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TIGHTNESS OF PROBABILITIES ON $C([0, 1]; \mathscr{S}')$ AND $D([0, 1]; \mathscr{S}')$

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Let $C_{\mathscr{S}'} = C([0, 1]; \mathscr{S}')$ be the space of all continuous mappings of [0, 1] to $\mathscr{S}'$, where $\mathscr{S}'$ is the topological dual of the Schwartz space $\mathscr{S}$ of all rapidly decreasing functions. Let C be the Banach space of all continuous functions on [0, 1]. For each $\varphi \in \mathscr{S}$, Π_φ is defined by $\Pi_\varphi: x \in C_{\mathscr{S}'} \to x.(\varphi) \in C$. Given a sequence of probability measures $\{P_n\}$ on $C_{\mathscr{S}'}$ such that for each $\varphi \in \mathscr{S}$, $\{P_n \Pi_\varphi^{-1}\}$ is tight in C, we prove that $\{P_n\}$ itself is tight in $C_{\mathscr{S}'}$. A similar result is proved for the space of all right continuous mappings of [0, 1] to $\mathscr{S}'$.

1. Introduction. Recently some types of limit theorems for $\mathscr{S}'$-valued stochastic processes connected with the system of infinite particles have been studied by several authors [3], [4], [6], [10] and others. In this paper, motivated by their works, we will give a simple sufficient condition for the tightness of a certain class of $\mathscr{S}'$-valued stochastic processes. We will discuss our purposes in a context of E'-valued stochastic processes, where E' is the topological dual of a nuclear Fréchet space E. Of course $\mathscr{S}'$-valued stochastic processes are typical examples of them.

Let $C_{E'} = C([0, 1]; E')$ be the space of all continuous mappings of [0, 1] to E'. For $\xi \in E$, we denote by Π_ξ the mapping of $C_{E'}$ to C(= the space of real continuous functions) defined by

$$\Pi_\xi: x \in C_{E'} \to \langle x., \xi \rangle \in C,$$

where $\langle x, \xi \rangle$ denotes the canonical bilinear form on $E' \times E$. We are concerned with the tightness of a sequence of probability measures $\{P_n\}$ on $C_{E'}$. We will prove in Section 3 that if for each $\xi \in E$, the sequence $\{P_n \Pi_\xi^{-1}\}$ is tight in C, $\{P_n\}$ itself is tight in $C_{E'}$. In the course of the proof, the nuclear property of the space E plays an essential role like in the case of the Minlos-Sazonov theorem. A similar result will be discussed for the space of all right continuous mappings of [0, 1] to E'.

As an application of these results, we will discuss the convergence of sequences of E'-valued stochastic processes (Theorem 5.3).

2. Spaces of $C([0, 1]; E')$ and $D([0, 1]; E')$. Let E be a Fréchet space whose topology is defined by an increasing sequence of Hilbertian semi-norms $\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots \le \|\cdot\|_p \le \cdots$. Let E_p be the completion of E by $\|\cdot\|_p$, E'_p the topological dual of E_p and $\|\cdot\|_{-p}$ the dual norm of E'_p. The space E is called nuclear if for each $n \in \mathbf{N}$ (natural numbers) there exists a natural number $m > n$ such that the canonical mapping ι_{mn}; $E_m \to E_n$ is nuclear (Schaefer [8]). We always assume that E is a nuclear Fréchet space.

Since E is separable, there exists a countable dense subset $\{\xi_i\}$ of E. For each $n \in \mathbf{N}$, we choose a complete orthonormal system $\{e_j^n\}$ of E_n by the Schmidt orthogonalization of $\{\xi_i\}$ successively. Then it is evident that

$$\xi_i = \sum_{j=1}^{m(n,i)} a_j^n(i) e_j^n + \theta_i^n, \text{ where } m(n, i) \le i \quad (2.1)$$
$$\text{and } \|\theta_i^n\|_n = 0.$$

Let $C_{E'} = C([0, 1); E')$ and $C_{E'_p} = C([0, 1]; E'_p)$ be the spaces of all continuous mappings of [0, 1] to E' with the strong topology and of [0, 1] to E'_p with the $\|\cdot\|_{-p}$-topology

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respectively. Let $\{|\cdot|_\lambda; \lambda \in \Lambda\}$ be the set of semi-norms defining the strong topology of E'. Set

$$|||x|||_\lambda = \sup_t |x_t|_\lambda,\ x \in C_{E'}.$$

(Except for the case where we write the index set of t, the supremum is taken over [0, 1].) We will introduce on $C_{E'}$ the projective limit topology of $\{|||\cdot|||_\lambda; \lambda \in \Lambda\}$. Then $C_{E'}$ becomes a completely regular topological space. Of course $C_{E'_p}$ is the Banach space with the uniform norm topology.

To characterize the compact sets of $C_{E'}$, we prepare the following moduli. Let C be the Banach space of all real continuous functions on [0, 1].

The modulus of continuity of $f \in C$ is defined by

$$W_f(\delta) = \sup_{|t-s|<\delta} |f(t) - f(s)|, \quad 0 < \delta < 1.$$

For $g \in C_{E'}$ and $h \in C_{E'_p}$, the moduli are defined similarly as follows;

$$W_g(\delta; \xi) = \sup_{|t-s|<\delta} |\langle g_t, \xi\rangle - \langle g_s, \xi\rangle|, \quad 0 < \delta < 1, \xi \in E,$$

$$W_h(\delta; p) = \sup_{|t-s|<\delta} \|h_t - h_s\|_{-p}, \quad 0 < \delta < 1.$$

Now we will show:

Proposition 2.1. *If A is compact in $C_{E'}$, then there exists a $p \in \mathbf{N}$ such that A is compact in $C_{E'_p}$.*

