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APPROXIMATE LOCAL LIMIT THEOREMS FOR LAWS OUTSIDE DOMAINS OF ATTRACTION

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Let $\{X_k\}$ be a sequence of independent, identically distributed, nondegenerate random variables and $S_n = X_1 + \cdots + X_n$. Define $G(x) = P\{|X| > x\}$, $K(x) = x^{-2}\int_{|y|\le x} y^2\,dF(y)$, $Q(x) = G(x) + K(x)$ for $x > 0$, and $\{a_n\}$ by $Q(a_n) = n^{-1}$ for large n. Let (A) denote the condition: $\limsup_{x\to\infty} G(x)/K(x) < \infty$. We show that (A) implies the following: there exist $\varepsilon > 0$, $C > 0$, such that for each $M > 0$ a sequence $\{b_n\}$ and a positive constant c can be found for which $c \le a_n P\{S_n \in (x-\varepsilon, x+\varepsilon)\} \le C$ whenever $|x - b_n| \le Ma_n$ and n is sufficiently large. In fact, the upper bound is valid for all x. We also prove that (A) is necessary for either the upper bound result or the lower bound result so that these results are equivalent. Feller had shown that (A) is equivalent to the existence of $\{\gamma_n\}$, $\{\delta_n\}$ such that the sequence $\{(S_n - \delta_n)/\gamma_n\}$ is stochastically compact.

1. Introduction. Let $X_1, X_2, \cdots$ be independent, identically distributed, nondegenerate random variables taking values in $\mathbb{R}^1$ and $S_n = X_1 + \cdots + X_n$. Let X be a random variable with the same distribution as X_1, F its distribution function, and for $x > 0$ define

$$G(x) = P\{|X| > x\}, \quad K(x) = x^{-2}\int_{|y|\le x} y^2\,dF(y),$$

$$Q(x) = G(x) + K(x) = E(x^{-1}|X| \wedge 1)^2. \tag{1.1}$$

The function Q is continuous and strictly decreasing once the support of F is reached. Thus for sufficiently large n, we may define a sequence $\{a_n\}$ by

$$Q(a_n) = \frac{1}{n}. \tag{1.2}$$

The sequence $\{a_n\}$ is increasing, tends to infinity, and may be used to normalize $\{S_n\}$ for weak convergence whenever X is in the domain of attraction of a stable law; but we will see that it also describes the distribution of $\{S_n\}$ quite well much more generally.

The basic analytic condition we will assume is

$$\limsup_{x\to\infty} \frac{G(x)}{K(x)} < \infty. \tag{A}$$

This condition was introduced by Feller [2] who showed it to be equivalent to

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the existence of a centering sequence $\{b_n\}$ such that $\{(S_n - b_n)/a_n\}$ is stochastically compact which means that it is a tight sequence and all its limit laws are nondegenerate. It is also worth noting here that Jain and Orey ([5]; Theorem 2.5) showed that stochastic compactness is equivalent to

$$(B_1) \qquad \lim_{\rho\to\infty} \sup_n nG(\rho a_n) = 0.$$

Thus (B_1) is equivalent to (A). Another equivalent analytic condition is:

(B_2) There exists $\lambda > 0$ such that $x^\lambda Q(x)$ eventually decreases.

This follows from Lemma 2.1 of [7]. If X is in the domain of attraction of a stable law of index α then

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = \frac{2-\alpha}{\alpha}$$

so we are considering a class of distributions much larger than the class of distributions in the domain of attraction of a stable law.

We will show that the condition (A) implies that there exist $\varepsilon > 0$, $C > 0$ such that for each $M > 0$ a centering sequence $\{b_n\}$ and $c > 0$ can be found for which

$$(1.3) \qquad c \le a_n P\{S_n \in (x - \varepsilon, x + \varepsilon)\} \le C$$

for all n sufficiently large, the upper bound holding for all x and the lower bound for $x \in (b_n - Ma_n, b_n + Ma_n)$. We will also show that by taking M large the mass S_n assigns to the complement of $(b_n - Ma_n, b_n + Ma_n)$ can be made arbitrarily small uniformly in n. (Of course, in the lattice case ε must be taken large enough so that the interval includes a lattice point and some obvious modifications must be made to allow for periodicity.) Furthermore, (A) is necessary for (1.3); in fact, it is necessary for either the upper bound result or the lower bound result so that these results imply each other! We will also show that (A) is equivalent to tightness of the sequence $\{(S_n - b_n)/a_n\}$ for an appropriate centering sequence.

The pair of inequalities (1.3) gives a weakened version of the local limit theorem (see [11]). By summing the bounds in (1.3) over x one also obtains bounds for long intervals, and by using this together with the tightness of $\{(S_n - b_n)/a_n\}$, one also obtains a weakened version of the central limit theorem under (A). In many applications these estimates are all one needs. Feller [2] does make some comments about the local limit theorem in this setting but he has neglected the possibility that the limit density may be zero and so his observations are incorrect.

The main results are proved in Theorem 1 in Section 2. The equivalence of (A) to stochastic compactness is due to Feller, and the equivalence of (A) to tightness is easy via (B_1). That (A) implies the upper bound in (1.3) for all x is in [3] and has also been shown independently by Hall [12]. The heart of the matter here is the lower bound results. We will explain a little of the difficulty at this point. The centering sequence to be used for tightness or stochastic compactness is not too critical and can clearly be altered by addition of any multiple of a_n. However, for the lower bound in (1.3) the centering is much more delicate and in fact may need to be changed if M is increased. Centering at the mean, if it exists, may not even work for tightness and stochastic compactness, and even centering at the median of S_n may not work for the lower bound! In the proof we will work with the limits of subsequences of $\{(S_n - b_n)/a_n\}$. These limits do have densities but some may have support on $[0, \infty)$ and others on $(-\infty, 0]$. The crux of the problem is to find a centering sequence $b_n(M)$ so that *all* these limit densities will be positive on $[-M, M]$. We give a constructive definition of $b_n(M)$ in terms of F in Section 3. We will also show that in certain nice situations one may in fact center at 0 or at ES_n even for the lower bound. This is often important in applications. Finally, in Section 4 we will prove that (A) implies

$$|P\{S_n = x\} - P\{S_n = y\}| \le \frac{C|x-y|}{a_n^2}$$

in the lattice case under the right assumption about periodicity. This is often quite useful in conjunction with the upper bound in (1.3).

