The Annals of Probability
1983, Vol. 11, No. 1, 78–101

SMALL DEVIATIONS IN THE FUNCTIONAL CENTRAL LIMIT THEOREM WITH APPLICATIONS TO FUNCTIONAL LAWS OF THE ITERATED LOGARITHM

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We prove a small deviation theorem of a new form for the functional central limit theorem for partial sums of independent, identically distributed finite-dimensional random vectors. The result is applied to obtain a functional form of the Chung-Jain-Pruitt law of the iterated logarithm which is also a strong speed of convergence theorem refining Strassen's invariance principle.

1. Introduction. In his classical paper [5] Chung proved the following remarkable results:

(CI) If $\{W(t): t \geq 0\}$ is real-valued Brownian motion, then

$$\liminf_{t\to\infty}\left(\frac{\mathrm{LL}t}{t}\right)^{1/2} \max_{s\leq t}|W(s)| = \pi/8^{1/2}$$

(here and below LL stands for "log log");

(CII) If $\{X_j\}$ are real-valued, independent, identically distributed random variables such that $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^3 < \infty$ and if $S_k = \sum_{j=1}^k X_j$, then

$$\liminf_{n\to\infty}\left(\frac{\mathrm{LL}n}{n}\right)^{1/2} \max_{k\leq n}|S_k| = \pi/8^{1/2}.$$

Since the appearance of Chung's paper in 1948, in successive papers by several authors the condition $E|X_1|^3 < \infty$ was gradually relaxed; this development culminated in 1975 in the work of Jain and Pruitt [12] (see references therein), where it was proved that a finite second moment is enough for (CII). The result (CII) is sometimes referred to as the other law of the iterated logarithm, in contrast to the Hartman-Wintner law.

In a separate line of reseach, Strassen obtained in a now classical work [20] the following deep results on the (usual) law of the iterated logarithm:

(SI) (Strassen's functional law of the iterated logarithm for Brownian motion) If $\{W(t): t \geq 0\}$ is real-valued Brownian motion, then

$$P\{\lim_{t\to\infty}\inf_{f\in K}\|(2t\mathrm{LL}t)^{-1/2}W((\cdot)t) - f\|_\infty = 0\} = 1 \quad \text{and}$$

$$P\{\text{for all } f\in K, \liminf_{t\to\infty}\|(2t\mathrm{LL}t)^{-1/2}W((\cdot)t) - f\|_\infty = 0\} = 1,$$

where $\|g\|_\infty = \sup_{0\leq t\leq 1}|g(t)|$ for $g \in C[0, 1]$ and

$$K = \left\{f \in C[0, 1]: f(0) = 0,\ f \text{ is absolutely continuous and } \int_0^1 (f'(t))^2\,dt \leq 1\right\};$$

Received October 1981; revised March 1982.
AMS 1980 *subject classifications.* Primary 60F15.
Key words and phrases. Small deviations, other law of the iterated logarithm, Strassen's invariance principle.

(SII) (Strassen's invariance principle) If $\{X_j\}$ are real valued, independent, identically distributed random variables such that $EX_1 = 0$, $EX_1^2 = 1$, $S_k = \sum_{j=1}^{k} X_j$, and Z_n is the $C[0, 1]$-valued random vector defined by setting $Z_n(t) = S_k$ for $t = k/n$ $(k = 0, \cdots, n)$ (here $S_0 \equiv 0$ by definition) and extending by linear interpolation in each interval $[(k-1)n^{-1}, kn^{-1}]$, then

$$P\{\lim_{n\to\infty}\inf_{f\in K}\|(2n\text{LL}n)^{-1/2}Z_n - f\|_\infty = 0\} = 1 \quad \text{and}$$

$$P\{\text{for all } f \in K,\ \liminf_{n\to\infty}\|(2n\text{LL}n)^{-1/2}Z_n - f\|_\infty = 0\} = 1,$$

where K is as in (SI).

In a very recent paper, Csáki [7] proved some interesting results connecting (CI) and (SI). His results are of the following nature: for certain functions $f \in K$ with $\int (f')^2 < 1$,

$$\liminf_{t\to\infty}(\text{LL}t)\,\|(2t\text{LL}t)^{-1/2}W((\cdot)t) - f\|_\infty = c(f)\pi/8^{1/2} \quad \text{a.s.,} \tag{1.1}$$

where $0 < c(f) < \infty$. Thus for $f \equiv 0$, Csáki's result reduces to (CI) (here $c(f) = 2^{-1/2}$), while in general it gives a strong rate of convergence result in (SI) for a certain subclass of K (see Section 6 below for more precise statements). Csáki obtains also some partial results when $\int (f')^2 = 1$. The arguments in [7] are based on the well known asymptotic behavior of the lower tail of one-dimensional Wiener measure, the Cameron-Martin translation formula and variations of well-known techniques in iterated logarithm proofs.

The following question arises naturally: is it possible to prove an invariance principle corresponding to (1.1), connecting (CII) and (SII)? In this paper we answer this question affirmatively, proving a strong invariance principle under second moment conditions, for a broad class of functions in the set K (Theorems 5.1 and 5.5; also Theorem 5.4). The basic tools—apart from well-known iterated logarithm arguments—used in proving the invariance principles are some delicate small deviation results obtained in Section 4 (Theorems 4.3, 4.5, 4.7; Corollary 4.6); these results are of a character which appears to be new in the literature.

It should be remarked that under the sole assumption of a finite second moment, even in the one-dimensional case our results on the other law of the iterated logarithm are not directly accessible to a.s. approximation methods. That is, our Theorems 5.1 and 5.5 cannot be proved by combining Csáki's result for Brownian Motion (or rather, the improved statement given in Theorem 6.1 of the present paper) with an appropriate strong approximation theorem; in fact, the normalization factor $(\text{LL}n/n)^{1/2}$ requires an a.s. approximation rate which cannot be achieved (see [9], pages 93 and 121). On the other hand, if a moment assumption more stringent than $E|X_1|^2 < \infty$ is introduced, then Theorems 5.1 and 5.5 can be obtained from Theorem 6.1 via an appropriate strong approximation theorem. (I thank N. Jain, J. Kuelbs and W. Philipp for some exchanges on this point.)

All our results are proved for random vectors taking values in a finite-dimensional Banach space.

We proceed now to describe the contents of each section. Section 2 is of a preliminary character. The definition of the reproducing kernel Hilbert space H_μ of a Gaussian measure μ on a Banach space and the abstract form of the Cameron-Martin translation formula are recalled. Some useful inequalities are derived from the formula. Then we describe the specific form of these notions in the case when μ is the Wiener measure associated to a vector-valued Brownian motion.

Section 3 is devoted to proving some limit theorems for vector-valued Brownian motion. Theorem 3.1 generalizes a well-known asymptotic estimate of the lower tail of the maximum of the absolute value of one-dimensional Brownian motion to the case of a vector-valued Brownian motion based on a Gaussian measure γ on a finite dimensional Banach space endowed with an arbitrary norm. Theorems 3.3 and 3.4 give related results for translated vector-valued Brownian motion.

The basic results of this paper—the small deviation theorems—are contained in Section

4. The aim is to find asymptotic estimates for probabilities of the form

$$P\{\| n^{-1/2}Z_n - a_n f \|_\infty \leq a_n^{-1} r\},$$

where $\{a_n\}$ is any real sequence satisfying $0 < a_n \to \infty$ and $n^{-1}a_n^2 \to 0$, $r > 0$, $f \in H_\mu$ (μ is the Wiener measure associated to a vector-valued Brownian motion) and Z_n is as in (SII). The upper bound result—Theorem 4.3—is proved for all $f \in H_\mu$ and is essentially in final form. We prove the lower bound result—Theorem 4.5—for a large subset of H_μ: those f in H_μ such that f' belongs to L^∞. However, we have been unable to overcome certain technical difficulties and prove what appears to be the natural conjecture: namely that Theorem 4.5 is true for all $f \in H_\mu$. For one-dimensional random variables and $f \equiv 0$ our results reduce to a small deviation result of Mogulskii [16].

In Section 5 we combine the results of Section 4 with variations of well-known arguments appearing in the literature on laws of the iterated logarithm—such as those in Csáki's paper [7]—and prove a strong invariance principle connecting the Chung-Jain-Pruitt theorem and Strassen's invariance principle, embodied in the lower and upper bounds given by Theorems 5.1 and 5.5. By taking $f \equiv 0$, we obtain in Corollary 5.6 a generalization of the Chung-Jain-Pruitt theorem for random vectors taking values in a Banach space of arbitrary finite dimension. The idea of obtaining the (one-dimensional) Chung-Jain-Pruitt theorem from a small deviation result appears in Csörgő and Révész [8]. Related results appear in [17].

Finally, in Section 6 we improve some of the results of Csáki [7] for one-dimensional Brownian motion; at the same time, we generalize the results to the case of vector-valued Brownian motion. In particular, we give a complete answer to one of the questions in Csáki's paper concerning the class of functions for which (1.1) is valid. As compared to [7], our improvement is achieved by a sharper use of the Cameron-Martin formula.

NOTATION. Throughout the paper, B will denote a finite-dimensional real Banach space endowed with a norm p. We will write $I = [0, 1]$. For $f \in C(I, B)$, the space of B-valued continuous functions defined on I, we will write $\| f \|_\infty = \sup_{t \in I} p(f(t))$. We shall write "LL" for "log log". For a real number x, $[x]$ will denote the integer part of x.

2. Preliminaries on Gaussian measures, the Cameron-Martin formula and the Wiener measure associated to vector-valued Brownian motion. Let E be a separable Banach space, μ a centered Gaussian measure on E. We recall the definition of the Hilbert space of μ (also called the reproducing kernel Hilbert space of μ) and some related notions (see e.g. [2], [14]). Since $\int \xi^2\, d\mu < \infty$ for each $\xi \in E'$, there is a natural map π from E' into $L^2(E, \mu)$, or more briefly, $L^2(\mu)$; π is the restriction to E' of the canonical map of the space $\mathscr{L}^2(\mu)$ of μ-square integrable functions into the quotient space $L^2(\mu)$. We will write E'_μ for the closure in $L^2(\mu)$ of $\pi(E')$. Let

$$H_\mu = \{h \in E : \xi \to \xi(h) \text{ is } \mathscr{L}^2(\mu)\text{-continuous on } E'\}$$

and let $\phi : H_\mu \to E'_\mu$ be defined by

$$\xi(h) = \langle \pi(\xi), \phi(h) \rangle_{L^2(\mu)}$$

for $\xi \in E'$, $h \in H_\mu$. It turns out that the map ϕ is linear and bijective. An inner product is introduced in H_μ by setting

$$\langle h, k \rangle_\mu = \langle \phi(h), \phi(k) \rangle_{L^2(\mu)} \quad \text{for} \quad h, k \in H_\mu;$$

then $\langle H_\mu, \langle \cdot, \cdot \rangle_\mu \rangle$ is a Hilbert space, the *Hilbert space of* μ. The inclusion map from H_μ into E is compact and therefore $K_\mu = \{h \in H_\mu : \| h \|_\mu \leq 1\}$ is a compact subset of E. If $S(\mu)$ is the topological support of μ, then $S(\mu) = \overline{H_\mu}$ (closure in E). The inverse ϕ^{-1} of ϕ has a useful description, as follows. For each $\eta \in E'_\mu$ the measure $\eta(x)d\mu(x)$ possesses a barycenter $\Delta(\eta) = \int x\eta(x)\, d\mu(x) \in E$ (the integral may be interpreted either in the Pettis or in the Bochner sense) and one has $\Delta : E'_\mu \to E$ is linear, injective, $\Delta(E'_\mu) = H_\mu$ and $\Delta = \phi^{-1}$. We observe that if $\hat{\Delta} = \Delta \circ \pi$, then $\hat{\Delta}(E')$ is dense in H_μ.

The following well-known result was first proved by Cameron and Martin [4] in the case when μ is the standard Wiener measure on $C[0, 1]$ (under certain restrictions on h). For the sake of completeness we give a short, direct proof, which J. Samur helped us construct.

PROPOSITION 2.1. *Let μ be a centered Gaussian measure on E. For $x \in E$ define $\mu_x(\cdot) = \mu(\cdot - x)$. Then for all $h \in H_\mu$ one has $\mu_h \ll \mu$ and*

$$d\mu_h = \exp\{\phi(h) - (1/2) \| h \|_\mu^2\} \, d\mu.$$

PROOF. In order to simplify the notation we will put $\| \cdot \|_2 = \| \cdot \|_{L^2(\mu)}$. Let us first remark that for $\eta, \zeta \in E'_\mu$, $\mathscr{L}_\mu(\eta) = N(0, \| \eta \|_2^2)$ and $\mathscr{L}_\mu(\eta, \zeta)$ is a centered Gaussian measure on $\mathbb{R}^2$; this follows from the fact that both statements are true for the elements of E'.

