The Annals of Probability
1983, Vol. 11, No. 2, 393–402

A SIMPLE CRITERION FOR TRANSIENCE OF A REVERSIBLE MARKOV CHAIN

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An old argument of Royden and Tsuji is modified to give a necessary and sufficient condition for a reversible countable state Markov chain to be transient. This Royden criterion is quite convenient and can, on occasion, be used as a substitute for the criterion of Nash-Williams [6]. The result we give here yields a very simple proof that the Nash-Williams criterion implies recurrence. The Royden criterion also yields as a trivial corollary that a recurrent reversible random walk on a state space X remains recurrent when it is constrained to run on a subset X' of X. An apparently weaker criterion for transience is also given. As an application, we discuss the transience of a random walk on a horn shaped subset of $\mathbb{Z}^d$.

0. Introduction. In [7] Royden gave a necessary and sufficient condition for a covering surface of a compact Riemann Surface to have a Green function. Roughly, the criterion consisted of triangulating the surface and asking whether a flow could be constructed on the dual graph with the following properties: with the exception of one vertex, the flow into the vertex equals the flow out; at one vertex the flow out exceeds the flow in; the whole flow should have "finite energy".

Tsuji [8, Theorems X.44 and X.9] gave a simple proof of Royden's result, and Mori [5] used it to prove that a $\mathbb{Z}^2$ cover of a compact surface admits no Green function while a $\mathbb{Z}^3$ cover does. Considerably more recently, and independently, Debaun [1] has proved analogous results for a Riemannian manifold.

The extension of Royden's criterion to the setting of a reversible Markov chain is in Section 1 and is straightforward. Royden's criterion has a "physical" interpretation and this is described in Section 2. Section 3 mentions some related probabilistic results; Section 4 includes, as a simple application of the recurrence part of the criterion, a simple proof of Nash Williams' test for recurrence.

Section 5 shows that a somewhat relaxed form of Royden's condition also implies transience.

Loosely it says that if there is a vector field u over our state space with $\int |\nabla u|^2 < \infty$, $\int |\operatorname{div} u| < \infty$, and $\int \operatorname{div} u \neq 0$ then the Markov chain is transient.

In the final section we discuss the problem of determining how much one has to fatten a 1 or 2 dimensional simple random walk into 3 space before it becomes transient. Our results are sharp.

1. Royden's criterion. We now state and prove the formulation of Royden's criterion appropriate to a Markov chain. Throughout this paper denote by (X, Y_n, p_{ij}) the Markov chain with countable state space X, process Y, and transition probabilities $p_{ij}\,(i, j \in X)$. We will use $\mathbb{P}^i$ and $\mathbb{E}^i$ to denote the probability and expectation obtained when Y_0 is conditioned to be $i \in X$. The chain $(X, Y, \mathbf{p})$ is reversible if there exist strictly positive weights $\pi_i\,(i \in X)$ such that

$$\pi_i p_{ij} = \pi_j p_{ji}$$

Received December 1981; revised July 1982.

AMS 1980 *subject classifications.* Primary 60J10; secondary 60J45, 31C25, 31C12.

Key words and phrases. Recurrence, transience, Markov chain, symmetric Markov chain, energy.

in which case we will denote $\pi_i p_{ij}$ by a_{ij} for brevity. Observe that $p_{ij} = a_{ij}/\sum_k a_{ik}$ and $\pi_i = \sum_j a_{ij}$. Many of the usual Markov chains satisfy this reversibility condition. Henceforth we assume that a Markov chain is reversible.

THEOREM. *(X, Y, p_{ij}, π_i) is a transient reversible Markov chain if and only if we may find real numbers $u_{ij}(i, j \in X)$ with the following properties:*

(i) $u_{ij} = -u_{ji}$,
(ii) *there exists* $i_0 \in X$ *such that* $\sum_j u_{i_0 j} \neq 0$ *and* $\sum_j u_{ij} = 0$ *for all* $i \neq i_0$,
(iii) $\sum_{i,j} u_{ij}^2/a_{ij} < \infty$, *where we adopt the convention that* $0/0 = 0$ *and* $x/0 = \infty$ *if* $x \neq 0$.

We will refer to any sequence u_{ij} of real numbers as a flow on X if (i), (ii) apply and if $u_{ij} = 0$ whenever $p_{ij} = 0$. If a flow satisfies property (iii) we will say it has finite energy.