Proof. For each ξ in E the set $\{\langle x_\cdot, \xi\rangle; x \in A\}$ is compact in C by the assumption. By the Ascoli-Arzelá theorem, the following properties hold:

$$\sup_{x\in A}\sup_t |\langle x_t, \xi\rangle| < +\infty. \tag{2.2}$$

$$\lim_{\delta\to 0}\sup_{x\in A} W_x(\delta; \xi) = 0. \tag{2.3}$$

Then the Banach-Steinhaus theorem and (2.2) tells us that there exist a $q \in \mathbf{N}$ and an $L > 0$ such that

$$\sup_{x\in A}\sup_t |\langle x_t, \xi\rangle| \le L\|\xi\|_q.$$

Since E is nuclear, there exists a natural number $r > q$ such that $\sum_{j=1}^\infty \|e_j^r\|_q^2 < +\infty$, so that we have

$$\begin{aligned}\sup_{x\in A}\sup_t \|x_t\|_{-r}^2 &= \sup_{x\in A}\sup_t (\textstyle\sum_{j=1}^\infty \langle x_t, e_j^r\rangle^2)\\ &\le \textstyle\sum_{j=1}^\infty L^2\|e_j^r\|_q^2 = l < +\infty.\end{aligned} \tag{2.4}$$

Since $\sup_{x\in A} W_x(\delta; e_j^r)^2 \le 4L^2\|e_j^r\|_q^2$, then by (2.3) and the Lebesgue convergence theorem we get

$$\begin{aligned}\lim_{\delta\to 0}\sup_{x\in A} W_x(\delta; r) &= \lim_{\delta\to 0}\sup_{x\in A}\sup_{|t-s|<\delta} (\textstyle\sum_{j=1}^\infty \langle x_t - x_s, e_j^r\rangle^2)^{1/2}\\ &\le \lim_{\delta\to 0}(\textstyle\sum_{j=1}^\infty \sup_{x\in A} W_x(\delta; e_j^r)^2)^{1/2}\\ &= (\textstyle\sum_{j=1}^\infty \lim_{\delta\to 0}\sup_{x\in A} W_x(\delta; e_j^r)^2)^{1/2} = 0.\end{aligned} \tag{2.5}$$

Again since E is nuclear, there exists a natural number $p > r$ such that $\sum_{j=1}^\infty \|e_j^p\|_r^2 < +\infty$. Then it follows from (2.4) that

$$\lim_{N\to\infty} \sum_{j=N}^\infty \sup_{x\in A}\sup_t \langle x_t, e_j^p\rangle^2 \le \lim_{N\to\infty} \sum_{j=N}^\infty l\|e_j^p\|_r^2 = 0,$$

so that

(2.6) the set $\{x_t; x \in A\}$ has compact closure in E'_p for each $t \in [0, 1]$.

Since $\|\cdot\|_{-r} \geq \|\cdot\|_{-p}$, by (2.5) we get

$$\lim_{\delta\to 0}\sup_{x\in A} W_x(\delta; p) = 0. \tag{2.7}$$

Therefore by (2.6), (2.7) and the Ascoli-Arzelà theorem, we obtain that A has compact closure in $C_{E'_p}$. But A is automatically closed in $C_{E'_p}$ by definition of the toplology on $C_{E'}$. Thus the proof is completed.

Let $D_{E'} = D([0, 1]; E')$ be the space of all mappings of [0, 1] to E' that are right continuous and have left-hand limits in the strong topology of E', and $D_{E'_p} = D([0, 1]; E'_p)$ be the complete separable metric space with the Skorohod topology of all mappings of [0, 1] to E'_p that are right continuous and have left-hand limits in the $\|\cdot\|_{-p}$-topology. Let Φ be the set of all strictly increasing continuous mappings of [0, 1] onto itself. Following P. Billingsley [2], (page 112), set

$$d_\lambda(x, y) = \inf_{\phi\in\Phi}\left\{\sup_t |x_t - y_{\phi(t)}|_\lambda + \sup_{t\neq s, t,s\in[0,1]}\left|\log\frac{\phi(t)-\phi(s)}{t-s}\right|\right\}, \quad x, y\in D_{E'}.$$

We will introduce on $D_{E'}$ the projective limit topology of $\{d_\lambda(\cdot, \cdot); \lambda\in\Lambda\}$. Then $D_{E'}$ also becomes a completely regular topological space.

To characterize the compact sets of $D_{E'}$ we prepare the following moduli. Let D be the usual Skorohod space of all real right continuous functions with left-hand limits on [0, 1].

For $f\in D$, a modulus corresponding to the role of the modulus of continuity in C is defined by

$$W'_f(\delta) = \inf_{\{t_i\}}\max_{1\leq i\leq n}\sup\{|f(t) - f(s)|; t, s\in[t_{i-1}, t_i)\}, \quad 0<\delta<1,$$

where the infimum is taken over the finite sets $\{t_i\}$ of points satisfying

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_i - t_{i-1} > \delta, \quad i = 1, 2, \cdots, n.$$

For $g\in D_{E'}$, the moduli are defined similarly as follows;

$$W'_g(\delta; \xi) = \inf_{\{t_i\}}\max_{1\leq i\leq n}\sup\{|\langle g_t, \xi\rangle - \langle g_s, \xi\rangle|; t, s\in[t_{i-1}, t_i)\},$$
$$0<\delta<1, \quad \xi\in E,$$

$$W'_g(\delta; \langle\xi_1, \xi_2, \cdots, \xi_m\rangle) = \inf_{\{t_i\}}\max_{1\leq i\leq n}\sup\{(\textstyle\sum_{j=1}^m \langle g_t - g_s, \xi_j\rangle^2)^{1/2}; t, s\in[t_{i-1}, t_i)\},$$
$$0<\delta<1, \xi_j\in E, j = 1, 2, \cdots, m.$$

PROPOSITION 2.2 *If A is compact in $D_{E'}$, then there exists a $p\in\mathbf{N}$ such that A is compact in $D_{E'_p}$.*