Let $d\nu = \exp\{\phi(h) - (1/2) \| h \|_\mu^2\} \, d\mu$. In order to have $\nu = \mu_h$ it is enough to prove that their characteristic functionals are equal, $\hat{\nu} = \hat{\mu}_h$. Let $\xi \in E'$. Putting $\eta = \pi(\xi)$, $\alpha = \phi(h)$ we have the orthogonal decomposition in E'_μ

$$\eta = t\alpha + \beta \tag{2.1}$$

where $t \in \mathbb{R}$, $\langle \alpha, \beta \rangle_2 = 0$. Thus α and β are μ-independent and therefore, since $\| h \|_\mu = \| \alpha \|_2$,

$$\begin{aligned} \hat{\nu}(\xi) &= \int_E \exp(i\eta)\exp\left\{\alpha - \frac{1}{2} \| h \|_\mu^2\right\} d\mu \\ &= \exp\left\{-\frac{1}{2} \| h \|_\mu^2\right\}\left(\int_E \exp(i\beta) \, d\mu\right)\left(\int_E \exp((it + 1)\alpha) \, d\mu\right) \\ &= \exp\left\{-\frac{1}{2} \| \alpha \|_2^2 - \frac{1}{2} \| \beta \|_2^2\right\} \int_{\mathbb{R}} \exp(i(t - i)x) \, d\lambda(x), \end{aligned} \tag{2.2}$$

where $\lambda = N(0, \| \alpha \|_2^2)$. By an elementary calculation,

$$\int_{\mathbb{R}} \exp(i(t - i)x) \, d\lambda(x) = \exp\left\{-\frac{1}{2} (t - i)^2 \| \alpha \|_2^2\right\}. \tag{2.3}$$

From (2.1)–(2.3), we have

$$\begin{aligned} \hat{\nu}(\xi) &= \exp\left\{-\frac{1}{2} \| \eta \|_2^2 + it \| \alpha \|_2^2\right\} = \exp\left\{-\frac{1}{2} \| \eta \|_2^2 + i\langle \eta, \alpha \rangle_2\right\} \\ &= \exp\left\{i\xi(h) - \frac{1}{2} \int_E \xi^2 \, d\mu\right\} = \hat{\mu}_h(\xi). \quad \square \end{aligned}$$

We prove next some inequalities which are consequences of Proposition 2.1. Similar but somewhat simpler inequalities were used by Borell in [3].

PROPOSITION 2.2. (1) *Let V be a convex, symmetric, measurable subset of E. Then for all $h \in H_\mu$, $z \in E$, $\xi \in E'$*

$$\mu(h + z + V) \le \mu(V)\exp\{-(1/2)(\| h \|_\mu^2 - \| h - g \|_\mu^2) - \xi(z) + \sup_{x \in V}\xi(x)\},$$

where $g = \hat{\Delta}(\xi)$.

(2) *Let A be a Borel subset of E, V as in* (1). *Then for all $h \in H_\mu$, $\xi \in E'$,*

$$\mu(V)\mu(h + A \cap V) \ge (\mu(A \cap V))^2 \exp\{-(1/2)(\| h \|_\mu^2 + \| h - g \|_\mu^2) + \inf_{x \in V}\xi(x)\},$$

where $g = \hat{\Delta}(\xi)$.

PROOF. By Proposition 2.1, for any $\xi \in E'$

$$\begin{aligned} \mu(h + z + V) &= \exp\left\{-\frac{1}{2}\|h\|_\mu^2\right\}\int_{z+V}\exp\{-\phi(h)\}\,d\mu. \\ &\le \exp\left\{-\frac{1}{2}\|h\|_\mu^2 - \inf_{x\in z+V}\xi(x)\right\}\int_{z+V}\exp\{\xi - \phi(h)\}\,d\mu. \end{aligned} \tag{2.4}$$

Now

$$\inf_{x\in z+V}\xi(x) = \xi(z) - \sup_{y\in V}\xi(y), \tag{2.5}$$

and by Proposition 2.1,

$$\int_{z+V}\exp\{\xi - \phi(h)\}\,d\mu = \mu(g - h + z + V)\exp\left\{\frac{1}{2}\|g - h\|_\mu^2\right\}. \tag{2.6}$$

By a well-known property of Gaussian measures (see e.g. [2]), $\mu(x + V) \le \mu(V)$ for all $x \in E$; hence (2.4)–(2.6) imply the first inequality.

In order to prove the second inequality we observe first that by Proposition 2.1 and Jensen's inequality,

$$\begin{aligned} \mu(h + A \cap V) &= \exp\left\{-\frac{1}{2}\|h\|_\mu^2\right\}\int_{A\cap V}\exp\{-\phi(h)\}\,d\mu \\ &\ge \exp\left\{-\frac{1}{2}\|h\|_\mu^2\right\}\mu(A\cap V)\exp\left\{-(\mu(A\cap V))^{-1}\int_{A\cap V}\phi(h)\,d\mu\right\}. \end{aligned} \tag{2.7}$$

Putting $(-h)$ instead of h in (2.7) and multiplying the two inequalities, we obtain

$$\mu(h + A\cap V)\mu(-h + A\cap V) \ge \exp\{-\|h\|_\mu^2\}(\mu(A\cap V))^2. \tag{2.8}$$

By (1)

$$\mu(-h + A\cap V) \le \mu(-h + V) \le \mu(V)\exp\{-(½)(\|h\|_\mu^2 - \|h - g\|_\mu^2) + \sup_{x\in V}\xi(x)\}. \tag{2.9}$$

Combining (2.8) and (2.9) we get (2). □

Let B be a finite dimensional Banach space with norm p and let γ be a centered Gaussian measure on B. Let $\{W(t) : t \ge 0\}$ be a B-valued γ-Brownian motion; that is, $\{W(t) : t \ge 0\}$ is a B-valued stochastic process with stationary independent increments, $W(0) = 0$ a.s., W has continuous paths and $\mathscr{L}(W(1)) = \gamma$. Writing $I = [0, 1]$ and $W = \{W(t) : t \in I\}$, we define $\mu_\gamma = \mathscr{L}(W)$; this is the *Wiener measure on $C(I, B)$ associated with γ-Brownian motion.* Let us remark that since B is finite dimensional, the topological support $S(\gamma)$ of γ coincides with H_γ; in particular, $\gamma(H_\gamma) = 1$. This implies that $P\{W(t) \in H_\gamma \text{ for all } t \ge 0\} = 1$, and therefore $\mu_\gamma(C(I, H_\gamma)) = 1$. When no confusion may arise we shall write μ instead of μ_γ.

We will give next a description of H_μ which will be useful later on. This description is somewhat more concrete than that given in Kuelbs and Le Page [15]. We shall denote by $\mathscr{M}(I, F)$ the Banach space of vector-valued measures defined on the Borel σ-algebra of I, taking values in the finite dimensional Banach space F, endowed with the total variation norm $\|\cdot\|_v$. Let the space $C(I, B)$ be endowed with the norm $\|f\|_\infty = \sup_{t\in I}p(f(t))$. On B' we put the norm dual to p. Then the dual space of $C(I, B)$ may be described by the following well-known (see [19], page 193) proposition.

PROPOSITION 2.3. *For each $\xi \in \mathscr{M}(I, B')$ define $\tilde\xi \in (C(I, B))'$ by*

$$\tilde\xi(f) = \int f\,d\xi, \qquad (f \in C(I, B)).$$

Then the map $\xi \to \tilde{\xi}$ *is an isometry of* $\mathscr{M}(I, B')$ *onto* $(C(I, B))'$.

We shall denote by Δ_γ, $\hat{\Delta}_\gamma$ (resp., Δ_μ, $\hat{\Delta}_\mu$) the Δ, $\hat{\Delta}$ maps associated with the Gaussian measure γ (resp., μ) earlier in this section. We shall omit the proof of the next result; the arguments are similar to those of the one-dimensional case, which are well-known.

PROPOSITION 2.4. *Let B be a finite dimensional Banach space, γ a centered Gaussian measure on B. Let μ be the Wiener measure on $C(I, B)$ associated to γ-Brownian motion. Then*

(1) $H_\mu = \{f \in C(I, B) : f(0) = 0,\ f \text{ is absolutely continuous and } f' \in L^2(I, H_\gamma)\}$,

$$\langle f, g\rangle_\mu = \int_I \langle f'(t), g'(t)\rangle_\gamma \, dt \quad \text{for} \quad f, g \in H_\mu.$$

(2) *For all* $\xi \in \mathscr{M}(I, B')$,

$$(\hat{\Delta}_\mu \xi)(t) = \int_0^t \hat{\Delta}_\gamma(\xi[s, 1]) \, ds.$$

REMARK. Although we will not need this fact, it may be of interest to point out that one may show:

$$\hat{\Delta}_\mu(\mathscr{M}(I, B')) = \{f \in H_\mu : f' \text{ is of bounded variation}\}.$$

The following technical lemma will be useful in Section 4.

LEMMA 2.5. *Let* $0 \le \alpha < \beta < 1$.
(1) *For* $f \in H_\mu$, *define*

$$f_{\alpha,\beta}(t) = f(\alpha + (\beta - \alpha)t) - f(\alpha), \quad (t \in I).$$

Then $f_{\alpha,\beta} \in H_\mu$ *and* $\|f_{\alpha,\beta}\|_\mu^2 = (\beta - \alpha) \int_\alpha^\beta \|f'(t)\|_\gamma^2 \, dt$.
(2) *For* $\xi \in \mathscr{M}(I, B')$, *define*

$$\xi_{\alpha,\beta}(A) = (\beta - \alpha)\{\xi(\alpha + (\beta - \alpha)A) + \xi(\beta, 1]\delta_1(A)\}, \quad A \text{ Borel in } I.$$

Then $\xi_{\alpha,\beta} \in \mathscr{M}(I, B')$, $\|\xi_{\alpha,\beta}\|_v \le 2(\beta - \alpha)\|\xi\|_v$ *and* $\hat{\Delta}_\mu(\xi_{\alpha,\beta}) = (\hat{\Delta}_\mu(\xi))_{\alpha,\beta}$.

PROOF. (1) follows by a routine check. The first two statements of (2) follow at once from the definitions. For the proof of the second statement, we have from Proposition 2.4(2)

$$\begin{aligned}
(\hat{\Delta}_\mu(\xi_{\alpha,\beta}))(t) &= \int_0^t \hat{\Delta}_\gamma(\xi_{\alpha,\beta}([s, 1])) \, ds \\
&= (\beta - \alpha) \int_0^t \hat{\Delta}_\gamma\{\xi([\alpha + (\beta - \alpha)s, \beta]) + \xi(\beta, 1]\} \, ds \\
&= (\beta - \alpha) \int_0^t \hat{\Delta}_\gamma(\xi([\alpha + (\beta - \alpha)s, 1])) \, ds \\
&= \int_\alpha^{\alpha + (\beta - \alpha)t} \hat{\Delta}_\gamma(\xi([u, 1])) \, du \\
&= (\hat{\Delta}_\mu \xi)(\alpha + (\beta - \alpha)t) - (\hat{\Delta}_\mu \xi)(\alpha) = (\hat{\Delta}_\mu \xi)_{\alpha,\beta}(t). \square
\end{aligned}$$

We close the section by stating for ready reference an immediate corollary of Propositions 2.2 and 2.4.

COROLLARY 2.6. *Let μ be the Wiener measure on $C(I, B)$ associated to γ-Brownian motion. Let $V = \{\varphi \in C(I, B) : \|\varphi\|_\infty \leq 1\}$, $r > 0$. Then*
(1) *For all $h \in H_\mu$, $\varphi \in rV$, $\xi \in \mathscr{M}(I, B')$,*

$$\mu(h + \varphi + rV) \leq \mu(rV)\exp\{-(½)(\|h\|_\mu^2 - \|h - g\|_\mu^2) + 2r\|\xi\|_v\},$$

where $g = \hat{\Delta}_\mu \xi$.
(2) *For all Borel sets A in $C(I, B)$, $h \in H_\mu$, $\xi \in \mathscr{M}(I, B')$,*

$$\mu(rV)\mu(h + A \cap (rV)) \geq (\mu(A \cap (rV)))^2\exp\{-(½)(\|h\|_\mu^2 + \|h - g\|_\mu^2) - r\|\xi\|_v\},$$

where $g = \hat{\Delta}_\mu \xi$.

3. Some limit theorems for vector-valued Brownian motion. The constant $c_{\gamma,p}$ given by the following result will play an important role in Sections 4 and 5.

THEOREM 3.1. *Let B be a finite dimensional Banach space with norm p, γ a centered Gaussian measure on B. Let W be defined as in Section 2. Then*

$$c_{\gamma,p} = -\lim_{\rho\to 0}\rho^2\log P\{\|W\|_\infty \leq \rho\}$$

exists and $0 < c_{\gamma,p} < \infty$.

PROOF. We first show: for all $r > 0$, all $k \in \mathbb{N}$,

$$P\{\|W\|_\infty \leq r\} \leq (P\{\|W\|_\infty \leq k^{1/2}r\})^k. \tag{3.1}$$

For $x \in B$, let P_x be the probability measure on $C(I, B)$ defined by $P_x = \mathscr{L}(x + W)$; in particular, $P_0 = \mu$. By the Markov property of Brownian motion, for all $r > 0$, $k \in \mathbb{N}$,

$$\begin{aligned} P\{\|W\|_\infty \leq r\} &= P\{\sup_{0\leq t\leq (k-1)k^{-1}} p(W(t)) \leq r, \sup_{(k-1)k^{-1}\leq t\leq 1} p(W(t)) \leq r\} \\ &= \int_A P_{Y(\omega)}(E)P(d\omega), \end{aligned}$$

where $Y = W((k-1)k^{-1})$, $A = \{\sup_{0\leq t\leq (k-1)k^{-1}} p(W(t)) \leq r\}$ and $E = \{f \in C(I, B) : \sup_{0\leq t\leq k^{-1}} p(f(t)) \leq r\}$. It follows that

$$P\{\|W\|_\infty < r\} \leq P(A) \sup_{p(x)\leq r} P_x(E).$$

By a well-known property of Gaussian measures (see e.g. [2]), applied to μ, we have

$$\sup_{p(x)\leq r} P_x(E) = \mu(E) = P\{\sup_{0\leq t\leq k^{-1}} p(W(t)) \leq r\},$$

and therefore

$$P\{\|W\|_\infty \leq r\} \leq P(A)P\{\sup_{0\leq t\leq k^{-1}} p(W(t)) \leq r\}.$$

Iterating the same procedure, we obtain: for all $r > 0$, $k \in \mathbb{N}$,

$$P\{\|W\|_\infty \leq r\} \leq (P\{\sup_{0\leq t\leq k^{-1}} p(W(t)) \leq r\})^k.$$

Now (3.1) follows upon transforming the right hand side by the scaling property of Brownian motion.