An interesting question which arises in this context is to characterize the class of limit laws which may be obtained as subsequential limits in this way. The restriction of stochastic compactness (distributions satisfying (A)) rules out much of the unusual behavior that can otherwise arise. In particular, one consequence of our proof will be that these limit laws must have C^∞ densities! This class of limit laws is described in [8] in terms of a condition analogous to (A) on the Lévy measure. For some other work related to stochastic compactness, see Jain and Orey [5] and Doeblin's original paper [1].

Bounds analogous to (1.3) for random variables taking values in $\mathbb{R}^n$ are obtained in [3] under some simplifying assumptions. In fact, this work was the motivation for the present paper. The problem in $\mathbb{R}^n$ is harder for two reasons: the tail of the distribution may die at different rates in different directions (even the two directions in $\mathbb{R}^1$ cause difficulties) and the centering question becomes more complicated.

We should also mention that these results have been used in two applications, one in the transient case and one in the recurrent case. In the former, Griffin [4] obtains an integral test for the rate of escape problem for transient random walk. In the latter, Jain and Pruitt [6] obtain the lim sup behavior for the local time of the random walk.

2. Main results. We start with a few observations about the function Q defined in (1.1). It is clear that

$$x^2 Q(x) \uparrow. \tag{2.1}$$

Furthermore, if (A) is satisfied then there is a $\lambda > 0$ and an x_0 such that

$$x^\lambda Q(x) \downarrow \quad \text{for} \quad x \ge x_0 \tag{2.2}$$

(see Lemma 2.4 of [7].) It then follows that there is a positive c_0 such that

$$x^\lambda Q(x) \ge c_0 y^\lambda Q(y) \quad \text{for} \quad \pi^{-1} \le x \le y. \tag{2.3}$$

It is a consequence of (2.1) and (2.2) that for given $\eta < 1$, there exists an n_0 such that with a_n as in (1.2)

$$\eta^{-\lambda} \le nQ(\eta a_n) \le \eta^{-2}, \quad n \ge n_0 \tag{2.4}$$

and also if $M > 1$ then

$$M^{-2} \le nQ(Ma_n) \le M^{-\lambda}, \quad n \ge n_0. \tag{2.5}$$

Although a_n may not be defined for small n, this will not matter since the results are for large n.

We first prove a lemma and a proposition which describe some of the consequences of (A) failing to hold. The proposition is of some independent interest and helps explain the equivalence of the upper and lower bounds. Below X_i^s will denote the symmetrization of X_i, and the superscript s will always mean that quantities correspond to these symmetrized random variables.

LEMMA 1. *If* (A) *fails and* G *is not slowly varying, then there exist integer sequences* $\{m_j\}$ *and* $\{n_j\}$ *such that*

$$m_j < n_j, \quad m_j/n_j \to 1, \quad \text{and} \quad a_{m_j}/a_{n_j} \to 0.$$

Furthermore, if $a_{m_j} \le x_j < 2x_j \le a_{n_j}$, *then there exists an integer* r_j *such that* $a_{r_j} \in [x_j, 2x_j]$, *and*

$$K(x_j) = o(G(x_j)). \tag{2.6}$$

PROOF. By Lemma 2.5 [7] G is not slowly varying iff

$$\limsup_{x\to\infty} K(x)/G(x) > 0, \tag{2.7}$$

and (A) fails iff

$$\liminf_{x\to\infty} K(x)/G(x) = 0. \tag{2.8}$$

Let $0 < \varepsilon_j \to 0$, and pick $\{y_j\}$ so that

$$\varepsilon_j^{-3} Q(y_j) \to 0. \tag{2.9}$$

Let

$$z_j = \inf\left\{x > y_j : \frac{K(y)}{G(y)} > \varepsilon_j \quad \text{for some} \quad y \in (y_j, x), \frac{K(x)}{G(x)} \le \varepsilon_j^3\right\}$$

and

$$\theta_j = \sup\left\{y : y_j < y < z_j, \frac{K(y)}{G(y)} > \varepsilon_j\right\}.$$

In view of (2.7) and (2.8) the z_j's exist and since K/G is right continuous and jumps up, whenever it jumps, we must have

$$K(z_j)/G(z_j) = \varepsilon_j^3, \; K(\theta_j)/G(\theta_j) = \varepsilon_j.$$

Now, if $x \in (\varepsilon_j z_j, z_j]$, then since $x^2K(x)\uparrow$

$$\frac{K(x)}{G(x)} \le \frac{z_j^2}{x^2}\frac{K(z_j)}{G(z_j)} < \varepsilon_j. \tag{2.10}$$

It follows that $\theta_j \le \varepsilon_j z_j$; in particular

$$y_j \le \varepsilon_j z_j. \tag{2.11}$$

It is also clear that

$$\frac{K(x)}{G(x)} \ge \varepsilon_j^3 \quad \text{on} \quad [\varepsilon_j z_j, z_j]. \tag{2.12}$$

By (2.12) we have $x^{\rho_j}\, Q(x)\downarrow$ on $[\varepsilon_j z_j, z_j]$ by Lemma 2.4 [7], where $\rho_j = 2\varepsilon_j^3/(1+\varepsilon_j^3)$, and so if $\varepsilon_j z_j \le x_j < 2x_j \le z_j$,

$$\frac{1}{Q(2x_j)} - \frac{1}{Q(x_j)} \ge (2^{\rho_j}-1)\frac{1}{Q(x_j)} \sim \frac{\rho_j \log 2}{Q(x_j)} \sim \frac{2\varepsilon_j^3 \log 2}{Q(x_j)} \ge \frac{\varepsilon_j^3}{Q(y_j)} \to \infty,$$

where (2.11) is used at the last inequality and then (2.9). Since $1/Q(a_n) = n$, this means r_j exists such that $a_{r_j} \in [x_j, 2x_j]$. We now pick m_j and n_j so that $a_{m_j} \in [\varepsilon_j z_j, 2\varepsilon_j z_j]$, $a_{n_j} \in [z_j/2, z_j]$. Then $a_{m_j}/a_{n_j} \to 0$. By (2.10) and Lemma 2.4 [7], $x^{2\varepsilon_j}\, Q(x)\uparrow$ on $[\varepsilon_j z_j, z_j]$, so

$$\frac{n_j}{m_j} = \frac{Q(a_{m_j})}{Q(a_{n_j})} \le \frac{Q(\varepsilon_j z_j)}{Q(z_j)} \le \varepsilon_j^{-2\varepsilon_j} \to 1.$$

Finally, (2.6) follows from (2.10). This completes the proof of Lemma 1.

