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OPERATOR-STABLE LAWS: MULTIPLE EXPONENTS AND ELLIPTICAL SYMMETRY

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We characterize the class of linear operators on a finite dimensional inner product space which are the exponents of a full operator-stable law. This answers a question of Paulauskas [6] concerning those operator-stable laws whose characteristic functions are the exponential of quadratic forms. The symmetry group of such laws must be conjugate to the group of all orthogonal transformations on the space.

1. Introduction. Operator-stable distributions are the analogues of stable distributions in n dimensions. Let V be a finite-dimensional real inner-product space. A nondegenerate distribution μ on V is called operator-stable if there exist independent identically distributed random vectors $\{X_n\}$ taking values in V, nonsingular linear operators $\{A_n\}$, and vectors $\{a_n\}$ in V such that the sequence $\{A_n \sum_1^n X_k - a_n\}$ converges in law to μ. In this work, attention is restricted to full measure, that is, measures which are not concentrated on a hyperplane in V. In his fundamental paper [8], Sharpe proved several results concerning full operator-stable measures μ. He proved that μ is infinitely divisible. Thus if $\hat{\mu}(y)$ denotes the characteristic function of μ and if $t > 0$, then $\hat{\mu}(y)^t$ is the characteristic function of the infinitely divisible distribution μ^t. Sharpe showed that there is a nonsingular linear operator A on V and there is a function a: $(0, \infty) \to V$ such that for all $t > 0$

$$\mu^t = t^A \mu * \delta(a(t))$$

where $t^A = \exp(A \ln t) = \sum_{k=0}^{\infty} (A \ln t)^k / k!$ and $t^A \mu = \mu t^{-A}$. Such an A is called an exponent of μ. In general, this exponent is not unique. In this paper we give necessary and sufficient conditions for an operator-stable distribution to have a unique exponent. We also study a class of operator-stable distributions with multiple exponents. These results are direct extensions of those in Hudson and Mason [3].

Let $\mathscr{S}(\mu)$, the symmetry group of μ, be the set of all nonsingular linear operators B on V such that for some $b \in V$, $\mu = B\mu * \delta(b)$. ($B\mu$ denotes μB^{-1} throughout.) It follows from Theorem 1 of Billingsley [1] that if μ is full, then $\mathscr{S}(\mu)$ is a compact subgroup of the general linear group, $GL(V)$. A classical result (see for example Theorem 5 of Billingsley) says that there exists a closed subgroup $\mathscr{O}_0$ of the group $\mathscr{O}$ of orthogonal linear operators on V and there exists a positive-definite self-adjoint linear operator W such that $\mathscr{S}(\mu) = W\mathscr{O}_0 W^{-1}$. (Any compact subgroup of $GL(V)$ is of this form.) In other words, $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}_0$. Let $\mathscr{E}(\mu)$ denote the set of all linear operators on V which are exponents of μ. In our first result, we relate $\mathscr{E}(\mu)$ to the tangent space $T(\mathscr{S}(\mu))$ of $\mathscr{S}(\mu)$; $T(\mathscr{S}(\mu))$ is the set of all linear operators A on V such that $A = \lim d_n^{-1}(D_n - I)$ for some sequence $\{D_n\} \subset \mathscr{S}(\mu)$ and some positive real numbers $d_n \to 0$. (I denotes the identity operator on V.)

THEOREM 1. *Let μ be full and operator-stable on V. Let B be any exponent for μ. Then*

$$\mathscr{E}(\mu) = B + T(\mathscr{S}(\mu)).$$

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As an easy corollary to this theorem, we have the following.

COROLLARY 1. *Let μ be full and operator-stable on V. Then μ has exactly one exponent if and only if $\mathscr{S}(\mu)$ is discrete.*

PROOF. From Theorem 1 we see that μ has exactly one exponent if and only if $T(\mathscr{S}(\mu)) = \{0\}$. Now it is well-known ([7] page 40 or the appendix) that the image of $T(G)$, where G is a closed subgroup of $GL(V)$, under the exponential map contains a neighborhood of I. Furthermore, $T(G)$ is a vector space and the exponential map is continuous. Thus $\{I\}$ is open in $\mathscr{S}(\mu)$ if and only if $T(\mathscr{S}(\mu)) = \{0\}$. □

Our next two results consider the case where $\mathscr{S}(\mu)$ is conjugate to the full orthogonal group $\mathscr{O}$, i.e. $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$ for some positive-definite self-adjoint linear operator W on V.

THEOREM 2. *Let μ be full and operator-stable on V and assume $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$. Then there is a real number $c \geq 1/2$ such that cI is an exponent of μ.*

THEOREM 3. *Let μ be full and operator-stable on V and assume $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$. Then there are a in V, γ in $(0, 2]$ and $\beta > 0$ such that for $y \in V$*

$$\hat{\mu}(y) = \exp\{i(a, y) - \beta \mid Wy \mid^{\gamma}\}.$$

Conversely, if $\hat{\mu}$ is of this form, then $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$, i.e. $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$.

COROLLARY 2. *Assume $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$ and let c be as given in Theorem 2. Then $c = 1/2$ implies μ is purely Gaussian and $c > 1/2$ implies μ has no Gaussian component.*

Theorem 3 gives an answer to the question raised by Paulauskasin [6], pages 362–363. He said: "Thus the theorem and examples show that there are many stable distributions, the characteristic function of which cannot be expressed by means of quadratic forms. What is more, it is very difficult (at any rate it seems so to us) to describe all the cases when we can do it." By Theorems 2 and 3, if μ is full and operator-stable and if $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$, then μ is multivariate stable (i.e., norming only by multiples of I is permitted) and $\hat{\mu}$ may be expressed in terms of nonnegative-definite quadratic forms. Conversely, if $\hat{\mu}$ may be expressed in terms of such quadratic forms, then $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$.