Let

$$L = \limsup_{\rho\to 0}\rho^2\log P\{\|W\|_\infty \leq \rho\},$$

$$\ell = \liminf_{\rho\to 0}\rho^2\log P\{\|W\|_\infty \leq \rho\}.$$

We prove now: $L = \ell$. In fact, let $\{r_n\}$, $\{s_n\}$ be two positive sequences such that $r_n \to 0$, $s_n \to 0$, $r_n s_n^{-1} \to 0$ and

$$\lim_n r_n^2 \log P\{\| W\|_\infty \le r_n\} = L, \quad \lim_n s_n^2 \log P\{\| W\|_\infty \le s_n\} = \ell.$$

Then by (3.1),

$$P\{\| W\|_\infty \le r_n\} \le (P\{\| W\|_\infty \le [s_n r_n^{-1}] r_n\})^{[s_n r_n^{-1}]^2} \le (P\{\| W\|_\infty \le s_n\})^{[s_n r_n^{-1}]^2},$$

$$r_n^2 \log P\{\| W\|_\infty \le r_n\} \le s_n^2 \log P\{\| W\|_\infty \le s_n\}((r_n s_n^{-1})^2 [s_n r_n^{-1}]^2),$$

implying $L \le \ell$ and consequently $L = \ell$. Let $-c_{\gamma,p}$ be the value of the limit.

If $(k+1)^{-1} \le r < k^{-1}$, we have by (3.1)

$$\begin{aligned} r^2 \log P\{\| W\|_\infty \le r\} &\le r^2 \log P\{\| W\|_\infty \le k^{-1}\} \le r^2 \log (P\{\| W\|_\infty \le 1\})^{k^2} \\ &= r^2 k^2 \log P\{\| W\|_\infty \le 1\} \le (k+1)^{-2} k^2 \log P\{\| W\|_\infty \le 1\}. \end{aligned}$$

It follows that $-c_{\gamma,p} \le \log P\{\| W\|_\infty \le 1\} < 0$.

It remains to show: $c_{\gamma,p} < \infty$. By the Markov property of Brownian motion, for all $r > 0$, $k \in \mathbb{N}$, $\varepsilon > 0$,

$$\begin{aligned} P\{\| W\|_\infty \le rk^{-1/2}\} &\ge P\{\| W\|_\infty \le rk^{-1/2}, p(W(1)) \le \varepsilon k^{-1/2}\} \\ &\ge P\{\sup_{0\le t\le (k-1)k^{-1}} p(W(t)) \le rk^{-1/2}, p(Y) \le \varepsilon k^{-1/2}, \\ &\qquad \sup_{(k-1)k^{-1}\le t\le 1} p(W(t)) \le rk^{-1/2}, p(W(1)) \le \varepsilon k^{-1/2}\} \\ &= \int_A P_{Y(\omega)}(E) P(d\omega), \end{aligned} \tag{3.2}$$

where $Y = W((k-1)k^{-1})$,

$$A = \{\sup_{0\le t\le (k-1)k^{-1}} p(W(t)) \le rk^{-1/2}, p(Y) \le \varepsilon k^{-1/2}\},$$

$$E = \{f \in C(I, B) \colon \sup_{0\le t\le k^{-1}} p(f(t)) \le rk^{-1/2}, p(f(k^{-1})) \le \varepsilon k^{-1/2}\}.$$

Now for any $\varepsilon > 0$, by an argument in [21] it is possible to choose $r > 0$ so that

$$\delta = \inf_{x\in S(\gamma), p(x)\le\varepsilon} P_x\{\| W\|_\infty \le r, p(W(1)) \le \varepsilon\} > 0, \tag{3.3}$$

where $S(\gamma)$ is the topological support of γ. In fact, by compactness

$$\eta = \inf_{x\in S(\gamma), p(x)\le\varepsilon} P_x\{p(W(1)) \le \varepsilon\} > 0.$$

Choose now $r > 0$ so that $P\{\| W\|_\infty > r - \varepsilon\} < \eta/2$. Then

$$P_x\{\| W\|_\infty \le r, p(W(1)) \le \varepsilon\} \ge P_x\{p(W(1)) \le \varepsilon\} - P\{\| W\|_\infty > r - \varepsilon\} \ge \eta/2$$

for all $x \in S(\gamma)$, $p(x) \le \varepsilon$, proving (3.3).

Next, for $p(x) \le \varepsilon k^{-1/2}$, by the scaling property of Brownian motion,

$$\begin{aligned} &P\{\sup_{0\le t\le k^{-1}} p(x + W(t)) \le rk^{-1/2}, p(x + W(k^{-1})) \le \varepsilon k^{-1/2}\} \\ &\qquad = P\{\| k^{1/2} x + W\|_\infty \le r, p(k^{1/2} x + W(1)) \le \varepsilon\} \ge \delta. \end{aligned} \tag{3.4}$$

Now (3.2) and (3.4) give

$$P\{\| W\|_\infty \le rk^{-1/2}\} \ge \delta P(A).$$

Iterating the same procedure, we get: for all $k \in \mathbb{N}$,

$$P\{\| W\|_\infty \le rk^{-1/2}\} \ge \delta^k. \tag{3.5}$$

If $rk^{-1} \le \rho < r(k-1)^{-1}$, then by (3.5)

$$\rho^2 \log P\{\|W\|_\infty \le \rho\} \ge \rho^2 \log P\{\|W\|_\infty \le rk^{-1}\} \ge \rho^2 k^2 \log \delta \ge r^2(k-1)^{-2}k^2 \log \delta.$$

It follows that $-c_{\gamma,p} \ge r^2 \log \delta > -\infty$. □

REMARKS. (1) It is well-known (see e.g. [5]) that if $B = \mathbb{R}^1$, $\gamma = N(0, 1)$, then $c_{\gamma,p} = \pi^2/8$. In fact, in this case there are sharp inequalities yielding

$$P\{\|W\|_\infty \le r\} \sim (4/\pi)\exp\{-(\pi^2/8)r^{-2}\} \quad \text{as } r \to 0,$$

which is a more precise result than Theorem 3.1 applied to this specific case.

(2) If $B = \mathbb{R}^k$, $p = \|\cdot\|_\infty$ on $\mathbb{R}^k$, γ is the canonical Gaussian measure, then it follows easily from the previous remark that $c_{\gamma,p} = k(\pi^2/8)$.

(3) In connection with the exact value of $c_{\gamma,p}$ when $B = \mathbb{R}^k$, p is the standard Euclidean norm and γ is the canonical Gaussian measure, see [6].

We shall need the following result in Section 4. The notation is as in Theorem 3.1.

LEMMA 3.2. *For every* $\alpha > 0$, $0 < \varepsilon < \alpha/2$, $\delta > 0$,

$$\liminf_{\rho\to 0} \rho^2 \log(\inf_{x\in S(\gamma), p(x)\le\varepsilon\rho} P\{\|W\|_\infty \le \alpha\rho, p(x + W(1)) \le \delta\rho\}) \ge -c_{\gamma,p}(\alpha - 2\varepsilon)^{-2}.$$

PROOF. Let us first prove the following fact: if $\beta + 2\varepsilon < \alpha$, then for any $\delta > 0$

$$\sigma = \inf_{z,w\in S(\gamma), p(z)\le\beta, p(w)\le\varepsilon} P\{\|z + W\|_\infty \le \alpha, p(z + w + W(1)) \le \delta\} > 0. \tag{3.6}$$

In fact, suppose $p(z) \le \beta$, $p(w) \le \varepsilon$, $z, w \in S(\gamma)$. Then by [18], page 46, proof of Proposition 7.1 applied to $f(t) = z + w(t \in I)$, we have

$$P\{\|z + w + W\|_\infty < \alpha - \varepsilon, p(z + w + W(1)) < \delta\} > 0,$$

$$P\{\|z + W\|_\infty \le \alpha, p(z + w + W(1)) \le \delta\} > 0;$$

the local compactness of B and the continuity of the map

$$(z, w) \to P\{\|z + W\|_\infty \le \alpha, p(z + w + W(1)) \le \delta\}$$

imply now (3.6).

By the Markov property, for $0 < \eta < 1$

$$\begin{aligned} g(x, \rho) &= P\{\|W\|_\infty \le \alpha\rho, p(x + W(1)) \le \delta\rho\} \\ &= P\{\sup_{0\le t\le 1-\eta} p(W(t)) \le \alpha\rho, \sup_{1-\eta\le t\le 1} p(W(t)) \le \alpha\rho, p(x + W(1)) \le \delta\rho\} \\ &= \int_A P_{Y(\omega)}(E_x) P(d\omega), \end{aligned} \tag{3.7}$$

where $Y = W(1-\eta)$, $A = \{\sup_{0\le t\le 1-\eta} p(W(t)) \le \alpha\rho\}$,

$$E_x = \{f \in C(I, B): \sup_{0\le t\le\eta} p(f(t)) \le \alpha\rho, p(x + f(\eta)) \le \delta\rho\}.$$

Put $\eta = \rho^2$ and let $\beta < \alpha - 2\varepsilon$. If $y \in S(\gamma)$, $p(y) \le \beta\rho$, then by the scaling property of Brownian motion

$$\begin{aligned} P_y(E_x) &= P\{\sup_{0\le t\le\rho^2} p(y + W(t)) \le \alpha\rho, p(y + x + W(\rho^2)) \le \delta\rho\} \\ &= P\{\|\rho^{-1}y + W\|_\infty \le \alpha, p(\rho^{-1}(y + x) + W(1)) \le \delta\}. \end{aligned}$$

It follows that if $p(y) \le \beta\rho$, $p(x) \le \varepsilon\rho$, $x, y \in S(\gamma)$ then $P_y(E_x) \ge \sigma$. From (3.7) we have: if $p(x) \le \varepsilon\rho$, $x \in S(\gamma)$, then

$$g(x, \rho) \geq \int_{A \cap \{p(Y) \leq \beta\rho\}} P_{Y(\omega)}(E_x) P(d\omega) \geq \sigma P(A \cap \{p(Y) \leq \beta\rho\})$$

$$\geq \sigma P\{\sup_{0 \leq t \leq 1-\rho^2} p(W(t)) \leq \beta\rho\} = \sigma P\{\| W\|_\infty \leq \beta\rho(1-\rho^2)^{-1/2}\}.$$

Therefore, by Theorem 3.1,

$$\liminf_{\rho\to 0}\rho^2 \log\{\inf_{x \in S(\gamma), p(x) \leq \varepsilon\rho} g(x, \rho)\}$$

$$\geq \liminf_{\rho\to 0}\rho^2 \log \sigma + \liminf_{\rho\to 0}\rho^2 \log P\{\| W\|_\infty \leq \beta\rho(1-\rho^2)^{-1/2}\}$$

$$= -c_{\gamma,p}\beta^{-2}.$$

Since β is any number smaller than $\alpha - 2\varepsilon$, the result follows. □

The next result is a simple consequence of Theorem 3.1 and Corollary 2.6. A closely related result has been proved by Borell ([3], Theorem 2.3).

THEOREM 3.3. *Let W be as in Section 2 and let μ be the associated Wiener measure on $C(I, B)$. Then for every $r > 0$, $f \in H_\mu$,*

$$\lim_{\lambda\to\infty}\lambda^{-2}\log P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} = -c_{\gamma,p} r^{-2} - (1/2)\| f\|_\mu^2.$$

PROOF. By Corollary 2.6 (1) with $z = 0$,

$$P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} = \mu(\lambda f + \lambda^{-1} r V)$$

$$\leq \mu(\lambda^{-1} r V)\exp\{-(1/2)(\|\lambda f\|_\mu^2 - \|\lambda f - \lambda g\|_\mu^2) + 2\lambda^{-1} r \|\lambda\xi\|_v\}$$

and therefore by Theorem 3.1

$$\limsup_{\lambda\to\infty}\lambda^{-2}\log P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} \leq -c_{\gamma,p} r^{-2} - (1/2)\| f\|_\mu^2 + (1/2)\| f - g\|_\mu^2.$$

Since $g = \hat{\Delta}_\mu(\xi)$ can be chosen arbitrarily $\|\cdot\|_\mu$-close to f, we get

$$\limsup_{\lambda\to\infty}\lambda^{-2}\log P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} \leq -c_{\gamma,p} r^{-2} - (1/2)\| f\|_\mu^2.$$

In order to get an inequality in the opposite direction we use (2.7) with $A = \lambda^{-1} r V$, taking the convex symmetric measurable set in (2.7) to be $\lambda^{-1} r V$ (V as in Corollary 2.6). Since $\int_{\lambda^{-1} r V}\phi(f)\, d\mu = 0$, we have

$$P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} = \mu(\lambda f + \lambda^{-1} r V)$$

$$\geq \exp\{-(1/2)\|\lambda f\|_\mu^2\}\mu(\lambda^{-1} r V).$$

Theorem 3.1 gives now

$$\liminf_{\lambda\to\infty}\lambda^{-2}\log P\{\| W - \lambda f\|_\infty \leq \lambda^{-1} r\} \geq - c_{\gamma,p} r^{-2} - (1/2)\| f\|_\mu^2. \square$$

The following slightly more general form of Theorem 3.3 will be useful in Section 6. In order to avoid repetition, we omit the proof, which follows from Theorem 3.3 in the same manner as that of Theorem 4.7 follows from Theorems 4.3 and 4.5.