PROOF. For each ξ in E the set $\{\langle x_\cdot, \xi\rangle; x\in A\}$ is compact in D by the assumption. Then by Theorem 14.3 of [2], we have

$$\sup_{x\in A}\sup_t|\langle x_t, \xi\rangle| < +\infty \tag{2.8}$$

and

$$\lim_{\delta\to 0}\sup_{x\in A} W'_x(\delta; \xi) = 0. \tag{2.9}$$

Therefore by (2.8) and the argument quite similar to the above proof we get that there exists a $p\in\mathbf{N}$ such that

$$\sup_{x\in A}\sup_t\|x_t\|_{-p} < +\infty, \tag{2.10}$$

$$\lim_{N\to\infty}\textstyle\sum_{j=N}^\infty \sup_{x\in A}\sup_t\langle x_t, e_j^p\rangle^2 = 0. \tag{2.11}$$

By making use of (2.9) we will prove that for each $m\in\mathbf{N}$,

$$\lim_{\delta\to 0}\sup_{x\in A} W'_x(\delta; \langle e_1^p, e_2^p, \cdots, e_m^p\rangle) = 0. \tag{2.12}$$

For $g \in D_{E'}$ we will define the following moduli;

$$W''_g(\delta;\xi) = \sup\{\min(|\langle g_t, \xi\rangle - \langle g_{t_1}, \xi\rangle|, |\langle g_{t_2}, \xi\rangle - \langle g_t, \xi\rangle|); \quad 0 \le t_1 \le t \le t_2 \le 1, |t_1 - t_2| \le \delta\}, 0 < \delta < 1, \xi \in E,$$

$$W''_g(\delta; \langle \xi_1, \xi_2, \cdots, \xi_m\rangle) = \sup\{\min((\textstyle\sum_{j=1}^m \langle g_t - g_{t_1}, \xi_j\rangle^2)^{1/2}, (\sum_{j=1}^m \langle g_{t_2} - g_t, \xi_j\rangle^2)^{1/2}); \quad 0 \le t_1 \le t \le t_2 \le 1, |t_1 - t_2| \le \delta\}, \quad 0 < \delta < 1, \xi_j \in E, j = 1, 2, \cdots, m.$$

Then by Lemma (A.28) (page 391) of R. Holley and D. W. Stroock [3], for each $m \in \mathbf{N}$ there exist τ_i^m, $i = 1, 2, \cdots, k_m$ such that

$$W''_g(\delta; \langle e_1^p, e_2^p, \cdots, e_m^p\rangle) \le 2 \sup_{i=1,2,\dots,k_m} W''_g(\delta; \tau_i^m), \tag{2.13}$$

where $\tau_i^m = \sum_{j=1}^m \alpha_{i,j}^m e_j^p$ and $\alpha_{i,j}^m$, $j = 1, 2, \cdots, m$ are real numbers. Since $W''_g(\delta;\xi) \le W'_g(\delta;\xi)$ (page 119 of [2]), then for each $m \in \mathbf{N}$, by (2.9) and (2.13) we have

$$\begin{cases} \lim_{\delta\to 0}\sup_{x\in A} W''_x(\delta; \langle e_1^p, e_2^p, \cdots, e_m^p\rangle) = 0, \\ \lim_{\delta\to 0}\sup_{x\in A}\sup\{(\sum_{j=1}^m \langle x_t - x_s, e_j^p\rangle^2)^{1/2}; t, s \in [0, \delta)\} = 0, \\ \lim_{\delta\to 0}\sup_{x\in A}\sup\{(\sum_{j=1}^m \langle x_t - x_s, e_j^p\rangle^2)^{1/2}; t, s \in [1-\delta, 1)\} = 0. \end{cases} \tag{2.14}$$

Hence by the argument similar with the proof of Theorem 14.4 (page 119) of [2] and (2.14), we get (2.12).

Now we will prove that A is totally bounded with respect to the metric $d(\cdot,\cdot)$ defined by

$$d(x, y) = \inf_{\phi\in\Phi}\{\sup_t \|x_t - y_{\phi(t)}\|_{-p} + \sup_t|\phi(t) - t|\}, \quad x, y \in D_{E'_p}.$$

For any $\varepsilon > 0$, by (2.11) there exists an $N_0 \in \mathbf{N}$ such that

$$\textstyle\sum_{j=N_0}^\infty \sup_{x\in A}\sup_t\langle x_t, e_j^p\rangle^2 < \dfrac{\varepsilon^2}{16}. \tag{2.15}$$

If we change ε, α and $W'_x(\delta)$ in the proof of Theorem 14.3 of [2] for $\varepsilon/4N_0$, $\sup_{x\in A}\sup_t\|x_t\|_{-p}$ and $W'_x(\delta; \langle e_1^p, e_2^p, \cdots, e_{N_0-1}^p\rangle)$ and follow that proof, we see that there exists a finite subset $B \subset D$ and for each $x \in A$ there exists a $\phi \in \Phi$ satisfying the following property:
(2.16) For each $\langle x_., e_j^p\rangle$, $(j = 1, 2, \cdots, N_0 - 1)$, there exists a $y_j \in B$ such that

$$\sup_t|y_j(t) - \langle x_{\phi(t)}, e_j^p\rangle| + \sup_t|\phi(t) - t| < \frac{3\varepsilon}{4N_0}.$$

