and $V_i^0 = V_i - \{\text{endpoints of } V_i\}$.

Recall that $P_1(\Gamma_T \neq \varnothing)$ is the probability that a contour prevents the occurrence of a 1 at site 0 at time T. The numerical bounds for $P_1(\Gamma_T \neq \varnothing)$, to be derived in the next two sections, will be based on the following result. In its statement, and *throughout the remainder of the paper*, we will write

$$\beta = \beta_{01} + \beta_{10}, \qquad \delta = \delta_{01} + \delta_{10}.$$

LEMMA 8. *Let* (ξ_t) *be an attractive spin system with flip rates* β_{ij} *and* δ_{ij}, *starting from* **1**. *Assume that* $\beta_{00} = 0$. *Given any* $T > 0$ *and* $\mathbf{\Delta} \in \mathscr{D}_N$, *if* $\mu_T^{\mathbf{\Delta}}$ *is defined by* (19), *then* $\mu_T^{\mathbf{\Delta}}$ *is absolutely continuous with respect to N-dimensional Lebesgue measure* ν_N. *Moreover,*

$$\frac{d\mu_T^{\mathbf{\Delta}}}{d\nu_N}(\boldsymbol{\lambda}) \le \rho(\mathbf{\Delta}) f(\mathbf{\Delta}, \boldsymbol{\lambda}) \qquad \boldsymbol{\lambda} \in \mathscr{L}(\mathbf{\Delta})$$
$$= 0 \qquad \boldsymbol{\lambda} \notin \mathscr{L}(\mathbf{\Delta}), \tag{22}$$

where $\rho(\mathbf{\Delta})$ *is given by* (20) *and for* $\alpha = (\delta - \delta_{00})^+$,

$$f(\mathbf{\Delta}, \boldsymbol{\lambda}) = \exp\{-(\beta + \delta - \alpha) \textstyle\sum_{i \in I_d(\mathbf{\Delta})} \lambda_i\} \exp\{-(\beta_{11} - \beta + \epsilon) b(\mathbf{\Delta}, \boldsymbol{\lambda})\}. \tag{23}$$

PROOF. Since $\mu_T^{\mathbf{\Delta}}(\mathscr{L}(\mathbf{\Delta})) = 1$, it suffices to prove (22). Fix $\boldsymbol{\lambda} \in \mathscr{L}(\mathbf{\Delta})$, and let γ be the unique curve in $\mathscr{G}_T$ such that $\mathbf{\Delta}(\gamma) = \mathbf{\Delta}$, $\boldsymbol{\lambda}(\gamma) = \boldsymbol{\lambda}$ and $p_0(\gamma) = (\tfrac{1}{2}, T)$. Put $b(\mathbf{\Delta}, \boldsymbol{\lambda}) = b$. For each $x \in \mathbb{Z}$, define

$$U_x^{\pm} = \{(x, t) : \exists\, i \in I_u(\mathbf{\Delta}) \quad \text{with} \quad (x \pm \tfrac{1}{2}, t) \in V_i^0(\gamma)\},$$

$$D_x^{\pm} = \{(x, t) : \exists\, i \in I_d(\mathbf{\Delta}) \quad \text{with} \quad (x \pm \tfrac{1}{2}, t) \in V_i^0(\gamma)\},$$

$$A_x = D_x^- \cap U_x^+, \qquad B_x = D_x^+ \cap U_x^-.$$

These sets divide those points on $\{x\} \times \mathbb{R}^+$ that are next to a vertical segment of Γ_T into six classes according to the values of $\xi_t(x-1)$, $\xi_t(x)$, and $\xi_t(x+1)$. By Lemmas 6 and 7, $\{\Gamma_T = \gamma\}$ is contained in the intersection over all $x \in \mathbb{Z}$ of the events

$$E_x = \{(U_x^- - B_x) \cap \mathscr{B}_{10}(x) = \varnothing\} \cap \{(D_x^+ - B_x) \cap \mathscr{B}_{01}(x) = \varnothing\}$$
$$\cap \{B_x \cap \mathscr{B}_{11}(x) = \varnothing\} \cap \{(U_x^+ - A_x) \cap \mathscr{D}_{10}(x) = \varnothing\}$$
$$\cap \{(D_x^- - A_x) \cap \mathscr{D}_{01}(x) = \varnothing\} \cap \{A_x \cap \mathscr{D}_{00}(x) = \varnothing\}.$$

The six events which define E_x are independent since they evaluate the P.p.ℓ.'s over disjoint sets, and all the E_x's are independent as well. Letting ν_x be 1-dimensional Lebesque measure on $\{x\} \times [0, \infty)$, we have

$$P_1(\cap_{x\in\mathbb{Z}} E_x) = \prod_{x\in\mathbb{Z}}[\exp\{-\beta_{10}\nu_x(U_x^- - B_x)\}\exp\{-\beta_{01}\nu_x(D_x^+ - B_x)\}\exp\{-\beta_{11}\nu_x(B_x)\}$$
$$\cdot\exp\{-\delta_{10}\nu_x(U_x^+ - A_x)\}\exp\{-\delta_{01}\nu_x(D_x^- - A_x)\}\exp\{-\delta_{00}\nu_x(A_x)\}].$$

Since γ is a simple closed curve,

$$\textstyle\sum_x \nu_x(U_x^+) = \sum_x \nu_x(U_x^-) = \sum_x \nu_x(D_x^+) = \sum_x \nu(D_x^-),$$

and the common value is $\sum_{i\in I_u(\Delta)} \lambda_i$. Note also that $\sum_x \nu_x(B_x) = b$. Thus, one easily obtains the bound:

$$(24)\qquad P_1(\cap_{x\in\mathbb{Z}} E_x) \le \exp\{-(\beta + \delta - \alpha) \textstyle\sum_{i\in I_d(\Delta)} \lambda_i\}\exp\{-(\beta_{11} - \beta + \alpha)b\}.$$

Next, abbreviate $(x_i(\gamma), t_i(\gamma)) = (x_i, t_i)$, and observe that by Lemma 5, $\{\Gamma_T \ne \varnothing\}$ is also contained in the event $\cap_{i=1}^N F_i$, where $F_i = \{(x_i, t_i) \in B_{jk}(x_i)\}$ if $\rho_i(\mathbf{\Delta}) = \beta_{jk}$, and $F_i = \{(x_i, t_i) \in \mathscr{D}_{jk}(x_i)\}$ if $\rho_i(\mathbf{\Delta}) = \delta_{jk}$.