THEOREM 3.4. *In the set-up of Theorem 3.3, let $0 < \lambda_n \to \infty$ and let $\{g_n\} \subset C(I, B)$ be such that $\lambda_n^2 \| f - g_n\|_\infty \to 0$ as $n \to \infty$. Let $\alpha_n \sim \lambda_n$, $\beta_n \sim \lambda_n$. Then*

$$\lim_{n\to\infty}\lambda_n^{-2}\log P\{\| W - \alpha_n g_n\|_\infty \leq \beta_n^{-1} r\} = -c_{\gamma,p} r^{-2} - (1/2)\| f\|_\mu^2.$$

4. The small deviation theorems. Let γ be a centered Gaussian measure on the finite dimensional Banach space B. Let $\{X_j\}$ be a sequence of independent, identically distributed mean zero B-valued random vectors, $S_k = \sum_{j=1}^{k} X_j\,(S_0 \equiv 0)$. Assume that $E\varphi(X_1)\psi(X_1) = \int \varphi\psi\, d\gamma$ for all $\varphi, \psi \in B'$; then $\mathscr{L}(X_1)$ belongs to the domain of normal attraction of γ and the functional central limit theorem—Donsker's invariance principle—holds (see e.g. [1], [10]; also [13]): if Z_n is the $C(I, B)$-valued random vector defined by setting

$$(4.1)\qquad Z_n(t) = \begin{cases} S_k \quad \text{if} \quad t = k/n \quad (k = 0, \cdots, n) \\ \text{extended by linear interpolation in each interval} \\ \qquad [(k-1)n^{-1}, kn^{-1}], \end{cases}$$

then

$$(4.2)\qquad \mathscr{L}(n^{-1/2}Z_n) \to_w \mu,$$

where μ is the Wiener measure on $C(I, B)$ associated to γ-Brownian motion (see Section 2) and $\to_w$ stands for weak convergence on the space of probability measures on $C(I, B)$.

Our aim in this section is to find asymptotic estimates for probabilities of the form

$$(4.3)\qquad P\{\|n^{-1/2}Z_n - a_n f\|_\infty \le a_n^{-1} r\}$$

where $\{a_n\}$ is any real sequence satisfying $0 < a_n \to \infty$, $n^{-1}a_n^2 \to 0$, $r > 0$, $f \in H_\mu$. Our method of proof exploits the independent increment property of the process $\{S_n : n \in \mathbb{N}\}$; in the one-dimensional case, this idea appears in [11] and [16]. The situation we deal with here, involving a function f in H_μ, is, however, considerably more complicated.

LEMMA 4.1. *Let W be as in Section 2, Z_n as in* (4.1). *Suppose K is a compact subset of $S(\mu)$ and $0 < \rho_n \to \rho$ with $0 < \rho < \infty$.*

(1) *For every $\delta > 0$, $\eta > 0$, there exists $n_0 = n_0(\delta, \eta)$ such that $n \ge n_0$ implies*:

$$P\{\|y_n + n^{-1/2}Z_n - g\|_\infty \le \rho_n + \eta\} \le (1+\delta)P\{\|y_n + W - g\|_\infty \le \rho_n + \eta\}$$

for all $g \in K$, all $\{y_n\} \subset B$ such that $p(y_n) \le \rho_n (n \in \mathbb{N})$.

(2) *Assume $0 < r_n \to r$, $0 < r < \infty$, $\rho \ge 2r$. Then for every $\delta > 0$, there exists $n_1 = n_1(\delta)$ such that $n \ge n_1$ implies*:

$$P\{\|n^{-1/2}Z_n - g\|_\infty \le \rho_n, p(x_n + n^{-1/2}Z_n(1) - g(1)) \le r_n\}$$
$$\ge (1-\delta)P\{\|W - g\|_\infty \le \rho_n, p(x_n + W(1) - g(1)) \le r_n\}$$

for all $g \in K$, all $\{x_n\} \subset B$ such that $p(x_n) \le r_n$, $x_n \in S(\gamma)(n \in \mathbb{N})$.

PROOF. (1) Suppose the statement is false. Then there exist $\delta > 0$, $\eta > 0$ and sequences $0 < n_k \to \infty$, $\{g_k\} \subset K$, $\{y_k\} \subset B$ with $p(y_k) \le \rho_{n_k}$ such that for all $k \in \mathbb{N}$

$$(4.4)\qquad P\{\|y_k + n_k^{-1/2}Z_{n_k} - g_k\|_\infty \le \rho_{n_k} + \eta\} > (1+\delta)P\{\|y_k + W - g_k\|_\infty \le \rho_{n_k} + \eta\}.$$

By compactness we have by passing to an appropriate subsequence (which we denote like the initial sequence) $g_k \to g \in K$, $y_k \to y$ with $p(y) \le \rho$. Then it easily follows from (4.2) and elementary properties of W that

$$(4.5)\qquad \begin{aligned} &P\{\|y_k + n_k^{-1/2}Z_{n_k} - g_k\|_\infty \le \rho_{n_k} + \eta\} \to P\{\|y + W - g\|_\infty \le \rho + \eta\}, \\ &P\{\|y_k + W - g_k\|_\infty \le \rho_{n_k} + \eta\} \to P\{\|y + W - g\|_\infty \le \rho + \eta\}. \end{aligned}$$

But $P\{\|y + W - g\|_\infty \le \rho + \eta\} \ge P\{\|W - g\|_\infty \le \eta\} > 0$ because $g \in S(\mu)$ and thus (4.5) contradicts (4.4).

The proof of (2) is similar. We will just point out that for $p(x) \le r$, $x \in S(\gamma)$, $g \in S(\mu)$, if we define $\varphi_x(t) = tx$ $(t \in I)$, then

$$P\{\|W - g\|_\infty \le \rho, p(x + W(1) - g(1)) \le r\} \ge P\{\|W + \varphi_x - g\|_\infty \le r\} > 0$$

because $g - \varphi_x \in S(\mu)$. □

We first obtain an asymptotic upper bound for the probabilities (4.3).

LEMMA 4.2. *Let $n \in \mathbb{N}$ and let $\{t_k: k = 0, \cdots, m\}$ be a subdivision of I such that $nt_k \in \mathbb{N}$ for all k. Let $f \in C(I, B)$ and for $k = 1, \cdots, m$, let*

$$f_k(u) = f(t_{k-1} + u(t_k - t_{k-1})) - f(t_{k-1}), \quad (u \in I).$$

Then for all $\alpha > 0$,

$$P\{\|Z_n - f\|_\infty \le \alpha\} < \textstyle\prod_{k=1}^m \sup_{p(x_k)\le\alpha} P\{\|x_k + Z_{n(t_k - t_{k-1})} - f_k\|_\infty \le \alpha\}.$$

PROOF. Let $Z'_n = Z_n - f$, $A = \{\sup_{0\le t\le t_{m-1}} p(Z'_n(t)) \le \alpha\}$,

$$E = \{g \in C([0, t_{m-1}], B): \sup_{0\le t\le t_{m-1}} p(g(t)) \le \alpha\},$$

$$F = \{(g, h): g \in C([0, t_{m-1}], B), h \in C([t_{m-1}, t_m], B),$$
$$\sup_{t_{m-1}\le t\le t_m} p(g(t_{m-1}) + h(t)) \le \alpha\},$$

$$V_n(t) = Z'_n(t) - Z'_n(t_{m-1})(t \in [t_{m-1}, t_m]).$$

We have

$$P\{\|Z_n - f\|_\infty \le \alpha\} = P(A \cap \{\sup_{t_{m-1}\le t\le t_m} p(Z'_n(t_{m-1}) + V_n(t)) \le \alpha\})$$
$$= P\{Z'_n | [0, t_{m-1}] \in E, (Z'_n | [0, t_{m-1}], V_n) \in F\}.$$

Since $\{Z'_n(t): 0 \le t \le t_{m-1}\}$ is independent of $\{V_n(t): t_{m-1} \le t \le t_m\}$, we have by Fubini's theorem

$$P\{\|Z_n - f\|_\infty \le \alpha\}$$

(4.6)
$$= \int_E P\{\sup_{t_{m-1}\le t\le t_m} p(g(t_{m-1}) + V_n(t)) \le \alpha\}\, d\mathscr{L}(Z'_n | [0, t_{m-1}])(g).$$

For $x \in B$,

$$\sup_{t_{m-1}\le t\le t_m} p(x + V_n(t)) = \sup_{0\le u\le 1} p(x + Z'_n(t_{m-1} + u(t_m - t_{m-1})) - Z'_n(t_{m-1}))$$
$$= \sup_{0\le u\le 1} p(x + Z_n(t_{m-1} + u(t_m - t_{m-1})) - Z_n(t_{m-1}) - f_m(u)).$$

Observing that

$$\mathscr{L}(\{Z_n(t_{m-1} + u(t_m - t_{m-1})) - Z_n(t_{m-1}): 0 \le u \le 1\} = \mathscr{L}(\{Z_{n(t_m - t_{m-1})}(u): u \in I\}),$$

we have

$$P\{\sup_{t_{m-1}\le t\le t_m} p(x + V_n(t)) \le \alpha\} = P\{\|x + Z_{n(t_m - t_{m-1})} - f_m\|_\infty \le \alpha\},$$

and therefore we have from (4.6)

$$P\{\|Z_n - f\|_\infty \le \alpha\} \le P(A) \sup_{p(x)\le\alpha} P\{\|x + Z_{n(t_m - t_{m-1})} - f_m\|_\infty \le \alpha\}.$$

Iterating the same procedure gives the assertion. □

THEOREM 4.3. *Let Z_n be as in (4.1), μ as in (4.2), $c_{\gamma,p}$ as in Theorem 3.1. Let $0 < a_n \to \infty$, $n^{-1}a_n^2 \to 0$. Then for all $f \in H_\mu$, $r > 0$,*

$$\limsup_{n\to\infty} a_n^{-2} \log P\{\|n^{-1/2} Z_n - a_n f\|_\infty \le a_n^{-1} r\} \le -c_{\gamma,p} r^{-2} - (1/2)\|f\|_\mu^2.$$

PROOF. Let ρ be a fixed positive number. For $n \in \mathbb{N}$, $k = 0, \cdots, [\rho^2 a_n^2]$, let $t_{n,k} = kn^{-1}[n/\rho^2 a_n^2]$. If $t_{n,[\rho^2 a_n^2]} = 1$, put $b_n = [\rho^2 a_n^2]$; otherwise, put $t_{n,k} = 1$ for $k = [\rho^2 a_n^2] + 1$ and $b_n = [\rho^2 a_n^2] + 1$. Then $\{t_{n,k}: k = 0, \cdots, b_n\}$ is a subdivision of I such that $nt_{n,k} \in \mathbb{N}$ for all k.

Let $d\nu(t) = \|f'(t)\|_\gamma^2 \, dt$. For $M > 0$, let

$$G_n(M) = \{k: k = 1, \cdots, b_n - 1 \quad \text{and} \quad b_n\nu([t_{n,k-1}, t_{n,k}]) > M\}.$$

Then (i) card $G_n(M) \le b_n \|\nu\| M^{-1}$,

(ii) $\lim_{M\to\infty} \sup_n \sum_{k\in G_n(M)} \nu([t_{n,k-1}, t_{n,k}]) = 0$.

In fact,

$$\|\nu\| \ge \textstyle\sum_{k\in G_n(M)} \nu([t_{n,k-1}, t_{n,k}]) \ge M\, b_n^{-1} \text{ card } G_n(M),$$

proving (i). In order to prove (ii), let us observe that for $k < b_n$,

$$t_{n,k} - t_{n,k-1} = q_n n^{-1}$$

where $q_n = [n/\rho^2 a_n^2]$. Given $\varepsilon > 0$, let τ be such that

$$\int_{\{t:\|f'(t)\|_\gamma > \tau\}} \|f'(t)\|_\gamma^2 \, dt < \varepsilon/2.$$

Now choose M_0 so that $\tau^2 \|\nu\| (\sup_n b_n q_n n^{-1}) M_0^{-1} < \varepsilon/2$. Then for $M \ge M_0$ and all $n \in N$, putting $I_k = [t_{n,k-1}, t_{n,k}]$, we have

$$\begin{aligned}\textstyle\sum_{k\in G_n(M)} \int_{I_k} \|f'(t)\|_\gamma^2 \, dt &= \textstyle\sum_{k\in G_n(M)} \int_{I_k \cap \{t:\|f'(t)\|_\gamma \le \tau\}} + \sum_{k\in G_n(M)} \int_{I_k\cap\{t:\|f'(t)\|_\gamma > \tau\}} \\ &\le \tau^2 (q_n n^{-1}) \text{ card } G_n(M) + (\varepsilon/2) \\ &\le \tau^2 (q_n n^{-1}) b_n \|\nu\| M^{-1} + (\varepsilon/2) \\ &< (\varepsilon/2) + (\varepsilon/2) = \varepsilon.\end{aligned}$$

This proves (ii).