COROLLARY 3. *If $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$, i.e. $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$ where W is a positive-definite self-adjoint linear operator on V, then*

$$\mathscr{E}(\mu) = cI + W\mathscr{Q}W^{-1},$$

for some $c \geq 1/2$, where $\mathscr{Q}$ is the set of all skew-symmetric linear operators Q on V, i.e. $Q + Q^* = 0$.

PROOF. By Theorem 1, $\mathscr{E}(\mu) = T(\mathscr{S}(\mu)) + B$ for any B in $\mathscr{E}(\mu)$. By Theorem 2, $cI \in \mathscr{E}(\mu)$ for some $c \geq 1/2$. Set $B = cI$. Since $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$, $\mathscr{S}(W^{-1}\mu) = \mathscr{O}$. Hence $W\mathscr{S}(W^{-1}\mu)W^{-1} = \mathscr{S}(\mu)$. Therefore $T(\mathscr{S}(\mu)) = WT(\mathscr{S}(W^{-1}\mu))W^{-1} = WT(\mathscr{O})W^{-1} = W\mathscr{Q}W^{-1}$. The last equality follows from the well-known fact that $T(\mathscr{O}) = \mathscr{Q}$. (This may be seen as follows. Let $D \in \mathscr{Q}$. Then $e^{tD}e^{tD^*} = e^{t(D+D^*)} = I$ for all t. Hence $e^{tD} \in \mathscr{O}$ for all t. By differentiation, $D \in T(\mathscr{O})$. Now let $D \in T(\mathscr{O})$. Then $e^{tD} \in \mathscr{O}$ for all t. Hence $I = e^{tD}e^{tD^*} = e^{t(D+D^*)}$. By differentiation, $D \in \mathscr{Q}$. Therefore $T(\mathscr{O}) = \mathscr{Q}$.) □

According to Theorem 4 of Sharpe [8] and Theorem 1 of Hudson and Mason [4], V is the direct sum of two independent subspaces V_1 and V_2. Furthermore, μ is the convolution of a Gaussian measure μ_1 concentrated on V_1 and a measure μ_2 concentrated on V_2 having no Gaussian component. If $\mathscr{S}(\mu)$ is conjugate to $\mathcal{O}$, then by Corollary 3 every exponent B of μ is of the form $B = cI + WQW^{-1}$ where Q is skew-symmetric. It follows that every eigenvalue of B has real part equal to c. From Theorem 1 of [4] it is easy to see that if $c = \frac{1}{2}$, then μ is purely Gaussian, and if $c > \frac{1}{2}$, then μ has no Gaussian component.

From this decomposition we also obtain the following Corollary to Theorem 3, using the above notation.

COROLLARY 4. *Assume $\mathscr{S}(\mu_2)$ (on V_2) is conjugate to the group of orthogonal linear operators on V_2. Then*

$$\hat{\mu}(y) = \exp\{i(a, y) - (Cy, y) - \beta \mid \tilde{W}y \mid^{\gamma}\}$$

where $a \in V$, $\gamma \in (0, 2)$, $\beta \geq 0$, C is a linear operator on V such that C restricted to V_1 is positive-definite and self-adjoint on V_1 and $\tilde{W}$ is a linear operator on V such that $\tilde{W}$ restricted to V_2 is positive-definite and self-adjoint on V_2.

REMARK. The operator C does not necessarily vanish on V_2 nor does $\tilde{W}$ necessarily vanish on V_1, but dim range C = dim V_1.

PROOF. We have $\mu = \mu_1 * \mu_2$. As in [8] or [4], we have $\hat{\mu}_1(y) = \exp\{i(a_1, y) - (Cy, y)\}$ where $a_1 \in V_1$ and C is a linear operator on V such that the restriction of C to V_1 is positive-definite and self-adjoint on V_1. Since μ_2 is full and operator-stable on V_2 and satisfies the hypothesis of Theorem 3, there is a positive-definite self-adjoint linear operator W on V_2 such that $\hat{\mu}_2(y) = \exp\{i(a_2, y) - \beta \mid Wy \mid^{\gamma}\}$ for y in V_2, where $a_2 \in V_2$, $\beta > 0$, $\gamma \in (0, 2)$. Define $\tilde{W}$ on V by $\tilde{W}(y) = W(y)$ for $y \in V_2$ and $\tilde{W}(y) = 0$ for $y \in V_2^{\perp}$. □

Sections 2, 3 and 4 are devoted to the proofs of Theorems 1, 2, and 3. In Section 5 we discuss a special case, namely $V = R^3$. The case $V = R^2$ was fully discussed in Hudson and Mason [3]. For the convenience of the reader, we provide an appendix with some information concerning the Lie theory of matrix groups which is used throughout this paper. All the results in the appendix are well-known.

2. Proof of Theorem 1. For each $t > 0$ let G_t be the set of all linear operators A on V such that for some a in V, $\mu^t = A\mu * \delta(a)$. Set $G = \cup_{t>0} G_t$. Define a map η: $G \to (0, \infty)$ by $\eta(A) = t$ if $A \in G_t$. Sharpe showed that G is a closed subgroup of linear operators and that η is a continuous homomorphism from G onto the multiplicative group of positive numbers ([8], page 58). The tangent space $T(G)$ of G is the set of all linear operators D such that $D = \lim_{n\to\infty}(g_n - I)/d_n$ in the operator norm topology where $\{g_n\} \subset G$, I is the identity map on V and $\{d_n\}$ is a sequence of positive numbers converging to zero. It is well-known ([7] page 40 or the appendix) that $T(G)$ is a vector space and that the exponential map E defined by $E(A) = \sum_{k=0}^{\infty} A^k/k!$ for A in $T(G)$ is analytic and takes $T(G)$ into G. Define a map L: $T(G) \to R^1$ by $L(A) = \ln(\eta(E(A)))$. We will show that L is a linear operator on $T(G)$.