Let $\{x_j^p\}$ be a sequence of elements in E'_p such that $\langle x_j^p, e_i^p\rangle = \delta_{ji}$, where $\delta_{ji} = 1$ if $j = i$ and $\delta_{ji} = 0$ if $j \ne i$. Set $y = \sum_{j=1}^{N_0-1} y_j x_j^p$, then $y \in D_{E'_p}$ and by (2.15) and (2.16) we have

$$\begin{aligned} d(x, y) &\le \sup_t(\textstyle\sum_{j=1}^{N_0-1} (y_j(t) - \langle x_{\phi(t)}, e_j^p\rangle)^2 + \sum_{j=N_0}^\infty \langle x_{\phi(t)}, e_j^p\rangle^2)^{1/2} + \sup_t|\phi(t) - t| \\ &\le \textstyle\sum_{j=1}^{N_0-1} (\sup_t|y_j(t) - \langle x_{\phi(t)}, e_j^p\rangle| + \sup_t|\phi(t) - t|) + (\sum_{j=N_0}^\infty \sup_t \langle x_{\phi(t)}, e_j^p\rangle^2)^{1/2} \\ &< (N_0 - 1)\frac{3\varepsilon}{4N_0} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

This shows the totally boundedness. The rest of proving that A has compact closure in $D_{E'_p}$ is quite similar to that in the proof of Theorem 14.3 of [2], which completes the proof similarly as before.

Before we proceed to the following sections we give the definitions of weak convergence and tightness of probability measures P_n, $n \in \mathbf{N}$ and P on $\mathscr{B}_Z$ which denotes the Borel field on a topological space Z. Let $\mathbf{X}^n$, $n \in \mathbf{N}$ and $\mathbf{X}$ be Z-valued random variables.

If $\int_Z f dP_n \to \int_Z f dP$ for every bounded continuous real function f on Z, we say that P_n

converges weakly to P and write $P_n \Rightarrow P$. If the distribution P_n of $\mathbf{X}^n \Rightarrow$ the distribution P of $\mathbf{X}$, we say that $\mathbf{X}^n$ converges in law to $\mathbf{X}$ and write $\mathbf{X}^n \to_{\mathscr{L}} \mathbf{X}$. The sequence $\{P_n\}$ is said to be tight in Z if for any $\varepsilon > 0$ there exists a compact set K of Z such that $P_n(K) \geq 1 - \varepsilon$ for all $n \geq 1$.

3. Tightness in $C([0, 1]; E')$. Let $\{P_n\}$ be a sequence of probability measures on $(C_{E'}, \mathscr{B}_{C_{E'}})$. For each ξ in E we denote by Π_ξ the mapping of $C_{E'}$ to C defined by

$$\Pi_\xi : x \in C_{E'} \to \langle x., \xi \rangle \in C.$$

Then we have

THEOREM 3.1. *Suppose that for each ξ in E the sequence $\{P_n\Pi_\xi^{-1}\}$ is tight in C. Then the sequence $\{P_n\}$ itself is tight in $C_{E'}$.*

PROOF. Since the sequence $\{P_n\Pi_\xi^{-1}\}$ is tight in C, by Theorem 8.2 of [2] the following two conditions hold:

(3.1) For each $\varepsilon > 0$ there exists an $a = a_\xi$ such that

$$P_n\Pi_\xi^{-1}(f \in C; \sup_t |f(t)| > a)$$
$$= P_n(x \in C_{E'}; \sup_t |\langle x_t, \xi \rangle| > a) \leq \varepsilon, \quad n \geq 1.$$

(3.2) For each $\varepsilon > 0$ and $\rho > 0$, there exist a $\delta = \delta_\xi$, $(0 < \delta < 1)$ and an $n_0 = n_0(\xi) \in \mathbf{N}$ such that

$$P_n\Pi_\xi^{-1}(f \in C;\ W_f(\delta) \geq \varepsilon) = P_n(x \in C_{E'};\ W_x(\delta; \xi) \geq \varepsilon) \leq \rho, \quad n \geq n_0.$$

By (3.1) we get

LEMMA 3.2. *For any $\varepsilon > 0$ there exist an $r \in \mathbf{N}$ and an M, $(0 < M < +\infty)$ such that*

$$P_n(x \in C_{E'}; \sup_t \| x_t \|_{-r} \leq M) \geq 1 - \varepsilon/2, \quad n \geq 1. \tag{3.3}$$

This lemma is proved along the same line of the proof of Theorem 1 of [7] so that we give a sketch of the proof.

To prove this lemma we use

LEMMA 3.3. *For any $\rho > 0$ there exist a $q \in \mathbf{N}$ and a $\delta > 0$ such that*

$$\sup_n \int_{C_{E'}} \sup_t |1 - e^{i\langle x_t, \xi \rangle}|\, dP_n \leq \rho + 2\frac{\| \xi \|_q^2}{\delta^2}. \tag{3.4}$$

PROOF. To prove the lemma we will introduce the following;

$$M(\xi) = \sup_n \int_{C_{E'}} \frac{\sup_t |\langle x_t, \xi \rangle|}{1 + \sup_t |\langle x_t, \xi \rangle|}\, dP_n, \ \xi \in E.$$

Then $M(\xi)$ has the following properties.

1) $M(\xi) \geq 0$ and $M(-\xi) = M(\xi)$.
2) $M(\xi + \eta) \leq M(\xi) + M(\eta)$ for any ξ, η in E.
3) $M(\xi)$ is a lower semi-continuous function on E.
4) $\lim_{n\to\infty} M(\xi/n) = 0$.

Properties 1), 2) and 3) are proved by a manner similar to that of [7]. For the proof of 4), we proceed as follows. For any $\varepsilon > 0$, by (3.1) there exists an $m_0 = m_0(\xi) \in \mathbf{N}$ such that

$\sup_n P_n(x \in C_{E'}; \sup_t |\langle x_t, \xi \rangle| > \sqrt{m_0}) < \varepsilon$. Then if $m > m_0$,

$$M\left(\frac{\xi}{m}\right) = \sup_n\left(\int_{\{x \in C_{E'}; \sup_t|\langle x_t, \xi\rangle| \leq \sqrt{m}\}} \frac{\sup_t |\langle x_t, \xi/m\rangle|}{1 + \sup_t|\langle x_t, \xi/m\rangle|} dP_n + \int_{\{x \in C_{E'}; \sup_t|\langle x_t, \xi\rangle| > \sqrt{m}\}} \frac{\sup_t |\langle x_t, \xi/m\rangle|}{1 + \sup_t|\langle x_t, \xi/m\rangle|} dP_n\right)$$

$$\leq \frac{1}{\sqrt{m}+1} + \sup_n P_n(x \in C_{E'}; \sup_t |\langle x_t, \xi\rangle| > \sqrt{m}) < \frac{1}{\sqrt{m}} + \varepsilon.$$

Letting $m \to \infty$, 4) is proved.