Exploiting the independence properties of P.p.ℓ.'s, we see that $\cap_{i=1}^N F_i$ has probability density $\rho(\mathbf{\Delta})\, d\lambda_1\, d\lambda_2 \cdot \cdots \cdot d\lambda_N$, and moreover, we can combine this with (24) to get the upper bound in (22) for $\dfrac{d\mu_T^\Delta}{d\nu_N}$. Further details are left to the reader. □

4. Proof of the Theorem assuming (12). We now prove (7) in the special case when (12) holds. Our argument will be based on Lemma 8 and some simple combinatorics. The next three lemmas are preparatory to the main result of the section.

LEMMA 9. *Let (ξ_t) have flip rates β_{ij} and δ_{ij} such that*

$$\beta_{00} = 0, \qquad \beta_{11} \ge \beta.$$

For any $\mathbf{\Delta} \in \mathscr{D}_N$, with $\alpha = (\delta - \delta_{00})^+$,

$$P_1(p_0(\Gamma_T) = (1/2, T) \text{ and } \mathbf{\Delta}(\Gamma_T) = \mathbf{\Delta}) \le \rho(\mathbf{\Delta}) \binom{N}{|I_d(\mathbf{\Delta})|} (\beta + \delta - \alpha)^{-N}.$$

PROOF. Note that

$$\mathscr{L}(\mathbf{\Delta}) \subset \mathscr{L}'(\mathbf{\Delta}) = \{\boldsymbol{\lambda} \in (R^+)^N : \textstyle\sum_{i\in I_d(\Delta), i\ne N+1} \lambda_i > \sum_{i\in I_u(\Delta)} \lambda_i\}.$$

Hence, by Lemma 8 and the extra hypothesis (12),

$$\mu_T^\Delta(\mathscr{L}(\mathbf{\Delta})) \le \mu_T^\Delta(\mathscr{L}'(\mathbf{\Delta})) \le \rho(\mathbf{\Delta}) \int_{\mathscr{L}'(\Delta)} \exp\{-(\beta + \delta - \alpha) \textstyle\sum_{i\in I_d(\Delta)} \lambda_i\}\, d\boldsymbol{\lambda}.$$

The integral equals

$$\binom{N-1}{|I_d(\mathbf{\Delta})| - 1}(\beta + \delta - \alpha)^{-N},$$

as can be seen by making the change of variables

$$\boldsymbol{\lambda} \to (\textstyle\sum_{i\in I_d(\Delta)} \lambda_i, \lambda_2, \cdots, \lambda_N).$$

Since $\binom{n-1}{k-1} \le \binom{n}{k}$, the proof is complete. □

At this point it is convenient to introduce, for any $\mathbf{\Delta} \in \mathscr{D}_N$, the quantities:

$$L(\mathbf{\Delta}) = |\{i : \Delta_i = (d, \ell, d)\}|, \qquad R(\mathbf{\Delta}) = |\{i : \Delta_i = (d, r, d)\}|,$$
$$M(\mathbf{\Delta}) = |\{i : \Delta_i = (d, \ell, u)\}|, \qquad W(\mathbf{\Delta}) = \text{the length of } H_0$$

($H_0 = H_0(\gamma)$ for any γ such that $\mathbf{\Delta}(\gamma) = \mathbf{\Delta}$). Define

$$\bar{\mathscr{D}} = \mathscr{D}(N, W, L, R, M) = \{\mathbf{\Delta} \in \mathscr{D}_N : W(\mathbf{\Delta}) = W, L(\mathbf{\Delta}) = L, R(\mathbf{\Delta}) = R, M(\mathbf{\Delta}) = M\}.$$

We can now evaluate $\rho(\mathbf{\Delta})$ as follows.

LEMMA 10. *For $\mathbf{\Delta} \in \bar{\mathscr{D}}$, and $\rho(\mathbf{\Delta})$ given by* (20),

$$\rho(\mathbf{\Delta}) = \delta_{01}^L \beta_{01}^R \delta_{10}^{\frac{N+W}{2} - L - 2M + 1} \beta_{10}^{\frac{N-W}{2} - R} \delta_{00}^{M-1} \delta_{11}^M.$$

PROOF. Write $N_1 = |\{i : \Delta_i = (u, \ell, u)\}|$, $N_2 = |\{i : \Delta_i = (u, r, u)\}|$, $N_3 = |\{i : \Delta_i = (u, \ell, d)\}|$. According to (20), we need only verify

$$(25) \qquad N_1 = \frac{N+W}{2} - L - 2M + 1, \qquad N_2 = \frac{N-W}{2} - R, \qquad N_3 = M - 1.$$

Let γ satisfy $\mathbf{\Delta}(\gamma) = \mathbf{\Delta}$. Since $V_1(\gamma)$ is directed down and $V_{N+1}(\gamma)$ is directed up, γ changes directions from down to up one more time than it changes direction from up to down. The triples (d, ℓ, u) correspond to changes of the former type, (u, ℓ, d) to changes of the latter type. Hence $N_3 = M - 1$. The values of N_1 and N_2 are now determined by the equations

$$L + R + M + N_1 + N_2 + N_3 = N \quad \text{and} \quad L + M + N_1 + N_3 - R - N_2 = W. \qquad \square$$

The final lemma of this section estimates the number of direction vectors $\mathbf{\Delta}$ having given values of N, W, L, R and M.

LEMMA 11. $|\bar{\mathscr{D}}| \le C \Big/ \binom{N}{L+R+M}$, *where*

$$(26) \quad C = C(N, W, L, R, M) = \frac{N!}{L!R!M!(M-1)!\left(\frac{N-W}{2} - R\right)!\left(\frac{N+W}{2} - 2M - L + 1\right)!}.$$

(Interpret the right side as 0 if any of the terms in parentheses is negative.)

PROOF. Let $\mathbf{\Delta}_d$ be the subsequence of $\mathbf{\Delta}$ comprised of $\Delta_i \in I_d$, $\mathbf{\Delta}_u$ the subsequence of $\Delta_i \in I_u$. It is easy to see that any pair $(\mathbf{\Delta}_d, \mathbf{\Delta}_u)$ determines a unique $\mathbf{\Delta}$, since the two subsequences fit together in only one possible way. Thus, with N_1, N_2 and N_3 as in the previous lemma,

$$|\bar{\mathscr{D}}| \le |\{\mathbf{\Delta}_d\{|\cdot|\}\mathbf{\Delta}_u\}| = \frac{(L+R+M)!}{L!R!M!} \cdot \frac{(N_1+N_2+N_3)!}{N_1!N_2!N_3!}.$$

The claim follows. $\square$

We are now ready to prove that (10) implies (7) in the special case when (12) holds. Thus, let (ξ_t) satisfy the hypotheses of the Theorem. There is no loss of generality by assuming, in addition, that

$$(27) \qquad \delta_{00} \ge \delta_{10} + \delta_{01};$$

for if (27) fails, then **1** is "less stable" in the modified system $(\bar{\xi}_t)$ with $\bar{\delta}_{00} = \delta$, and with all other flip rates unchanged. This can be proved by using the "basic coupling" of Section 2.1 of [10] to construct (ξ_t^ε) and $(\bar{\xi}_t^\varepsilon)$ on the same probability space in such a way that

$$\bar{\xi}_t^\varepsilon(x) \le \xi_t^\varepsilon(x), \qquad \forall x, t, \varepsilon.$$

Then stability for $(\bar{\xi}_t)$, which satisfies (27), implies stability for (ξ_t).

