The Annals of Probability
1983, Vol 11, No. 4, 867–875

HYDRODYNAMICS OF THE VOTER MODEL

By Errico Presutti and Herbert Spohn

Università di Roma and Universität München

We study the voter model on $\mathbb{Z}^d$, $d \geq 3$, for a sequence μ^ε of initial states which have a gradient in the mean magnetization of the order ε, $\varepsilon \to 0$. We prove that the magnetization field $m^\varepsilon(f, t) = \varepsilon^d \sum f(\varepsilon x)\eta(x, \varepsilon^{-2}t)$ tends to a deterministic field $m(f, t) = \int dq f(q) m(q, t)$ as $\varepsilon \to 0$. $m(q, t)$ is the solution of the diffusion equation. The fluctuations of $m^\varepsilon(f, t)$ around its mean are given by an infinite dimensional, non-homogeneous Ornstein-Uhlenbeck process. In the limit as $\varepsilon \to 0$, locally, i.e. around $(\varepsilon^{-1}q, \varepsilon^{-2}t)$, the voter model is in equilibrium with parameter $m(q, t)$.

1. Introduction. Systems with many components, as spins or particles, often admit a macroscopic description. The presumably best known example is a fluid. Microscopically the fluid consists of many small molecules moving according to the mechanical laws of motion. On a macroscopic scale the state of the fluid is specified by the mass, velocity and energy density and the time evolution of these fields is governed by the hydrodynamic equations, a set of five non-linear partial differential equations. Since the time evolution of the five hydrodynamic fields only partially reflects the time evolution of the exact microscopic state of the fluid, hydrodynamics can be derived from microscopic dynamics only within a certain approximation. This approximation consists in considering microscopic states with slow variation in the hydrodynamic fields, i.e. with small (average) mass, velocity and energy density gradients. Then locally the fluid is almost in thermal equilibrium with the local equilibrium parameters governed approximately by the hydrodynamic equations.

In recent years it has been realized that the *hydrodynamic picture* of the time evolution of a many particle system can be made precise for certain stochastic interacting particle systems. The favorite model of investigation is the simple symmetric exclusion process in one dimension [3, 4, 5]. Also the asymmetric exclusion process and harmonic oscillators with random exchanges of energy have been investigated [9, 11]. In this paper we will study the hydrodynamics of the voter model [2, 6]. We are interested in the voter model because its microscopic structure differs in essential points from the one of the simple exclusion process.

(i) In the simple exclusion process, the number of particles is locally conserved, i.e. the number of particles in the bounded region Δ can change only through the boundary of Δ. In essence this local conservation law is the reason that on a macroscopic scale the density of particles is governed by the diffusion equation. For the voter model the magnetization (or, say, the number of up spins) is *not* locally conserved. Nevertheless, since in dimension $d \geq 3$ the voter model has a one-parameter family of extremal stationary (= equilibrium) states, the hydrodynamic picture provides a valid description. Namely, on a macroscopic scale the voter model is locally, in space-time, in equilibrium and the local equilibrium parameters change according to the solution of its hydrodynamic equation which turns out to be the diffusion equation.

(ii) In the simple exclusion process, the equilibrium states have correlations of short range. For the voter model the static correlations *decay slowly* [1]. This difference is reflected in the fluctuation fields for the two models.

Received December 1981; revised January 1983.

Partially supported by the Nato Research Grant No. 040.82.

AMS 1970 *subject classifications.* Primary 60K35; secondary, 82A05.

Key words and phrases. Voter model, states of local equilibrium, fluctuations of the magnetization field.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability.* www.jstor.org

Holley and Stroock [8] studied the fluctuation field of the voter model in the particular case where the initial state is given by a Bernoulli measure with equal probability for spin up and spin down. They "hope that someone else will take up the voter model and explain to them what is going on." With this paper, we try to accomplish this by proving that the voter model allows for a hydrodynamic description. We expect that the hydrodynamic picture of time evolution is valid for many other stochastic particle systems.

To give a brief outline of the paper: In Section 2 we define the voter model, settle the notation and state the main results. In the remaining sections we prove the structure of the magnetization field and its fluctuations and local equilibrium.

2. Summary of the results. We consider spins on the lattice $\mathbb{Z}^d$. The space of spin configurations is $\{-1, 1\}^{\mathbb{Z}^d} = \Omega$. Ω is equipped with the usual product topology. A configuration is denoted by $\eta \in \Omega$. $\eta(x)$ is its value at the lattice site $x \in \mathbb{Z}^d$, $\eta(x) \in \{-1, 1\}$. The voter model is defined by the flip rates

$$c(x, \eta) = d - \tfrac{1}{2} \textstyle\sum_{y, |x-y|=1} \eta(x)\eta(y). \tag{2.1}$$

We restrict ourselves to the simplest voter model with only symmetric nearest neighbor interactions. The generator of the flip dynamics is then given by

$$(Lf)(\eta) = \textstyle\sum_{x\in\mathbb{Z}^d} c(x, \eta)(f(\eta^x) - f(\eta)) \tag{2.2}$$

acting on strictly local functions $f\colon \Omega \to \mathbb{R}$, where η^x denotes the configuration η with the spin at site x reversed. The closure of L generates the unique Markov semigroup $\{e^{Lt} \mid t \geq 0\}$ on $C(\Omega)$, the space of bounded and continuous functions on Ω. In turn, it follows from the general theory of Markov processes that e^{Lt} determines a unique Markov jump process $\eta(t)$ with state space Ω. The canonical path space of this process is $D([0, \infty), \Omega)$, the space of functions $t \mapsto \eta(t) \in \Omega$ which are right continuous and have left hand limits. If μ is the starting measure for this process, then P_μ denotes its path measure on $D([0, \infty), \Omega)$ and E_μ its expectation. We denote by $\eta(x, t)$ the value of the spin at site x at time t.