Therefore Lemma 1.2.3. (page 386) of D. Xia [11] tells us that the properties 1), 2), 3) and 4) imply that $M(\xi)$ is continuous at 0 in E. Thus the rest of the proof is similar to that of Lemma 1 of [7].

PROOF OF LEMMA 3.2. For $\varepsilon > 0$ set $\rho = ((\sqrt{e} - 1)/8\sqrt{e})\varepsilon$ in Lemma 3.3. Since E is nuclear, there exists a natural number $r > q$ such that $\sum_{j=1}^{\infty} \| e_j^r \|_q^2 < +\infty$. Then by the first half of the proof of Lemma 2 of [7], it holds for any $n \in \mathbf{N}$ that

$$P_n(x \in C_{E'}; \sup_t \textstyle\sum_{j=1}^{\infty} \langle x_t, e_j^r\rangle^2 > h^2) \leq \frac{\sqrt{e}}{\sqrt{e}-1}\left(\rho + \frac{2}{\delta^2}\left(\frac{\sum_{j=1}^{\infty} \| e_j^r \|_q^2}{h^2}\right)\right).$$

Letting h tend to sufficiently large M, we get

$$\sup_n P_n(x \in C_{E'}; \sup_t \textstyle\sum_{j=1}^{\infty} \langle x_t, e_j^r\rangle^2 > M^2) \leq 2\frac{\sqrt{e}}{\sqrt{e}-1}\rho \leq \frac{\varepsilon}{4}. \tag{3.5}$$

By changing e_j^r for θ_j^r in the above estimation, we have

$$\inf_n P_n(x \in C_{E'}; \sup_t \textstyle\sum_{j=1}^{\infty} \langle x_t, \theta_j^r\rangle^2 = 0) \geq 1 - (\varepsilon/4). \tag{3.6}$$

By (3.5) and (3.6) we have

$$\inf_n P_n(x \in C_{E'}; \sup_t \| x_t \|_{-r} \leq M) \geq 1 - (\varepsilon/2).$$

This completes the proof of Lemma 3.2.

We will now return to the proof of Theorem 3.1. Let ε, r and M be the same as those in Lemma 3.2. Take a natural number $p > r$ such that $\sum_{j=1}^{\infty} \| e_j^p \|_r^2 < +\infty$. For each e_j^p, by (3.2) choose $K_j \subset C_{E'}$ such that

$$P_n(K_j) \geq 1 - \frac{\varepsilon}{2^{j+1}}, \quad n \geq 1, \tag{3.7}$$

$$\lim_{\delta \to 0} \sup_{x \in K_j} W_x(\delta; e_j^p) = 0. \tag{3.8}$$

Put $K = \{x \in C_{E'}; \sup_t \| x_t \|_{-r} \leq M\} \cap \{\cap_{j=1}^{\infty} K_j\}$. Then we get

$$P_n(K) \geq 1 - \varepsilon, \quad n \geq 1. \tag{3.9}$$

Then by the argument similar with the proof of Proposition 2.1, we get

$$\sup_{x \in K} \sup_t \| x_t \|_{-r} \leq M < +\infty$$

and

$$\lim_{\delta \to 0} \sup_{x \in K} W_x(\delta; p) = 0.$$

Thus K has compact closure in $C_{E_p'}$. Since the injection of $C_{E_p'}$ into $C_{E'}$ is continuous, the closure of K in $C_{E_p'}$ is compact in $C_{E'}$. This, together with (3.9), completes the proof of Theorem 3.1.

4. Tightness in $D([0, 1]; E')$. Let $\{P_n\}$ be a sequence of probability measures on $(D_{E'}, \mathscr{B}_{D_{E'}})$. For each ξ in E we also denote by Π_ξ the mapping of $D_{E'}$ to D defined by

$$\Pi_\xi : x \in D_{E'} \to \langle x., \xi \rangle \in D.$$

Then we have

THEOREM 4.1. *Suppose that for each ξ in E the sequence $\{P_n \Pi_\xi^{-1}\}$ is tight in D. Then the sequence $\{P_n\}$ itself is tight in $D_{E'}$.*

PROOF. By the assumption of the theorem and by Theorem 15.2 of [2], the following two conditions hold:

(4.1) For each $\varepsilon > 0$ there exists an $a = a_\xi$ such that

$$P_n \Pi_\xi^{-1}(f \in D; \sup_t | f(t) | > a) = P_n(x \in D_{E'}; \sup_t | \langle x_t, \xi \rangle | > a) \leq \varepsilon, \quad n \geq 1.$$

(4.2) For each $\varepsilon > 0$ and $\rho > 0$, there exist a $\delta = \delta_\xi$, $(0 < \delta < 1)$ and an $n_0 = n_0(\xi) \in \mathbf{N}$ such that

$$P_n \Pi_\xi^{-1}(f \in D; W_f'(\delta) \geq \varepsilon) = P_n(x \in D_{E'}; W_x'(\delta; \xi) \geq \varepsilon) \leq \rho, \quad n \geq n_0.$$

By making use of (4.2), for each $j \in \mathbf{N}$ we choose $\hat{K}_j \subset D_{E'}$ which plays the rôle of K_j in the proof of Theorem 3.1 as it satisfies the following properties;

$$\lim_{\delta \to 0} \sup_{x \in \hat{K}_j} W_x'(\delta; e_j^p) = 0,$$

$$\lim_{\delta \to 0} \sup_{x \in \hat{K}_j} W_x'(\delta; \tau_i^j) = 0, \quad i = 1, 2, \cdots, k_j.$$

Then the proofs of Proposition 2.2 and Theorem 3.1 tells us that the sequence $\{P_n\}$ is tight in $D_{E'}$, which completes the proof.