For a given $\varepsilon > 0$, choose now M so that card $G_n(M) \le b_n\varepsilon$ and $\sum_{k\in G_n(M)} \nu(I_k) < \varepsilon/2$ for all $n \in N$. Let

$$F_n = ([1, b_n - 1] \cap N) \backslash G_n(M).$$

Next we apply Lemma 4.2 with $\alpha = a_n^{-1} n^{1/2} r$ and $(a_n n^{1/2}) f$ instead of f. Putting $c_n = a_n n^{1/2} q_n^{-1/2}$, $d_n = a_n^{-1} n^{1/2} q_n^{-1/2}$,

$$f_{n,k}(u) = f(t_{n,k-1} + u(t_{n,k} - t_{n,k-1})) - f(t_{n,k-1}), \quad (u \in I),$$

we have by Lemma 4.2

$$\begin{aligned} &P\{\|n^{-1/2} Z_n - a_n f\|_\infty \le a_n^{-1} r\} \\ &\qquad \le \textstyle\prod_{k\in F_n} \sup_{p(y_k)\le d_n r} P\{\|y_k + q_n^{-1/2} Z_{q_n} - c_n f_{n,k}\|_\infty \le d_n r\}. \end{aligned} \tag{4.7}$$

For $k \in F_n$, by Lemma 2.5 (1) $f_{n,k} \in H_\mu$ and

$$\|f_{n,k}\|_\mu^2 = (t_{n,k} - t_{n,k-1}) \int_{t_{n,k-1}}^{t_{n,k}} \|f'(t)\|_\gamma^2 \, dt = q_n n^{-1} \nu(I_k) \le q_n n^{-1} b_n^{-1} M,$$

and therefore

$$\|c_n f_{n,k}\|_\mu^2 \le (a_n^2 n q_n^{-1})(q_n n^{-1} b_n^{-1} M) = a_n^2 b_n^{-1} M,$$

showing that $\sup\{\|c_n f_{n,k}\|_\mu: n \in N, k \in F_n\} < \infty$ and consequently $\{c_n f_{n,k}: n \in N, k \in F_n\}$ is a relatively compact subset of $S(\mu)$ for the $\|\cdot\|_\infty$-metric on $C(I, B)$.

Since $q_n \to \infty$ and $d_n \to \rho$, for given $\delta > 0$, $\eta > 0$ there exists by Lemma 4.1 (1) a number $n_0 = n_0(\delta, \eta)$ such that $n \ge n_0$ implies

$$\begin{aligned} &\sup_{p(y)\le d_n r} P\{\|y + q_n^{-1/2} Z_{q_n} - c_n f_{n,k}\|_\infty < d_n r + \eta\} \\ &\qquad \le (1 + \delta) \sup_{p(y)\le d_n r} P\{\|y + W - c_n f_{n,k}\|_\infty \le d_n r + \eta\} \end{aligned} \tag{4.8}$$

for all $k \in F_n$.

By Corollary 2.6 (1), applied with $h = c_n f_{n,k}$, $\varphi(t) = y_k$ for $t \in I$, we have

$$P\{\|y_k + W - c_n f_{n,k}\|_\infty \leq d_n r + \eta\} = \mu(c_n f_{n,k} - \varphi + (d_n r + \eta)V)$$

$$\leq \mu((d_n r + \eta)V)\exp\{-(½)(\| c_n f_{n,k}\|_\mu^2 - \| c_n(g_{n,k} - f_{n,k})\|_\mu^2) + 2(d_n r + \eta)\| c_n \xi_{n,k}\|_v\}, \tag{4.9}$$

where $g = \hat{\Delta}_\mu(\xi)$ with $\xi \in \mathscr{M}(I, B')$,

$$g_{n,k}(u) = g(t_{n,k-1} + u(t_{n,k} - t_{n,k-1})) - g(t_{n,k-1})(u \in I),$$

$$\xi_{n,k}(A) = (t_{n,k} - t_{n,k-1})\{\xi(t_{n,k-1} + (t_{n,k} - t_{n,k-1})A) + \xi(t_{n,k}, 1]\delta_1(A)\}$$

($k = 1, \cdots, b_n$; A Borel in I); recall that $\hat{\Delta}_\mu(\xi_{n,k}) = g_{n,k}$ by Lemma 2.5 (2).

The next step is to find appropriate bounds for the terms in the exponent in the right hand side of (4.9). By the absolute continuity of ν, there exists $\beta > 0$ such that $\lambda(E) < \eta$ implies $\nu(E) < \varepsilon/2$, where λ is Lebesgue measure on I. Let n_1 be such that $n_1 \geq n_0$ and $q_n n^{-1} \leq \beta$ for $n \geq n_1$. Then for $n \geq n_1$

$$\begin{aligned} &\textstyle\sum_{k\in F_n} \| c_n f_{n,k}\|_\mu^2 = \sum_{k\in F_n} c_n^2 q_n n^{-1}\nu([t_{n,k-1}, t_{n,k}]) \\ &= a_n^2(\|\nu\| - \nu(I_{b_n}) - \textstyle\sum_{k\in G_n(M)} \nu(I_k)) \geq a_n^2(\|\nu\| - (\varepsilon/2) - (\varepsilon/2)) = a_n^2(\|f\|_\mu^2 - \varepsilon). \end{aligned} \tag{4.10}$$

Also,

$$\begin{aligned} \textstyle\sum_{k\in F_n} \| c_n(g_{n,k} - f_{n,k})\|_\mu^2 &= c_n^2 q_n n^{-1} \textstyle\sum_{k\in F_n} \displaystyle\int_{I_k} \|g' - f'\|_\gamma^2 \, d\lambda \\ &\leq a_n^2 \|g - f\|_\mu^2. \end{aligned} \tag{4.11}$$

By Lemma 2.5 (2),

$$\textstyle\sum_{k\in F_n} d_n c_n \|\xi_{n,k}\|_v \leq n q_n^{-1} \sum_{k=1}^{b_n} \|\xi_{n,k}\|_v \leq n q_n^{-1}(q_n n^{-1}) 2b_n \|\xi\|_v = 2b_n \|\xi\|_v, \tag{4.12}$$

and $\sum_{k\in F_n} c_n \|\xi_{n,k}\|_v \leq 2b_n d_n^{-1} \|\xi\|_v$.

Now by (4.7) – (4.12), we have for $n \geq n_1$

$$P\{\|n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\} \leq (1 + \delta)^{b_n}\{\mu((d_n r + \eta)V)\}^{b_n(1-\varepsilon)-1}$$

$$\cdot\exp\{-(½)a_n^2(\|f\|_\mu^2 - \varepsilon + \|g - f\|_\mu^2) + 4b_n(r + d_n^{-1}\eta)\|\xi\|_v\},$$

$$\begin{aligned} a_n^{-2}\log P\{\|n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\} &\leq a_n^{-2} b_n \log(1 + \delta) \\ &+ a_n^{-2}(b_n(1 - \varepsilon) - 1)\log \mu((d_n r + \eta)V) \\ &- (½)(\|f\|_\mu^2 - \varepsilon + \|g - f\|_\mu^2) \\ &+ 4(a_n^{-2} b_n)(r + d_n^{-1}\eta)\|\xi\|_v, \end{aligned}$$

and therefore, since $a_n^{-2} b_n \to \rho^2$, $d_n \to \rho$,

$$\begin{aligned} L &= \limsup_{n\to\infty} a_n^{-2}\log P\{\|n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\} \\ &\leq \rho^2\log(1 + \delta) + \rho^2(1 - \varepsilon)\log \mu((\rho r + \eta)V) \\ &\quad - (½)(\|f\|_\mu^2 - \varepsilon + \|g - f\|_\mu^2) + 4\rho^2(r + \rho^{-1}\eta)\|\xi\|_v. \end{aligned}$$

Since ε, δ, η are arbitrary, it follows that for all $\rho > 0$

$$L \leq \rho^2\log \mu(\rho r V) - (½)(\|f\|_\mu^2 - \|g - f\|_\mu^2) + 4r\rho^2 \|\xi\|_v. \tag{4.13}$$

By Theorem 3.1, letting $\rho \to 0$ in (4.13) we have

$$L \leq -c_{\gamma,\rho} r^{-2} - (½)(\|f\|_\mu^2 - \|g - f\|_\mu^2).$$

Finally, since $g = \hat{\Delta}_\mu(\xi)$ may be chosen arbitrarily $\|\cdot\|_\mu$-close to f, we conclude

$$L \leq -c_{\gamma,p} r^{-2} - (½)\|f\|_\mu^2. \square$$

We obtain now a lower bound for the probabilities (4.3).

LEMMA 4.4. *Let $n \in \mathbb{N}$ and let $\{t_k: k = 0, \cdots, m\}$ be a subdivision of $[0, a]$ with $a \geq 1$, such that $nt_k \in \mathbb{N}$ for all k. Let $f \in C([0, a], B)$ and for $k = 1, \cdots, m$, let*

$$f_k(u) = f(t_{k-1} + u(t_k - t_{k-1})) - f(t_{k-1}), \quad (u \in I).$$

Then for all $\alpha > 0$, all $0 < \varepsilon < 1$,

$$P\{\sup_{t\in[0,a]} p(\tilde{Z}_n(t) - f(t)) \leq \alpha\} \geq \prod_{k=1}^m \inf_{p(x_k)\leq\varepsilon\alpha, x_k\in S(\gamma)} P\{\| Z_{n(t_k-t_{k-1})} - f_k \|_\infty \leq (1-\varepsilon)\alpha, p(x_k + Z_{n(t_k-t_{k-1})}(1) - f_k(1)) \leq \varepsilon\alpha\},$$

where $\tilde{Z}_n$ is the extension of Z_n to $[0, a]$ obtained by putting $\tilde{Z}_n(k/n) = S_k (k \leq na)$ and interpolating linearly.

PROOF. Let $Z'_n = \tilde{Z}_n - f$, $A = \{\sup_{0\leq t\leq t_{m-1}} p(Z'_n(t)) \leq \alpha, p(Z'_n(t_{m-1})) \leq \varepsilon\alpha\}$,

$$E = \{g \in C([0, t_{m-1}], B): \sup_{0\leq t\leq t_{m-1}} p(g(t)) \leq \alpha, p(g(t_{m-1})) \leq \varepsilon\alpha\},$$

$$F = \{(g, h): g \in C([0, t_{m-1}], B), h \in C([t_{m-1}, t_m], B), \sup_{t_{m-1}\leq t\leq t_m} p(g(t_{m-1}) + h(t)) \leq \alpha, p(g(t_{m-1}) + h(t_m)) \leq \varepsilon\alpha\},$$

$$V_n(t) = Z'_n(t) - Z'_n(t_{m-1}) (t \in [t_{m-1}, t_m]).$$

We have

$$\begin{aligned} P\{\sup_{t\in[0,a]} p(\tilde{Z}_n(t) - f(t)) \leq \alpha\} &\geq P\{\sup_{t\in[0,a]} p(Z'_n(t)) \leq \alpha, p(Z'_n(t_m)) \leq \varepsilon\alpha\} \\ &\geq P(A \cap \{\sup_{t_{m-1}\leq t\leq t_m} p(Z'_n(t_{m-1}) + V_n(t)) \leq \alpha, p(Z'_n(t_{m-1}) + V_n(t_m)) \leq \varepsilon\alpha\}) \\ &= P\{Z'_n \mid [0, t_{m-1}] \in E, (Z'_n \mid [0, t_{m-1}], V_n) \in F\}. \end{aligned}$$

Reasoning as in the proof of Lemma 4.2, we have

$$P\{\sup_{t\in[0,a]} p(\tilde{Z}_n(t) - f(t)) \leq \alpha\} \geq \int_E P\{\sup_{t_{m-1}\leq t\leq t_m} p(g(t_{m-1}) + V_n(t)) \leq \alpha, p(g(t_{m-1}) + V_n(t_m)) \leq \varepsilon\alpha\}\, d\nu(g), \tag{4.14}$$

where $\nu = \mathscr{L}(Z'_n \mid [0, t_{m-1}])$.