As already noted, it suffices to check (18). Fix $\varepsilon > 0$, $T > 0$, and write $P = P_1^\varepsilon$, $\Gamma = \Gamma_T^\varepsilon$, $p = (\frac{1}{2}, T)$. Check that Lemmas 9, 10 and 11 apply to (ξ_t^ε) (i.e. the system with flip rates (8)). By translation invariance and those results,

$$P(\Gamma \neq \varnothing) \leq \textstyle\sum_{W=1}^{\infty} W \cdot P(W(\mathbf{\Delta}(\Gamma)) = W, p_0(\Gamma) = p) \tag{28}$$
$$\leq \textstyle\sum_{W,N,L,R,M} W \cdot C(W, N, L, R, M)\, \Pi^\varepsilon\, (W, N, L, R, M),$$

where C is given by (25) and

$$\Pi^\varepsilon = \frac{(\delta_{01} + \varepsilon)^L \beta_{01}^R (\delta_{10} + \varepsilon)^{\frac{N+W}{2} - 2M + 1} \beta_{10}^{\frac{N-W}{2} - R} (\delta_{00} + \varepsilon)^{M-1} \varepsilon^M}{(\beta + \delta + \epsilon)^N}.$$

Now observe that C can be written as

$$C = \binom{N}{\frac{N-W}{2}} \binom{\frac{N+W}{2}}{2M-1} \binom{2M-1}{M} \binom{\frac{N-W}{2}}{R} \binom{\frac{N+W}{2} - 2M + 1}{L}.$$

Hence, summing on L and R in (28), and using the bounds

$$\binom{N}{\frac{N-W}{2}} \leq 2^N \quad \text{and} \quad \binom{2M-1}{M} \leq 2^{2M-1},$$

we get

$$P(\Gamma \neq \varnothing) \leq \textstyle\sum_{W,N,M} W \cdot 2^N \binom{\frac{N+W}{2}}{2M-1} 2^{2m-1} \frac{(\delta + 2\varepsilon)^{\frac{N+W}{2} - 2M + 1} \beta^{\frac{N-W}{2}} (\delta_{00} + \varepsilon)^{M-1} \varepsilon^M}{(\beta + \delta + \varepsilon)^N}.$$

Rewrite the sum as

$$\textstyle\sum_{W,N} W \cdot 2^N$$
$$\cdot \sqrt{\frac{\varepsilon}{(\delta_{00} + \varepsilon)}} \textstyle\sum_M \left[\binom{\frac{N+W}{2}}{2M-1} (2\sqrt{\varepsilon(\delta_{00} + \varepsilon)})^{2M-1} (\delta + 2\varepsilon)^{\frac{N+W}{2} - (2M-1)} \right] \beta^{\frac{N-W}{2}} (\beta + \delta + \varepsilon)^N$$

$$\leq \sqrt{\frac{\varepsilon}{(\delta_{00} + \varepsilon)}} \textstyle\sum_{W,N} W \cdot 2^N (2\sqrt{\varepsilon(\delta_{00} + \varepsilon)} + \delta + 2\varepsilon)^{\frac{N+W}{2}} \beta^{\frac{N-W}{2}} (\beta + \delta + \varepsilon)^{-N}$$

$$= \sqrt{\frac{\varepsilon}{(\delta_{00} + \varepsilon)}} \left\{ \textstyle\sum_N \left[\frac{2\sqrt{\beta}\,(2\sqrt{\varepsilon(\delta_{00} + \varepsilon)} + \delta + 2\varepsilon)^{1/2}}{\beta + \delta + \varepsilon} \right]^N \right\}$$
$$\cdot \left\{ \textstyle\sum_W W \cdot \left[\left(\frac{2\sqrt{\varepsilon(\delta_{00} + \varepsilon)} + \delta + 2\varepsilon}{\beta} \right)^{1/2} \right]^W \right\}$$

$$= \theta \{\textstyle\sum_{N=1}^{\infty} \sigma_1^N\} \{\textstyle\sum_{W=1}^{\infty} W \cdot \sigma_2^W\} = \theta \frac{\sigma_1}{1 - \sigma_1} \frac{\sigma_2}{(1 - \sigma_2)^2},$$

this last provided $\sigma_1 < 1$ and $\sigma_2 < 1$. (Here, $\theta = \sqrt{\epsilon/(\delta_{00} + \epsilon)}$ and σ_1 and σ_2 are the quantities in the square brackets.) If $0 < \varepsilon < \beta - \delta$, then $(\sigma_2/\sigma_1) = (\beta + \delta + \varepsilon)/2\beta < 1$, so $\sigma_2 < \sigma_1$. Thus, to establish (18), it suffices to check that

$$\lim_{\varepsilon \to 0} \sigma_1(\varepsilon) < 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta(\varepsilon) \sigma_1(\varepsilon) = 0. \tag{29}$$

If $\delta_{00} > 0$, then

$$\lim_{\varepsilon\to 0} \sigma_1(\varepsilon) = \frac{2\sqrt{\beta\delta}}{\beta+\delta} < 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \theta(\varepsilon) = 0;$$

if $\delta_{00} = 0$, then $\delta = 0$ by (27), so

$$\lim_{\sigma\to 0} \sigma_1(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \theta(\varepsilon) = 1.$$

In either case (29) holds. Thus, assuming (12), the theorem is proved. □

REMARK. By a standard percolation argument (see, for example, Toom [13]) it can be shown that **1** is stable for any $\epsilon < \epsilon^*$, where

$$\varepsilon^* = \sup\{\varepsilon \in (0, \beta - \delta) : \sigma_1(\varepsilon) < 1\}.$$

For example, if (ξ_t^ε) is the basic contact process with flip rates (4), and if

$$\sigma_1^2(\varepsilon) = \frac{32\varepsilon}{(2+\varepsilon)^2} < 1,$$

then **1** is stable. This yields the bound (13) for the critical constant mentioned in the introduction.