From now on we restrict ourselves to $d \geq 3$. (We comment briefly on $d = 1, 2$ at the end of this section.) In this case the extremal time invariant states are given by $\{\mu_m \,\|\, m \mid \leq 1\}$ [6]. They are also translation invariant and are parametrized by the magnetization,

$$E_{\mu_m}(\eta(x)) = m. \tag{2.3}$$

Their covariance decays slowly,

$$E_{\mu_m}(\eta(x)\eta(y)) - m^2 = (1 - m^2)p(x - y) \cong (1 - m^2)(1 - p(e))\,|x - y|^{-d+2} \tag{2.4}$$

for large $|x - y|$ [1]. Here $p(x)$ is the probability that a simple, symmetric random walker on $\mathbb{Z}^d$ who starts at x will ever hit the origin and $e \in \mathbb{Z}^d$ with $|e| = 1$.

The object of our study is the *magnetization field* $m(f, t)$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions. Then we define for every $f \in \mathcal{S}(\mathbb{R}^d)$

$$m(f, t) = \textstyle\sum_{x\in\mathbb{Z}^d} f(x)\eta(x, t). \tag{2.5}$$

$m(f, t)$ is considered as an $\mathcal{S}'(\mathbb{R}^d)$-valued process. We are interested in the global and local structure of $m(f, t)$ in a situation where initially the gradient of the average magnetization is *small* on the scale set by $\mathbb{Z}^d$. We therefore choose a sequence μ^ε, $\varepsilon \to 0$, of starting measures on Ω such that their magnetization gradient is of the order ε. We will show then that the magnetization gradient remains of order ε at any later time. More precisely, for each ε, $0 < \varepsilon \leq 1$, we choose a starting measure μ^ε on Ω with the following property.

(C1) There exists a continuous function $m\colon \mathbb{R}^d \to [-1, 1]$ such that

$$\lim_{\varepsilon\to 0} \sup_{q\in\mathbb{R}^d} |E_{\mu^\varepsilon}(\eta([\varepsilon^{-1}q])) - m(q)| = 0.$$

Here $[a]$ denotes the integer part of $a \in \mathbb{R}^d$.

In addition to (C1) the sequence of initial states has to satisfy certain cluster properties.

(C2) There exist functions Φ_k: $\mathbb{N} \to \mathbb{R}_+$ with $\lim_{n\to\infty}\Phi_k(n) = 0$, $k = 2, 3, \cdots$, such that for every $\varepsilon > 0$, $k \geq 2$, $x_1, \cdots, x_k \in \mathbb{Z}^d$,

$$|E_{\mu^\varepsilon}(\prod_{i=1}^k \eta(x_i)) - \prod_{i=1}^k E_{\mu^\varepsilon}(\eta(x_i))| \leq \Phi_k(\min_{i\neq j}|x_i - x_j|).$$

The rescaled magnetization field is defined by

$$m^\varepsilon(f, t) = \varepsilon^d \sum_{x\in\mathbb{Z}^d} f(\varepsilon x)\eta(x, \varepsilon^{-2}t) \tag{2.6}$$

with starting measure μ^ε. We will show that the limit magnetization field is deterministic.

THEOREM 1. *Let P^ε be the path measure of $m^\varepsilon(f, t)$ on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$. Let P be the measure on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$ which is concentrated on the single history*

$$t \mapsto \int dqf(q)(e^{t\Delta}m)(q).$$

Then weak-$\lim_{\varepsilon\to 0} P^\varepsilon = P$.

As a next step one would like to know the fluctuations around the deterministic limit. Let us define the rescaled fluctuation field by

$$\xi^\varepsilon(f, t) = \varepsilon^{(d+2)/2} \sum_{x\in Z^d} f(\varepsilon x)(\eta(x, \varepsilon^{-2}t) - E_{\mu^\varepsilon}(\eta(x, \varepsilon^{-2}t))). \tag{2.7}$$

The time scale in (2.7) is set through the one of the magnetization field. The power of the prefactor is determined by the scaling properties of the equilibrium fluctuations (2.4). We will prove that the limiting fluctuations are governed by an infinite-dimensional, inhomogeneous Ornstein-Uhlenbeck process.

THEOREM 2. *Let P^ε be the path measure of $\xi^\varepsilon(f, t)$ on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$ and let $(e^{\Delta t}m)(q) = m(q, t)$.*

(i) *Let P be the Gaussian measure on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$ with mean zero and covariance*

$$E(\xi(f, t)\xi(g, s)) = 2\, d(1 - p(e)) \int_0^{t\wedge s} ds' \int dq(1 - m(q, s')^2)f(q, t - s')g(q, s - s') \tag{2.8}$$

for all $f, g \in \mathscr{S}(\mathbb{R}^d)$ with $(e^{\Delta t}f)(q) = f(q, t)$. If Φ_2 in (C2) *is integrable, $\sum_{x\in Z^d}\Phi_2(|x|) < \infty$, then*

$$\text{weak-}\lim_{\varepsilon\to 0} P_\varepsilon = P.$$

(ii) *Let $\mu^\varepsilon = \mu_m$ for $0 < \varepsilon \leq 1$ and let P be the Gaussian measure on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$ with mean zero and covariance*

$$E(\xi(f, t)\xi(g, s)) = (1 - m^2)(1 - p(e)) \int dqf(q, |t - s|)(\Delta^{-1}g)(q) \tag{2.9}$$

for all $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\text{weak-}\lim_{\varepsilon\to 0} P^\varepsilon = P.$$