Proceeding as in the proof of Lemma 4.2, we obtain for all $x \in B$

$$\begin{aligned} &P\{\sup_{t_{m-1}\leq t\leq t_m} p(x + V_n(t)) \leq \alpha, p(x + V_n(t_m)) \leq \varepsilon\alpha\} \\ &= P\{\| x + Z_{n(t_m-t_{m-1})} - f_m \|_\infty \leq \alpha, p(x + Z_{n(t_m-t_{m-1})}(1) - f_m(1)) \leq \varepsilon\alpha\}. \end{aligned} \tag{4.15}$$

Since $\mathscr{L}(X_1)$ has the same covariance structure as γ, it follows that $P\{S_k \in S(\gamma)$ for all $k\} = 1$, and therefore $P\{\tilde{Z}_n \in C([0, a], S(\gamma))\} = 1$. From (4.14) and (4.15) we have now

$$\begin{aligned} P\{\sup_{t\in[0, a]} p(\tilde{Z}_n(t) - f(t)) \leq \alpha\} &\geq P(A) \cdot \inf_{p(x)\leq\varepsilon\alpha, x\in S(\gamma)} P\{\| x + Z_{n(t_m-t_{m-1})} - f_m \|_\infty \\ &\qquad \leq \alpha, p(x + Z_{n(t_m-t_{m-1})}(1) - f_m(1)) \leq \varepsilon\alpha\} \cdot \\ &\leq P(A) \cdot \inf_{p(x)\leq\varepsilon\alpha, x\in S(\gamma)} P\{\| Z_{n(t_m-t_{m-1})} - f_m \|_\infty \\ &\qquad \leq (1-\varepsilon)\alpha, p(x + Z_{n(t_m-t_{m-1})}(1) - f_m(1)) \leq \varepsilon\alpha\}. \end{aligned}$$

Iterating the same procedure gives the result. □

THEOREM 4.5. *Let Z_n be as in (4.1), μ as in (4.2), $c_{\gamma,p}$ as in Theorem 3.1. Let $0 < a_n \to \infty$, $n^{-1}a_n^2 \to 0$. Then for all $f \in H_\mu$ such that $f' \in L^\infty(I, H_\gamma)$, $r > 0$,*

$$\liminf_{n\to\infty} a_n^{-2} \log P\{\| n^{-1/2} Z_n - a_n f \|_\infty \leq a_n^{-1} r\} \geq -c_{\gamma,p} r^{-2} - (½)\, \|f\|_\mu^2.$$

PROOF. Let ρ be a fixed positive number. For $n \in \mathbb{N}$, $k \in \mathbb{N}$, let $t_{n,k} = kn^{-1}[n/\rho^2 a_n^2]$ and let $k_n = [n/[n/\rho^2 a_n^2]]$. If $t_{n,k_n} = 1$, put $b_n = k_n$. If $t_{n,k_n} < 1$, put $b_n = k_n + 1$; it is easily checked that $t_{n,b_n} \geq 1$. Then $\{t_{n,k}: k = 0, \cdots, b_n\}$ is a subdivision of $[0, t_{n,b_n}]$ such that $nt_{n,k} \in \mathbb{N}$ and $t_{n,k} - t_{n,k-1} = q_n n^{-1}$ for all k, where $q_n = [n/\rho^2 a_n^2]$.

We define t: $[0, t_{n,b_n}] \to B$ by setting $\tilde{f}(t) = f(t)$ for $t \in I$, $\tilde{f}(t) = f(1)$ for $t \in [1, t_{n,b_n}]$. We shall apply now Lemma 4.4 with $a = t_{n,b_n}$, $\alpha = a_n^{-1} n^{1/2} r$ and $(a_n n^{1/2})\tilde{f}$ instead of f. Putting $c_n = a_n n^{1/2} q_n^{-1/2}$, $d_n = a_n^{-1} n^{1/2} q_n^{-1/2}$,

$$\tilde{f}_{n,k}(u) = \tilde{f}(t_{n,k-1} + u(t_{n,k} - t_{n,k-1})) - \tilde{f}(t_{n,k-1}), \quad (u \in I),$$

we have

$$\begin{aligned} &P\{\| n^{-1/2} Z_n - a_n f \|_\infty \leq a_n^{-1} r\} \geq P\{\sup_{t \in [0,a]} p(n^{-1/2} \tilde{Z}_n(t) - a_n \tilde{f}(t)) \leq a_n^{-1} r\} \\ &\geq \textstyle\prod_{k=1}^{b_n} \inf_{p(x_k) \leq \varepsilon r d_n, x_k \in S(\gamma)} P\{\| q_n^{-1/2} Z_{q_n} - c_n \tilde{f}_{n,k} \|_\infty \\ &\qquad \leq (1-\varepsilon) r d_n, p(x_k + q_n^{-1/2} Z_{q_n}(1) - c_n \tilde{f}_{n,k}(1)) \leq \varepsilon r d_n\}. \end{aligned} \tag{4.16}$$

Since

$$\| \tilde{f}_{n,k} \|_\mu^2 = q_n n^{-1} \int_{t_{n,k-1}}^{t_{n,k}} \| f'(t) \|_\gamma^2 \, dt \leq (q_n n^{-1})^2 M,$$

where $M = (\text{ess. sup}_{t \in I} \| f'(t) \|_\gamma)^2$ we have for $k = 1, \cdots, b_n$

$$\| c_n \tilde{f}_{n,k} \|_\mu^2 = (a_n^2 n q_n^{-1})(q_n n^{-1})^2 M = a_n^2 q_n n^{-1} M,$$

showing that $\sup\{\| c_n \tilde{f}_{n,k} \|_\mu : n \in \mathbb{N}, k = 1, \cdots, b_n\} < \infty$ and consequently $\{c_n f_{n,k}: n \in \mathbb{N}, k = 1, \cdots, b_n\}$ is a relatively compact subset of $S(\mu)$ for the $\| \cdot \|_\infty$-metric on $C(I, B)$.

Since $q_n \to \infty$ and $d_n \to \rho$, given $\delta > 0$ there exists by Lemma 4.1 (2) a number $n_0 = n_0(\delta)$ such that $n \geq n_0$ implies

$$\begin{aligned} &\inf_{p(x) \leq \varepsilon r d_n, x \in S(\gamma)} P\{\| q_n^{-1/2} Z_{q_n} - c_n \tilde{f}_{n,k} \|_\infty \leq (1-\varepsilon) r d_n, p(x + q_n^{-1/2} Z_{q_n}(1) - c_n \tilde{f}_{n,k}(1)) \leq \varepsilon r d_n\} \\ &\geq (1-\delta) \inf_{p(x) \leq \varepsilon r d_n, x \in S(\gamma)} P\{\| W - c_n \tilde{f}_{n,k} \|_\infty \\ &\qquad \leq (1-\varepsilon) r d_n, p(x + W(1) - c_n \tilde{f}_{n,k}(1)) \leq \varepsilon r d_n\}. \end{aligned} \tag{4.17}$$

Let $A_n(x) = \{\varphi \in C(I, B): \| \varphi \|_\infty \leq (1-\varepsilon) r d_n, p(x + \varphi(1)) \leq \varepsilon r d_n\}$, $A_n = (1-\varepsilon) r d_n V$, where V is as in Corollary 2.6. By Corollary 2.6, applied with $h = c_n \tilde{f}_{n,k}$, $A = A_n(x)$, we have for $k = 1, \cdots, b_n - 1$

$$\begin{aligned} &\mu(A_n)\mu(c_n \tilde{f}_{n,k} + A_n(x)) \\ &\geq \{\mu(A_n(x))\}^2 \exp\{-(½)(\| c_n \tilde{f}_{n,k} \|_\mu^2 - \| c_n(g_{n,k} - \tilde{f}_{n,k}) \|_\mu^2) - d_n r \| c_n \xi_{n,k} \|_v\}, \end{aligned} \tag{4.18}$$

where $g = \hat{\Delta}_\mu(\xi)$ with $\xi \in \mathscr{M}(I, B')$,

$$g_{n,k}(u) = g(t_{n,k-1} + u(t_{n,k} - t_{n,k-1})) - g(t_{n,k-1}), \quad (u \in I),$$

$$\xi_{n,k}(A) = (t_{n,k} - t_{n,k-1})\{\xi(t_{n,k-1} + (t_{n,k} - t_{n,k-1})A) + \xi(t_{n,k}, 1]\delta_1(A)\}$$

($k = 1, \cdots, b_n - 1$; A Borel in I); recall that $\hat{\Delta}_\mu(\xi_{n,k}) = g_{n,k}$ by Lemma 2.5 (2).

Proceeding as in the proof Theorem 4.3, we have

$$\textstyle\sum_{k=1}^{b_n-1} \| c_n \tilde{f}_{n,k} \|_\mu^2 \leq a_n^2 \| f \|_\mu^2, \tag{4.19}$$

$$\textstyle\sum_{k=1}^{b_n-1} \| c_n(g_{n,k} - \tilde{f}_{n,k}) \|_\mu^2 \leq a_n^2 \| g - f \|_\mu^2 \tag{4.20}$$

$$\textstyle\sum_{k=1}^{b_n-1} d_n c_n \| \xi_{n,k} \|_v \leq 2 b_n \| \xi \|_v. \tag{4.21}$$

Also, we have

$$\begin{aligned} &\inf_{p(x) \leq \varepsilon r d_n, x \in S(\gamma), n \in \mathbb{N}} P\{\| W - c_n \tilde{f}_{n,b_n} \|_\infty \\ &\qquad \leq (1-\varepsilon) r d_n, p(x + W(1) - c_n \tilde{f}_{n,b_n}(1)) \leq \varepsilon r d_n\} = \eta > 0; \end{aligned} \tag{4.22}$$

this may be proved directly, using the compactness of $\{c_n \tilde{f}_{n,b_n}: n \in \mathbb{N}\}$, or from Corollary 2.6 (2) and Lemma 3.2.

By (4.16)-(4.22), we have for $n \geq n_0$:

$$P\{\| n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\} \geq (1-\delta)^{b_n}\eta\{\inf_{p(x)\leq \varepsilon r d_n, x\in S(\gamma)} \mu(A_n(x))\}^{2b_n}\{\mu(A_n)\}^{-b_n}$$
$$\exp\{-(½)a_n^2(\|f\|_\mu^2 - \|g-f\|_\mu^2) - 2rb_n\|\xi\|_v\},$$

and since $a_n^{-2}b_n \to \rho^2$, $d_n \to \rho$, we have for all $\rho > 0$

$$\ell = \liminf_{n\to\infty} a_n^{-2}\log P\{\| n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\}$$
$$\geq \rho^2\log(1-\delta) + 2\rho^2\log\{\inf_{p(x)\leq\varepsilon r\rho, x\in S(\gamma)}\mu(A(x,\rho))\} - \rho^2\log\mu((1-\varepsilon)r\rho V) \tag{4.23}$$
$$- (½)(\|f\|_\mu^2 - \|g-f\|_\mu^2) - 2r\rho^2\|\xi\|_v,$$

where $A(x,\rho) = \{\varphi\in C(I,B): \|\varphi\|_\infty \leq (1-\varepsilon)r\rho, p(x+\varphi(1)) \leq \varepsilon r\rho\}$.

Since δ is arbitrary, the first term on the right may be omitted. By Theorem 3.1 and Lemma 3.2, respectively, we have

$$\lim_{\rho\to 0}\rho^2\log\mu((1-\varepsilon)r\rho V) = -c_{\gamma,p}((1-\varepsilon)r)^{-2},$$
$$\liminf_{\rho\to 0}\rho^2\log\{\inf_{p(x)\leq\varepsilon r\rho, x\in S(\gamma)}\mu(A(x,\rho))\} \geq -c_{\gamma,p}(r(1-3\varepsilon))^{-2}.$$

Therefore letting $\rho \to 0$ in (4.23) we obtain

$$\ell \geq -2c_{\gamma,p}(r(1-3\varepsilon))^{-2} + c_{\gamma,p}(r(1-\varepsilon))^{-2} - (½)(\|f\|_\mu^2 - \|g-f\|_\mu^2). \tag{4.24}$$

Finally, since $\hat{\Delta}_\mu(\mathscr{M}(I,B'))$ is dense in H_μ and ε is an arbitrary positive number, we get from (4.24) the desired inequality. □

Combining Theorems 4.3 and 4.5, we obtain the following.

COROLLARY 4.6. *Let Z_n be as in* (4.1), *μ as in* (4.2), *$c_{\gamma,p}$ as in Theorem* 3.1. *Let $0 < a_n \to \infty$, $n^{-1}a_n^2 \to 0$. Then for all $f\in H_\mu$ such that $f' \in L^\infty(I, H_\gamma)$, $r > 0$,*

$$\lim_{n\to\infty} a_n^{-2}\log P\{\| n^{-1/2}Z_n - a_n f\|_\infty \leq a_n^{-1} r\} = -c_{\gamma,p}r^{-2} - (½)\|f\|_\mu^2.$$

Our next result is a slight extension of Theorems 4.3 and 4.5.

THEOREM 4.7. *Assume $0 < n_k \uparrow \infty$, $n_k \in N$; $0 < a_k \to \infty$, $a_k^2 n_k^{-1} \to 0$; $0 < b_k$, $0 < c_k$, $b_k \sim a_k$, $c_k \sim a_k$ as $k\to\infty$. Let $f\in H_\mu$ and let $\{g_k\}\subset C(I,B)$ be such that $a_k^2\|g_k - f\|_\infty \to 0$. Then*

(1) $\limsup_{k\to\infty} a_k^{-2}\log P\{\| n_k^{-1/2}Z_{n_k} - b_k g_k\|_\infty \leq c_k^{-1}r\} \leq -c_{\gamma,p}r^{-2} - (½)\|f\|_\mu^2$,

(2) *if $f' \in L^\infty(I, H_\gamma)$, then*

$$\liminf_{k\to\infty} a_k^{-2}\log P\{\| n_k^{-1/2}Z_{n_k} - b_k g_k\| \leq c_k^{-1}r\} \geq -c_{\gamma,p}r^{-2} - (½)\|f\|_\mu^2.$$

PROOF. For $n_k \leq n < n_{k+1}$, *let* $\alpha_n = a_k$, $\beta_n = b_k$, $\gamma_n = c_k$, $h_n = g_k$. Then $0 < \alpha_n \to \infty$, $\alpha_n^2/n \to 0$, $\alpha_n \sim \beta_n$, $\alpha_n \sim \gamma_n$ and $\alpha_n^2\|h_n - f\|_\infty \to 0$.