5. Proof of the Theorem when $\beta_{11} < \beta_{01} + \beta_{10}$. Our task in this final section is to prove the theorem when

$$\delta_{00} \geq \delta \qquad \text{and} \quad \beta_{11} < \beta.$$

The first inequality is (27); as already noted, there is no loss of generality by making this assumption. The second inequality identifies the case when (12) fails. The difficulty here is due to the presence of the term

$$\exp\{-(\beta_{11} - \beta + \varepsilon) b(\mathbf{\Delta}, \mathbf{\lambda})\} > 1$$

in $f(\mathbf{t}, \mathbf{\Delta})$, which is contributed by the sets B_x of the proof of Lemma 8. We call these sets *bottlenecks*. Each bottleneck is a rectangular set of width 1. Depending on the position of a given bottleneck, it plays the role of either an isthmus or a peninsula in the region bounded by Γ_t. To deal with the bottlenecks, we treat them as connectors between successive generations of a branching structure, as described below.

Given $\gamma \in \mathcal{G}_t$, let $\mathcal{O}$ be the inside of γ (i.e., the bounded component of γ^c). Every line $\mathbb{R} \times \{t\}$ intersects $\mathcal{O}$ in a (possibly empty) set of disjoint horizontal line segments of integer length. Let

$$\mathcal{B} = \text{the union of all such segments of length one.}$$

Then $\bar{\mathcal{B}}$ is a disjoint union of closed rectangles of width one. These rectangles are the *bottlenecks* of γ; $b(\gamma) = b(\mathbf{\Delta}(\gamma), \mathbf{\lambda}(\gamma))$ is the sum of the heights of the bottlenecks. Let

$$\mathcal{B}' = \text{the union of the bottom segments (i.e., the bases) of the rectangles in } \mathcal{B}.$$

The segments in $\mathcal{B}'$ cut $\mathcal{O}$ into smaller pieces; let

$$G = \{\text{the boundaries of these pieces}\} \ (\text{i.e., } \{\text{the boundaries of the bounded components of } (\gamma \cup \mathcal{B}')^c\}).$$

We group the curves in G into generations as follows:

$$\gamma_0 = \text{the unique curve in } G \text{ which contains } (0, T),$$

$$G_0 = \{\gamma_0\},$$

and continuing inductively, for $n \geq 0$,

$$G_{n+1} = \{\text{all curves in } G - \cup_{k\leq n} G_k \text{ which have nonempty intersection with some curve in } G_n\}.$$

Also, let

$$m = m(\gamma) = \text{the cardinality of } G_1,$$
$$g = g(\gamma) = \max\{n : G_n \neq \emptyset\}.$$

Our interest will center on γ_0, the m curves in G_1, and the m curves representing the total progeny of each of these m curves in G_1. Thus, write

$$\mathscr{B}'' = \mathscr{B}' \cap \gamma_0.$$

The set $(\gamma\backslash\gamma_0) \cup \mathscr{B}''$ consists of m disjoint simple closed curves which we call $\gamma_1, \gamma_2, \cdots, \gamma_m$ (use the ordering inherited from the orientation of γ). Each of these curves surrounds one of the curves in G_1 together with all of its offspring. Furthermore, $\gamma_0 \in \mathscr{G}_t$, while each of the curves $\gamma_1, \cdots, \gamma_m$ is a translate of a curve in $\mathscr{G}_T$. Therefore, any of the quantities we have considered which depend only on the shape of $\gamma \in \mathscr{G}_T$ can be defined as well for γ_0, $\gamma_1, \cdots, \gamma_m$.

We will eventually be able to derive a contour estimate $p_n(\varepsilon)$ for the probability that any one of the curves $\gamma_1, \cdots, \gamma_m$ has at most n generations. The multiplicative nature of these estimates will allow us to mimic branching process techniques, in order to show that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} p_n(\varepsilon) = 0,$$

and then to prove (18). Very roughly speaking, we view the sets G_n as generations of a branching process which has gone extinct. When ε is small, this branching process becomes supercritical so that the probability of extinction (and hence the probability that there is a finite contour) is less than 1.

We will start with a precise definition of the functions $p_n(\varepsilon)$. For $\mathbf{\Delta} \in \mathscr{D}_n$, $N \geq 1$, and $n \geq 0$, let

$$\mathscr{L}_n(\mathbf{\Delta}) = \{t \in \mathscr{L}(\mathbf{\Delta}) : \exists\, \gamma \in \mathscr{G} \quad \text{with} \quad g(\gamma) \leq n \quad \text{and} \quad \boldsymbol{\lambda}(\gamma) = \mathbf{t}\}.$$

Let ρ^ε and f^ε be the functions ρ and f corresponding to the flip rates for (ξ_t^ε), as defined in Section 3, and put

$$\textstyle\sum_n^\varepsilon(W, M) = \sum_{\Delta: W(\Delta)=W, M(\Delta)=M} \rho^\varepsilon(\mathbf{\Delta}) \displaystyle\int_{\lambda\in\mathscr{L}_n(\Delta)} f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda})\, d\boldsymbol{\lambda}.$$

Then we define

$$p_n(\varepsilon) = \textstyle\sum_M \sum_n^\varepsilon(2, M) \qquad \text{for} \quad n \geq 0. \tag{30}$$

The following lemma establishes an inequality between $\sup_{T\geq 0} P_1^\varepsilon(\Gamma_T \neq \emptyset)$ and the functions $p_n(\varepsilon)$ which is similar in spirit to the simple fact from the theory of branching processes that the probability of extinction equals the limit as $n \to \infty$ of the probability of extinction within n generations.

LEMMA 12. *For $m \geq 0$, let*

$$Z_m^\varepsilon = \left(\frac{\beta}{\varepsilon}\right)^m \textstyle\sum_W W \sum_{M\geq m} \displaystyle\binom{M}{m} \textstyle\sum_0^\varepsilon (W, M).$$

If $p_n(\varepsilon)$ is given by (30), *then for any $\varepsilon > 0$,*

$$P_1^\varepsilon(\Gamma_T \neq \emptyset) \leq \lim_{n\to\infty} \textstyle\sum_{m=0}^\infty Z_m^\varepsilon (p_n(\varepsilon))^m, \qquad T > 0. \tag{31}$$

PROOF. For ε, $T > 0$, write $\Gamma = \Gamma_T^\varepsilon$ and $p = (\tfrac{1}{2}, T)$. Then

$$P_1^\varepsilon(\boldsymbol{\Gamma} \neq \emptyset) = \lim_{n\to\infty} \textstyle\sum_{m=0}^\infty \sum_W W \cdot P(W(\mathbf{\Delta}(\boldsymbol{\Gamma})) = W, p_0(\boldsymbol{\Gamma}) = p, g(\boldsymbol{\Gamma}) \leq n+1, m(\boldsymbol{\Gamma}) = m). \tag{32}$$