Theorems 1 and 2 provide the expected picture. The magnetization is deterministic with small Gaussian fluctuations around it. However, in comparison to other models the fluctuations are huge because of slowly decaying static correlations. To contrast we should mention that for the simple exclusion process the analogues of Theorems 1 to Theorem 3 are true in any dimension. Because of rapidly decaying static correlations, the fluctuation field is now defined by

$$\xi^\varepsilon(f, t) = \varepsilon^{(d/2)} \sum_{x\in Z^d} f(\varepsilon x)(\eta(x, \varepsilon^{-2}t) - E_{\mu^\varepsilon}(\eta(x, \varepsilon^{-2}t))).$$

Its limit is a Gaussian field with mean zero and covariance

$$\langle \xi(f, t)\xi(g, s)\rangle = \langle \xi(f(t), 0)\xi(g(s), 0)\rangle$$

$$+ \int_0^{t\wedge s} ds' \int dq\,(1 - m(q, s')^2)\,\mathrm{grad}\, f(q, t - s')\cdot\mathrm{grad}\, g(q, s - s'), \tag{2.10}$$

$t, s \geq 0, f(t) = e^{t\Delta}f$.

Theorem 1 provides us only with the information that $m(q, t)$ indicates the value of the magnetization "around" position $\varepsilon^{-1}q$ at time $\varepsilon^{-2}t$ and that its fluctuations are negligible in a first approximation as ε goes to zero. A more refined study shows that the path measure in the neighborhood of $(\varepsilon^{-1}q, \varepsilon^{-2}t)$ is approximately the equilibrium path measure $P_{\mu_{m(q,t)}}$.

THEOREM 3. *For every* $q \in \mathbb{R}^d$, $t > 0$, $n \geq 1$, $x_1, \cdots, x_n \in \mathbb{Z}^d$, $s_1, \cdots, s_n \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0}\sup_{q\in\mathbb{R}^d} | E_{\mu^\varepsilon}(\textstyle\prod_{j=1}^n \eta([\varepsilon^{-1}q] + x_j, \varepsilon^{-2}t + s_j)) - E_{\mu_{m(q,t)}}(\prod_{j=1}^n \eta(x_j, s_j))| = 0. \tag{2.11}$$

An alternative way to read Theorem 3 is that $m(q, t)$ is the parameter of that equilibrium path measure which at time $\varepsilon^{-2}t$ "best" approximates the actual measure around $\varepsilon^{-1}q$. This interpretation is further strengthened in Theorem 4 of Section 4.

We may picture the voter model as being patched together, in space-time, out of many pieces in equilibrium whose parameter changes on the macroscopic scale according to its hydrodynamic equation, i.e. the diffusion equation. The fact that the hydrodynamics of the voter model is governed by a linear equation reflects the simplicity of the model. For example, in the simple symmetric exclusion process with speed change we expect that the hydrodynamic equation governing the density of particles, $\rho(q, t)$, is a non-linear diffusion equation of the form $(\partial/\partial t)\rho(q, t) = \mathrm{div}\, D(\rho(q, t))\,\mathrm{grad}\,\rho(q, t)$ with some $d \times d$ matrix valued function D on $[0, 1]$.

REMARK. For $d = 1, 2$ the voter model has only two extremal time invariant states, namely those concentrated on the configurations either $\eta(x) = 1$ or $\eta(x) = -1$ for all x. This indicates a breakdown of the hydrodynamic picture. For $d = 1$ the limit magnetization field exists but it is not deterministic. For $d = 2$, Theorem 1 holds. We did not investigate the nature of the fluctuations around the deterministic limit. Locally the voter model is in a superposition of the two extremal time invariant states.

3. Global structure of the magnetization field. In this section we prove Theorems 1 and 2.

DEFINITION 3.1. *The dual of the voter model.* The voter model has a very simple dual, the annihilating random walk process. Given $\vec{x} = (x_1, \cdots, x_n)$, $x_i \in \mathbb{Z}^d$, $i = 1, \cdots, n$, $P_{\vec{x}}$ denotes the law of the process $\vec{x}(t) = (x_1(t), \cdots, x_n(t))$ of n independent random walks on $\mathbb{Z}^d$ starting from $\vec{x}$. Each particle moves at independent Poisson times of mean 1 and jumps with equal probability on the nearest neighbor sites. The random variables $\sigma_i(t)$, $i = 1, \cdots, n$, can take values either 0 (the particle is "alive") or 1 (the particle is "dead"). All particles start alive, i.e. $\sigma_i(0) = 0$, $i = 1, \cdots, n$, and whenever two alive particles are at the same site they both die. Dead particles remain dead. We will employ the following shorthand notation: for $A \subset \{1, \cdots, n\}$

$$\{\sigma_A(t) = 0\} = \{\sigma_i(t) = 0 \text{ iff } i \in A, i = 1, \cdots, n\}$$

$$\{\sigma_A(\infty) = 0\} = \{\sigma_i(t) = 0\ \forall t \in \mathbb{R}_+ \text{ iff } i \in A, i = 1, \cdots, n\}.$$

Let $E_{\vec{x}}$ be the expectation with respect to $P_{\vec{x}}$. We have then

$$E_\mu(\textstyle\prod_{i=1}^n \eta(x_i, t)) = \sum_{A\subset\{1,\cdots,n\}} E_{\vec{x}}[E_\mu(\prod_{i\in A}\eta(x_i(t), 0))\mathbf{1}(\{\sigma_A(t) = 0\})], \tag{3.1}$$

where $\mathbf{1}(\{\cdot\})$ is the characteristic function of $\{\cdot\}$.