5. Application. For elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in $[0, 1]$, let $\Pi_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m}$ be the mapping that carries the point x of $C_{E'}$ or $D_{E'}$ to the point $(\langle x_{t_1}, \xi_1 \rangle, \langle x_{t_2}, \xi_2 \rangle, \cdots, \langle x_{t_m}, \xi_m \rangle)$ of $\mathbb{R}^m$ where $\mathbb{R}^m$ is the m-dimensional Euclidean space. Then we have

PROPOSITION 5.1. *Let $\{P_n\}$ be a sequence of probability measures on $C_{E'}$. If for each ξ in E the sequence $\{P_n \Pi_\xi^{-1}\}$ is tight in C and for any finite elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in $[0, 1]$,*

$$P_n(\Pi_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m})^{-1} \Rightarrow Q_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m},$$

where $Q_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m}$ is the probability measure on $\mathbb{R}^m$, then there exists a unique probability measure P on $C_{E'}$ such that $P_n \Rightarrow P$.

PROOF. By the assumption that $\{P_n \Pi_\xi^{-1}\}$ is tight in C and by Theorem 3.1, we get $\{P_n\}$ is tight in $C_{E'}$. Proposition 2.1 implies that compact subsets of the completely regular topological space $C_{E'}$ are all metrizable, so that by Theorem 2 of Section 5 of Smolyanov and Fomin [9], each subsequence of $\{P_n\}$ contains a further subsequence converging weakly. Take two subsequences $\{P_{n^1}\}$ and $\{P_{n^2}\}$ of $\{P_n\}$. Then $\{P_{n^1}\}$ contains a subsequence $\{P_{n_\nu^1}\}$ converging weakly to Q_1 and $\{P_{n^2}\}$ contains a subsequence $\{P_{n_\mu^2}^\nu\}$ converging weakly to Q_2. By the hypothesis that

$$P_n(\Pi_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m})^{-1} \Rightarrow Q_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m}$$

for any finite elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in $[0, 1]$, we have

$$Q_1(\Pi_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m})^{-1} = Q_2(\Pi_{t_1, t_2, \cdots, t_m}^{\xi_1, \xi_2, \cdots, \xi_m})^{-1}.$$

Taking $C_{E'} = \cup_{p=1}^{\infty} C_{E'_p}$, which is easily derived from the argument in the proof of Proposition 2.1, into account, it is easily seen that the class of all cylinder sets having the form $\{x \in C_{E'}; (\langle x_{t_1}, \xi_1\rangle, \langle x_{t_2}, \xi_2\rangle, \cdots, \langle x_{t_m}, \xi_m\rangle) \in A, A \in \mathscr{B}_{\mathbb{R}^m}\}$ generates $\mathscr{B}_{C_{E'}}$. Thus we have $Q_1 = Q_2$, which completes the proof together with Theorem 2.3 of [2].

By Theorem 4.1, similarly we have

PROPOSITION 5.2. *Let $\{P_n\}$ be a sequence of probability measures on $D_{E'}$. If for each ξ in E, the sequence $\{P_n \Pi_\xi^{-1}\}$ is tight in D and for any finite elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in* [0, 1],

$$P_n(\Pi^{\xi_1,\xi_2,\cdots,\xi_m}_{t_1,t_2,\cdots,t_m})^{-1} \Rightarrow Q^{\xi_1,\xi_2,\cdots,\xi_m}_{t_1,t_2,\cdots,t_m},$$

where $Q^{\xi_1,\xi_2,\cdots,\xi_m}_{t_1,t_2,\cdots,t_m}$ is a probability measure on $\mathbb{R}^m$, then there exists a unique probability measure P on $D_{E'}$ such that $P_n \Rightarrow P$.

Now we will give the theorem of convergence in law for a sequence $\{\mathbf{X}^n = \{X_t^n; t \in [0, 1]\}\}$ of E'-valued stochastic processes. For each ξ in E we denote by X_ξ^n the real stochastic process $\{\langle X_t^n, \xi\rangle; t \in [0, 1]\}$. Then we have

THEOREM 5.3. 1) *Suppose that the sample paths of $\mathbf{X}^n$ are elements in $C_{E'}$ for every $n \in \mathbf{N}$. Further suppose that for each ξ in E the sequence of distributions of X_ξ^n is tight in C and for any finite elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in* [0, 1], *the distribution of $(\langle X_{t_1}^n, \xi_1\rangle, \langle X_{t_2}^n, \xi_2\rangle, \cdots, \langle X_{t_m}^n, \xi_m\rangle)$ converges in law to some m-dimensional probability distribution. Then there exists the limit process* **X** *whose sample paths are elements in $C_{E'}$ such that $\mathbf{X}^n \to_{\mathscr{L}} \mathbf{X}$.*

2) *Suppose that the sample paths of $\mathbf{X}^n$ are elements in $D_{E'}$ for every $n \in \mathbf{N}$. Further suppose that for each ξ in E the sequence of distributions of X_ξ^n is tight in D and for any finite elements $\xi_1, \xi_2, \cdots, \xi_m$ in E and points $t_1, t_2, \cdots, t_m$ in* [0, 1] *the distribution of $(\langle X_{t_1}^n, \xi_1\rangle, \langle X_{t_2}^n, \xi_2\rangle, \cdots, \langle X_{t_m}^n, \xi_m\rangle)$ converges in law to some m-dimensional probability distribution. Then there exists the limit process* **X** *whose sample paths are elements in $D_{E'}$ such that $\mathbf{X}^n \to_{\mathscr{L}} \mathbf{X}$.*