Now consider a curve $\gamma \in \mathscr{G}$ such that $g(\gamma) \le n+1$ and $m(\gamma) = m$. As in the proof of Lemma 8, we associate a probability density $\rho^\varepsilon(\mathbf{\Delta}) f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda})\, d\boldsymbol{\lambda}$ with the event $\{\Gamma = \gamma\}$, where $\mathbf{\Delta} = \mathbf{\Delta}(\gamma)$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\gamma)$. Of course the density involves a Radon-Nikodym derivative; for the sake of brevity and clarity we omit the formal details. Let $\gamma_i = \gamma_i(\gamma)$, $0 \le i \le m$, and put $\mathbf{\Delta}_i = \mathbf{\Delta}(\gamma_i)$, $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}(\gamma_i)$. It is easy to see that

(33) $$f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda}) = \textstyle\sum_{i=0}^m f^\varepsilon(\mathbf{\Delta}_i, \boldsymbol{\lambda}_i).$$

We also need to determine the relationship between $\rho^\varepsilon(\mathbf{\Delta})$ and the $\rho^\varepsilon(\mathbf{\Delta}_i)$. This involves the manner in which γ_0 is connected to each of the γ_i, $i \ge 1$, and so is not quite multiplicative. Let $H_i' = \gamma_0 \cap \gamma_i$, $i \ge 1$. Then H_i' is a horizontal segment of length one belonging to a unique horizontal segment H_{i0} of length two in γ_i. If we write

$$\begin{aligned} \beta_i &= \beta_{01} && \text{when } H_i' \text{ is the left half of } H_{i0}, \\ &= \beta_{10} && \text{when } H_i' \text{ is the right half of } H_{i0}, \end{aligned}$$

then, in fact,

(34) $$\rho^\varepsilon(\mathbf{\Delta}) = \varepsilon^{-m} \rho^\varepsilon(\mathbf{\Delta}_0) \textstyle\prod_{i=1}^m \beta_i \rho^\varepsilon(\mathbf{\Delta}_i).$$

The term ε^{-m} is present because $\mathbf{\Delta}_0$ contains m triples (d, ℓ, u) which do not correspond to triples (d, ℓ, u) in $\mathbf{\Delta}$. We note in passing that the factorizations in (33) and (34) will help us to mimic a branching process analysis. Combining these two equations, one gets

(35) $$\rho^\varepsilon(\mathbf{\Delta}) f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda}) = \varepsilon^{-m} \rho^\varepsilon(\mathbf{\Delta}_0) f^\varepsilon(\mathbf{\Delta}_0, \boldsymbol{\lambda}_0) \textstyle\prod_{i=1}^m \beta_i \rho^\varepsilon(\mathbf{\Delta}_i) f^\varepsilon(\mathbf{\Delta}_i, \boldsymbol{\lambda}^i).$$

Write $\lambda_{ij} = \lambda_j(\gamma_i)$. Since $\sum_{i=0}^m N(\gamma_i) = N(\gamma)$, and since the change of variables

$$\boldsymbol{\lambda} \to (\lambda_{01}, \cdots, \lambda_{0N(\gamma_0)}, \cdots, \lambda_{m1}, \cdots, \lambda_{mN(\gamma_m)})$$

is piecewise differentiable and measure preserving, it follows that the probability density of $\{\Gamma = \gamma\}$ is bounded by the right side of (35) times $d\lambda_{01} \cdots d\lambda_{mN(\gamma_m)}$. The question now arises: given $\gamma_0 \in \mathscr{G}_{T_0}$, $\gamma_1, \cdots, \gamma_m \in \mathscr{G}$, how many ways can we translate the γ_i, $i \ge 1$, to get γ_i', $i \ge 1$, such that there is a $\gamma \in \mathscr{G}_{T_0}$ with $\gamma_0(\gamma) = \gamma_0$ and $\gamma_i(\gamma) = \gamma_i'$ for $i \ge 1$? Clearly there are at most $(m^{M(\Delta_0)})$ choices for the segments $H_i'(\gamma)$, $1 \le i \le m$. Introduce the equivalence relation between curves: $\gamma' \sim \gamma''$ if γ'' is a translate of γ'. Then these considerations show that the event

$$\{\gamma_0(\Gamma) = \gamma_0, m(\Gamma) = m, \gamma_i(\Gamma) \sim \Gamma_i, \beta_i(\Gamma) = \beta_i, 1 \le i \le m\}$$

has probability density bounded by

$$\binom{M(\mathbf{\Delta}_0)}{m} \varepsilon^{-m} \textstyle\prod_{i=1}^m \beta_i \prod_{i=0}^m [\rho^\varepsilon(\mathbf{\Delta}_i) f^\varepsilon(\mathbf{\Delta}_i, \boldsymbol{\lambda}_i)\, d\boldsymbol{\lambda}_i]$$

Sum over the possible β_i to see that the density of

$$\{\gamma_0(\Gamma) = \gamma_0, m(\Gamma) = m, \gamma_i(\Gamma) \sim \gamma_i, 1 \le i \le m\}$$

is bounded by

(36) $$\left(\frac{\beta}{\varepsilon}\right)^m \binom{M(\mathbf{\Delta}_0)}{m} \textstyle\prod_{i=0}^m [\rho^\varepsilon(\mathbf{\Delta}_i) f^\varepsilon(\mathbf{\Delta}_i, \boldsymbol{\lambda}_i)\, d\boldsymbol{\lambda}_i].$$

Note that if $g(\gamma) \le n+1$, then $g(\gamma_i) \le n$ for $i \ge 1$, while $g(\gamma_0) = 0$. Thus $\boldsymbol{\lambda}_0 \in \mathscr{L}_0(\mathbf{\Delta}_0)$ and $\boldsymbol{\lambda}_i \in \mathscr{L}_n(\mathbf{\Delta}_i)$, $i \ge 1$. Also, for $1 \le i \le m$, each γ_i must be such that $W(\mathbf{\Delta}_i) = 2$. Thus, integrating (36) over all such $\boldsymbol{\lambda}$ and $\mathbf{\Delta}$ and consulting definition (30) yields

$$\begin{aligned} &P(W(\mathbf{\Delta}(\Gamma)) = W, p_0(\Gamma) = p, g(\Gamma) \le n+1, m(\Gamma) = m) \\ &\quad \le \left(\frac{\beta}{\varepsilon}\right)^m \textstyle\sum_{M \ge m} \sum_0^\varepsilon \binom{\mathrm{M}}{\mathrm{m}} (W, M)(p_n(\varepsilon))^m. \end{aligned}$$

The desired result now follows from (32). □

The next lemma is reminiscent of the familiar recursion formula from branching processes that relates the probability of extinction within $n+1$ generations to a power series in the probability of extinction within n generations. The proof is similar to the proof of Lemma 12, so we omit it.