PROOF OF THEOREM 1. Since $|\eta(x, t)| = 1$, we have for $0 < \varepsilon \le 1$

$$E_{\mu^\varepsilon}(m^\varepsilon(f, t)^2) \le (\varepsilon^d \sum_{x\in\mathbb{Z}^d} |f(\varepsilon x)|)^2 \le (c \sup_{q\in\mathbb{R}^d} (1 + |q|^2)^d |f(q)|)^2 \tag{3.2}$$

with c a constant independent of f and ε. By the results of [7] this implies that the family $\{P^\varepsilon, 0 < \varepsilon \le 1\}$ is tight.

We regard $f \mapsto m(f, t)$ as a point in $\mathcal{S}'(\mathbb{R}^d)$ and $t \mapsto m(\cdot, t)$ as a path in $\mathcal{S}'(\mathbb{R}^d)$. For any $F \in C_0^\infty(\mathbb{R})$

$$F(m(f, t)) - \int_0^{\varepsilon^{-2}t} ds \sum c(x, \eta(s))\{F[m(f, \varepsilon^2 s) - 2\varepsilon^d f(\varepsilon x)\eta(x, s)] - F(m(f, \varepsilon^2 s))\} \tag{3.3}$$

is a P^ε martingale relative to the σ-algebra $\sigma\{m(\cdot, s), 0 \le s \le t\}$.

Here $\eta(x, s)$ is identified with $m(g_{x,\varepsilon}, \varepsilon^2 s)$, where $g_{x,\varepsilon} \in \mathcal{S}(\mathbb{R}^d)$, $g_{x,\varepsilon}(\varepsilon x) = \varepsilon^{-d}$ and $g_{x,\varepsilon}$ has a support around εx of diameter less than ε.

We will show that, for any limit point P^0 of $\{P^\varepsilon \mid 0 < \varepsilon \le 1\}$ as $\varepsilon \to 0$,

$$F(m(f, t)) - \int_0^t ds\, F'(m(f, s))m(\Delta f, s) \tag{3.4}$$

is a P^0 martingale relative to $\sigma\{m(\cdot, s), 0 \le s \le t\}$ for all $F \in C_0^\infty(\mathbb{R})$, $f \in \mathcal{S}(\mathbb{R}^d)$. Since by (C1) and (C2)

$$P^0(\{m(f, 0) = \int dq m(q)f(q)\}) = 1, \tag{3.5}$$

$P^0 = P$. Therefore we need only to check that (3.4) is a P^0 martingale. We expand up to second order

$$(3.3) = F(m(f, t)) - \int_0^{\varepsilon^{-2}t} ds\, [\varepsilon^2 F'(m(f, \varepsilon^2 s))m(\Delta^\varepsilon f, \varepsilon^2 s) + O(\varepsilon^{2d}) \textstyle\sum_x f(\varepsilon x)^2]$$

$$\begin{aligned} &= F(m(f, t)) - \int_0^t ds\, F'(m(f, s))m(\Delta f, s) \\ &\quad - \int_0^t ds\, F'(m(f, s))m(\Delta^\varepsilon f - \Delta f, s) + O(\varepsilon^{d-2})\varepsilon^d \textstyle\sum_x f(\varepsilon x)^2 \end{aligned} \tag{3.6}$$

with Δ^ε the lattice Laplacian on $\varepsilon\mathbb{Z}^d$, i.e.

$$(\Delta^\varepsilon f)(x) = \varepsilon^{-2} \textstyle\sum_{e,|e|=1} [f(x + \varepsilon e) - f(x)]. \tag{3.7}$$

By (3.2) the error term in (3.6) vanishes in distribution as $\varepsilon \to 0$ □

PROOF OF THEOREM 2. We prove only Theorem 2(i), since (ii) is proven along the same lines. Our proof follows closely the one of Holley and Stroock [8]. However, in our case $E_{\mu^\varepsilon}(\eta(x, t)) \ne 0$ and therefore some additional terms have to be estimated. Let

$$p(x_1 - x_2, t) = P_{(x_1,x_2)}(\{\sigma_i(t) = 1, i = 1, 2\}), \quad p(x) = \lim_{t\to\infty} p(x, t).$$