PROPOSITION 1. (i) *If* (A) *fails, then there exists a subsequence* $\{r_i\}$ *of the integers and a centering sequence* $\{b_{r_i}\}$ *such that* $(S_{r_i} - b_{r_i})/a_{r_i} \to \lambda\delta_0$ *vaguely, where* $0 < \lambda < 1$ *and* δ_0 *is the probability measure concentrated at* 0. (ii) *If* (A) *fails, then there exists a subsequence* $\{r_i\}$ *of the integers such that* $S^s_{r_i}/a_{r_i} \to \lambda\delta_0$ *vaguely for some* $0 < \lambda < 1$.

PROOF. The failure of (A) means that $\liminf_{x\to\infty} K(x)/G(x) = 0$. We consider two cases. If $\lim_{x\to\infty} K(x)/G(x) = 0$, then G is slowly varying (see Lemma 2.5 [7]) and then $S_n/a_n \to e^{-1}\delta_0$ vaguely (Proposition 7.2 [13]). Furthermore, when G is slowly varying it is easy to check that $G^s(x) \sim 2G(x)$, so G^s is slowly varying and

$$nG^s(a_n) \sim 2nG(a_n) \sim 2nQ(a_n) = 2.$$

Thus $S^s_n/a_n \to e^{-2}\delta_0$ vaguely (Proposition 7.2 [13]).

We now consider the case when $\limsup_{x\to\infty} K(x)/G(x) > 0$. Choose sequences $\{m_i\}$ and $\{n_i\}$ as in Lemma 1 and suppose $m_i \le r_i \le n_i$. Define

$$b_{r_i} = r_i\, E(X1\{|X| \le a_{m_i}\}).$$

Then, for $0 < \varepsilon \le 1$, we have by truncating at $\pm a_{m_i}$ and using Chebyshev

$$P\{|S_{n_i} - b_{n_i} - S_{r_i} + b_{r_i}| \ge \varepsilon a_{m_i}\} \le \frac{(n_i - r_i)a_{m_i}^2 K(a_{m_i})}{\varepsilon^2 a_{m_i}^2} + (n_i - r_i)G(a_{m_i})$$

$$\le \varepsilon^{-2}(n_i - r_i)Q(a_{m_i}) = \varepsilon^{-2}\frac{n_i - r_i}{m_i}$$

and the last quantity $\to 0$ as $i \to \infty$. Essentially the same argument shows that $(S_{n_i}^s - S_{r_i}^s)/a_{m_i} \to 0$ in probability since $Q^s(x) \le CQ(x)$ for some $C > 0$. Now take a subsequence of $\{n_i\}$ so that $(S_{n_i} - b_{n_i})/a_{n_i}$ converges vaguely to V. To simplify the notation we do not rename the subsequence. Now it is possible to find $\{\gamma_i\}$ such that $\gamma_i/a_{n_i} \to 0$ and $(S_{n_i} - b_{n_i})/\gamma_i \to \lambda\delta_0$, where $\lambda = V(\{0\})$, i.e., the mass that was going to the origin still does and the rest of the mass necessarily goes to infinity. Since there is no harm in assuming $\gamma_i \ge a_{m_i}$ we may actually choose $\gamma_i = a_{r_i}$ for some $r_i \in [m_i, n_i]$ by Lemma 1. Now by the first part of the argument $(S_{r_i} - b_{r_i})/a_{r_i} \to \lambda\delta_0$. The same argument clearly gives part (ii) of the proposition. It only remains to show that $0 < \lambda < 1$. Let

$$T_{r_i} = \textstyle\sum_{k=1}^{r_i} X_k 1\{|X_k| \le a_{m_i}\} - b_{r_i}; \quad T'_{r_i} = \sum_{k=1}^{r_i} X_k 1\{|X_k| > a_{m_i}\}.$$

For $\varepsilon > 0$,

$$P\{|T_{r_i}| \ge \varepsilon a_{r_i}\} \le \frac{r_i a_{m_i}^2 K(a_{m_i})}{\varepsilon^2 a_{r_i}^2} \le \frac{1}{\varepsilon^2}\frac{r_i}{m_i}\frac{K(a_{m_i})}{G(a_{m_i})} \to 0$$

since $r_i \sim m_i$ and $K(a_{m_i}) = o(G(a_{m_i}))$ by (2.6). Thus

$$P\{|S_{r_i} - b_{r_i}| \le \varepsilon a_{r_i}\} = P\{|T_{r_i} + T'_{r_i}| \le \varepsilon a_{r_i}\}$$

$$\ge P\{|T_{r_i}| \le \varepsilon a_{r_i}, |X_k| \le a_{m_i}, 1 \le k \le r_i\}$$

$$\ge P\{|X_k| \le a_{m_i}, 1 \le k \le r_i\} - P\{|T_{r_i}| > \varepsilon a_{r_i}\},$$

and there must be mass at least

$$(1 - G(a_{m_i}))^{r_i} \sim \exp(-r_i G(a_{m_i})) \sim \exp(-m_i Q(a_{m_i})) = e^{-1}$$

at 0 for the limit law. On the other hand, if exactly one of the r_i summands exceeds a_{m_i} in absolute value and it in fact exceeds a_{n_i} then we will have (when $|T_{r_i}| \le \varepsilon a_{r_i}$)

$$|S_{r_i} - b_{r_i}| \ge a_{n_i} - |T_{r_i}| \ge (1 - \varepsilon)a_{r_i}$$

and this mass goes to infinity. So the mass going to infinity must be asymptotically at least

$$r_i G(a_{n_i})(1 - G(a_{m_i}))^{r_i - 1} \sim r_i Q(a_{n_i})e^{-1} \sim e^{-1}.$$

The proof that $0 < \lambda < 1$ in (ii) is similar; one needs the inequality

$$K^s(x) \le 8K(2x) + 2G(x)G(2x).$$

This proves the proposition.