LEMMA 13. *For $m \geq 0$, let*

$$Y_m^\varepsilon = \left(\frac{\beta}{\varepsilon}\right)^m \sum_{M\geq m} \binom{M}{m} \Sigma_0^\varepsilon (2, M).$$

Then with $p_n(\varepsilon)$ given by (30),

$$p_{n+1}(\varepsilon) \leq \sum_{m=0}^{\infty} Y_m^\varepsilon (p_n(\varepsilon))^m.$$

In order to apply Lemmas 12 and 13, we need some manageable estimates for Z_m^ε and Y_m^ε. To get these, we first estimate the integral in the definitions of Z_m^ε and Y_m^ε (Lemma 14), and then we use that result to show that Z_m^ε and Y_m^ε are bounded by A^m for some constant $A < \infty$ (Lemma 15).

LEMMA 14. *For $\mathbf{\Delta} \in \mathscr{D}(N, W, L, R, M)$ and any $\varepsilon > 0$,*

$$\int_{\lambda\in\mathscr{L}_0(\Delta)} f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda})\, d\boldsymbol{\lambda} \leq C_1^M \binom{N}{L+R+M} (\beta+\delta)^{-N},$$

where $C_1 = 2(\beta+\delta)/(\beta_{11}+\delta)$.

PROOF. Fix $\mathbf{\Delta} \in \bar{\mathscr{D}}$ and $\lambda \in (\mathrm{R}^+)^N$, and introduce:

$$I_{du}(\mathbf{\Delta}) = \{i : \Delta_i = (d, \ell, u)\}, \qquad I_{du}^+ = \{i+1 : \Delta_i = (d, \ell, u)\},$$

$$J_d(\mathbf{\Delta}, \boldsymbol{\lambda}) = \{i \in I_{du} : \lambda_i \leq \lambda_{i+1}\}, \qquad J_u(\mathbf{\Delta}, \boldsymbol{\lambda}) = \{i \in I_{du}^+ : \lambda_i \leq \lambda_{i-1}\},$$

$$K_d(\mathbf{\Delta}, \boldsymbol{\lambda}) = I_{du} \setminus J_d, \qquad K_u(\mathbf{\Delta}, \boldsymbol{\lambda}) = I_{du}^+ \setminus J_u,$$

$$I_b(\mathbf{\Delta}, \boldsymbol{\lambda}) = J_d \cup J_u.$$

If $g(\gamma) = g(\mathbf{\Delta}, \boldsymbol{\lambda}) = 0$, then $b(\mathbf{\Delta}, \boldsymbol{\lambda}) = \sum_{i\in I_b} \lambda_i$. Since in this case, $|I_b| = |I_{du}| = M$, there are 2^M possible sets $J \subset I_{du}(\mathbf{\Delta})$ such that $J_d(\mathbf{\Delta}, \boldsymbol{\lambda}) = J$ and once J_d is determined, so are J_u, K_d, K_u and I_b. Now note that $\mathscr{L}_0(\mathbf{\Delta}) \subset \mathscr{L}_0'(\mathbf{\Delta})$, where

$$\mathscr{L}_0'(\mathbf{\Delta}) = \{\boldsymbol{\lambda} \in (R^+)^N : \textstyle\sum_{i\in I_u\setminus I_{du}^+, i\neq N+1} \lambda_i + \sum_{i\in K_u, i\neq N+1} \lambda_i - \lambda_{i-1} \leq \sum_{i\in I_d\setminus I_{du}} \lambda_i + \sum_{i\in K_d} \lambda_i - \lambda_{i+1}\}.$$

Make the change of variables

$$\begin{aligned} \lambda_i &\to \lambda_i - \lambda_{i+1}, && i \in K_d,\\ &\to \lambda_i - \lambda_{i-1}, && i \in K_u,\\ &\to \lambda_i && \text{otherwise} \end{aligned}$$

followed by the change of variables

$$\begin{aligned} \lambda_{i_0} &\to g = \textstyle\sum_{i\in I_d\setminus J_d} \lambda_i, && i_0 = \min\{i : i \in I_d \setminus J_d\},\\ \lambda_{j_0} &\to h = \textstyle\sum_{i\in I_b} \lambda_i, && j_0 = \min\{i : i \in J_d\},\\ \lambda_i &\to \lambda_i && \text{otherwise.} \end{aligned}$$

Then for each possible $J = J_d$, estimation of $f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda})$ yields

$$\int_{\lambda\in\mathscr{L}_0(\Delta):J_d(\Delta,\lambda)=J} f^\varepsilon(\mathbf{\Delta}, \boldsymbol{\lambda})\, d\boldsymbol{\lambda} \leq \int_{g=0}^{\infty} e^{-(\beta+\delta+\varepsilon)g} \int_{h=0}^{\infty} e^{-(\beta_{11}+\delta+2\varepsilon)h} \left[\int_{\lambda\in S(g,h)} \textstyle\prod_{i\neq i_0, j_0} d\lambda_i\right] dh\, dg$$

where

$$S(g, h) = \{\lambda : \sum_{i \in I_b, i \neq j_0} \lambda_i \leq h; \sum_{i \in I_d \setminus J_d, i \neq i_0} \lambda_i \leq g; \sum_{i \in I_u \setminus J_u, i \neq N+1} \lambda_i \leq g\}.$$

Evaluating the integrals and summing over J, we get

$$\int_{\lambda \in \mathscr{L}_0(\Delta)} f^\varepsilon(\Delta, \lambda)\, d\lambda \leq \left(\frac{2}{\beta_{11} + \delta + 2\varepsilon}\right)^M \binom{N}{|I_d|} (\beta + \delta + \varepsilon)^{-(N-M)}.$$

$$\leq \left(\frac{2}{\beta_{11} + \delta}\right)^M \binom{N}{|I_d|} (\beta + \delta)^{-(N-M)}.$$

This proves the lemma, with $C_1 = \dfrac{2(\beta + \delta)}{\beta_{11} + \delta}$. □

LEMMA 15. *Let Z_m^ε and Y_m^ε be defined as in Lemmas* 12 *and* 13. *Then $Y_m^\varepsilon \leq Z_m^\varepsilon$ for all $m \geq 0$, $\varepsilon > 0$, and there is a constant $A < \infty$, depending only on β_{ij} and δ_{ij}, such that $Z_m^\varepsilon \leq A^m$, $m \geq 0$, for all sufficiently small $\varepsilon > 0$.*