First, we show that for any real number t, $L(tA) = tL(A)$. Let n be an integer. Then $E(nA) = E(A)^n$, so $\eta(E(nA)) = (\eta(E(A)))^n$. Thus, $L(nA) = nL(A)$. Now, let n be a nonzero integer. Then $L(A) = L(n(A/n)) = nL(A/n)$, and so $L((1/n)A) = (1/n)L(A)$. Thus $L(tA) = tL(A)$ holds for all rational numbers t and by the continuity of L, the equation holds for all t.

Next, we show that if x, y are in $T(G)$, then $L(x + y) = L(x) + L(y)$. To do this, we select an open neighborhood N of zero in $T(G)$ such that E^{-1} exists and is continuously differentiable on $E(N)\cdot E(N) = \{E(x)E(y) : x, y \in N\}$. This is possible since $E'(0) = I$ is invertible and multiplication in G is continuous ([7] page 40 or the appendix). Define Z: $N \times N \to T(G)$ by $Z(x, y) = E^{-1}(E(x)E(y))$. We claim

$$\lim_{t\to 0} \frac{1}{t} Z(tx, ty) = x + y.$$

To see this, note that Z is differentiable at (0, 0). Now $Z(0, 0) = 0$ so

$$\frac{|Z(x, 0) - Z(0, 0) - x|}{|x|} = 0$$

and hence $Z'(0, 0)(x, 0) = x$. Similarly, $Z'(0, 0)(0, y) = y$. Since $Z'(0, 0)$ is linear, $Z'(0, 0) \cdot (x, y) = x + y$. Now

$$\lim_{t \to 0} \frac{|Z(tx, ty) - Z(0, 0) - Z'(0, 0)(tx, ty)|}{|(tx, ty)|} = 0,$$

so $\lim |(1/t)Z(tx, ty) - (x + y)| = 0$. This proves our claim. Since L is continuous,

$$\begin{aligned} L(x + y) &= \lim_{t \to 0} L((1/t)Z(tx, ty)) \\ &= \lim_{t \to 0}(1/t)L(Z(tx, ty)) = \lim_{t \to 0}(1/t)\ln(\eta(E(tx)E(ty))) \\ &= \lim_{t \to 0}(1/t)(L(tx) + L(ty)) = L(x) + L(y). \end{aligned}$$

This shows that L is a linear operator on $T(G)$.

We know that $\mathscr{E}(\mu) \subset T(G)$ since $A \in \mathscr{E}(\mu)$ implies that t^A is in G for all $t > 0$ and A is the derivative of t^A at $t = 1$. We now show $\mathscr{E}(\mu) = \{A \in T(G) : L(A) = 1\}$. It is clear that for $A \in \mathscr{E}(\mu)$, $L(A) = 1$. Let $A \in T(G)$ with $L(A) = 1$. By linearity, $L((\ln t)A) = \ln t$ for all $t > 0$ and thus $\eta(t^A) = t$ which implies $A \in \mathscr{E}(\mu)$. Thus $\mathscr{E}(\mu) = \{A \in T(G) : L(A) = 1\}$.

Finally, we show that $T(\mathscr{S}(\mu)) = \ker L$. Since $\mathscr{S}(\mu) = G_1$ and $E(T(\mathscr{S}(\mu))) \subset \mathscr{S}(\mu)$, $T(\mathscr{S}(\mu)) \subset \ker L$. Let $A \in \ker L$. Then $L(tA) = 0$ for all $t > 0$. Hence, $\eta(E(tA)) = 1$ which implies $e^{tA} \in G_1$. Thus $A \in T(\mathscr{S}(\mu))$.

These last two results show that $\mathscr{E}(\mu) = T(\mathscr{S}(\mu)) + B$ for any B in $E(\mu)$. □

3. Proof of Theorem 2. Assume $\mathscr{S}(\mu) = \mathscr{O}$. Let $\mathscr{Q}$ denote the set of all skew-symmetric linear operators on V, i.e. $Q \in \mathscr{Q}$ if and only if $Q + Q^* = 0$. Let L, G, $T(G)$ and $Z(x, y)$ be as in the proof of Theorem 1. For $x, y \in T(G)$, set $[x, y] = xy - yx$. For real numbers s and t sufficiently small, $Z(sx, ty)$ has a power series expansion given by the Campbell-Baker-Hausdorff formula (appendix or [7], page 61) and the coefficient of st in that expansion is a constant multiple of $[x, y]$. On the other hand, $L(Z(sx, ty)) = L(sx) + L(ty) = sL(x) + tL(y)$, so $L([x, y]) = 0$. Thus for all x, y in $T(G)$, $[x, y] \in T(\mathscr{S}(\mu)) = T(\mathscr{O}) = \mathscr{Q}$.

Now let $B \in \mathscr{E}(\mu)$ and $Q \in \mathscr{Q}$. Then $[B, Q] \in \mathscr{Q}$ so $[B, Q] + [B, Q]^* = 0$. That is, $BQ - QB + (-QB^* + B^*Q) = 0$, or $(B + B^*)Q = Q(B + B^*)$. Thus $B + B^*$ commutes with every $Q \in \mathscr{Q}$ and hence with every rotation, since every rotation is of the form e^Q for some $Q \in \mathscr{Q}$. (This latter fact is easy to see from page 274 of Curtice [2].) The only subspaces invariant under all rotations are $\{0\}$ and V, so by Schur's Lemma (Lang [5], page 173) $B + B^* = cI$ for some number c. Define Q to be $(c/2)I - B$ and note that $Q \in \mathscr{Q}$. By Theorem 1, $(c/2)I = B + Q$ is an exponent for μ.