In order to discuss the lattice case we must first say a few words about periodicity. We assume throughout that $\mathbb{Z}$, the integers, is the right lattice for the random walk. This means that $S_n \in \mathbb{Z}$ a.s. for all n and this is not the case for any sublattice. (Note that this assumption just amounts to a change of scale.) It still may be the case that the symmetrized random walk S_n^s with summands having the distribution of $X_1 - X_2$ may live on a sublattice. For example, the symmetrization of simple random walk lives on the even integers. We will let p denote the largest integer such that $S_n^s \in p\mathbb{Z}$ a.s. for all n. Then the original random walk has the property that $S_{kp} \in p\mathbb{Z}$ a.s. for all k while S_{kp+i} will be in some coset C_i of $p\mathbb{Z}$ in $\mathbb{Z}$ a.s. for all k, $i = 1, \cdots, p-1$. More details can be found in Spitzer [10].

We begin by listing the various equivalent statements. These statements are first formulated for the lattice case; where necessary, the modifications needed for the non-lattice case are given afterwards.

(A) $$\limsup_{x\to\infty} \frac{G(x)}{K(x)} < \infty.$$

(U) There is a positive C and an n_0 such that with $\{a_n\}$ as in (1.2)

$$P\{S_n = x\} \le \frac{C}{a_n} \quad \text{for all} \quad x \in \mathbb{Z},\ n \ge n_0.$$

In the non-lattice case, for every $\eta > 0$, there is a positive C and n_0 such that

$$P\{|S_n - x| \le \eta\} \le \frac{C}{a_n} \quad \text{for all} \quad x \in \mathbb{R}^1,\ n \ge n_0.$$

(L_1) For every $\varepsilon > 0$, there exist $c > 0$, n_0, and sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$P\{S_n = x\} \ge \frac{c}{a_n} \quad \text{for all} \quad n \ge n_0 \tag{2.13}$$

and all $x \in [\alpha_n, \beta_n]$ such that $x \in C_i$ where $n \equiv i \pmod p$ and

$$P\{S_n \notin [\alpha_n, \beta_n]\} \le \varepsilon.$$

Furthermore, the dependence of α_n, β_n on ε is such that

$$\lim_{n\to\infty,\varepsilon\to 0} \frac{\beta_n(\varepsilon) - \alpha_n(\varepsilon)}{a_n} = \infty.$$

In the non-lattice case, replace (2.13), (2.14) by $P\{|S_n - x| \le \eta\} \ge c a_n^{-1}$ where c depends on η.

(L_2) For every $M > 0$, there exist $c > 0$, n_0, and a sequence $\{\delta_n\}$ such that

$$P\{S_n = x\} \ge \frac{c}{a_n} \quad \text{for all} \quad n \ge n_0 \tag{2.14}$$

and all x such that $|x - \delta_n| \le M a_n$ and $x \in C_i$ where $n \equiv i \pmod p$.

(L_3) Same as (L_2) with "For every $M > 0$" replaced by "There exists $M > 0$".

(T_1) There exist a sequence $\{\delta_n\}$, a $C > 0$ and an n_0 such that

$$P\{|S_n - \delta_n| \geq \theta a_n\} \leq \frac{C}{\theta^\lambda} \quad \text{for all} \quad \theta,\ n \geq n_0$$

where λ is as in (2.2).

(T_2) There exists a sequence $\{\delta_n\}$ such that $\{(S_n - \delta_n)/a_n\}$ is tight.

(C) There exist sequences $\{\gamma_n\}$, $\{\delta_n\}$ such that for every subsequence of $\{(S_n - \delta_n)/\gamma_n\}$ there is a further subsequence which converges weakly to a nondegenerate limit.

In applications, it is essential that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ be constructed knowing only the distribution of X. We will do this in the next section. It will also be apparent that for any $\varepsilon > 0$, $(\alpha_n + \beta_n)/2$ may be used for δ_n in (T_1), (T_2), and (C).

THEOREM 1. *The statements* (A), (U), (L_1), (L_2), (L_3), (T_1), (T_2), *and* (C) *are equivalent.*

PROOF. The proofs go via the analytic condition (A). Feller proved that (A) $\Leftrightarrow$ (C) so we will not repeat this. Also (A) $\Rightarrow$ (U) is in [3, Theorem 3.6] and has been proved independently by Hall [12]. The necessity of (A) for (U) and (T_2) follows easily from Proposition 1. Since (L_1) $\Rightarrow$ (L_2) $\Rightarrow$ (L_3) and (T_1) $\Rightarrow$ (T_2) we only need to prove (A) $\Rightarrow$ (T_1), (A) $\Rightarrow$ (L_1), and (L_3) $\Rightarrow$ (A). We will need an estimate on the characteristic function φ of X which is contained in Theorem 2.10 of [3]. Under (A), there is a $c > 0$ such that

$$(2.15) \qquad |\varphi(u)| \leq 1 - c\, Q\left(\frac{1}{|u|}\right), \quad 0 < |u| \leq \frac{\pi}{p}.$$

In the non-lattice case, the bound holds for $0 < |u| \leq M$ for any M where c may depend on M.

(A) $\Rightarrow$ (T_1). For $\theta > 0$ we let

$$U_n(\theta) = \textstyle\sum_{i=1}^n X_i 1\{|X_i| \leq \theta a_n\}$$

and $\delta_n = EU_n(2^{1/\lambda})$. Then for $\theta \geq 2^{1/\lambda}$ we have for large n

$$|EU_n(\theta) - \delta_n| \leq n \int_{2^{1/\lambda} a_n < |x| \leq \theta a_n} |x|\, dF \leq n\theta a_n G(2^{1/\lambda} a_n)$$

$$\leq n\theta a_n Q(2^{1/\lambda} a_n) \leq \frac{\theta}{2}\, a_n$$

by (2.5). Then by Chebyshev's inequality and (2.5)

$$P\{|S_n - \delta_n| \geq \theta a_n\} \leq P\{S_n \neq U_n(\theta)\} + P\left\{|U_n(\theta) - EU_n(\theta)| \geq \frac{\theta}{2} a_n\right\}$$

$$\leq nG(\theta a_n) + n\frac{4\theta^2 a_n^2 K(\theta a_n)}{\theta^2 a_n^2} \leq 4nQ(\theta a_n) \leq \frac{4}{\theta^\lambda} .$$

The bound is trivial for small θ.