By duality, Equation (3.1) and (C2), we have

$$\begin{aligned} E_{\mu^\varepsilon}(\xi^\varepsilon(f, t)^2) &= \varepsilon^{d+2} \textstyle\sum_{x_1,x_2} f(\varepsilon x_1)f(\varepsilon x_2)E_{(x_1,x_2)}(\mathbf{1}(\{\sigma_i(\varepsilon^{-2}t) = 0, i = 1, 2\}) \\ &\quad [E_{\mu^\varepsilon}(\eta(x_1(\varepsilon^{-2}t), 0)\eta(x_2(\varepsilon^{-2}t), 0)) - E_{\mu^\varepsilon}(\eta(x_1(\varepsilon^{-2}t), 0))E_{\mu^\varepsilon}(\eta(x_2(\varepsilon^{-2}t), 0))] \\ &\quad + \mathbf{1}(\{\sigma_i(\varepsilon^{-2}t) = 1, i = 1, 2\})[1 - E_{\mu^\varepsilon}(\eta(x_1(\varepsilon^{-2}t), 0))E_{\mu^\varepsilon}(\eta(x_2(\varepsilon^{-2}t), 0))]) \\ &\le \varepsilon^{d+2} \textstyle\sum_{x_1,x_2} |f(\varepsilon x_1)f(\varepsilon x_2)| \{E_{(x_1,x_2)}(\Phi_2(|x_1(\varepsilon^{-2}t) - x_2(\varepsilon^{-2}t)|)) + 2p(x_1 - x_2, \varepsilon^{-2}t)\}. \end{aligned} \tag{3.8}$$

From [8] for all $k \in \mathbb{R}^d$

$$|\sum_x e^{ikx} p(x, t)| \leq (1 + 4t).$$

By assumption, for all $k \in \mathbb{R}^d$

$$|\sum_{x_1} e^{ikx} E_{x_1}(\Phi_2(|x_1(t)|))| \leq \sum_x \Phi_2(|x|) = b < \infty.$$

Therefore

$$\begin{aligned} E_{\mu^\varepsilon}(\xi^\varepsilon(f, t)^2) &\leq \varepsilon^2(1 + 4\varepsilon^{-2}t + b)\varepsilon^d \sum_x |f(\varepsilon x)|^2 \\ &\leq (1 + 4t + b)c \sup_{q \in \mathbb{R}^d}(1 + |q|^2)^d |f(q)|^2. \end{aligned} \tag{3.9}$$

This implies, [7], that the family $\{P^\varepsilon, 0 < \varepsilon \leq 1\}$ is tight.

We regard $f \to \xi(f, t)$ as a point in $\mathscr{S}'(\mathbb{R}^d)$ and $t \to \xi(\cdot, t)$ as a path in $\mathscr{S}'(\mathbb{R}^d)$. For $F \in C_0^\infty(R)$ we think of

$$F(\xi^\varepsilon(f, t)) = F(m^\varepsilon(\varepsilon^{(-d+2)/2} f, t) - \varepsilon^{(d+2)/2} \sum_x f(\varepsilon x) E_{\mu^\varepsilon}(\eta(x, \varepsilon^{-2} t)))$$

as a time dependent function of the magnetization. Then as in (3.3)

$$\begin{aligned} F(\xi(f, t)) - \int_0^{\varepsilon^{-2}t} ds\{\textstyle\sum_x c(x, \eta(s))\{F(\xi(f, \varepsilon^2 s) - 2\varepsilon^{(d+2)/2} f(\varepsilon x)\eta(x, s)) - F(\xi(f, \varepsilon^2 s))\} \\ - \varepsilon^{(d+2)/2} F'(\xi(f, \varepsilon^2 s)) \frac{d}{ds} \textstyle\sum_x f(\varepsilon x) E_{\mu^\varepsilon}(\eta(x, s))\} \end{aligned} \tag{3.10}$$

is a P^ε-martingale relative to $\sigma\{\xi(\cdot, s), 0 \leq s \leq t\}$. Here $\eta(x, s)$ is identified with $\xi(g_{x,\varepsilon}, \varepsilon^2 s) + E_{\mu^\varepsilon}(\eta(x, s))$, where $g_{x,\varepsilon} \in \mathscr{S}(\mathbb{R}^d)$, $g_{x,\varepsilon}(\varepsilon x) = \varepsilon^{-(d+2)/2}$ and $g_{x,\varepsilon}$ has a support around εx of diameter less than ε. We will show that, for any limit point P^0 of $\{P^\varepsilon, 0 < \varepsilon \leq 1\}$ as $\varepsilon \to 0$,

$$\begin{aligned} F(\xi(f, t)) - \int_0^t ds\, \{F'(\xi(f, s))\xi(\Delta f, s) \\ + 2d(1 - p(e))\left[\int dq\, (1 - m(q, t)^2) f(q)^2\right] F''(\xi(f, s))\} \end{aligned} \tag{3.11}$$

is a P^0 martingale relative to $\sigma\{\xi(\cdot, s), 0 \leq s \leq t\}$ for all $F \in C_0^\infty(R^d)$, $f \in \mathscr{S}(\mathbb{R}^d)$. By (C2) and since Φ_2 is assumed to be integrable, $P^0(\{\xi(f, 0) = 0\}) = 1$ and therefore $P^0 = P$.