PROOF. We first prove that the existence of a flow with finite energy implies that Y is transient. The initial step is an elementary Hilbert space argument. Let H be the Hilbert space of all sequences $(v_{ij})_{i,j\in X}$ satisfying (iii), and define an inner product on H by

$$(\mathbf{v}, \mathbf{w}) = \sum_i \sum_j \frac{v_{ij} w_{ij}}{a_{ij}}.$$

Condition (ii) can be expressed as follows: for each $i \in X$ define $\ell^i \in H$ by

$$(\ell^i)_{jk} = \delta_{ij} a_{ik},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise. Trivial computation shows that $\langle \ell^i, \ell^i \rangle = \pi_i$ and justifies our claim that ℓ^i is in H. A vector $\mathbf{u} \in H$ has property (ii) if and only if

$$(\ell^{i_0}, \mathbf{u}) \neq 0, \quad \langle \ell^i, \mathbf{u} \rangle = 0 \; \forall \; i \neq i_0.$$

By hypothesis there is a vector $\mathbf{u} \in H$ with $\langle \ell^{i_0}, \mathbf{u} \rangle = 1$, $\langle \ell^i, \mathbf{u} \rangle = 0$ for all $i \in X$ with $i \neq i_0$, and $u_{ij} = -u_{ji}$ for all $i, j \in X$. Let E denote the affine space of all vectors in H with these three properties. Let $\mathbf{w}$ denote the unique vector in E which minimises $\langle \mathbf{w}, \mathbf{w} \rangle$. By a standard argument from Hilbert spaces, $\mathbf{w}$ exists and moreover is characterised by the following property:

$$\langle \mathbf{w}, \mathbf{w} - \mathbf{e} \rangle = 0$$

for all $\mathbf{e}$ in E. This second fact will allow us to construct a function W on X (rather than on $X \times X$) such that

$$a_{ij}(W_j - W_i) = w_{ij}.$$

We construct W_i. Let $i_0, i_1, \cdots, i_n = i$ be a path in X with $p_{i_k i_{k+1}} > 0$ for each $k < n$. Define W_i by

$$W_i = \sum_{k=0}^{n-1} \frac{w_{i_k i_{k+1}}}{a_{i_k i_{k+1}}}.$$

To show that W_i is well defined and independent of our choice of path we exploit $\langle \mathbf{w}, \mathbf{w} - \mathbf{e} \rangle = 0$. Let $j_0, j_1, \cdots, j_n = j_0$ be an irreducible chain of vertices with $p_{j_k j_{k+1}} \neq 0$ for all $k < n$ and $j_k \neq j_m$ if $k, m < n$. It suffices to prove that

$$\sum_{k=0}^{n-1} \frac{w_{j_k j_{k+1}}}{a_{j_k j_{k+1}}} = 0.$$

Define (f_{ij}) by $f_{j_k j_{k+1}} = -f_{j_{k+1}, j_k} = 1$ for all $k < n$, and by $f = 0$ otherwise. Then computation shows that

$$\langle \mathbf{f}, \mathbf{w} \rangle = \sum_{k=0}^{n-1} \frac{w_{j_k j_{k+1}}}{a_{j_k j_{k+1}}} + \sum_{k=0}^{n-1} \frac{-w_{j_{k+1} j_k}}{a_{j_{k+1} j_k}};$$

but $w_{ij} = -w_{ji}$ and $a_{ij} = a_{ji}$, so it follows that

$$\langle \mathbf{f}, \mathbf{w} \rangle = 2 \sum_{k=0}^{n-1} \frac{w_{j_k j_{k+1}}}{a_{j_k j_{k+1}}}.$$

On the other hand $\langle \mathbf{f}, \ell^i \rangle = 0$ for all $i \in X$ and so $w + f$ is in E. Therefore $\langle \mathbf{f}, \mathbf{w} \rangle = 0$ as required.

We are halfway to our objective. We have constructed a function W_i on X with the following properties:

(i) $W_{i_0} = 0$,
(ii) W is not identically zero (because $\langle \mathbf{w}, \ell^{i_0} \rangle = 1$),
(iii) $W_i = \sum_{j \in X} p_{ij} W_j$ for all $i \neq i_0$,
(iv) $\sum_{i,j \in X} a_{ij} (W_i - W_j)^2 = \sum_{i,j \in X} \pi_i p_{ij} (W_i - W_j)^2 < \infty$.

We will now demonstrate that the existence of such a W is not compatible with the hypothesis that Y be recurrent. Suppose Y is recurrent. Let T denote the first time Y hits i_0. Then $W(Y_{k \wedge T})$ is a $\mathbb{P}^i$-martingale and moreover it converges almost surely to zero as k tends to ∞. We will prove that

$$\sup_k \mathbb{E}^i((W(Y_{k \wedge T}) - W(Y_0))^2) < \infty,$$

and so that $W(Y_{k \wedge T}) \equiv 0$. This in turn would imply that $W_i = 0$ for all i giving us the desired contradiction. Let us obtain the mean square estimate: by the martingale property we have

$$\begin{aligned} \mathbb{E}^i([W(Y_{k \wedge T}) - W(Y_0)]^2) &= \mathbb{E}^i(\textstyle\sum_{j=0}^{k-1} [W(Y_{(j+1) \wedge T}) - W(Y_{j \wedge T})]^2) \\ &\leq \mathbb{E}^i(\textstyle\sum_0^\infty [W(Y_{(j+1) \wedge T}) - W(Y_{j \wedge T})]^2). \end{aligned}$$