Now assume $\mathscr{S}(\mu)$ is conjugate to $\mathscr{O}$, i.e. for some positive-definite self-adjoint W, $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$. Then $\mathscr{S}(W^{-1}\mu) = \mathscr{O}$. Hence, for some real number c, cI is an exponent for $W^{-1}\mu$. Thus, $W(cI)W^{-1}$ is an exponent for μ. But, $W(cI)W^{-1} = cI$.

We know that $c \geq ½$ since the eigenvalues of an exponent for μ must have real parts greater than or equal to ½. □

4. Proof of Theorem 3. First assume $\mathscr{S}(\mu) = \mathscr{O}$. By Theorem 2, there is a number $c \geq ½$ such that cI is an exponent for μ. We consider two cases, $c > ½$ or $c = ½$. When $c > ½$, μ has no Gaussian component, so let M be its Lévy measure. By Theorem 2 of [4], there exist a vector $a \in V$ and a finite Borel measure K on the unit sphere U in V such that for every $y \in V$

$$\hat{\mu}(y) = \exp\left\{ i(a, y) + \int_0^\infty \int_U \left(e^{i(t^c u, y)} - 1 - \frac{i(t^c u, y)}{1 + t^{2c}} \right) t^{-2} K(du)\, dt \right\}.$$

Since $\mathcal{S}(\mu) = \mathcal{O}$, $\mu = O\mu*\delta(r)$ for all $O \in \mathcal{O}$, where $r = r(O) \in V$. It follows that $M = OM$ and hence $K = OK$ for all $O \in \mathcal{O}$. But this implies that K is proportional to the Haar measure on U, i.e. $K(du) = \gamma_1 du$ for some $\gamma_1 > 0$, where du is Haar measure on U.

Let $J(y)$ denote the integral in the above representation of $\hat{\mu}(y)$. It suffices to show $J(y) = -\beta|y|^\gamma$ for some $\beta > 0$, where $\gamma = 1/c \in (0, 2)$. By a change of variable, we have

$$J(y) = \gamma\gamma_1 \int_0^\infty \int_U \left(e^{it(u,y)} - 1 - \frac{it(u,y)}{1+t^2}\right) t^{-(1+\gamma)}\, du\, dt.$$

To evaluate this integral we consider three cases: $\gamma < 1$, $\gamma > 1$, $\gamma = 1$.

First, $\gamma < 1$. Since

$$\int_U (u,y)\, du = 0, \quad J(y) = \gamma\gamma_1 \int_0^\infty \int_U (e^{it(u,y)} - 1) t^{-(1+\gamma)} dt.$$

By interchanging the order of integration, we find

$$J(y) = \gamma\gamma_1 \left\{\gamma_2 \int_{U_1} (u,y)^\gamma du + \bar{\gamma}_2 \int_{U_2} |(u,y)|^\gamma\, du\right\}$$

where real part of γ_2 is negative, $U_1 = \{u \in U: (u,y) \geq 0\}$ and $U_2 = \{u \in U: (u,y) < 0\}$. So,

$$J(y) = \gamma\gamma_1 |y|^\gamma \left\{\gamma_2 \int_{U_1} (u,y')^\gamma du + \bar{\gamma}_2 \int_{U_2} |(u,y')|^\gamma\, du\right\}$$

where $y' = y/|y|$. But these two integrals have the same value and they are independent of $y' \in U$. Letting $\beta = -2\gamma\gamma_1(Re\ \gamma_2) \int_{U_1} (u,y')^\gamma\, du$, we obtain $J(y) = -\beta|y|^\gamma$ with $\beta > 0$ as desired.

For $\gamma > 1$, an easy calculation shows that

$$J(y) = \gamma\gamma_1 \left\{i\gamma_3 \int_U (u,y)\, du + \gamma_4 \int_{U_1} (u,y)^\gamma\, du + \bar{\gamma}_4 \int_{U_2} |(u,y)|^\gamma\, du\right\}$$

where $\gamma_3 \in R^1$, $Re\ \gamma_4 < 0$. As before, this yields $J(y) = -\beta|y|^\gamma$ for some $\beta > 0$.

Now for $\gamma = 1$, another calculation shows that

$$\int_0^\infty \left(e^{it(u,y)} - 1 - \frac{it(u,y)}{1+t^2}\right) t^{-2} dt = -\frac{\pi}{2}|(u,y)| - i(u,y)\ln|(u,y)| + i\gamma_5(u,y)$$

where $\gamma_5 \in R^1$. Upon integrating with respect to u, the second and third terms vanish while the first term yields a negative constant times $|y|$. Hence, again $J(y) = -\beta|y|$, for some $\beta > 0$.

Therefore, $\hat{\mu}(y) = \exp\{i(a,y) - \beta|y|^\gamma\}$ when $c > ½$ and $\mathcal{S}(\mu) = \mathcal{O}$.

Now, assume $c > ½$ and $\mathcal{S}(\mu) = W\mathcal{O}W^{-1}$ for some positive-definite self-adjoint linear operator W. Then $\mathcal{S}(W^{-1}\mu) = \mathcal{O}$. Hence, $(W^{-1}\mu)^\wedge(y) = \exp\{i(a',y) - \beta|y|^\gamma\}$ for some $a' \in V$, $\beta > 0$, $\gamma \in (0, 2)$. But, $\hat{\mu}(y) = (W^{-1}\mu)^\wedge(Wy)$. Therefore, $\hat{\mu}(y) = \exp\{i(a,y) - \beta|Wy|^\gamma\}$, where $a = Wa'$.