PROOF. It is easy to see from the definitions that $Y_m^\varepsilon \leq Z_m^\varepsilon$ for all $m \geq 0$, $\varepsilon > 0$. By Lemma 14 and the analysis of Section 4,

$$Z_m^\varepsilon \leq \bar{Z}_m^\varepsilon \equiv \left(\frac{\beta}{\varepsilon}\right)^m \sum_W W \sum_{M \geq m} 2^M \sum_{N,L,R} \sum_{\Delta \in \mathscr{D}} \rho^\varepsilon(\Delta) C_1^M \binom{N}{L + R + M} (\beta + \delta)^{-N}$$

$$\leq \left(\frac{\beta}{\varepsilon}\right)^m \sum_W W \sum_{M \geq m} (2C_1)^M \sum_{N,L,R} C \cdot \bar{\Pi}^\varepsilon,$$

where $\bar{\Pi}^\varepsilon$ is $\prod^\varepsilon$ with $(\beta + \delta + \varepsilon)^N$ replaced by $(\beta + \delta)^N$ in the denominator. For $m = 0$, the last expression above is majorized by

$$\sum_{W,N,L,R,M} W \cdot C \cdot \bar{\Pi}^{2C_1\varepsilon} \leq \theta(2C_1\varepsilon)\{\sum_N \bar{\sigma}_1^N(2C_1\varepsilon)\}\{\sum_W W \cdot \sigma_2^W(2C_1\varepsilon)\},$$

where $\theta(\varepsilon)$, $\sigma_2(\varepsilon)$ are as in Section 4, and $\bar{\sigma}_1(\varepsilon)$ is $\sigma_1(\varepsilon)$ with $\beta + \delta + \varepsilon$ replaced by $\beta + \delta$ in the denominator. Since

$$\frac{\sigma_2}{\bar{\sigma}_1} < 1, \lim_{\varepsilon \to 0} \bar{\sigma}_1(\varepsilon) < 1, \qquad \text{and } \lim_{\varepsilon \to 0} \theta(\varepsilon)\bar{\sigma}_1(\varepsilon) = 0,$$

we conclude that there is an $\varepsilon_0 > 0$ such that

$$\bar{Z}_0^\varepsilon < 1 \qquad \text{for all} \quad \varepsilon \in (0, \varepsilon_0]. \tag{37}$$

Next, note that

$$\left(\frac{\varepsilon_0}{\beta}\right)^m \bar{Z}_m^{\varepsilon_0} \downarrow 0 \qquad \text{as } m \to \infty,$$

these terms being the tails of a convergent series. Thus, for suitable $\varepsilon_0 > 0$, depending only on β_{ij} and δ_{ij}, we have

$$\left(\frac{\varepsilon_0}{\beta}\right)^m \bar{Z}_m^{\varepsilon_0} < 1 \qquad \text{for all} \quad m \geq 0.$$

Finally, since $\bar{Z}_m^\varepsilon$ is evidently increasing in ε, it follows that

$$Z_m^\varepsilon \leq \bar{Z}_m^\varepsilon \leq \bar{Z}_m^{\varepsilon_0} < \left(\frac{\beta}{\varepsilon_0}\right)^m$$

for all $\varepsilon \in (0, \varepsilon_0]$. The lemma is proved, with $A = \beta/\varepsilon_0$. □

Lemmas 12 and 15 have the following easy corollary:

(38) $$\sup_{T>0} P_1^\varepsilon(\Gamma_T \neq \emptyset) \le Z_0^\varepsilon + \lim_{n\to\infty} \sum_{m=1}^\infty A^m p_n^m(\varepsilon),$$

where A is a finite positive constant defined in Lemma 15. Therefore, (18) follows from (37) and (38) if $\lim_{\varepsilon\to 0} \sup_n p_n(\varepsilon) = 0$. Since (18) implies stability, we are done once we have proved:

LEMMA 16.

(39) $$\lim_{\varepsilon\to 0} \sup_n p_n(\varepsilon) = 0.$$

PROOF. By Lemmas 13 and 15,

$$p_{n+1}(\varepsilon) \le Y_0^\varepsilon + p_n(\varepsilon) Y_1^\varepsilon + \sum_{m\ge 2} A^m (p_n(\varepsilon))^m.$$

Supposing we could show that

(40) $$\lim_{\varepsilon\to 0} Y_0^\varepsilon = 0 \qquad \text{and}$$

(41) $$\limsup_{\varepsilon\to 0} Y_1^\varepsilon < 1.$$

Then by taking some ideas from the theory of branching processes, we could prove (39) as follows. Put $\bar{p}_0(\varepsilon) = p_0(\varepsilon)$, and for $n \ge 0$, define $\bar{p}_n(\varepsilon)$ inductively by

$$\bar{p}_{n+1}(\varepsilon) = Y_0^\varepsilon + Y_1^\varepsilon \bar{p}_n(\varepsilon) + \sum_{m\ge 2} A^m (\bar{p}_n(\varepsilon))^m.$$

Then

(42) $$p_n(\varepsilon) \le \bar{p}_n(\varepsilon) \qquad \text{for all} \quad n, \varepsilon.$$

Introduce $\Phi_\varepsilon(x) = Y_0^\varepsilon + Y_1^\varepsilon x + \sum_{m\ge 2} A^m x^m$. By (40) and (41),

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(0) = 0, \qquad \limsup_{\varepsilon\to 0} \Phi_\varepsilon'(0) < 1.$$

Hence, for sufficiently small positive ε, there is a least positive solution x_ε of $\Phi_\varepsilon(x) = x$, such that

(43) $$\lim_{\varepsilon\to 0} x_\varepsilon = 0.$$

An argument analogous to the fixed point analysis for Galton-Watson processes yields

(44) $$\lim_{n\to\infty} \bar{p}_n(\varepsilon) = x_\varepsilon.$$

The desired result (39) now follows from (42), (43) and (44).

Thus, it remains to prove (40) and (41). The proof of (40) is quite easy:

$$Y_0^\varepsilon \le Z_0^\varepsilon = \left(\frac{\varepsilon}{\beta}\right) Z_1^\varepsilon \le \frac{\varepsilon}{\beta} A$$

by Lemma 15. Now let $\varepsilon \to 0$ to get (40).