For the last two assertions, we will give the details of the proof in the lattice case, and then indicate the necessary modifications in the nonlattice case.

(A) $\Rightarrow$ (L_1). We assume that α_n, β_n satisfy

$$(2.16)\qquad \begin{aligned} c_1 \leq P\{S_n \leq \alpha_n\}, \quad P\{S_n < \alpha_n\} \leq \frac{\varepsilon}{2} \\ c_1 \leq P\{S_n \geq \beta_n\}, \quad P\{S_n > \beta_n\} \leq \frac{\varepsilon}{2} \end{aligned}$$

for some $c_1 > 0$. In the next section we will see how to construct these sequences. But they certainly exist even if we take $c_1 = \varepsilon/2$. We will show that any $\{\alpha_n\}$, $\{\beta_n\}$ satisfying (2.16) will work in (L_1). Choose $x_n \in [\alpha_n, \beta_n]$ to satisfy

$$P\{S_n = x_n\} = \min_{\{x \in [\alpha_n, \beta_n] \cap C_i\}} P\{S_n = x\}$$

where $n \equiv i (\mathrm{mod}\, p)$, and let

$$c = \liminf_{n \to \infty} a_n P\{S_n = x_n\}.$$

We need to prove $c > 0$. Choose a subsequence along which $a_n P\{S_n = x_n\} \to c$. Since (A) $\Rightarrow$ (T_1) if we use $\{\delta_n\}$ as above, we can find θ so that

$$P\{|S_n - \delta_n| \geq \theta a_n\} < c_1$$

and so $[\alpha_n, \beta_n] \subset [\delta_n - \theta a_n, \delta_n + \theta a_n]$. By taking further subsequences we may thus assume that

$$(2.17)\qquad \frac{\beta_n - \delta_n}{a_n} \to \beta, \quad \frac{\alpha_n - \delta_n}{a_n} \to \alpha, \quad \frac{x_n - \delta_n}{a_n} \to z$$

where

$$-\theta \leq \alpha \leq z \leq \beta \leq \theta.$$

Furthermore, by Helly's compactness theorem, we may assume that

$$(2.18)\qquad \frac{S_n - \delta_n}{a_n} \to H \text{ vaguely}$$

along the subsequence. For the remainder of this proof, we restrict n to be in this subsequence. Since we have already proved that (A) $\Rightarrow$ (T_2) we know that H is a probability distribution. It is necessarily infinitely divisible. We let ψ denote its characteristic function. By the inversion formula, we have

$$
\begin{aligned}
a_n P\{S_n = x_n\} &= a_n \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-iux_n)\varphi^n(u)\, du \\
&= a_n \frac{p}{2\pi} \int_{-\pi/p}^{\pi/p} \exp(-iux_n)\varphi^n(u)\, du
\end{aligned} \tag{2.19}
$$

since $\exp(-iux_n)\varphi^n(u)$ is the characteristic function of the random variable $S_n - x_n$ which must have period $2\pi/p$ because it takes its values in $p\mathbb{Z}$. Now we need a bound for the integrand. By (2.3) we have for $1 \le |v| \le \pi p^{-1} a_n$

$$
\left(\frac{a_n}{|v|}\right)^{\lambda} Q\left(\frac{a_n}{|v|}\right) \ge c_0 a_n^{\lambda} Q(a_n) \quad \text{so that} \quad nQ\left(\frac{a_n}{|v|}\right) \ge c_0 |v|^{\lambda}.
$$

Now, by (2.15)

$$
\begin{aligned}
|\varphi^n(va_n^{-1})| &\le \left(1 - cQ\left(\frac{a_n}{|v|}\right)\right)^n \\
&\le \exp(-cnQ(a_n|v|^{-1})) \le \exp(-c_2|v|^{\lambda}), \quad 1 \le |v| \le \frac{\pi}{p}\, a_n.
\end{aligned} \tag{2.20}
$$

Recalling (2.17), (2.18), and (2.19), and using dominated convergence

$$
\begin{aligned}
a_n P\{S_n = x_n\} &= \frac{p}{2\pi} \int_{-\pi a_n/p}^{\pi a_n/p} \exp(-iva_n^{-1}(x_n - \delta_n))\varphi^n(va_n^{-1})\exp(-iva_n^{-1}\delta_n)\, dv \\
&\to \frac{p}{2\pi} \int_{-\infty}^{\infty} e^{-ivz}\psi(v)\, dv = ph(z)
\end{aligned}
$$

where h is the density for H. In fact, (2.20) implies that $|\psi(v)| \le \exp\{-c_2|v|^{\lambda}\}$ for $|v| \ge 1$ so that h not only exists but has derivatives of all orders. We note that this is true for any subsequential limit of $\{(S_n - b_n)/a_n\}$ as mentioned in the introduction. To complete the proof we must show that h is positive on $[\alpha, \beta]$. For this we apply a result of Sharpe [9] which asserts that an infinitely divisible law with characteristic function in L^p for all p has a density which is either never zero or else zero on a closed half-line. Note that ψ is in L^p for all p by the bound on $|\psi|$. Finally since

$$
\lim_{n\to\infty} P\left\{\frac{S_n - \delta_n}{a_n} \ge \beta\right\} = \lim_{n\to\infty} P\left\{\frac{S_n - \delta_n}{a_n} \ge \frac{\beta_n - \delta_n}{a_n}\right\} \ge c_1
$$

with a similar argument applying to the left tail, we see that H has mass on both sides of $[\alpha, \beta]$ and so h must be positive on $[\alpha, \beta]$. To prove the last statement, suppose there is a sequence $\varepsilon_k \to 0$ and a subsequence $\{n_k\}$ such that

$$
\frac{\beta_{n_k}(\varepsilon_k) - \alpha_{n_k}(\varepsilon_k)}{a_{n_k}} \to M < \infty. \tag{2.21}
$$