Now, assume $c = ½$. Then μ is purely Gaussian. Hence, $\hat{\mu}(y) = \exp\{i(a,y) - (Cy,y)\}$, where C is a positive-definite self-adjoint operator on V. Assume $a = 0$ and $\mathcal{S}(\mu) = \mathcal{O}$. Then for every $O \in \mathcal{O}$, $\hat{\mu}(y) = \hat{\mu}(O^*y) = \exp(-(OCO^*y, y))$. Therefore, $OCO^* = C$, i.e. $OC = CO$ for every $O \in \mathcal{O}$. By Schur's Lemma, there is a real number β such that $C = \beta I$. Since C is positive-definite, $\beta > 0$. Therefore, $\hat{\mu}(y) = \exp(-\beta|y|^2)$. In case $a \neq 0$, by the above $(\mu*\delta(-a))^\wedge(y) = \exp(-\beta|y|^2)$, so $\hat{\mu}(y) = \exp(i(a,y) - \beta|y|^2)$.

The final case to consider is when $c = \frac{1}{2}$ and $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$. Arguments like the preceding yield $\hat{\mu}(y) = \exp(i(a, y) - \beta \mid Wy \mid^2)$.

Now we prove the converse of the theorem. Since $(W^{-1}\mu * \delta(-W^{-1}a))\hat{}(y) = \exp(-\beta \mid y \mid^{\gamma})$, $\mathscr{S}(W^{-1}\mu) = \mathscr{O}$. Therefore, $W^{-1}\mu * \delta(-W^{-1}a) = OW^{-1}\mu * \delta(-OW^{-1}a)$ which implies $\mu = WOW^{-1}\mu * \delta(a - WOW^{-1}a)$ for all $O \in \mathscr{O}$. Thus, $\mathscr{S}(\mu) = W\mathscr{O}W^{-1}$. □

5. V = R³. We know that $\mathscr{S}(\mu)$ is conjugate to a closed subgroup $\mathscr{O}_0$ of $\mathscr{O}$.

LEMMA 1. *The dimension of $T(\mathscr{O}_0)$ is either* 0, 1 *or* 3.

PROOF. We know that $T(\mathscr{O})$ is $\mathscr{Q}$, the set of all skew-symmetric operators. Since $\mathscr{O}_0$ is a closed subgroup contained in $\mathscr{O}$, $T(\mathscr{O}_0)$ is closed under $[\cdot, \cdot]$ and contained in $T(\mathscr{O})$. Define the linear transformation f from R^3 onto $\mathscr{Q}$ by

$$f(a, b, c) = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}.$$

Note that for x and y in R^3, $[f(x), f(y)] = f(x \times y)$, where $\times$ is the cross product on R^3. Hence, $(R^3, \times)$ is isomorphic to $\mathscr{Q}$. Therefore, $T(\mathscr{O}_0) = \{0\}$, $\dim T(\mathscr{O}_0) = 1$ or $\mathscr{O}_0 = 0$. □

When $\dim T(\mathscr{O}_0) = 0$, μ has a unique exponent. When $\mathscr{O}_0 = \mathscr{O}$, Theorem 3 and Corollary 3 describe $\hat{\mu}$ *and* $\mathscr{E}(\mu)$. We now consider the case where $\dim T(\mathscr{O}_0) = 1$.

LEMMA 2. *Let $\mathscr{O}_0$ be a closed subgroup of $\mathscr{O}$ with* $\dim \mathscr{O}_0 = 1$. *Then*

$$T(\mathscr{O}_0) = \left\{ \begin{bmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : c \in R^1 \right\}$$

with respect to some orthonormal basis for R^3.

PROOF. We know $\dim T(\mathscr{O}_0) = 1$, so let $\{Q\}$ be a basis for $T(\mathscr{O}_0)$. Then $Q \in \mathscr{Q}$. Since $\det Q = \det(-Q^*) = (-1)^3 \det Q$, $\det Q = 0$, i.e. Q is singular. Let $u \in R^3$ be such that $\mid u \mid = 1$ and $Qu = 0$. Let this u be the third member of an orthonormal basis for R^3. This is the basis needed in the Lemma. By skew-symmetry the third row and third column are all zeros, and by skew-symmetry the diagonal is all zeros and the (1, 2)-element is the negative of the (2, 1)-element. Since $\{Q\}$ is a basis for $T(\mathscr{O}_0)$, we have the stated result. □

Since $\mathscr{S}(\mu)$ is conjugate to such an $\mathscr{O}_0$, select W to be positive-definite and self-adjoint so that

$$T(\mathscr{S}(W^{-1}\mu)) = \left\{ \begin{bmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : c \in R^1 \right\}$$

with respect to some orthonormal basis for R^3.

THEOREM 4. *If $T(\mathscr{S}(\mu))$ is neither* $\{0\}$ *nor conjugate to $\mathscr{Q}$, then there is a positive-definite self-adjoint linear operator W on R^3 and there are real numbers $a, b \geq \frac{1}{2}$ such that*

$$\mathscr{E}(W^{-1}\mu) = \left\{ \begin{bmatrix} a & -c & 0 \\ c & a & 0 \\ 0 & 0 & b \end{bmatrix} : c \in R^1 \right\}$$

with respect to some orthonormal basis for R^3.