Given $\varepsilon > 0$, let n_0 be such that $n \geq n_0$ implies

$$\alpha_n^2\|h_n - f\|_\infty < \varepsilon(1+\varepsilon)^{-2},\ |\beta_n^2\alpha_n^{-2} - 1| < \varepsilon,\ |\gamma_n\beta_n^{-1} - 1| < \varepsilon.$$

Then for $n \geq n_0$,

$$\beta_n^2\|h_n - f\|_\infty = \beta_n^2\alpha_n^{-2}(\alpha_n^2\|h_n - f\|_\infty) \leq (1+\varepsilon)\varepsilon(1+\varepsilon)^{-2},\quad \beta_n\|h_n - f\|_\infty \leq \varepsilon(1+\varepsilon)^{-1}\beta_n^{-1}.$$

Now $\| n^{-1/2}Z_n - \beta_n f\|_\infty \leq (r-\varepsilon)(1+\varepsilon)^{-1}\beta_n^{-1}$ implies: for $n \geq n_0$

$$\| n^{-1/2}Z_n - \beta_n h_n\|_\infty \leq \| n^{-1/2}Z_n - \beta_n f\|_\infty + \beta_n\|f - h_n\|_\infty \leq r(1+\varepsilon)^{-1}\beta_n^{-1} \leq r\gamma_n^{-1},$$

and therefore

$$\{\| n^{-1/2}Z_n - \beta_n f\|_\infty \leq (r-\varepsilon)(1+\varepsilon)^{-1}\beta_n^{-1}\} \subset \{\| n^{-1/2}Z_n - \beta_n h_n\|_\infty \leq r\gamma_n^{-1}\}. \tag{4.25}$$

Similarly, for $n \geq n_0$

(4.26) $\{\|n^{-1/2}Z_n - \beta_n h_n\|_\infty \leq r\gamma_n^{-1}\} \subset \{\|n^{-1/2}Z_n - \beta_n f\|_\infty \leq r((1-\varepsilon)^{-1} + \varepsilon)\beta_n^{-1}\}$.

Let $u(\varepsilon) = (r - \varepsilon)(1 + \varepsilon)^{-1}$. Since $\beta_n \sim \alpha_n$, Theorem 4.5 and (4.25) give

$$\liminf_{n\to\infty} \alpha_n^{-2} \log P\{\|n^{-1/2} Z_n - \beta_n h_n\|_\infty \leq r\gamma_n^{-1}\}$$
$$\geq \liminf_{n\to\infty} \alpha_n^{-2} \log P\{\|n^{-1/2}Z_n - \beta_n f\|_\infty \leq u(\varepsilon)\beta_n^{-1}\}$$
$$\geq -c_{\gamma,p}(u(\varepsilon))^{-2} - (1/2)\|f\|_\mu^2.$$

Since $u(\varepsilon) \to r$ as $\varepsilon \to 0$, (2) is proved. Statement (1) follows in a similar way from Theorem 4.3 and (4.26). □

5. An invariance principle for the other law of the iterated logarithm and a refinement of Strassen's invariance principle. We will apply the results of Section 4 to obtain Theorems 5.1 and 5.5, which taken together may be regarded as a functional form of the law of the iterated logarithm of Chung [5] and Jain-Pruitt [12]. At the same time, these theorems give a strong speed of convergence result refining Strassen's invariance principle [20] (see also [10]). Theorems 5.1 and 5.5 deal with the case $\|f\|_\mu < 1$; a partial result for $\|f\|_\mu = 1$ is given by Theorem 5.4, but this case is essentially different and will require a separate investigation.

Throughout the section, Z_n will be as in (4.1), μ as in (4.2), and $c_{\gamma,p}$ will be the constant given by Theorem 3.1.

THEOREM 5.1. *For all $f \in H_\mu$ with $\|f\|_\mu < 1$,*

$$\liminf_{n\to\infty} (\mathrm{LL}n)\|(2n\mathrm{LL}n)^{-1/2}Z_n - f\|_\infty \geq (c_{\gamma,p}/2)^{1/2}(1 - \|f\|_\mu^2)^{-1/2} \quad \text{a.s.}$$

We shall need the following lemmas.

LEMMA 5.2. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be such that*

(i) *φ is decreasing and $\varphi(k) \to 0$,*
(ii) *$k\varphi(k)$ is eventually strictly increasing and $k\varphi(k) \uparrow \infty$,*
(iii) *$(\log k)^{-1}(\log \varphi(k)) \to 0$,*
(iv) *$(\log k)^2\varphi(k) \to 0$.*

Let $n_k = [\exp(k\varphi(k))]$. Then

(1) *for all $a > 1$, $\sum_k \exp\{-a\mathrm{LL}n_k\} < \infty$,*
(2) *$n_{k+1}^{-1} n_k \to 1$,*
(3) *$(\mathrm{LL}n_{k+1})^2(n_{k+1} - n_k)/n_{k+1} \to 0$,*
(4) *$\mathrm{LL}n_{k+1}(\mathrm{LL}n_{k+1} - \mathrm{LL}n_k) \to 0$.*

PROOF. (1) follows easily from assumption (ii). Observe next: for $k \geq k_0$ (say), by assumptions (i) and (ii)

(5.1) $0 \leq (k+1)\varphi(k+1) - k\varphi(k) = k\{\varphi(k+1) - \varphi(k)\} + \varphi(k+1) \leq \varphi(k+1)$.

Now (2) follows from assumption (i) and (5.1). By (5.1) and the elementary inequality $1 - e^{-x} \leq x$, we have

$$(n_{k+1} - n_k)/n_{k+1} \sim 1 - \exp(-\{(k+1)\varphi(k+1) - k\varphi(k)\}) \leq \varphi(k+1);$$

thus

$$(\mathrm{LL}n_{k+1})^2(n_{k+1} - n_k)/n_{k+1} \lesssim \log^2(k+1)\left\{1 + \frac{\log\varphi(k+1)}{\log(k+1)}\right\}^2 \varphi(k+1) \to 0$$

by assumptions (iii) and (iv).

From (i) we obtain $\mathrm{LL}n_{k+1} - \mathrm{LL}n_k \gtrsim \log(k^{-1}(k+1))$ and (4) follows easily from here. □

Thus the conditions imposed on φ imply that $\{n_k\}$ grows slightly slower than a first-order exponential function. If $\alpha > 2$ and $\varphi(k) = (\log k)^{-\alpha}$, then φ satisfies (i)–(iv) of Lemma 5.1.

LEMMA 5.3. *Let $m, n, r \in N$, $m \leq n \leq r$. Then for all $f \in H_\mu$,*

$$(\mathrm{LL}n)\|(2n\mathrm{LL}n)^{-1/2}Z_n - f\|_\infty \geq \left\{\frac{(\mathrm{LL}r)m}{(\mathrm{LL}m)r}\right\}^{1/2}(\mathrm{LL}m)\|(2m\mathrm{LL}m)^{-1/2}Z_m - f\|_\infty$$
$$- \{(\mathrm{LL}r)^2(r-m)/r\}^{1/2}M\|f\|_\mu$$
$$- \{(\mathrm{LL}r)^2(r-m)/r + (\mathrm{LL}r)(\mathrm{LL}r - \mathrm{LL}m)m/r\}\|f\|_\infty,$$

where $M = \sup\{p(x) : x \in H_\gamma, \|x\|_\gamma \leq 1\}$.

PROOF. It is easily checked that $Z_n(mt/n) = Z_m(t)$ $(t \in I)$. Putting $a = (2m\mathrm{LL}m)^{1/2}$, $b = (2r\mathrm{LL}r)^{1/2}$, $c = (\mathrm{LL}r/r)^{1/2}$, $g = f((m/n)(\cdot))$ on I, we have

$$(5.2)\qquad \begin{aligned}(\mathrm{LL}n)\|(2n\mathrm{LL}n)^{-1/2}Z_n - f\|_\infty &\geq \left(\frac{\mathrm{LL}n}{n}\right)^{1/2}\sup_{t\in I}p(Z_n(mt/n) - (2n\mathrm{LL}n)^{1/2}f(mt/n))\\ &\geq c\,\|Z_m - (2n\mathrm{LL}n)^{1/2}g\|_\infty.\end{aligned}$$

Now

$$(5.3)\qquad \|Z_m - (2n\mathrm{LL}n)^{1/2}g\|_\infty \geq \|Z_m - af\|_\infty - b\,\|f-g\|_\infty - (b-a)\|f\|_\infty.$$

Since $f \in H_\mu$, for all $t \in I$

$$(5.4)\qquad \begin{aligned}p(f(t) - g(t)) \leq M\,\|f(t) - g(t)\|_\gamma = M\left\|\int_{(m/n)t}^t f'(s)\,ds\right\|_\gamma \leq M\int_{(m/n)t}^t \|f(s)\|_\gamma\,ds\\ \leq M(t - (m/n)t)^{1/2}\|f\|_\mu \leq ((r-m)/r)^{1/2}M\,\|f\|_\mu.\end{aligned}$$

Also

$$(5.5)\qquad b - a \leq (2r\mathrm{LL}r - 2m\mathrm{LL}m)^{1/2} = \{2(r-m)\mathrm{LL}r + 2m(\mathrm{LL}r - \mathrm{LL}m)\}^{1/2}.$$

Combining (5.2)–(5.5) we get the desired inequality. □

PROOF OF THEOREM 5.1. Let $n_k = [\exp(k\varphi(k))]$, with φ as in Lemma 5.2. We first prove: for all $\varepsilon > 0$, if $b_k = (2n_k\mathrm{LL}n_k)^{1/2}$,

$$(5.6)\qquad \liminf_{k\to\infty} d^{-1}(\mathrm{LL}n_k)\|b_k^{-1}Z_{n_k} - f\|_\infty \geq 1 - \varepsilon \quad \text{a.s.},$$

where $d = (c_{\gamma,p}/2)^{1/2}(1 - \|f\|_\mu^2)^{-1/2}$.

Let $A_k = \{d^{-1}(\mathrm{LL}n_k)\|b_k^{-1}Z_{n_k} - f\|_\infty \leq 1 - \varepsilon\}$. Then in order to prove (5.6) it is enough to prove $\sum_k P(A_k) < \infty$, by the Borel-Cantelli lemma. By Theorem 4.3, applied with $a_n = (\mathrm{LL}n)^{1/2}$, we have, putting $c = c_{\gamma,p}$

$$\limsup_{n\to\infty}(\mathrm{LL}n)^{-1}\log P\{\|n^{-1/2}Z_n - (2\mathrm{LL}n)^{1/2}f\|_\infty \leq d(1-\varepsilon)2^{1/2}(\mathrm{LL}n)^{-1/2}\}$$
$$\leq -c(d(1-\varepsilon)2^{1/2})^{-2} - (½)\|2^{1/2}f\|_\mu^2.$$

So given $\delta > 0$, there exists k_0 such that $k \geq k_0$ implies

$$(5.7)\qquad P(A_k) \leq \exp\{-\{c(d(1-\varepsilon)2^{1/2})^{-2} + \|f\|_\mu^2 - \delta\}\mathrm{LL}n_k\}.$$

Since $c(d(1-\varepsilon)2^{1/2})^{-2} + \|f\|_\mu^2 = (1-\|f\|_\mu^2)(1-\varepsilon)^{-2} + \|f\|_\mu^2 > 1$, choosing δ small enough we have from (5.7)

$$\sum_{k=k_0}^{\infty} P(A_k) \le \sum_{k=k_0}^{\infty} \exp\{-a\mathrm{LL}n_k\}$$

with $a > 1$. By Lemma 5.2 this series converges, proving (5.6).

Now (5.6) implies

$$\liminf_{k\to\infty}(\mathrm{LL}n_k)\| b_k^{-1}Z_{n_k} - f\|_\infty \ge d \quad \text{a.s.},$$

which together with Lemmas 5.2 and 5.3 imply the result. □

THEOREM 5.4. *For all* $f \in H_\mu$ *with* $\|f\|_\mu = 1$,

$$\lim_{n\to\infty}(\mathrm{LL}n)\|(2n\mathrm{LL}n)^{-1/2}Z_n - f\|_\infty = \infty, \quad \text{a.s.}$$

PROOF. Let $\{n_k\}$ be as in Theorem 5.3, $\lambda > 0$. Proceeding as in Theorem 5.3, one proves

$$\liminf_{k\to\infty}(\mathrm{LL}n_k)\| b_k^{-1}Z_{n_k} - f\|_\infty \ge \lambda, \quad \text{a.s.}$$

The proof is then completed as in Theorem 5.3. □

THEOREM 5.5. *For all* $f \in h_\mu$ *such that* $\|f\|_\mu < 1$ *and* $f' \in L^\infty(I, H_\gamma)$,

$$\liminf_{n\to\infty}(\mathrm{LL}n)\|(2n\mathrm{LL}n)^{-1/2}Z_n - f\|_\infty \le (c_{\gamma,p}/2)^{1/2}(1-\|f\|_\mu^2)^{-1/2} \quad \text{a.s.}$$

PROOF. It is enough to show: there exists a sequence $\{n_k\} \subset N$, $n_k \uparrow \infty$, such that for all $\varepsilon > 0$,

$$\liminf_{k\to\infty} d^{-1}(\mathrm{LL}n_k)\|(2n_k\mathrm{LL}n_k)^{-1/2}Z_{n_k} - f\|_\infty \le 1 + \varepsilon, \quad \text{a.s.}, \tag{5.8}$$

where $d = (c_{\gamma,p}/2)^{1/2}(1-\|f\|_\mu^2)^{-1/2}$.