By the choice of $\{\alpha_n\}$ and $\{\beta_n\}$ we have

$$\lim_{k\to\infty} P\left\{\frac{\alpha_{n_k}(\varepsilon_k) - \delta_{n_k}}{a_{n_k}} \le \frac{S_{n_k} - \delta_{n_k}}{a_{n_k}} \le \frac{\beta_{n_k}(\varepsilon_k) - \delta_{n_k}}{a_{n_k}}\right\} = 1.$$

Since $\left\{\frac{S_{n_k} - \delta_{n_k}}{a_{n_k}}\right\}$ is tight and (2.21) holds, we must have

$$\frac{\alpha_{n_k} - \delta_{n_k}}{a_{n_k}} \to \alpha_1, \quad \frac{\beta_{n_k} - \delta_{n_k}}{a_{n_k}} \to \beta_1$$

along a subsequence (where α_1, β_1 are finite). Thus along a further subsequence a limit law would have compact support $[\alpha_1, \beta_1]$, which is a contradiction. (Recall Sharpe's result.)

We now outline the proof in the non-lattice case. Define

$$k(u) = \frac{\sin u}{u}, \quad r(u) = (1 - |u|)^+, \quad R(x) = 2\,\frac{1 - \cos x}{x^2}.$$

By the inversion formula,

$$\frac{\rho}{2\eta} \int R(\rho(x_n - y)) a_n P\{|S_n - y| \le \eta\}\, dy = a_n \int \exp(-iux_n) r\left(\frac{u}{\rho}\right) k(u\eta) \varphi^n(u)\, du.$$

Split the integration on the left hand side into $|y - x_n| < \eta$ and $|y - x_n| \ge \eta$ and use (U) on the latter set to obtain an upper bound

$$\frac{\pi}{\eta} a_n P\{|S_n - x_n| < 2\eta\} + \frac{4C}{\rho\eta^2}.$$

For the right hand side, pick x_n to minimize $P\{|S_n - x_n| < 2\eta\}$ subject to $\alpha_n \le x_n \le \beta_n$ and proceed as in the lattice case to show convergence to $2\pi h(z) > 0$; in this case the bound for the integrand is only needed for $|v| \le \rho a_n$ because of r and the bound may be extended to this range in the non-lattice case. With η and z fixed we may choose ρ so that $4C/\rho\eta^2 < \pi h(z)$.

$(L_3) \Rightarrow (A)$. We shall symmetrize and use part (ii) of Proposition 1. If (L_3) holds for S_n, then for $|x| \le Ma_n/2$ and $x \in p\mathbb{Z}$

$$P\{S_n^s = x\} \ge \sum_{y \in C_i, |y - \delta_n| \le Ma_n/2} P\{S_n = y\} P\{S_n = x + y\}$$

(2.22)
$$\ge \left(\frac{c}{a_n}\right)^2 \frac{Ma_n}{p} = \frac{c^2 M}{p} \frac{1}{a_n}.$$

But this contradicts the conclusion of Proposition 1. The analogue of (2.22) in the non-lattice case is similar.

3. Centering. We start by observing that centering may be necessary for the lower bound even when it is not needed for stochastic compactness or tightness. For example, let X have a stable distribution of index $\alpha < 1$ which is supported on $[0, \infty)$. Then $n^{-1/\alpha}S_n$ still has the same distribution and so converges weakly. But (L_2) and (L_3) do not hold without centering since there is no mass on the negative axis. In fact, in this example δ_n must grow with M (essentially like Ma_n) so this shows that the dependence of δ_n on M cannot be dropped. Furthermore since the median of S_n is $n^{1/\alpha}$ times the median of X it is asymptotically a constant times a_n which shows that the median of S_n cannot be used for δ_n when M is large in this example.

A centering sequence for stochastic compactness or tightness is easy to construct. In fact the $\{\delta_n\}$ constructed in the proof of (A) $\Rightarrow$ (T_1) above suffices. After taking care of some preliminaries, we will construct the $\alpha_n(\varepsilon)$, $\beta_n(\varepsilon)$ sequences of (L_1). Their average, for appropriate ε, will also suffice for (L_2), (L_3), (T_1), (T_2), and (C).

We start by introducing some notation. For $a > 0$, let

$$Y_i = X_i \wedge a, \quad Z_i = -a \vee Y_i, \tag{3.1}$$

and

$$T_n = \textstyle\sum_{i=1}^n Y_i, \quad V_n = \sum_{i=1}^n Z_i.$$

Then by Lemma 3.2 of [7] we have for $C_1 \le 1/6$

$$P\{T_n \ge EV_n + C_1 naQ(a)\} \ge e^{-35nQ(a)} \tag{3.2}$$

provided that both a and $nQ(a)$ are sufficiently large and $EX = 0$ if $EX^2 < \infty$. A check of the proof shows that in fact $nQ(a) \ge 7$ is adequate. For future reference, we note that

$$\begin{aligned} EV_n = nEZ_1 &= n\left(\int_{|y|\le a} y\, dF(y) + aG_+(a) - aG_-(a)\right) \\ &= n\int_0^a (G_+(y) - G_-(y))\, dy \end{aligned} \tag{3.3}$$

where we are using $G_+(a) = P\{X > a\}$, $G_-(a) = P\{X < -a\}$.

The condition (A) is assumed in the rest of the section.