Let $\sum_k$ be the σ-field generated by $(W(Y_{j \wedge T}))_{j \leq k}$. Then the last expression may be rewritten as

$$\mathbb{E}^i(\textstyle\sum_0^\infty \mathbb{E}([W(Y_{(j+1) \wedge T}) - W(Y_{j \wedge T})]^2 \mid \sum_j)) = \sum_{n \in X} g(i, n) \sum_{m \in X} p_{nm} (W_m - W_n)^2,$$

where $g(i, n)$ is the $\mathbb{P}^i$ expected number of visits Y makes to n before hitting i_0. (If $i = n$ we count Y_0 in n as a visit to n.) A simple computation shows that $\pi_i g(i, j) = \pi_j g(j, i)$. Moreover $g(j, i) \leq g(i, i)$. Putting these two facts together we observe that

$$g(i, j) \leq \frac{\pi_j}{\pi_i} g(i, i).$$

Because Y eventually hits i_0 it follows that the $\mathbb{P}^i$ probability that $Y_n = i$ for some $n > 0$, denoted by f_i, is strictly less than one. But $g(i, i) = \sum_{k=0}^\infty (f_i)^k$ and so is finite. Therefore

$$\sum_{n \in X} g(i, n) \sum_{m \in X} p_{nm} (W_n - W_m)^2 \leq \frac{g(i, i)}{\pi_i} \sum_{n \in X} \sum_{m \in X} \pi_n p_{nm} (W_n - W_m)^2 < \infty,$$

as required.

The converse direction is easier. The idea is to take the gradient of the function $W_i = \mathbb{P}^i(Y_n = i_0$ for some $n \geq 0)$. This will surely be a nonconstant flow if Y_n is transient. Moreover it is easy to prove that the energy of the flow is at most $2\pi_{i_0}$. Although this fact is already known to probabilists, formula 2.2 in [4] is essentially what we require, we give an argument here for the convenience of the reader. Suppose (Y, X, p_{ij}) is transient. Fix i_0 and let $W_i = \mathbb{P}^i(Y_n = i_0$ for some $n \geq 0)$. We claim that $\mathbf{u}$ given by

$$u_{ij} = \pi_i p_{ij} (W_i - W_j)$$

has all the required properties. Certainly $\sum_j u_{i_0 j} > 0$ and $\sum_j u_{ij} = 0$ for all $i \neq i_0$. What needs to be proved is that

$$\sum_{i,j} (W_i - W_j)^2 \pi_i p_{ij} < \infty.$$

Let F be a finite set containing i_0 and $F^* = F\backslash\{i_0\}$. Define w_i (depending on F) by

$$w_i = \mathbb{P}^i(Y \text{ hits } i_0 \text{ before quitting } F).$$

Then

$$\sum_{i\in F^*}\sum_{j\in X}\pi_i p_{ij}(w_i - w_j)^2 = \sum_{i\in F^*}\sum_{j\in X}\pi_i p_{ij}(w_j^2 - w_i^2),$$

because $i \in F^*$ implies that

$$w_i = \sum_{j\in X} p_{ij} w_j.$$

The symmetry of $\pi_i p_{ij}$ allows us to cancel most of the terms on the right hand side. (This is Green's Theorem). Further observe that $w_{i_0} \geq w_i \geq w_j$ for all i in F, for all j in $X\backslash F$. Using these two facts, we obtain

$$\begin{aligned}\sum_{i\in F^*}\sum_{j\in X}\pi_i p_{ij}(w_i - w_j)^2 &= \sum_{i\in F^*}\sum_{j\in X\backslash F}\pi_i p_{ij}(w_j^2 - w_i^2)\\ &\quad + \sum_{i\in F^*}\pi_i p_{ii_0}(w_{i_0}^2 - w_i^2)\\ &\leq \sum_{i\in F^*}\pi_i p_{ii_0}(w_{i_0}^2 - w_i^2) \leq \pi_{i_0}.\end{aligned}$$

Now allow F to vary; let $F_1 \subset \cdots \subset F_n \subset \cdots$ be any collection of finite subsets X with $\cup F_i = X$ and let $w_i^{(n)}$ be $\mathbb{P}^i(Y \text{ hits } i_0 \text{ before leaving } F_n)$. Then $\lim_{n\to\infty} w_i^n = W_i$ and so by Fatou's theorem we obtain

$$\sum_{i\neq i_0}\sum_{j\in X}\pi_i p_{ij}(W_i - W_j)^2 \leq \pi_{i_0}.$$

On inserting i_0 into the sum we have

$$\sum_{i\in X}\sum_{j\in X}\pi_i p_{ij}(W_i - W_j)^2 \leq 2\pi_{i_0}.$$

The theorem is proved.

2. Physical interpretation. Imagine the points $i \in X$ to be nodes connected together by tubes of length 1, the cross-sectional area of the pipe from i to j being a_{ij}. Insist that $\sum_j a_{ij}$ the total cross-sectional area at any node is finite. Then the a_{ij} are symmetric and determine a reversible Markov chain in the usual way with $\pi_i = \sum_j a_{ij}$ and $p_{ij} = a_{ij}/\pi_i$.