The proof of (41) is much more involved. First note that

$$Y_1^\varepsilon = \left(\frac{\beta}{\varepsilon}\right) \sum_{M\ge 1} M \sum_0^\varepsilon (2, M)$$

while

$$Y_2^\varepsilon = \left(\frac{\beta}{\varepsilon}\right)^2 \sum_{M\ge 2} \binom{M}{2} \sum_0^\varepsilon (2, M).$$

Thus

$$Y_1^\varepsilon - 2\left(\frac{\varepsilon}{\beta}\right) Y_2^\varepsilon = \left(\frac{\beta}{\varepsilon}\right)\left[\sum_0^\varepsilon (2, 1) + \sum_{M\ge 2} (2M - M^2) \sum_0^\varepsilon (2, M)\right] \le \left(\frac{\beta}{\varepsilon}\right) \sum_0^\varepsilon (2, 1).$$

Hence, for ε small enough that Lemma 15 yields $Y_2^\varepsilon \le Z_2^\varepsilon \le A^2$,

$$Y_1^\varepsilon \le 2\left(\frac{\varepsilon}{\beta}\right)A^2 + \left(\frac{\beta}{\varepsilon}\right)\textstyle\sum_0^\varepsilon (2, 1).$$

To prove (41), it therefore suffices to show that

$$\limsup_{\varepsilon\to 0}\left(\frac{\beta}{\varepsilon}\right)\textstyle\sum_0^\varepsilon (2, 1) = 0. \tag{45}$$

Now if $\mathbf{\Delta} \in \mathscr{D}(2, N, L, R, 1)$ and $\boldsymbol{\lambda} \in \mathscr{L}_0(\mathbf{\Delta})$, then $\gamma = \gamma(\mathbf{\Delta}, \boldsymbol{\lambda})$ assumes a particularly simple form. $M = 1$, so $N_3 = 0$. Thus there is only one change of vertical direction. The transition takes place at a unique $i_* \in I_d$ such that $\Delta_{i_*} = (d, \ell, u)$. $I_d = \{i \le i_*\}$, $I_u = \{N \ge i > i_*\}$, $|I_d| = L + R + 1$ and $|I_u| = N - L - R - 1$. As in the proof of Lemma 14, we consider the cases $\lambda_{i_*} \le \lambda_{i_*+1}$ and $\lambda_{i_*} > \lambda_{i_*+1}$ separately. Denote $I_{dr} = \{i : \Delta_i = (d, r, d)\}$, $I_{d\ell} = \{i : \Delta_i = (d, \ell, d)\}$, $I_{ur} = \{i : \Delta_{i-1} = (u, r, u)\}$, $I_{u\ell} = \{i : \Delta_{i-1} = (u, \ell, u)\}$. (Note the use of Δ_{i-1} in the definitions of I_{ur} and $I_{u\ell}$. In particular, $i_{n+1} \in I_{ur} \cup I_{u\ell}$). Then if we abbreviate

$$L_d = \textstyle\sum_{i=1}^{i_*-1} \lambda_i, \qquad L_u = \sum_{i=i_*+2}^{N+1} \lambda_i,$$

we arrive at the equation,

$$\left(\frac{\beta}{\varepsilon}\right)\textstyle\sum_{0(2,1)}^\varepsilon = \beta\Big\{\sum_{N,L,R}\sum \displaystyle\int_{(\mathbf{\Delta},\boldsymbol{\lambda}):\mathbf{\Delta}\in\mathscr{D}(N,2,L,R,1),\mathbf{\Delta}\in\mathscr{L}_0(\boldsymbol{\lambda}),L_u<L_d} \textstyle\prod_{i\in I_{dr}} (\beta_{01}e^{-\beta_{01}\lambda_i}\, d\lambda_i) \tag{46}$$

$$\textstyle\prod_{i\in I_{d\ell}} ((\delta_{01} + \varepsilon)e^{-(\delta_{01}+\varepsilon)\lambda_i}\, d\lambda_i) \prod_{i\in I_{ur}} (\beta_{10}e^{-\beta_{10}\lambda_i}\, d\lambda_{i-1}) \prod_{i\in I_{u\ell}} ((\delta_{10} + \varepsilon)e^{-(\delta_{10}+\varepsilon)\lambda_i}\, d\lambda_{i-1})$$

$$\cdot e^{-(\lambda_{i_*+1}-\lambda_{i_*})(\beta_{10}+\delta_{10}+\varepsilon)}e^{-(\beta_{11}+\delta+2\varepsilon)\lambda_{i_*}}d\lambda_{i_*}\Big\}$$

$$+\beta\{\quad\}_2. \tag{47}$$

Here $\{\quad\}_2$ has the same form as the quantity in brackets in (46), except that L_u and L_d are interchanged, the exponent $-(\lambda_{i_*+1} - \lambda_{i_*})(\beta_{10} + \delta_{10} + \varepsilon)$ is replaced by $-(\lambda_{i_*} - \lambda_{i_*+1})(\beta_{01} + \delta_{01} + \varepsilon)$, and the exponent $-(\beta_{11} + \delta + 2\varepsilon)\lambda_{i_*}$ is replaced by $-(\beta_{11} + \delta + 2\varepsilon)\lambda_{i_*+1}$.

Now consider (X_t^-, X_t^+), a pair of independent continuous time random walks, starting from $(-1, 1)$, such that

X_t^+ has increments:	$+1$ at rate β_{01}
	-1 at rate $\delta_{01} + \varepsilon$,
X_t^- has increments:	-1 at rate β_{10},
	$+1$ at rate $\delta_{10} + \varepsilon$.

Let $\tau = \min\{t : X_t^+ - X_t^- = 1\}$. Then some thought reveals that (46) equals

$$\frac{\beta}{\beta_{11} + \delta + 2\varepsilon}\Pr(\tau < \infty, X_t^+ \text{ jumps at } t = \tau),$$

while (47) equals

$$\frac{\beta}{\beta_{11} + \delta + 2\varepsilon}\Pr(\tau < \infty, X_t^- \text{ jumps at } t = \tau).$$

Hence

$$\left(\frac{\beta}{\varepsilon}\right)\textstyle\sum_0^\varepsilon (2, 1) = \dfrac{\beta}{\beta_{11} + \delta + 2\varepsilon}\Pr(\tau < \infty).$$

Finally, $(X_t^+ - X_t^-)$ is a random walk with increments:

$$+1 \text{ at rate } \beta, \quad -1 \text{ at rate } \delta + 2\varepsilon,$$

so by the gambler's ruin formula (applied to the imbedded difference chain),

$$\Pr(\tau < \infty) = \Pr(X_t^+ - X_t^- = 1 \quad \text{for some} \quad t) = \max\left\{\frac{\delta + 2\varepsilon}{\beta}, 1\right\}.$$

Since $\beta > \delta$, we conclude that

$$\limsup_{\varepsilon \to 0} \left(\frac{\beta}{\varepsilon}\right) \sum_0^\varepsilon (2, 1) = \frac{\delta}{\beta_{11} + \delta} < 1.$$

($\beta > \delta$ implies $\beta_{11} > 0$.) Thus (45) holds, as desired. □

Added in Proof. The first author has now proved that (3) implies ergodicity for attractive systems.

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