LEMMA 2. *Assume* $EX^2 = \infty$. *If* $\eta < 1/3$, $(2\eta^\lambda)^{-1} \ge 7$ *where* λ *is as in* (2.2), $\rho \in (1, \eta^{-1})$, *and* C, ε *are fixed positive constants, then there is a* c (*which depends on* η, ρ, C, *and* ε) *such that*

$$P\{S_n \ge n\mu_n + Ca_n\} \ge c$$

for all n *sufficiently large which satisfy*

$$G_+(\rho\eta a_n) > \varepsilon n^{-1} \tag{3.4}$$

where

$$\mu_n = \int_0^{\eta a_n} (G_+(y) - G_-(y))\, dy.$$

PROOF. By (2.4), if $k \in [\frac{1}{2} n, n]$ and n is large we have

$$kQ(\eta a_n) \geq (2\eta^\lambda)^{-1} \geq 7$$

so that we may use (3.2) with $a = \eta a_n$. Thus by (3.3) and (2.4)

$$P\{T_k \geq k\mu_n\} \geq P\{T_k \geq k\mu_n + C_1 k\eta a_n Q(\eta a_n)\} \geq \exp(-35\eta^{-2}). \tag{3.5}$$

Now we take δ positive and small enough that

$$e^{-\delta} + \exp(-35\eta^{-2}) > 1 \quad \text{and} \quad \delta \leq \varepsilon. \tag{3.6}$$

Then we choose ξ so that $G_+(\xi a_n) = \delta n^{-1}$. (There are two ways of avoiding the possibility of G_+ jumping over δn^{-1}. One is to convolve X with a normal random variable so that G_+ is continuous for the new summands. Then since $a_n n^{-1/2} \to \infty$ when $EX^2 = \infty$ one may deduce the lemma for the given summands from the lemma for the convolved summands. It is also easy to check that this change does not alter a_n significantly. A disadvantage is that this changes μ_n and this complicates the definition of β_n. The alternative is to enrich the probability space by introducing a sequence of auxiliary Bernoulli variables, one for each X_i, to appropriately split the atom at ξa_n. If this is needed, then in what follows if we write $X_i > \xi a_n$ we really mean either $X_i > \xi a_n$ or $X_i = \xi a_n$ and the corresponding Bernoulli variable is one.) By (3.4) and (3.6)

$$G_+(\rho\eta a_n) > \varepsilon n^{-1} \geq \delta n^{-1}$$

so that $\xi \geq \rho\eta$. Now we take m so that

$$m(\rho - 1)\eta \geq C \tag{3.7}$$

and define $\Gamma(i_1, \cdots, i_m) = \Lambda(i_1, \cdots, i_m) \cap \Delta(i_1, \cdots, i_m)$ where

$$\Lambda(i_1, \cdots, i_m) = \{X_{i_1} > \xi a_n, \cdots, X_{i_m} > \xi a_n\},$$

$$\Delta(i_1, \cdots, i_m) =$$
$$\{\textstyle\sum'(X_j \wedge \eta a_n) \geq (n - m)\mu_n,\ X_j \leq \xi a_n \quad \text{for} \quad j \neq i_1, \cdots, i_m, j \leq n\},$$

where $\sum'$ denotes the sum over all $j \in [1, n]$ excluding $i_1, \cdots, i_m$. Now the events Λ and Δ are independent and

$$P(\Lambda) = \{G_+(\xi a_n)\}^m = \left(\frac{\delta}{n}\right)^m, \quad P(\Delta) \geq \exp(-35\eta^{-2}) + \left(1 - \frac{\delta}{n}\right)^n - 1 > c > 0$$

for large n by (3.5) and (3.6). Thus

$$P(\Gamma(i_1, \cdots, i_m)) \geq c\left(\frac{\delta}{n}\right)^m$$

and since the Γ's are disjoint

$$P(\cup_{i_1,\cdots,i_m}\Gamma(i_1, \cdots, i_m)) \geq \binom{n}{m} c \left(\frac{\delta}{n}\right)^m \sim c\,\frac{\delta^m}{m!}.$$

Now, on this union, we have by (3.7)

$$S_n \geq m\xi a_n + (n - m)\mu_n \geq m\rho\eta a_n + (n - m)\mu_n$$

$$\geq m(\rho - 1)\eta a_n + n\mu_n \geq Ca_n + n\mu_n$$

since $\xi \geq \rho\eta$ and $\mu_n \leq \eta a_n$. Thus we have proved the lemma.

Now we are ready to give the general construction of $\{\alpha_n\}$, $\{\beta_n\}$. An important special case where a simpler procedure works is obtained in Theorem 3. We exclude the case $EX^2 < \infty$ from Theorem 2 for simplicity; it will be covered in Theorem 3.

THEOREM 2. *Assume* $EX^2 = \infty$. *Given* $\varepsilon > 0$, *choose* $\eta < 1/3$ *so that*

$$(2\eta)^{-\lambda} \geq 14, \quad (2\eta)^{-\lambda} \geq 144 \log(4/\varepsilon) \tag{3.8}$$

where λ *is chosen so that* $(2 - \lambda)/\lambda$ *is larger than the* lim sup *in* (A) *and let*

$$\zeta_n = \int_0^{2\eta a_n} (G_+(y) - G_-(y))\, dy.$$

Define

$$\beta_n = \begin{cases} n\zeta_n + \frac{1}{3} n\eta a_n Q(2\eta a_n) & \text{if } G_+(2\eta a_n) \leq \frac{\varepsilon}{4n} \\ n\zeta_n + (4\varepsilon^{-1})^{1/\lambda}(1 + \eta^{-2})a_n & \text{if } G_+(2\eta a_n) > \frac{\varepsilon}{4n} \end{cases}$$

and

$$\alpha_n = \begin{cases} n\zeta_n - \frac{1}{3} n\eta a_n Q(2\eta a_n) & \text{if } G_-(2\eta a_n) \leq \frac{\varepsilon}{4n} \\ n\zeta_n - (4\varepsilon^{-1})^{1/\lambda}(1 + \eta^{-2})a_n & \text{if } G_-(2\eta a_n) > \frac{\varepsilon}{4n}. \end{cases}$$

Then (L_1) *is satisfied with these* $\{\alpha_n\}$, $\{\beta_n\}$ *sequences.*

PROOF. By Lemma 3.1 of [7] with $a = 2\eta a_n$, we have for all positive r and s

$$P\{T_n \geq n\zeta_n + \tfrac{1}{2} re^r n2\eta a_n Q(2\eta a_n) + sr^{-1}2\eta a_n\} \leq e^{-s} \tag{3.9}$$

and by (3.8) we may use (3.2) with $a = 2\eta a_n$ to obtain

$$P\{T_n \geq n\zeta_n + \tfrac{1}{3} n\eta a_n Q(2\eta a_n)\} \geq \exp(-35/4\eta^2). \tag{3.10}$$