Suppose that i_0 is a node of X, and that an incompressible fluid enters the system at i_0 at a constant rate, but can escape at no other vertex. Suppose also that the pipes are all full of the liquid. Then let u_{ij} be the volume rate at which fluid flows from i to j along the pipe between them. Then the incompressibility implies that $\sum_j u_{ij} = 0$ for $i \neq i_0$ and by hypothesis $\sum_j u_{i_0 j} \neq 0$. The mass of fluid in the pipe from i to j is just a_{ij}; its velocity is given by u_{ij}/a_{ij}. The total kinetic energy of the fluid is therefore given by

$$\sum_{i,j} a_{ij}\left(\frac{u_{ij}}{a_{ij}}\right)^2 = \sum_{i,j}\frac{u_{ij}^2}{a_{ij}}.$$

Royden's criterion for transience then reads: $(X, Y, \mathbf{a})$ is transient if and only if we can construct a flow through the network with finite kinetic energy.

3. General remarks. We should make some remarks concerning the connections between Royden's criterion and existing probabilistic results. The idea of using energy in Markov chain theory is not new. Doyle and Snell used the electrical network to motivate a similar approach to the problem of determining recurrence and transience [2, particularly page 125]. That paper also mentions several more classical sources.

Nash-Williams [6] has a criterion for recurrence. This criterion was extensively exploited by Griffeath and Liggett [4] (and a new proof of sufficiency of the criterion was given). A major purpose of [4] was to give a necessary and sufficient condition for recurrence for a class of reversible Markov chains. Royden's criterion gives an alternative proof of the transience result there and also yields a very quick proof of the sufficiency of the Nash-Williams criterion for recurrence. We will give the latter proof in the next section.

Let (X, Y_n, a_{ij}) be a reversible Markov process; let $\tilde{a}_{ij}$ be any sequence of real numbers satisfying $0 \le \tilde{a}_{ij} \le a_{ij}$ and $\tilde{a}_{ij} = \tilde{a}_{ji}$. Let $\tilde{X} = \{i : \tilde{\pi}_i = \sum_j \tilde{a}_{ij} > 0\}$, and $\tilde{Y}$ the Markov process with transition probabilities given by $\tilde{p}_{ij} = \tilde{a}_{ij}/\tilde{\pi}_i$ for i, j in $\tilde{X}$. We will call the process $(\tilde{X}, \tilde{Y}_n, \tilde{a}_{ij})$ **subordinate** to (X, Y_n, a_{ij}). If $X' \subset X$ is any subset of X, we may **restrict** the process to X' by taking $a'_{ij} = a_{ij}$ if $i, j \in X'$ and $a'_{ij} = 0$ otherwise.

A trivial consequence of Royden's criterion is the following: if (X, Y_n, a_{ij}) is recurrent and $(\tilde{X}, \tilde{Y}_n, \tilde{a}_{ij})$ is subordinate to it, then $(\tilde{X}, \tilde{Y}_n, \tilde{a}_{ij})$ is also recurrent. In the case of $X = \mathbb{Z}^2$ this was a problem of Feller [3, page 425]. This sort of result is not obvious if one uses traditional Fourier analysis techniques.

4. The Nash Williams result. All the essential ingredients of the Nash Williams result can already be seen in the following easy application of Royden's criterion.

EXAMPLE. The simple random walk on $\mathbb{Z}^2$ is recurrent.

PROOF. Let u_{ij} be a flow on the lattice $\mathbb{Z}^2$ with source at $\mathbf{0}$ of strength 1. Let $[A^n, A^{n+1}]$ denote the set of $8n + 4$ edges connecting the square $A^{(n)}$ of width $2n$ to the square $A^{(n+1)}$ of width $(n + 1)$. By hypothesis $\sum_{(i,j)\in[A^n,A^{n+1}]} u_{ij} = 1$. A simple computation shows that

$$\textstyle\sum_{(i,j)\in[A^n,A^{n+1}]} u_{ij} = 1 \quad \text{for all} \quad n.$$

By the Cauchy Schwartz inequality

$$\sum_{(i,j)\in[A^n,A^{n+1}]} \frac{u_{ij}^2}{a_{ij}} \ge \Big[\sum_{(i,j)\in[A^n,A^{n+1}]} |u_{ij}|\Big]^2 \Big[\sum_{(i,j)\in[A^n,A^{n+1}]} a_{ij}\Big]^{-1} \ge \Big[\sum_{(i,j)\in[A^n,A^{n+1}]} a_{ij}\Big]^{-1}.$$

To consider the simple random walk on $\mathbb{Z}^2$ we put $a_{ij} = 1$ if i and j are nearest neighbours and zero otherwise. Then we obtain

$$\sum_{(i,j)\in\mathbb{Z}^2} \frac{u_{ij}^2}{a_{ij}} \ge \sum_{n=0}^{\infty} \frac{1}{8n + 4} = \infty,$$

and the flow does not have finite energy. This completes the argument because the flow was arbitrary.