We have to check only that (3.11) is a P^0-martingale. We expand up to third order. Then

$$\begin{aligned} (3.10) = F(\xi(f, t)) - \int_0^{\varepsilon^{-2}t} ds\{F'(\xi(f, \varepsilon^2 s))\, [- 2\varepsilon^{(d+2)/2} \textstyle\sum_x (c(x, \eta(s)) f(\varepsilon x)\eta(x, s) \\ + \tfrac{1}{2} f(\varepsilon x)(d/ds)E_{\mu^\varepsilon}(\eta(x, s)))] + 2F''(\xi(f, \varepsilon^2 s))\varepsilon^{d+2}(\textstyle\sum_x c(x, \eta(s)) f(\varepsilon x)^2) \\ + O(\varepsilon^{3(d+2)/2}\varepsilon^{-d})\varepsilon^d(\textstyle\sum_x f(\varepsilon x)^3)\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} = F(\xi(f, t)) - \int_0^t ds\{F'(\xi(f, s))\xi(\Delta f, s) \\ + 2d(1 - p(e))\left[\int dq(1 - m(q, t)^2) f(q)^2\right] F''(\xi(f, s))\} \\ - \int_0^t ds\{2F''(\xi(f, s))[\varepsilon^d \textstyle\sum_{x_1} f(\varepsilon x_1)^2 \\ (d - \tfrac{1}{2} \textstyle\sum_{x_2, |x_1 - x_2| = 1} E_{\mu^\varepsilon}(\eta(x_1, \varepsilon^{-2}s)\eta(x_2, \varepsilon^{-2}s))) \\ - d(1 - p(e)) \int dq(1 - m(q, t)^2) f(q)^2] + F'(\xi(f, s))\xi(\Delta^\varepsilon f - \Delta f, s) \\ + F''(\xi(f, s))\varepsilon^d \textstyle\sum_{x_1, x_2, |x_1 - x_2| = 1} f(\varepsilon x_1)^2\, [E_{\mu^\varepsilon}(\eta(x_1, \varepsilon^{-2}s)\eta(x_2, \varepsilon^{-2}s)) \\ - \eta(x_1, \varepsilon^{-2}s)\eta(x_2, \varepsilon^{-2}s)]\} + O(\varepsilon^{(d+2)/2}). \end{aligned} \tag{3.13}$$

By Theorem 3 and (2.4)

$$\lim_{\varepsilon\to 0}\varepsilon^d \sum_{x_1,x_2,|x_1-x_2|=1} f(\varepsilon x_1)^2 E_{\mu^\varepsilon}(\eta(x_1, \varepsilon^{-2}s)\eta(x_2, \varepsilon^{-2}s))$$

(3.14)

$$= 2d \int dq f(q)^2 [m(q, t)^2 + (1 - m(q, t)^2)p(e)],$$

$|e| = 1$. Therefore the first error term in (3.13) tends to zero in distribution as $\varepsilon \to 0$. The second error term vanishes in distribution, since $E_{\mu^\varepsilon}(\xi^\varepsilon(\Delta^\varepsilon f - \Delta f, s)^2)$ tends to zero as $\varepsilon \to 0$ by (3.9).

To prove that the third error term vanishes in distribution as $\varepsilon \to 0$, we have to show that

$$\lim_{\varepsilon\to 0}\varepsilon^{2d} \sum_{x_1,x_2,|x_1-x_2|=1} \sum_{x_3,x_4,|x_3-x_4|=1} f(\varepsilon x_1)^2 f(\varepsilon x_3)^2$$

(3.15)

$$[E_{\mu^\varepsilon}(\textstyle\prod_{i=1}^4 \eta(x_i, \varepsilon^{-2}s)) - E_{\mu^\varepsilon}(\prod_{i=1}^2 \eta(x_i, \varepsilon^{-2}s)) E_{\mu^\varepsilon}(\prod_{i=3}^4 \eta(x_i, \varepsilon^{-2}s))] = 0.$$

For $s = 0$, (3.15) follows from (C2). Let us assume then $s > 0$. We use the duality (3.1) and introduce the four random walks $\vec{x}(t) = (x_1(t), \cdots, x_4(t))$ with law $P_{\vec{x}}$. To compute the first expectation in (3.15), we need to partition the paths according to which particles are still alive at time $\varepsilon^{-2}s$. The product of expectations in (3.15) can also be written as an expectation with respect to $P_{\vec{x}}$; the annihilation rule needs to be modified, however: the particles with label either 1 or 2 do not annihilate the particles with label either 3 or 4.

We break up the sum (3.15) into $|x_1 - x_3| < \varepsilon^{-1}\delta$ and $|x_1 - x_3| \geq \varepsilon^{-1}\delta$ with an arbitrary constant $\delta > 0$. In the latter part of the sum, the probability that a particle with label either 1 or 2 meets at any time a particle with either label 3 or 4 vanishes as $\varepsilon \to 0$. In this case we can neglect the difference between the two annihilation rules and conclude that this part tends to zero as $\varepsilon \to 0$. In the former part of the sum, i.e. when $|x_1 - x_3| < \varepsilon^{-1}\delta$, we bound the term in the square brackets by two, which implies that this part of the sum is bounded by const. δ. Because of the arbitrariness of δ, (3.15) follows. □

4. Local equilibrium. In this section, we examine the microscopic structure of the system and we first prove in Theorem 4 that any initial measure satisfying (C2) exhibits as time increases a local equilibrium structure. Related results are found in [6].