Let $n_k = k^k$, $a_k = (2n_k\mathrm{LL}n_k)^{1/2}$. Let $0 < \alpha \le 1$, $m \in N$. Then for any $g \in C(I, B)$, clearly

$$\begin{aligned} \| g - f\|_\infty \le{} & \sup_{t\in I} p(g(\alpha t) - f(\alpha t)) \\ & + \sup_{t\in I} p(g(\alpha + (1-\alpha)t) - g(\alpha) - \{f(\alpha + (1-\alpha)t) - f(\alpha)\}). \end{aligned} \tag{5.9}$$

Set $b_k = n_{k+1} - n_k$, $\alpha = n_k/n_{k+1}$ and for $t \in I$,

$$Y_k(t) = b_k^{-1/2}\{Z_{n_{k+1}}(\alpha + (1-\alpha)t) - Z_{n_{k+1}}(\alpha)\}, \qquad f_k(t) = f(\alpha + (1-\alpha)t) - f(\alpha).$$

Since $Z_{n_{k+1}}((n_k/n_{k+1})(\cdot)) = Z_{n_k}$, we have putting $g = a_{k+1}^{-1}Z_{n_{k+1}}$ in (5.9):

$$\| a_{k+1}^{-1}Z_{n_{k+1}} - f\|_\infty \le \| a_{k+1}^{-1}Z_{n_k} - f(\alpha(\cdot))\|_\infty + \| b_k^{1/2}a_{k+1}^{-1}Y_k - f_k\|_\infty. \tag{5.10}$$

By the law of the iterated logarithm in finite-dimensional spaces,

$$\limsup_{k\to\infty} a_k^{-1}\| Z_{n_k}\|_\infty = \limsup_{k\to\infty} a_k^{-1}\max_{j\le n_k} p(S_j) < \infty, \quad \text{a.s.}$$

Since $(\mathrm{LL}n_{k+1})a_k/a_{k+1} \to 0$, it follows that

$$(\mathrm{LL}n_{k+1})a_{k+1}^{-1}\| Z_{n_k}\|_\infty \to 0, \quad \text{a.s.} \tag{5.11}$$

Also, for all $t \in I$, if M is as in Lemma 5.3,

$$p(f(\alpha t)) \le M\| f(\alpha t)\|_\gamma = M\left\| \int_0^{\alpha t} f'(s)\, ds \right\|_\gamma \le M\int_0^{\alpha t} \| f'(s)\|_\gamma\, ds \le M(\alpha t)^{1/2}\| f\|_\mu,$$

and therefore $(\mathrm{LL}n_{k+1})\| f(\alpha(\cdot))\|_\infty \le M(\mathrm{LL}n_{k+1})(n_k/n_{k+1})^{1/2}\| f\|_\mu \to 0$.

From this last statement, (5.10) and (5.11), it follows that in order to prove (5.8) it is enough to prove:

(5.12) $$\limsup_{k\to\infty} d^{-1}(\mathrm{LL}n_{k+1})\| b_k^{1/2}a_k^{-1}Y_k - f_k \|_\infty \le 1+\varepsilon, \quad \text{a.s.}$$

But $\{Y_k\}$ is an independent family of $C(I, B)$ − valued random vectors (Y_k depends only on $\{X_j : n_k < j \le n_{k+1}\}$) and $\mathscr{L}(Y_k) = \mathscr{L}(b_k^{-1/2}Z_{b_k})$, as it is easily verified. Let

$$A_k = \{d^{-1}(\mathrm{LL}n_{k+1})\| b_k^{1/2}a_{k+1}^{-1}Y_k - f_k \|_\infty \le 1+\varepsilon\}.$$

By the Borel-Cantelli Lemma, (5.12) will follow if we prove that $\sum_k P(A_k) = \infty$.

Since for $t \in I$

$$p(f_k(t) - f(t)) \le M\| f_k(t) - f(t)\|_\gamma \le M\left\| \int_\alpha^{\alpha+(1-\alpha)t} f'(s)\,ds - \int_0^t f'(s)\,ds \right\|_\gamma$$

$$\le M\left\| \int_0^\alpha f'(s)\,ds \right\|_\gamma + M\left\| \int_t^{\alpha+(1-\alpha)t} f'(s)\,ds \right\|_\gamma$$

$$\le M\alpha^{1/2}\| f\|_\mu + M(\alpha(1-t))^{1/2}\| f\|_\mu$$

$$\le 2M\alpha^{1/2}\| f\|_\mu,$$

it follows that

(5.13) $$(\mathrm{LL}n_{k+1})\| f_k - f\|_\infty \le 2M\| f\|_\mu(\mathrm{LL}n_{k+1})(n_k/n_{k+1})^{1/2} \to 0.$$

Since $n_{k+1}b_k^{-1} \to 1$, $(\mathrm{LL}n_{k+1})b_k^{-1} \to 0$, we have from (5.13) and Theorem 4.7 (2), applied with $(\mathrm{LL}n_{k+1})^{1/2}$ in the role of a_k and b_k in the role of n_k, putting $c = c_{\gamma,p}$:

$$\liminf_{k\to\infty}(\mathrm{LL}n_{k+1})^{-1}\log P\{\| b_k^{-1/2}Z_{b_k} - b_k^{-1/2}a_{k+1}f_k \|_\infty \le d(1+\varepsilon)2^{1/2}b_k^{-1/2}n_{k+1}^{1/2}(\mathrm{LL}n_{k+1})^{-1/2}\}$$
$$\ge -c(d(1+\varepsilon)2^{1/2})^{-2} - (½)\| 2^{1/2}f\|_\mu^2.$$

So given $\delta > 0$, there exists k_0 such that $k \ge k_0$ implies

(5.14) $$P(A_k) \ge \exp\{-\{c(d(1+\varepsilon)2^{1/2})^{-2} + \| f\|_\mu^2 + \delta\}\mathrm{LL}n_{k+1}\}.$$

Since

$$c(d(1+\varepsilon)2^{1/2})^{-2} + \| f\|_\mu^2 = (1 - \| f\|_\mu^2)(1+\varepsilon)^{-2} + \| f\|_\mu^2 < 1,$$

choosing δ small enough we have from (5.14)

$$\sum_k P(A_k) \ge \sum_{k=k_0}^\infty \exp\{-a\mathrm{LL}n_{k+1}\} = \sum_{k=k_0+1} (k\log k)^{-a}$$

with $a < 1$ and therefore the series diverges, establishing (5.12) and thus completing the proof. □

COROLLARY 5.6. *Let $\{X_j\}$ be a sequence of independent, identically distributed B-valued random vectors. Assume $EX_1 = 0$ and for every $\varphi \in B'$, $E\varphi^2(X_1) < \infty$. Let γ be the centered Gaussian measure on B such that $\int \varphi^2\,d\gamma = E\varphi^2(X_1)$ for all $\varphi \in B'$. Then*

$$\liminf_{n\to\infty}\left(\frac{\mathrm{LL}n}{n}\right)^{1/2} \max_{k\le n} p(S_k) = c_{\gamma,p}^{1/2}, \quad \text{a.s.}$$

PROOF. Follow at once by observing that $\max_{k\le n}p(S_k) = \| Z_n\|_\infty$ and taking $f \equiv 0$ in Theorems 5.1 and 5.5. □

6. Improvement and generalization for vector-valued Brownian motion of some results of Csáki. The following result improves and generalizes a result of Csáki ([7], Theorem 2). In the case of one-dimensional Brownian motion, Csáki obtained the upper bound given below for all $f \in H_\mu$ with $\|f\|_\mu < 1$ and the lower bound for all $f \in H_\mu$ such that $\|f\|_\mu < 1$ and f' is of bounded variation (the question of the validity of the lower bound when f' is not of bounded variation is stated on pages 289 and 298 of [7]).

We use the definitions of Section 2 and 3.

THEOREM 6.1. *Let B be a finite dimensional Banach space and let γ be a centered Gaussian measure on B. Let $\{W(t): t \geq 0\}$ be γ-Brownian motion, μ the associated Wiener measure on $C(I, B)$. Then for all $f \in H_\mu$ such that $\|f\|_\mu < 1$,*

$$\liminf_{t\to\infty}(\mathrm{LL}t)\|(2t\mathrm{LL}t)^{-1/2}W((\cdot)t) - f\|_\infty = (c_{\gamma,p}/2)^{1/2}(1 - \|f\|_\mu^2)^{-1/2} \quad \text{a.s.}$$

PROOF. The two parts of the proof are similar to the proofs of Theorem 5.1 and 5.5, respectively. We will indicate the main steps.

(1) Just as in Theorem 5.1, we prove first: for every $\varepsilon > 0$,

$$\liminf_{k\to\infty} d^{-1}(\mathrm{LL}n_k)\|(2n_k\mathrm{LL}n_k)^{-1/2}W((\cdot)n_k) - f\|_\infty \geq 1 - \varepsilon, \quad \text{a.s.,} \tag{6.1}$$

where $d = (c_{\gamma,p}/2)^{1/2}(1 - \|f\|_\mu^2)^{-1/2}$ (of course, it is immaterial here whether n_k is defined as $\exp(k\varphi(k))$ or $[\exp(k\varphi(k))]$). Let

$$A_k = \{\| n_k^{-1/2}W((\cdot)n_k) - (2\mathrm{LL}n_k)^{1/2}f\|_\infty \leq d(1-\varepsilon)(2(\mathrm{LL}n_k)^{-1})^{1/2}\};$$

(6.1) will follow if we prove that $\sum_k P(A_k) < \infty$. Now

$$P(A_k) = P\{\| W - (2\mathrm{LL}n_k)^{1/2}f\|_\infty \leq d(1-\varepsilon)(2(\mathrm{LL}n_k)^{-1})^{1/2}\}$$

and by Theorem 3.3 we have

$$(\mathrm{LL}n_k)^{-1}\log P(A_k) \to -c_{\gamma,p}(d(1-\varepsilon)2^{1/2})^{-2} - (½)\|2^{1/2}f\|_\mu^2.$$

The proof of (6.1) is completed as in the proof of Theorem 5.1. In order to complete the proof of the lower bound result, we just point out that the inequality given in Lemma 5.3 is valid for Brownian motion when one takes $t \in \mathbb{R}^+$ instead of $n \in \mathbb{N}$ and $W((\cdot)t)$ instead of Z_n.

(2) In order to prove the upper bound result we follow the steps in the proof of Theorem 5.5 with certain modifications. As in Theorem 5.5, it is enough to show: there exists $\{n_k\} \subset \mathbb{N}$, $n_k \uparrow \infty$, such that for all $\varepsilon > 0$

$$\liminf_{k\to\infty} d^{-1}(\mathrm{LL}n_k)\|(2n_k\mathrm{LL}n_k)^{-1/2}W((\cdot)n_k) - f\|_\infty \leq 1 + \varepsilon. \quad \text{a.s.} \tag{6.2}$$

Take $n_k = k^k$, $a_k = (2n_k\mathrm{LL}n_k)^{1/2}$. Inequality (5.10) is substituted here by

$$\| a_{k+1}^{-1}W((\cdot)n_{k+1}) - f\|_\infty \leq \| a_{k+1}^{-1}W((\cdot)n_k) - f(\alpha(\cdot))\|_\infty + \| b_k^{1/2}a_{k+1}^{-1}W_k - f_k\|_\infty, \tag{6.3}$$

where α, b_k and f_k are as in (5.10) and

$$W_k(t) = b_k^{-1/2}\{W(n_k + tb_k) - W(n_k)\}, \quad (t \in I).$$

By the law of the iterated logarithm for a Brownian motion in a finite-dimensional space,

$$\limsup_{k\to\infty} a_k^{-1} \| W((\cdot)n_k)\|_\infty < \infty, \quad \text{a.s.}$$

It follows that

$$(\mathrm{LL}n_{k+1})a_{k+1}^{-1} \| W((\cdot)n_k)\|_\infty \to 0, \quad \text{a.s.} \tag{6.4}$$

Just as in Theorem 5.5,

(6.5) $$(\mathrm{LL}n_{k+1})\| f(\alpha(\cdot))\|_\infty \to 0.$$

From (6.3)–(6.5) it follows that in order to prove (6.2) it is enough to prove

(6.6) $$\liminf_{k\to\infty} d^{-1}(\mathrm{LL}n_{k+1})\| b_k^{1/2}a_k^{-1}W_k - f_k\|_\infty \leq 1+\varepsilon, \quad \text{a.s.}$$

But $\{W_k\}$ is an independent sequence of γ-Brownian motions over I; therefore if we define

$$A_k = \{d^{-1}(\mathrm{LL}n_{k+1})\| b_k^{1/2}a_{k+1}^{-1}W_k - f_k\|_\infty < 1+\varepsilon\},$$

by the Borel-Cantelli Lemma, (6.6) will follow if we can prove $\sum_k P(A_k) = \infty$. By Theorem 3.4 and (5.13), we have

$$(\mathrm{LL}n_{k+1})^{-1}\log P(A_k) \to -c_{\gamma,p}(d(1+\varepsilon)2^{1/2})^{-2} - (1/2)\| 2^{1/2}f\|_\mu^2.$$

The proof of (6.6) is completed by following the final steps in the proof of Theorem 5.5. □

The next result, corresponding to Theorem 5.4, is proved by an easy modification of the first half of the proof of Theorem 6.1. It improves a result of Csáki ([7], Theorem 1) for the case of one-dimensional Brownian motion.

THEOREM 6.2. *In the same framework of Theorem* 6.1, *for all* $f \in H_u$ *with* $\|f\|_\mu = 1$,

$$\lim_{t\to\infty}(\mathrm{LL}t)\|(2t\mathrm{LL}t)^{-1/2}W((\cdot)t) - f\|_\infty = \infty, \quad \text{a.s.}$$

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