We now prove the Nash Williams result.

THEOREM 6.4. *Suppose* $(X, Y, \mathbf{a})$ *is a reversible Markov chain and that* $X = \cup_{k=0}^{\infty} \Lambda^k$ *where the* Λ^k *are disjoint. Suppose further that* $i \in \Lambda^k$ *and* $a_{ij} > 0$ *together imply* $j \in \Lambda^{k-1} \cup \Lambda^k \cup \Lambda^{k+1}$, *and that for each* k *the sum* $\sum_{i\in\Lambda^k, j\in X} a_{ij} < \infty$. *Let* $[\Lambda^{k-1}, \Lambda^k]$ *denote the* (i, j) *such that* $i \in \Lambda^{k-1}, j \in \Lambda^k$. *The Markov chain* $(X, Y, \mathbf{a})$ *is recurrent if*

$$\sum_{k=0}^{\infty} \Big[\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} a_{ij}\Big]^{-1} = \infty.$$

PROOF. Assume, without loss of generality, that Λ^0 contains a single point i_0 and that u_{ij} is a flow on X with source i_0 and strength 1. (That is to say $\sum_j u_{i_0 j} = 1$ and $\sum_j u_{ij} = 0$ if $i \ne i_0$.) We must prove that u has infinite energy. Suppose that for some k the sum $\sum_{i\in\Lambda^k, j\in X} |u_{ij}|$ is infinite. Then the Cauchy Schwartz estimate used above and the hypothesis that $\sum_{i\in\Lambda^k, j\in X} a_{ij} < \infty$ together imply that $\sum_{i\in\Lambda^k, j\in X} (u_{ij}^2/a_{ij})$ is infinite and the proof is finished. So we may assume that $\sum_{i\in\Lambda^k, j\in X} u_{ij}$ is absolutely convergent for each k. In this case the sum is 1 for each k. To show this we reorder the sum. For $k > 0$ we see that summation over the X-coordinate first gives

$$\sum_{i\in\Lambda^k, j\in X} u_{ij} = 0.$$

On the other hand if i is in Λ^k then $u_{ij} \ne 0$ only if $j \in \Lambda^{k-1} \cup \Lambda^k \cup \Lambda^{k+1}$, moreover $u_{ij} = -u_{ji}$ for all i, j; combining these observations one obtains

$$\sum_{(i,j)\in[\Lambda^k,\Lambda^k+1]} u_{ij} = \sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} u_{ij}$$

and by induction

$$\sum_{(i,j)\in[\Lambda^k,\Lambda^k+1]} u_{ij} = 1 \quad \text{for all} \quad k.$$

Therefore the Cauchy-Schwartz estimate gives us for each k:

$$\left(\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} \frac{u_{ij}^2}{a_{ij}}\right) \geq \left(\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} | u_{ij} |\right)^2 \left(\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} a_{ij}\right)^{-1}$$

$$\geq \left(\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} a_{ij}\right)^{-1}.$$

Summing over k we obtain the theorem.

In [6] Nash-Williams gives a sort of converse result to this. He shows how any recurrent graph can be modified so that it is in the form required by the theorem. But life is not simple; there is a *recurrent* reversible Markov chain (X, Y_n, a_{ij}) and a partition of X into $\cup_0^\infty \Lambda^k$ (Λ^k disjoint) such that $|\Lambda^k| = 2$ and $\sum_k(\sum_{(i,j)\in[\Lambda^{k-1},\Lambda^k]} a_{ij})^{-1} < \infty$.

While this paper was being revised, Tom Liggett pointed out that a generalised form of Nash Williams criterion is true. Let X be a disjoint union of sets Λ^k, and suppose that $\sum_{i\in\Lambda^k, j\in X} a_{ij} < \infty$ for each k. Let $\tilde{a}_{km} = \sum_{i\in\Lambda^k, j\in\Lambda^m} a_{ij}$. If the random walk on the set of Λ^k induced by the $\tilde{a}_{km}$ is recurrent then the random walk on X is recurrent. The proof is essentially the same.

5. A better test for transience. To use Royden's criterion to prove transience of a particular Markov chain, one must find a flow with a single source and finite energy. This restriction can lead to considerable algebraic complexity in the description of the flow. In fact it suffices that there be a net influx of material into the network, without constraint on the number of sources or sinks.

THEOREM. *Let (X, Y_n, a_{ij}) be an irreducible and reversible Markov chain. Suppose there exist real numbers $(u_{ij})_{i,j\in X}$ with the following properties.*

(i) $u_{ij} = -u_{ji}$,

(ii) $\sum_{i\in X} | \sum_{j\in X} u_{ij} | < \infty$ and $\sum_{i\in X} (\sum_{j\in X} u_{ij}) \neq 0$,

(iii) $\sum_{i\in X} \sum_{j\in X} u_{ij}^2/a_{ij} < \infty$.

Then Y is transient. The converse follows trivially from Royden's criterion.