THEOREM 4. *Let M_0 be the set of probability measures for which* (C2) *of Section* 2 *holds. Then for every $\mu \in M_0$ there exists a function $m(x, t\,|\,\mu)$: $\mathbb{Z}^d \times \mathbb{R}_+ \to [-1, 1]$ such that for all $n \geq 1$, $x_1, \cdots, x_n \in \mathbb{Z}^d$,*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{Z}^d} |E_\mu(\textstyle\prod_{i=1}^n \eta(x_i + x, t)) - E_{\mu_{m(x,t|\mu)}}(\prod_{i=1}^n \eta(x_i))| = 0. \tag{4.1}$$

REMARK. (4.1) states that the system observed in a fixed region approaches equilibrium in the sense that its "distance" from the set $\mathscr{E} \equiv \{\mu_m,\ m \in [-1, 1]\}$ vanishes as t diverges. It does not necessarily converge to a point in $\mathscr{E}$, it might keep wandering closer and closer to $\mathscr{E}$. When another region is considered, its equilibrium may "initially" be different; a common behavior will eventually be established after some time, depending on the mutual distance between the regions. Only when the "observations" are suitably moved away in space along with time, they might keep showing some difference. Notice that (4.1) does not determine uniquely the function $m(x, t\,|\,\mu)$. Any other $m'(x, t\,|\,\mu)$ such that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{Z}^d} |m(x, t\,|\,\mu) - m'(x, t\,|\,\mu)| = 0$$

fulfills (4.1) if m does so. This ambiguity is in a sense removed in Theorem 3.

PROOF OF THEOREM 4. The extremal invariant measures μ_m, $|m| \le 1$ for the voter model are determined by

$$E_{\mu_m}(\prod_{i=1}^{n} \eta(x_i)) = \sum_{A\subset\{1,\cdots,n\}} P_{\vec{x}}(\{\sigma_A(\infty) = 0\})m^{|A|}. \tag{4.2}$$

We will prove Theorem 4 for

$$m(x, t\,|\,\mu) = E_x[E_\mu(\eta(x(t), 0))]. \tag{4.3}$$

We need to consider initial conditions of the form $x_i + x$, $i = 1, \cdots, n$, where $\vec{x} = (x_1, \cdots, x_n)$ is fixed while x varies in $\mathbb{Z}^d$. It is easy to convince oneself that the estimates we are going to prove are uniform in x, so for the sake of notational simplicity we set $x = 0$.

Having used (3.1), we want to exploit that the probability that $\sigma_i(t)$ changes after T is very small when T is large. We therefore introduce $t_\delta \in \mathbb{R}_+$, $\delta \in (0, 1)$ such that

$$0 < P_{\vec{x}}(\{\sigma_A(t_\delta) = 0\}) - P_{\vec{x}}(\{\sigma_A(\infty) = 0\}) < \delta, \tag{4.4}$$

for all $A \subset \{1, \cdots, n\}$. (3.1) and (4.4) imply then that for all $t \ge t_\delta$

$$E_\mu(\prod_{i=1}^{n} \eta(x_i, t)) = \sum_A E_{\vec{x}}[E_\mu(\prod_{i\in A}\eta(x_i(t), 0))\mathbf{1}(\{\sigma_A(t_\delta) = 0\})] + R \tag{4.5}$$

with $|R| \le \sum_A \delta$. We now introduce the process of n independent random walks all starting from the origin of $\mathbb{Z}^d$, its law being $P_{\vec{0}}$, and for each δ we couple the $P_{\vec{x}}$ and the $P_{\vec{0}}$ processes as follows: P denotes the law of $\vec{x}(t), \vec{y}(t)$, $2n$ independent random walks with $x_i(0) = x_i$, $y_i(0) = 0$, $i = 1, \cdots, n$, and E refers to its expectation. For $a \in \mathbb{Z}^d$ we set $a = (a^{(1)}, \cdots, a^{(d)})$ and given $i \in \{1, \cdots, n\}$, $u \in \{1, \cdots, d\}$ and δ we set

$$T(\delta, i, u) = \inf\{t > t_\delta \,|\, x_i^{(u)}(t) = y_i^{(u)}(t)\}. \tag{4.6}$$

We define then the coupled process by

$$\tilde{x}_i^{(u)}(t) = \begin{cases} x_i^{(u)}(t) & \text{if} \quad t \le T(\delta, i, u) \\ y_i^{(u)}(t) & \text{if} \quad t \ge T(\delta, i, u) \end{cases}, \quad \tilde{y}_i^{(u)}(t) = y_i^{(u)}(t). \tag{4.7}$$

The law of $\tilde{y}_i(t)$, $i = 1, \cdots, n$ is $P_{\vec{0}}$ while that of $\tilde{x}_i(t)$, $i = 1, \cdots, n$ is $P_{\vec{x}}$. Since $x_i^{(u)}(t)$, $y_i^{(u)}(t)$ are one dimensional random walks, the set

$$G_\delta(t) = \{T(\delta, i, u) < t \quad \text{for all} \quad i \in \{1, \cdots, n\}, u \in \{1, \cdots, d\}\}$$

is such that

$$\lim_{t\to\infty} P(G_\delta(t)) = 1. \tag{4.8}$$

Given δ we can then choose t so large that

$$P(G_\delta(t)) \ge 1 - \delta \tag{4.9}$$

and from (4.5) we get

$$\begin{aligned} &E_\mu(\prod_{i=1}^{n} \eta(x_i, t)) \\ &\quad = \sum_A E[E_\mu(\prod_{i\in A} \eta(\tilde{x}_i(t), 0))\mathbf{1}(\{\sigma_A(t_\delta) = 0\})\{\mathbf{1}(G_\delta(t)) + \mathbf{1}(G_\delta(t)^c)\}] + R \\ &\quad = \sum_A E[E_\mu(\prod_{i\in A} \eta(\tilde{x}_i(t), 0))\mathbf{1}(\{\sigma_A(t_\delta) = 0\})\mathbf{1}(G_\delta(t))] + R + R' \\ &\quad = \sum_A E[E_\mu(\prod_{i\in A}\eta(y_i(t), 0))\mathbf{1}(\{\sigma_A(t_\delta) = 0\})\mathbf{1}(G_\delta(t))] + R + R' \\ &\quad = \sum_A E[E_\mu(\prod_{i\in A} \eta(y_i(t), 0))\mathbf{1}(\{\sigma_A(t_\delta) = 0\})] + R + R' + R'' \end{aligned} \tag{4.10}$$

with $|R'|, |R''| \le \sum_A \delta$.