PROOF. Since $\mathscr{S}(\mu)$ is conjugate to a closed subgroup $\mathcal{O}_0$ of $\mathcal{O}$, select the W so that $\mathscr{S}(W^{-1}\mu) = \mathcal{O}_0$. By Lemma 2,

$$T(\mathscr{S}(W^{-1}\mu)) = \left\{ \begin{bmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : c \in R^1 \right\}$$

with respect to some orthonormal basis for R^3. Use this basis for all matrix representations. Let $B \in \mathscr{E}(W^{-1}\mu)$ with the representation (b_{ij}). As in the proof of Theorem 2, $BQ - QB \in \ker L = T(\mathscr{S}(W^{-1}\mu))$ for all $Q \in T(\mathscr{S}(W^{-1}\mu))$, i.e.

$$B\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $c \in R^1$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This implies that $b_{31} = b_{32} = b_{13} = b_{23} = 0$ and $b_{12} = b_{21}$. It only remains to show that $b_{11} = b_{22}$.

We now know that $B' = \text{diag.}\ (b_{11}, b_{22}, b_{33})$ is in $\mathscr{E}(W^{-1}\mu)$. Since B' commutes with every Q in $T(\mathscr{S}(W^{-1}\mu))$, as seen in the proof of Theorem 2, we have that diag. (b_{11}, b_{22}) commutes with every 2×2 skew-symmetric matrix. Again, as in the proof of Theorem 2, this implies that diag. (b_{11}, b_{22}) is a multiple of the identity. Hence, $b_{11} = b_{22}$. □

In our last result when $\dim T(\mathscr{S}(\mu)) = 1$, we characterize the Lévy measure. We say that a Lévy measure M is B-stable if B is a nonsingular linear operator and $t^B M = t \cdot M$ for all $t > 0$.

THEOREM 5. *Assume*

$$T(\mathscr{S}(\mu)) = \left\{ \begin{bmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : c \in R^1 \right\},$$

μ has an exponent B whose matrix representation is diag (a, a, b) *with respect to the usual basis, where a and b are both greater than* ½, *M is the Lévy measure of μ. Then we have that $\mathscr{S}(\mu) = \mathcal{O}'$, where $\mathcal{O}'$ is the subgroup of $\mathcal{O}$ generated by all orthogonal transformations which leave the z-axis invariant. Furthermore, we have that there is a finite Borel measure ν on $[-\pi/2, \pi/2]$ such that for Borel $A \subset R^3\backslash\{0\}$*

$$M(A) = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{\infty} I_A(t^B x(\theta, \varphi)) t^{-2} dt\, d\theta \nu(d\varphi),$$

where $x(\theta, \varphi) = (\cos\theta \cos\varphi, \sin\theta \cos\varphi, \sin\varphi)$.

Conversely, if ν is a finite Borel measure on $[-\pi/2, \pi/2]$ and if for Borel $A \subset R^3\backslash\{0\}$, $\bar{M}(A)$ is defined by the previous triple integral with $x(\theta, \varphi)$ as before and B as diag(a, a, b), *with a and b greater than* ½, *then $\bar{M}$ is a Lévy measure and is B-stable with $\mathscr{S}(\bar{M}) \subset \mathcal{O}'$.*

REMARK. In the converse part of this theorem, we can *not* conclude that $B \in \mathscr{E}(\mu)$ implies $\mathscr{S}(\mu) = \mathcal{O}'$. That is, μ may have an exponent which is diagonalizable even though $\mathscr{S}(\mu)$ is discrete.

PROOF. By the assumed form of $T(\mathscr{S}(\mu))$, we have $\mathscr{S}(\mu) = \mathcal{O}'$.

The first task is to construct ν. For A a Borel subset of U, let $K(A) = M(\{t^B x : x \in A, t > 1\})$. Since B is diagonal, each orbit of t^B hits U exactly once. Thus, K is a finite Borel measure on U. For $x \in U$, $t > 0$, define $T(x, t) = t^B x \in R^3\backslash\{0\}$. It is easy to see that T is a homeomorphism of $U \times (0, \infty)$ onto $R^3\backslash\{0\}$ when $R^3\backslash\{0\}$ has the usual topology and $U \times (0, \infty)$ has the product topology. Let $\tilde{M}$ be the measure on $U \times (0, \infty)$ given by $\tilde{M} = K \times \gamma$, where $d\gamma = t^{-2}dt$ on $(0, \infty)$.

LEMMA 3. $M = T\tilde{M}$.

PROOF. Let $D = \{t^B x : x \in C, r < t \le s\}$. Then

$$T\tilde{M}(D) = \int_{T^{-1}(D)} d(K \times \gamma) = \int_C \int_r^s t^{-2}dt\, dK = (1/r - 1/s)K(C) = M(D).$$

Hence, M and $T\tilde{M}$ agree on all sets of the form $C \times (r, s]$, so they agree everywhere. □

Next we show that there is a finite Borel measure ν on $[-\pi/2, \pi/2]$ such that $K = \lambda \times \nu$, where λ is Lebesque measure on $[0, 2\pi)$. Let $R(\theta)$ denote the counterclockwise rotation about the z-axis through an angle of θ radians. Let D and E be Borel subsets of $[0, 2\pi)$ and $[-\pi/2, \pi/2]$, respectively. Since $\mathscr{S}(M) = \mathscr{O}'$, $R(\theta)K(D \times E) = K(D \times E)$. Set $\alpha_E(D) = K(D \times E)$, for E fixed. Then α_E is a finite Borel measure on $[0, 2\pi)$ which is invariant under rotations, so $\alpha_E = a(E)\cdot\lambda$, where $a(E)$ is a constant depending on E. But, $a(\cdot)$ as a function of E is a finite Borel measure on $[-\pi/2, \pi/2]$. Set $\nu(E) = a(E)$. Clearly, $K = \lambda \times \nu$. This establishes the stated representation for M.