PROOF. Suppose such a flow u exists. By a simple modification of the flow we may assume that there exists an $i_0 \in X$ such that

$$\sum_j u_{i_0 j} > \sum_{i\neq i_0} | \sum_j u_{ij} |.$$

As before let H be the Hilbert space of all vectors satisfying (i) and (iii), and let $\ell^i \in H$ be the vector which gives

$$\langle \ell^i, \mathbf{v} \rangle = \sum_j v_{ij}.$$

Our object will be to construct $\mathbf{w} \in H$ such that

$$\langle \ell^i, \mathbf{w} \rangle = \langle \ell^i, \mathbf{u} \rangle \quad \forall\, i \neq i_0$$

and

$$|\langle \ell^{i_0}, \mathbf{w} \rangle| \leq \sum_{i\neq i_0} |\langle \ell^i, \mathbf{u} \rangle|.$$

The vector $\mathbf{u} - \mathbf{w}$ would satisfy Royden's criterion and the theorem would be proved. The vector $\mathbf{w}$ will now be constructed as a weak limit.

Choose a finite subset F of X containing i_0 and such that the restriction of a_{ij} to F is an irreducible Markov chain. Let F^* denote $F\backslash\{i_0\}$. Let $\mathbf{w}_F$ be the vector in H with minimal norm subject to the constraints

$$\langle \ell^i, \mathbf{w} \rangle = \langle \ell^i, \mathbf{u} \rangle \quad \forall\, i \in F^*.$$

As we allow F to vary, the norm of $\mathbf{w}_F$ stays uniformly bounded. Let F_n increase to X, and let $\mathbf{w}$ be a weak limit point of the $(\mathbf{w}_{F_n})$. Certainly $\langle \ell^i, \mathbf{w} \rangle = \langle \ell^i, \mathbf{u} \rangle$ for all i in $X\backslash\{i_0\}$. We will have completed our argument if we show that

$$|\langle \ell^{i_0}, \mathbf{w}_F \rangle| \le \sum_{i\in F^*} |\langle \ell^i, \mathbf{w}_F \rangle|$$

for any choice of F. The remainder of our argument will be a proof of this fact.

The minimality of $\mathbf{w}_F$ implies the existence of a function $\phi: X \to R$ such that if $(w_{ij})_{i,j\in X} = \mathbf{w}_F$ then

(a) $w_{ij} = a_{ij}[\phi(i) - \phi(j)]$,
(b) ϕ is zero on $X\backslash F^*$.

Property (a) follows from an identical argument to that used in the proof of Royden's criterion. Property (b) can be seen as follows—firstly ϕ as determined by (a) is only defined up to a constant; therefore we may as well assume $\phi(i_0) = 0$. Fix any $i \in X\backslash F$. If $a_{i_0 i} > 0$ then the fact that $\mathbf{w}_F$ has minimal norm implies that $w_{i_0 i} = 0$ and so $\phi(i) = \phi(i_0) = 0$. On the other hand if $a_{i_0 i} = 0$ we simply change it so that it is strictly positive. Imbedding the old Hilbert space (with $a_{i_0 i} = 0$) into the new one in the natural way, we see that if we do not change the constraint then $\mathbf{w}_F$ is minimal in this larger space also. Therefore $\phi(i) = \phi(i_0) = 0$ as required. To obtain our required bound on $|\langle \ell^{i_0}, \mathbf{w}_F \rangle|$ we decompose ϕ into two "potentials".

Let j be in F^* and define $g(i, j)$ to be the $\mathbb{P}^i$ expected number of visits Y makes to j before leaving F^*. (If $i = j$ then we count the starting point as the first visit). Then

$$\phi(i) = \sum_{j\in F^*} g(i,j) \frac{\langle \ell^j, \mathbf{w}_F \rangle}{\langle \ell^j, \ell^j \rangle}.$$

To prove this let $\tilde{\phi}$ denote the right hand expression and T the first exit time of Y from F^*. Then $(\phi - \tilde{\phi})(Y_{n\wedge T})$ is a martingale but $\phi = \tilde{\phi}$ is zero off F^* and so $\phi = \tilde{\phi}$. Because $\varphi(i_0) = 0$

$$\langle \ell^{i_0}, \mathbf{w}_F \rangle = -\sum_{j\in F^*} a_{i_0 j} \phi(j),$$

and so (because the function $g \ge 0$)

$$|\langle \ell^{i_0}, w_F \rangle| \le \sum_{j\in F^*} \sum_{k\in F^*} a_{i_0 j} g(j,k) \frac{|\langle \ell^k, \mathbf{w}_F \rangle|}{\langle \ell^k, \ell^k \rangle} \le \sum_k |\langle \ell^k, \mathbf{w}_F \rangle|$$

as required. This last inequality holds because

$$\sum_{i\notin F^*} \sum_{j\in F^*} a_{ij}\, g(j,k) = \langle \ell^k, \ell^k \rangle.$$