PROOF. By Propositions 5.1 and 5.2 the distribution of $\mathbf{X}^n$ converges weakly to the limit Q_1 (resp. Q_2) on $C_{E'}$ (resp. $D_{E'}$). If in Case 1) we take $(C_{E'}, \mathscr{B}_{C_{E'}}, Q_1)$ as the fundamental probability space $(\Omega, \mathscr{F}, P)$ and put $\mathbf{X} = \{X_t(\omega) = \omega_t; t \in [0, 1]\}$ and if in Case 2) we take $(D_{E'}, \mathscr{B}_{D_{E'}}, Q_2)$ as $(\Omega, \mathscr{F}, P)$ and define **X** similarly, then **X** has the desired properties. This completes the proof.

Finally we will apply this theorem to a limit theorem in K. Itô [4].

EXAMPLE. Independent Brownian motions. Let $\{B_k(t); t \in [0, 1]\}$, $k = 1, 2, \cdots$ be a sequence of independent 1-dimensional Brownian motions with $B_k(0) = 0$ for every $k \in \mathbf{N}$. We shall define a sequence of measure-valued stochastic processes $X_n(t, \cdot)$ as follows:

For a Borel subset $A \in \mathscr{B}_{\mathbb{R}^1}$

$$N_n(t, A) = \sum_{k=1}^n \chi_A(B_k(t))$$

and

$$X_n(t, A) = n^{-1/2}(N_n(t, A) - E[N_n(t, A)]),$$

where $E[\ ]$ denotes the mathematical expectation and $\chi_A(x)$ the indicator function of A.

Let $\mathscr{S}$ be the 1-dimensional Schwartz space. We can consider $X_n(t, \cdot)$ as an $\mathscr{S}'$-valued stochastic process $\mathbf{X}^n = \{X_t^n; t \in [0, 1]\}$ by setting

$$X_t^n(\varphi) = \int_{\mathbb{R}^1} \varphi(x) X_n(t, dx) = n^{-1/2} \sum_{k=1}^n (\varphi(B_k(t)) - E[\varphi(B_k(t))]), \varphi \in \mathscr{S}.$$

Then we have

PROPOSITION 5.4. *There exists an $\mathscr{S}'$-valued stochastic process* $\mathbf{X}$ *whose sample paths are elements in* $C_{\mathscr{S}'}$ *such that* $\mathbf{X}^n \to_{\mathscr{S}} \mathbf{X}$.

PROOF. First we prove the following inequality.

$$E[|X_t^n(\varphi) - X_{t_1}^n(\varphi)|^2 |X_t^n(\varphi) - X_{t_2}^n(\varphi)|^2] \le \alpha(\varphi)|t_1 - t_2|^2, \quad \varphi \in \mathscr{S} \tag{5.1}$$

for $t_1 \le t \le t_2$, where $\alpha(\varphi)$ is a positive constant.

For each $k \in \mathbf{N}$ set $F_k(t, \varphi) = \varphi(B_k(t)) - E[\varphi(B_k(t))]$.

Obviously

$$E[F_k(t, \varphi)] = 0. \tag{5.2}$$

Further we have

$$\begin{aligned} |F_k(t, \varphi) - F_k(s, \varphi)| &\le |\varphi(B_k(t)) - \varphi(B_k(s))| + E[|\varphi(B_k(t)) - \varphi(B_k(s))|] \\ &= \left| \int_{B_k(s)}^{B_k(t)} \varphi'(x)\, dx \right| + E\left[\left| \int_{B_k(s)}^{B_k(t)} \varphi'(x)\, dx \right| \right] \\ &\le \beta_1(\varphi)|B_k(t) - B_k(s)| + \beta_1(\varphi) E[|B_k(t) - B_k(s)|^2]^{1/2} \\ &= \beta_1(\varphi)(|B_k(t) - B_k(s)| + |t - s|^{1/2}), \text{ where } \beta_1(\varphi) = \sup_{x \in \mathbb{R}^1} |\varphi'(x)|. \end{aligned}$$

Using the above inequality we get

$$\begin{cases} \max\{E[|F_k(t, \varphi) - F_k(t_1, \varphi)|^2], E[|F_k(t, \varphi) - F_k(t_2, \varphi)|^2]\} \le \beta_2(\varphi)|t_1 - t_2|, \\ E[|F_k(t, \varphi) - F_k(t_1, \varphi)|^2 |F_k(t, \varphi) - F_k(t_2, \varphi)|^2] \le \beta_3(\varphi)|t_1 - t_2|^2, \end{cases} \tag{5.3}$$

where $\beta_2(\varphi)$ and $\beta_3(\varphi)$ are positive constants independent of k. Hence by making use of (5.2), (5.3) and the independence of the sequence $F_k(t, \varphi)$, $k = 1, 2, \cdots$, we obtain (5.1).

Therefore by Theorem 15.6 of [2] the sequence of distributions on D induced by $X_\varphi^n = \{X_t^n(\varphi); t \in [0, 1]\}$ is tight in D. However, since the sample paths of X_φ^n belong to C, the sequence of distributions of X_φ^n is tight in C. This, together with (C) of Theorem 6.1 of [4], shows that the conditions of 1) of Theorem 5.3 are satisfied, which completes the proof.