The expectation in (4.10) refers to a product of $\vec{x}(\cdot)$ and $\vec{y}(\cdot)$ measurable functions. Therefore it factorizes and we get

$$
\begin{aligned}
&E_\mu(\textstyle\prod_{i=1}^n \eta(x_i, t)) \\
(4.11)\quad &= \textstyle\sum_A E_{\vec{x}}(\mathbf{1}(\{\sigma_A(t_\delta) = 0\}))E_{\vec{0}}(E_\mu(\prod_{i\in A} \eta(y_i(t), 0))) + R + R' + R'' \\
&= \textstyle\sum_A E_{\vec{x}}\,(\mathbf{1}(\{\sigma_A(\infty) = 0\}))E_{\vec{\sigma}}(E_\mu(\prod_{i\in A} \eta(y_i(t), 0))) + R + R' + R'' + R'''
\end{aligned}
$$

with $|R'''| \le \sum_A \delta$. Given any $r > 0$

$$\lim_{t\to\infty} P_{\vec{0}}\,(\{|y_i(t) - y_j(t)| > r, \forall i \ne j\}) = 1. \tag{4.12}$$

From the assumption that the correlations of the initial measure μ decay, it follows then that for t large enough the E_μ expectation and consequently also the $E_{\vec{0}}$ expectation factorize. By comparing (4.11) with (4.2) and using (4.3) we obtain

$$|E_\mu(\textstyle\prod_{i=1}^n \eta(x_i, t)) - E_{\mu_{m(0,t|\mu)}}(\prod_{i=1}^n \eta(x_i, t))| < H\delta \tag{4.13}$$

for t large enough and with H independent of δ. This proves Theorem 4 because of the arbitrariness of δ. □

PROOF OF THEOREM 3. We first consider the case where $s_i = 0$, $i = 1, \cdots, n$. Then, using the uniformity of the factorization with respect to ε, by Theorem 4 we already know that the distribution with initial measure μ^ε is locally well approximated by the equilibrium ones with parameters $m(x, t\,|\,\mu^\varepsilon)$ uniformly in ε. It only remains to show then that $m(\varepsilon^{-1}q, \varepsilon^{-2}t\,|\,\mu^\varepsilon)$ converges to $m(q, t)$ as $\varepsilon \to 0$. But this follows from (3.1) and (4.3) together with (C1). The case where the s_i's are different from zero is completely analogous. In the dual process, the jth annihilating random walk now starts at time $-s_j$ at position x_j and it walks up to time $\varepsilon^{-2}t$. □

Acknowledgments. We are indebted to A. De Masi and N. Ianiro for many discussions which led to the proof of Theorem 4. We thank J. L. Lebowitz for his very kind hospitality at the Mathematics Department of Rutgers University where this work was done.

REFERENCES

[1] BRAMSON, M. and GRIFFEATH, D. (1978). Renormalizing the 3-dimensional voter model. *Ann. Probab.* **7** 418–432.
[2] CLIFFORD, P., and SUDBURY, A. (1973). A model for spatial conflict. *Biometrika* **60** 581–588.
[3] DE MASI, A., IANIRO, N. and PRESUTTI, E. (1982). Small deviations from local equilibrium for a process which exhibits hydrodynamical behavior. *J. Statist. Phys.* **29** 57–93.
[4] GALVES, A., KIPNIS, C., MARCHIRO, C., and PRESUTTI, E. (1981). Non-equilibrium measures which exhibit a temperature gradient study of a model. *Comm. Math. Phys.* **81** 127–147.
[5] GALVES, A., KIPNIS, C., and SPOHN, H. (1983). Fluctuation theory for the symmetric simple exclusion process. In preparation.
[6] HOLLEY, R., and LIGGETT, T. M. (1975). Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probab.* **3** 643–663.
[7] HOLLEY, R. and STROOCK, D. W. (1978). Generalized Ornstein-Uhlenbeck processes and infinite branching Brownian motions. Kyoto University Research Institute for Mathematical Sciences Publications A **14,** 741–788.
[8] HOLLEY, R. and STROOCK, D. W. (1979). Central limit phenomena for various interacting systems. *Ann. Math.* **110** 333–393.
[9] KIPNIS, C., MARCHIORO, C., and PRESUTTI, E. (1982). Heat flow in an exactly solvable model. *J. Statist. Phys.* **27** 65–74.
[10] LIGGETT, T. M. (1977). The stochastic evolution of infinite systems of interacting particles. *Lecture Notes in Math.* **598** 187–248. Springer Verlag, New York.
[11] ROST, H. (1981). Non-equilibrium behavior of a many particle process: density profile and local equilibria. *Z. Wahrsch. verw Gebiete* **58** 41–55.

ISTITUTO MATEMATICO
"GUIDO CASTELNUOVO"
00185 ROMA
ITALY

THEORETISCHE PHYSIK
THERESIENSTR. 37
8 MÜNCHEN 2
GERMANY