Crudely this expression says that the amount of fluid put into the finite system F^* at k is balanced by an equal amount coming out at the edges of F^*. It is proved by putting $u_{ij} = a_{ij}(g(i,k) - g(j,k))$. Then

$$\sum_{j\in F^*} \sum_{i\in X} u_{ji} = \sum_i u_{ki} = \pi_k = \langle \ell^k, \ell^k \rangle.$$

Because the sum is absolutely convergent we may cancel the terms $u_{ij} + u_{ji}$ which appear. If we do this we obtain the required formulae

$$\langle \ell_k, \ell_k \rangle = \sum_{j\in F^*} \sum_{i\notin F^*} a_{ji}[g(j,k) - g(i,k)] = \sum_{j\in F^*} \sum_{i\notin F^*} a_{ij} g(j,k)$$

and so

$$\langle \ell_k, \ell_k \rangle > \sum_{j\in F^*} a_{i_0 j} g(j,k)$$

as we required.

6. An application. It is well known that the random walk on $\mathbb{Z}^3$ is transient. Both forms of the criterion given here lead to simple proofs. An exact flow [5] can be constructed using the Green function $1/r$ for $\mathbb{R}^3$. Divide $\mathbb{R}^3$ up into disjoint unit cubes centered at the points of $\mathbb{Z}^3$. Put the flow between the adjacent points $\mathbf{u}, \mathbf{v}$ of $\mathbb{Z}^3$ equal to the integral of $(\nabla 1/r) \cdot (\mathbf{u} - \mathbf{v})$ across the common face of the cubes centered at $\mathbf{u}, \mathbf{v}$. The magnitude of the flow is of order $1/r^2$ and $\int_\varepsilon^\infty [1/r^2]^2 r^2\, dr < \infty$ so the flow has finite energy.

A more naive approach is to put the flow between $\mathbf{i}$ and $\mathbf{j}$ equal to $[1/|\mathbf{i}| - 1/|\mathbf{j}|] \wedge 2$ where $\mathbf{i}$ and $\mathbf{j}$ are adjacent points of $\mathbb{Z}^3$. It is easy to check that this flow satisfies all the requirements of our relaxed form of Royden's criterion described in the last section. The strength of the source at $\mathbf{j} \neq 0$ should be estimated using Taylor's theorem and the fact that $\Delta 1/r = 0$ (for $r \neq 0$); it is of order $1/|\mathbf{j}|^5$.

We wish to take the discussion further, and discuss the transience of random walks on some straightforward subsets of $\mathbb{Z}^d$. In particular, how much does one have to "fatten" a 1- or 2-dimensional walk before it becomes transient?

More precisely let $x^{(1)}(n), \cdots, x^{(k)}(n)$ be a collection of k positive increasing integer valued functions on $\{n \in \mathbb{Z}, n \geq 0\}$. Suppose further that $x^i(0) = 0$, and that $x^i(n+1) - x^i(n) \leq 1$ for each i. The following theorem is true.

THEOREM. *Let $\Omega = \{(\gamma_1, \cdots, \gamma_k, n) \in \mathbb{Z}^{k+1} \mid |\gamma_i| \leq x^{(i)}(n)\ \forall\, i\}$. Then the restriction of the simple random walk on $\mathbb{Z}^{k+1}$ to Ω is transient if and only if*

$$\sum_{n=0}^{\infty} \frac{1}{\prod_{i=1}^{k} (x^{(i)}(n) + 1)} < \infty.$$

REMARK. Interesting special cases of this are $x^{(1)}(n) = n$, $x^{(2)}(n) = n$. In this case Ω is a wedge in $\mathbb{Z}^3$, and $\sum_1^\infty 1/n^2 < \infty$ so the random walk on $\mathbb{Z}^3$ is again seen to be recurrent. Somewhat more interesting is to take $x^{(1)}(n) = n$, and $x^{(2)}(n)$ to be the integer part of $[\log(n+1)]^\alpha$. Then $\Omega \subset \mathbb{Z}^3$ will have a transient random walk of $\alpha > 1$. But Ω is just a slight fattening of the quadrant in $\mathbb{Z}^2$. This reflects the fact that the random walk on $\mathbb{Z}^2$ only just fails to be transient.

PROOF. The convergence of the sum is clearly required if the random walk on Ω is to be transient. That much follows directly from Nash-Williams' result. To complete the proof, we will sketch the construction of a flow on Ω whose energy is comparable with

$$\sum_{n=0}^{\infty} \frac{1}{\prod_{i=1}^{k} (x^{(i)}(n) + 1)}.$$

We do this in two stages. In the first we construct flows on the $\Omega^i = \{(\gamma, n) \mid |\gamma| \leq x^{(i)}(n)\}$ and then we will describe a method for taking products of such flows. (Of course no flow on Ω^i can have finite energy.)