Now, we prove the converse of the Theorem. We first show that $\bar{M}$ is a Lévy measure. Clearly, $\bar{M}$ is a measure on $R^3\backslash\{0\}$ so it remains to show that $\int (|x|^2 \wedge 1)\bar{M}(dx) < \infty$, where $c \wedge d$ means the minimum of c and d. Clearly, for $0 < t \le 1$, $\| t^B \| \le 3t^\alpha$, where $\alpha = a \wedge b$ and $\| \cdot \|$ is the operator norm. Thus

$$\int (|x|^2 \wedge 1)\bar{M}(dx) \le k_1 + k_2 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^1 (|t^B x(\theta, \varphi)|^2 \wedge 1)t^{-2}\, dt\, d\theta\, \nu(d\varphi)$$

for some constants k_1 and k_2. But

$$\int_0^1 (|t^B x(\theta, \varphi)|^2 \wedge 1)t^{-2}\, dt \le 9 \int_0^1 t^{2(\alpha-1)}\, dt < \infty$$

since $\alpha > ½$. Hence $\bar{M}$ is a Lévy measure.

Using Lemma 9 of [4] we obtain that $\bar{M}$ is B-stable. Finally, we have $\mathscr{S}(\bar{M}) \subset \mathscr{O}'$ since the defining integral is invariant under transformations in $\mathscr{O}'$. □

Appendix: Lie theory of matrix groups. Denote by $gl(n)$ the algebra of all $n \times n$ real matrices and by $GL(n)$ the group of invertible $n \times n$ matrices and choose a norm for $gl(n)$. If G is a subgroup of $GL(n)$, denote by $T(G)$ the subset of $gl(n)$ consisting of those x in $gl(n)$ such that $x = \lim_n e_n^{-1}(g_n - I)$ for some sequence $\{g_n\}$ in G and some sequence $\{e_n\}$ of positive numbers with limit zero, where I is the identity matrix. The set $T(G)$ is called the *tangent space* to G at I. The first theorems show that $T(G)$ plays the role of a Lie algebra to the Lie group G.

THEOREM 1A. *If G is closed relative to $GL(n)$ and x is in $T(G)$, then $E(tx)$ is in G for each number t. ($E : gl(n) \to GL(n)$ is the exponential function defined by $E(x) = \lim_n (I + x/n)^n = \lim_n \sum_{k=1}^n x^k/k!$.)*

PROOF. Let x be in $T(G)$ and choose sequences $\{g_n\}$ in G and $\{e_n\}$ of positive numbers with limit zero so that $x = \lim e_n^{-1}(g_n - I)$. Choose the integer sequence $\{n_k\}$ so that $n_k - 1 \le e_k^{-1} < n_k$ for all $k \ge 1$.

Define the sequence $\{d_k\}$ by $-d_k = x - n_k(g_k - I)$. Then $\{d_k\}$ has limit zero and

$$(I + n_k^{-1}(x + d_k))^{n_k} = g_k^{n_k}$$

is in G for each $k \geq 1$. Thus $\{g_k^{n_k}\}$ has limit $E(x)$ and since $E(x)$ is in $GL(n)$ and G is closed, we have $E(x)$ is in G.

It is clear that if x is in $T(G)$, then so is tx for each positive t. Hence, $E(tx)$ is in G for those t. The rest of Theorem 1A follows from the observation that $E(tx)^{-1} = E(-tx)$. □

From the definition of $T(G)$ we see that if f is a differentiable function from R^1 into G and $f(0) = I$, then $f'(0)$ is in $T(G)$. We now see that for each x and y in $T(G)$ we have the functions f and g defined by

$$f(t) = E(tx)E(ty), \qquad g(t) = E(\sqrt{t}\,x)E(\sqrt{t}\,y)E(-\sqrt{t}\,x)E(-\sqrt{t}\,y)$$

mapping into G. Hence, each of $x + y = f'(0)$ and $[x, y] \equiv xy - yx = g'(0)$ is in $T(G)$. Thus $T(G)$ is a linear subspace of $gl(n)$ which is closed under the "Lie bracket" or "commutator" product. The importance of [,] will be seen below.

The function E satisfies $E'(0) = I$ and hence by the inverse function theorem there is a neighborhood $\mathscr{U}$ of 0 so that $E \mid \mathscr{U}$ is reversibly continuously differentiable onto the neighborhood $E(\mathscr{U})$ of I.

Theorem 2A. *If G is closed relative to $GL(n)$, then $E(T(G))$ is a neighborhood of I in G.*

Proof. Suppose not. Choose the sequence $\{g_n\}$ in $G \cap E(\mathscr{U})\backslash E(T(G))$ so that $\{g_n\}$ has limit I. Let the sequence $\{x_k\}$ be defined by $E(x_k) = g_k$. Let p be the linear projection from $gl(n)$ onto $T(G)$. Note that

$$\lim \frac{\| E(x_k) - E(px_k)\|}{\| x_k - px_k \|} = 1$$

and

$$\lim \frac{\| E(x_k) - E(px_k) - E'(0)(x_k - px_k)\|}{\| x_k - px_k \|} = 0.$$

Also, the sequence $\{x_k - px_k\}$, and hence the sequence $\{E'(0)(x_k - px_k)\}$, has limit 0 since both $\{x_k\}$ and $\{px_k\}$ does. Since each x_k is not in $T(G)$, $x_k - pk_x$ is never 0.