6. Remarks.

(R.1). *Tightness in* $C([0, 1]; E_p')$ *and* $D([0, 1]; E_p')$. Let $\{P_n\}$ be a sequence of probability measures on $(C_{E'}, \mathscr{B}_{C_{E'}})$ or $(D_{E'}, \mathscr{B}_{D_{E'}})$. We say that $\{P_n\}$ is uniformly k-continuous if for any $\varepsilon > 0$ and $\rho > 0$ there exists a $\delta > 0$ such that

$$P_n(x \in C_{E'} (\text{or } D_{E'}); \sup_t |\langle x_t, \xi \rangle| > \varepsilon) \le \rho \quad \text{if} \quad \|\xi\|_k \le \delta, \quad n \ge 1.$$

If $\{P_n\}$ is uniformly k-continuous, then $M(\xi)$ defined in the proof of Lemma 3.3 is $\|\cdot\|_k$-continuous at 0 in E. Therefore, if we add the uniformly k-continuous conditions to Theorems 3.1 and 4.1 and Propositions 5.1 and 5.2, it follows from the proof that those theorems hold if E' is replaced by E_p'.

(R.2). *Interval* $[0, \infty)$. Let $C[j]$, $C_{E'}[j]$ and $C_{E_p'}[j]$ be the spaces of continuous mappings of $[0, j]$ to $\mathbb{R}^1$, E' and E_p' respectively. The topologies on these spaces are defined similarly as in Section 2. Let $C[\infty]$, $C_{E'}[\infty]$ and $C_{E_p'}[\infty]$ be the spaces of continuous mappings of $[0, \infty)$ to $\mathbb{R}^1$, E' and E_p' respectively. Further let $D[\infty]$, $D_{E'}[\infty]$ and $D_{E_p'}[\infty]$ be the spaces of right continuous mappings with left limits of $[0, \infty)$ to $\mathbb{R}^1$, E' and E_p' respectively.

(R.2.1). *Case* C. We will introduce on $C[\infty]$, $C_{E'}[\infty]$ and $C_{E'_p}[\infty]$ the projective limit topologies of $\{C[j]; j \in \mathbf{N}\}$, $\{C_{E'}[j]; j \in \mathbf{N}\}$ and $\{C_{E'_p}[j]; j \in \mathbf{N}\}$ respectively. Then it is shown along the line of W. Whitt [12] that Theorems 3.1 and 1) of 5.3, Propositions 5.1 and 5.4 hold even if the interval [0, 1] is replaced by $[0, \infty)$. We will also say that a sequence $\{P_n\}$ of probability measures on $C_{E'}[\infty]$ is uniformly k-continuous if for each $j \in \mathbf{N}$ and for any $\varepsilon > 0$ and $\rho > 0$ there exists a $\delta > 0$ such that

$$P_n(x \in C_{E'}[\infty]; \sup_{0 \le t \le j} |\langle x_t, \xi \rangle| > \varepsilon) \le \rho \quad \text{if} \quad \|\xi\|_k \le \delta, \quad n \ge 1.$$

Then (R.1) holds similarly for the interval $[0, \infty)$.

(R.2.2). *Case* D. Following T. Lindvall [5] we will introduce on $D_{E'}[\infty]$ a certain topology. Of course we will introduce on $D[\infty]$ and $D_{E'_p}[\infty]$ the Lindvall metrics.

For each $j \in \mathbf{N}$ define $g_j(t)$ by

$$g_j(t) = \begin{cases} 1 & \text{if } t \le j, \\ j + 1 - t & \text{if } j < t \le j + 1, \\ 0 & \text{if } t > j + 1. \end{cases}$$

For $0 \le t \le 1$ define $\psi(t)$ by

$$\psi(t) = \begin{cases} -\log(1 - t) & \text{if } 0 \le t < 1, \\ \infty & \text{if } t = 1. \end{cases}$$

For each $j \in \mathbf{N}$ we denote by $\hat{c}_j$ the mapping of $D_{E'}[\infty]$ to $D_{E'}$ by

$$\hat{c}_j : x \in D_{E'}[\infty] \to x_{\psi(t)} g_j(\psi(t)) \in D_{E'}.$$

Set

$$d_\lambda^\infty(x, y) = \textstyle\sum_{j=1}^\infty \frac{1}{2^j} d_\lambda(\hat{c}_j x, \hat{c}_j y)/1 + d_\lambda(\hat{c}_j x, \hat{c}_j y), \quad x, y \in D_{E'}[\infty].$$

We will introduce on $D_{E'}[\infty]$ the projective limit topology of $\{d_\lambda^\infty(\cdot, \cdot); \lambda \in \Lambda\}$.

Define $\hat{\Pi}_\xi$ and c_j by

$$\hat{\Pi}_\xi : x \in D_{E'}[\infty] \to \langle x., \xi \rangle \in D[\infty], \quad \xi \in E,$$

and

$$c_j : x \in D[\infty] \to x(\psi(t)) g_j(\psi(t)) \in D, \quad j \in \mathbf{N}.$$

Let $\{P_n\}$ be a sequence of probability measures on $(D_{E'}[\infty], \mathscr{B}_{D_{E'}[\infty]})$. The if for each ξ in E, $\{P_n \hat{\Pi}_\xi^{-1}\}$ is tight in $D[\infty]$, we have that $\{P_n \hat{\Pi}_\xi^{-1} c_j^{-1}\}$ is tight in D for each $j \in \mathbf{N}$. So taking $c_j \hat{\Pi}_\xi x = \Pi_\xi \hat{c}_j x$ for $x \in D_{E'}[\infty]$ into account, we have $\{P_n \hat{c}_j^{-1} \Pi_\xi^{-1}\}$ is tight in D for each ξ in E, so that $\{P_n \hat{c}_j^{-1}\}$ is tight in $D_{E'}$ by Theorem 4.1. Therefore $\{P_n\}$ is tight in $D_{E'}[\infty]$. Of course it is shown similarly as before that $D_{E'}[\infty]$ is a completely regular topological space whose compact subsets are all metrizable.

Thus Theorems 4.1 and 2) of 5.3, Proposition 5.2 and (R.1) hold similarly as in (R.2.1).

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