First construct flow $\mathbf{u}$ on the quadrant $\Omega^0 = \{(\gamma, n) \mid |\gamma| \leq n\}$ in $\mathbb{Z}^2$ with a single source at $(0, 0)$ and such that

$$u((\gamma, n), (\gamma, n+1)) = \frac{1}{2n+1} \qquad (\gamma, n) \in \Omega^0,$$

$$|u((\gamma, n), (\gamma', n))| \leq \frac{1}{2n+1} \qquad |\gamma - \gamma'| = 1,$$

and

$$|u((\gamma, n), (\gamma', n))| = 0 \qquad |\gamma - \gamma'| \neq 1.$$

We may "stretch" this flow to form a flow $\mathbf{u}^{(i)}$ on Ω^i by adding extra pipes parallel to the

n-axis. Then

$$u^{(i)}((\gamma, n), (\gamma, n+1)) = \frac{1}{2x^{(i)}(n)+1}, \qquad (\gamma, n) \in \Omega^i,$$

$$|u^{(i)}((\gamma, n), (\gamma', n))| \leq \frac{1}{2x^{(i)}(n)+1}, \qquad |\gamma - \gamma'| = 1,$$

and

$$|u^{(i)}((\gamma, n), (\gamma', n))| = 0 \qquad |\gamma - \gamma'| \neq 1.$$

At the end of this section we describe a method for constructing product flows. It will be clear that $\mathbf{w} = \mathbf{u}^{(1)} * \mathbf{u}^{(2)} * \cdots * \mathbf{u}^{(k)}$ is a flow on Ω with the following properties

$$\mathbf{w}((\mathbf{i}, n), (\mathbf{i}, n+1)) = \frac{2^k}{\prod_{m=1}^{k}(2x^{(m)}(n)+1)} \qquad (\mathbf{i}, n) \in \Omega,$$

$$|\mathbf{w}((\mathbf{i}, n), (\mathbf{j}, n))| \leq \frac{C_k}{\prod_{m=1}^{k}(2x^{(m)}(n)+1)} \qquad |\mathbf{i} - \mathbf{j}| = 1$$

and

$$|\mathbf{w}((\mathbf{i}, n), (\mathbf{j}, n))| = 0 \qquad |\mathbf{i} - \mathbf{j}| \neq 1.$$

To compute the energy of the flow $\mathbf{w}$ we see that we are considering, for each n, $O(\prod_{m=1}^{k}(2x^{(m)}(n)+1))$ terms of order $1/(\prod_{m=1}^{k}(2x^{(m)}(n)+1))^2$. Summing over n we see that the energy of the flow is comparable to

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{m=1}^{k}[2x^{(m)}(n)+1]}.$$

The construction of a product of two flows has nothing to do with probability. Let G, H be graphs. Define $G*H$ to be the graph whose vertex set is the Cartesian product of the vertex sets for G and H, and let two vertices (g, h) and (g', h') be connected by an edge if and only if $g = g'$, and h is connected to h' in H or if $h = h'$ and g is connected to g' in G. For example $\mathbb{Z}^d$ connected by the nearest neighbour relation is $\mathbb{Z}*\mathbb{Z}^{d-1}$.

Let $\mathbf{u}$ be a flow on $G*\mathbb{Z}$ and $\mathbf{v}$ be a flow on $H*\mathbb{Z}$. We now define a flow $\mathbf{w} = \mathbf{u}*\mathbf{v}$ on $G*H*\mathbb{Z}$ as follows:

$$w((g, h, n), (g, h, n \pm 1)) = \pm 2u((g, n), (g, n \pm 1))v((h, n), (h, n \pm 1)),$$

$$w((g, h, n), (g', h, n)) = u((g, n), (g', n))[v((h, n), (h, n+1)) - v((h, n), (h, n-1))],$$

$$w((g, h, n), (g, h', n)) = v((h, n), (h', n))[u((g, n), (g, n+1)) - u((g, n), (g, n-1))],$$

whenever g is connected to g' in G or h to h' in H. Set $w \equiv 0$ elsewhere.

The main point of this construction is that if $\mathbf{u}$ has no source or sink at (g, n) and $\mathbf{v}$ has none at (h, n) then $\mathbf{w}$ has none at (g, h, n); moreover the flows in the n-direction are easily calculable.

If $\mathbf{u}^{(i)}$ are the flows previously defined on $\Omega^{(i)}$, then extend them to $\mathbb{Z}*\mathbb{Z}$ by setting them equal to zero off the $\Omega^{(i)}$'s. Define $\mathbf{w}$ to be $\mathbf{u}^{(1)} * \cdots * \mathbf{u}^{(k)}$. Then because one of the $\mathbf{u}^{(i)}$ will be zero at any point not in Ω, it is clear that $\mathbf{w} = \mathbf{u}^{(1)} * \cdots * \mathbf{u}^{(k)}$ is zero off Ω and $\mathbf{w}$ is really supported on Ω; moreover its only source is $(0, \cdots, 0)$. The theorem now follows from the earlier remarks.

Acknowledgments. The author would like to thank Tom Liggett for many helpful conversations and for his critical comments on this manuscript, and also to thank the referee for some useful suggestions.

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