Thus, the sequence $\{(E(x_k) - E(px_k))/\| x_k - px_k \|\}$ is bounded and without loss of generality we may assume it converges to some y in $gl(n)$ with $\| y \| = 1$. But

$$y = \lim \frac{E(x_k)E(-px_k) - I}{\| x_k - px_k \|}$$

since $\{E(-px_k)\}$ has limit I, and from the above

$$y = \lim \frac{x_k - px_k}{\| x_k - px_k \|}.$$

Thus, from the first, y is in $T(G)$ (since $E(x_k)E(-px_k)$ is in G and $\| x_k - px_k \| \to 0$) and from the second, $py = 0$. This is impossible since $\| y \| = 1$. □

We now turn to the problem of stating the Campbell-Baker-Hausdorff Theorem. Let $\mathscr{U}$ be chosen as before and let $L = (E \mid \mathscr{U})^{-1}$. Since E is analytic, by the inverse function theorem so is L. The function V contained in $(gl(n) \times gl(n)) \times gl(n)$ defined by $V(x, y) = L(E(x)E(y))$ is hence analytic on some neighborhood of (0, 0) and has a power series expansion about (0, 0) with a positive radius of convergence. Theorem 3A, the Campbell-

Baker-Hausdorff Theorem, has a surprising thing to say about the coefficients in this power series expansion for V.

THEOREM 3A. *Independent of n, the typical coefficient $(1/k!)V^{(k)}(0, 0)(x, y)^k$ for $k \geq 2$ is a certain linear combination of summands of the form $[a_1, [a_2, \cdots [a_{k-1}, a_k] \cdots]]$ where each a_i is either x or y. For example, $V^{(2)}(0, 0)(x, y)^2 = [x, y]$ and $V^{(3)}(0, 0)(x, y)^3 = (½)[x, [x, y]] + (½)[y, [y, x]]$.*

Theorem 3A may be proved by actual calculations.

The most obvious consequence of Theorem 3A is that the function V's restriction to $G \times G$ is determined by a knowledge of $[\,,\,]$ restricted to $T(G) \times T(G)$ and hence the multiplication of G is determined near I by $[\,,\,]$. Thus, many nonlinear problems concerning $(G, \cdot)$ can be translated into linear problems concerning the Lie algebra $(T(G), +, [\,,\,])$.

For example, we have the following set up. Let G be a closed subgroup of $gl(n)$, H be a closed subgroup of $gl(m)$ and f be a continuous function from G into H. There is then defined a continuous F on some neighborhood of 0 in $T(G)$ into some neighborhood of 0 in $T(H)$ satisfying $E \circ F = f \circ E$. (One E denotes the exponential map on $gl(n)$ and the other on $gl(m)$.)

THEOREM 4A. *With f and F as above, f is a local group homomorphism ($f(xy) = f(x)f(y)$ for x and y near I) if and only if F is the restriction of a Lie algebra homomorphism T to some neighborhood of 0 in $T(G)$, (T is linear and $[Tx, Ty] = T([x, y])$).*

PROOF. Denote by V_G and V_H the restrictions of the appropriate V's to neighborhoods of (0, 0) in $T(G) \times T(G)$ and $T(H) \times T(H)$, respectively. If f is a local homomorphism, then $EF = fE$ implies

$$F(V_G(x, y)) = V_H(Fx, Fy)$$

for (x, y) sufficiently near (0, 0). It is elementary to see that since $f(E(tx))f(E(sx)) = f(E((t + s)x))$ for x in $T(G)$ and each s and t sufficiently near 0, we have for appropriate x (all x in some neighborhood of 0) and all t in $(-1, 1)$ that $f(E(tx)) = E(tF(x))$ and hence that $F(tx) = tF(x)$ for these x and t.

We have then for (x, y) in some neighborhod of (0, 0) that $F(V_G(tx, ty)) = V_H(tF(x), tF(y))$. But, $V_H(0, 0) = 0$ and $V_H'(0, 0)(a, b) = a + b$ so $\lim_{t\to 0}(1/t)V_H(tF(x), tF(y)) = F(x) + F(y)$. On the other hand, $F(x + y) = \lim_{t\to 0}(1/t)F(t(x + y)) = \lim_{t\to 0}F((1/t) \cdot V_G(tx, ty))$ since F is continuous. Hence, $F(x + y) = F(x) + F(y)$ and continuity ensures that F is locally linear. Finally, it follows that the coefficient of st in the power series expansion for $F(V_G(sx, ty))$ is $F([x, y]_G)$ and is also $[Fx, Fy]_H$. It is then clear that F is the restriction of a Lie algebra homomorphism.

Assume now that F is the restriction of an algebra homomorphism. Since F is linear, we see that

$$F(V_G(x, y)) = \sum_{k=1}^{\infty} (1/k!)F(V_G^{(k)}(0, 0)(x, y)^k).$$

But from the Campbell-Baker-Hausdorff Theorem and the fact that $F([x, y]) = [Fx, Fy]$, we see that $F(V_G^{(k)}(0, 0)(x, y)^k) = V_H^{(k)}(0, 0)(Fx, Fy)^k$ and f is a local homomorphism. □

We have applied theorem 4A in this paper in the following form. Let f be a continuous homomorphism from G to H and choose the Lie algebra homomorphism T from $T(G)$ to $T(H)$ so that $fE = ET$ near 0 in $T(G)$. It is clear that the L function maps a neighborhood of I in $\ker(f) = f^{-1}(I)$ onto a neighborhood of 0 in $\ker(T) = T^{-1}(0)$. On the other hand, $T([x, y]_G) = [Tx, Ty]_H = [0, y]_H = 0$ if x is in $\ker(T)$, so $\ker(T)$ absorbs algebra products on either side. This shows that each closed normal subgroup of G is paired with an algebra ideal in $T(G)$. It is harder to see that each algebra ideal in $T(G)$ is paired with a normal subgroup of G.

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