Choose r to solve $re^r = 1/6$ and use this and $s = \log(4/\varepsilon)$ in (3.9). Since

$$\frac{1}{6}\, nQ(2\eta a_n) \geq \frac{1}{6}\,(2\eta)^{-\lambda} \geq \frac{2s}{r}$$

for large n by (2.4) and (3.8), we may combine (3.9) and (3.10) to obtain

$$\exp(-35/4\eta^2) \leq P\{T_n \geq n\zeta_n + 1/3\; n\eta a_n Q(2\eta a_n)\} \leq \varepsilon/4.$$

Since $S_n \geq T_n$ and $P\{S_n \neq T_n\} \leq nG_+(2\eta a_n)$, this gives the necessary inequalities

$$c_1 \leq P\{S_n \geq \beta_n\} \leq \frac{\varepsilon}{2} \tag{3.11}$$

in case $G_+(2\eta a_n) \leq \varepsilon/4n$. For the other case we use the truncated variables as in (3.1) with $a = \theta a_n$ where $\theta = (4\varepsilon^{-1})^{1/\lambda}$. Then, using Chebyshev's inequality

$$P\{|S_n - EV_n| \geq \theta a_n\} \leq P\{S_n \neq V_n\} + P\{|V_n - EV_n| \geq \theta a_n\}$$
$$\leq nG(\theta a_n) + nQ(\theta a_n) \leq 2nQ(\theta a_n) \leq 2\theta^{-\lambda} = \frac{\varepsilon}{2}.$$

Moreover, by (3.3) and (2.4)

$$|EV_n - n\zeta_n| = n\left|\int_{2\eta a_n}^{\theta a_n} (G_+(y) - G_-(y))\, dy\right| \leq n\theta a_n Q(2\eta a_n) \leq \theta\eta^{-2} a_n$$

so that

$$P\{S_n \geq n\zeta_n + \theta(1 + \eta^{-2})a_n\} \leq P\{S_n \geq EV_n + \theta a_n\} \leq \frac{\varepsilon}{2}.$$

The lower bound in this case follows from Lemma 2 since

$$n\mu_n - n\zeta_n = n\int_{\eta a_n}^{2\eta a_n} (G_-(y) - G_+(y))\, dy \geq -n\eta a_n G(\eta a_n) \geq -\eta^{-1} a_n$$

by (2.4). Thus we have (3.11) in this case also. This and the analogous inequalities for α_n (which are similar) are all that were needed in the proof that (A) $\Rightarrow$ (L_1).

Finally, we describe one situation in which centering at the mean is adequate even for the lower bound. This includes the domain of attraction setting for $\alpha > 1$.

THEOREM 3. *If*

$$\limsup_{x\to\infty} \frac{G(x)}{K(x)} < 1, \tag{3.12}$$

then $E|X| < \infty$ *and one may use* $\delta_n = nEX$ *in Theorem* 1. *Furthermore, for every positive* M *there exist a positive* c *and an* n_0 *such that*

$$P\{S_n \geq nEX + Ma_n\} \geq c, \quad P\{S_n \leq nEX - Ma_n\} \geq c$$

for all $n \geq n_0$. *One may use*

$$\beta_n = nEX + Ma_n, \quad \alpha_n = nEX - Ma_n$$

in (L_1) *where* $M = (4\varepsilon^{-1})^{1/\lambda}(1 + \varepsilon\lambda/4(\lambda - 1))$ *and* λ *is chosen so that* $(2 - \lambda)/\lambda$ *is between the* lim sup *in* (3.12) *and* 1.

REMARK. There are examples in the domain of attraction of the Cauchy with $E|X| < \infty$, $EX = 0$, and $P\{S_n \geq 0\} \to 0$. Thus this result cannot be extended to include those random walks in the domain of attraction of the Cauchy having finite mean.

PROOF. If $EX^2 < \infty$ everything follows from the central limit theorem since $a_n \sim (EX^2)^{1/2}n^{1/2}$. Thus we assume $EX^2 = \infty$. If (3.12) holds, then $x^\lambda Q(x)\downarrow$ for large x for some $\lambda > 1$ by Lemma 2.4 of [7]. Then Q and G are integrable so $E|X| < \infty$. Next we have

$$\mu_n = \int_0^{\eta a_n} (G_+(y) - G_-(y))\,dy = EX - \int_{\eta a_n}^{\infty} (G_+(y) - G_-(y))\,dy \tag{3.13}$$

and

$$\int_{\eta a_n}^{\infty} G(y)\,dy \leq \int_{\eta a_n}^{\infty} Q(y)\,dy \leq (\eta a_n)^\lambda Q(\eta a_n) \int_{\eta a_n}^{\infty} y^{-\lambda}\,dy$$
$$\leq \frac{1}{\lambda - 1}\,\eta a_n Q(\eta a_n) \leq \frac{1}{\lambda - 1}\,\eta^{-1} a_n n^{-1}$$

by (2.4). This means that

$$n\mu_n - nEX = O(a_n). \tag{3.14}$$

We have for $\eta < 1$ by (2.4)

$$n\eta a_n Q(\eta a_n) \geq \eta^{1-\lambda} a_n$$

where $\lambda > 1$ here. Thus if we choose η small enough that

$$\eta^{-\lambda} \geq 7 \text{ and } \eta^{1-\lambda} \geq 24M$$

then by (3.2) with $a = \eta a_n$

$$P\{S_n \geq n\mu_n + 4Ma_n\} \geq c. \tag{3.15}$$

Next we let $\rho = 1 + M\eta$ and observe that by (2.4)

$$\int_{\eta a_n}^{\rho\eta a_n} G(y)\,dy \leq (\rho - 1)\eta a_n G(\eta a_n) \leq (\rho - 1)\eta a_n Q(\eta a_n)$$
$$\leq (\rho - 1)\eta^{-1} a_n \frac{1}{n} = Ma_n \frac{1}{n},$$

$$\int_{a_n}^{\infty} G(y)\,dy \le \int_{a_n}^{\infty} Q(y)\,dy \le \frac{1}{\lambda - 1}\,a_n Q(a_n)$$

$$\le \frac{1}{\lambda - 1}\,a_n\,\frac{1}{n} \le M a_n\,\frac{1}{n}$$

provided that $M \ge 1/(\lambda - 1)$. Thus by (3.13) we have

$$|\,nEX - n\mu_n - n\int_{\rho\eta a_n}^{a_n} (G_+(y) - G_-(y))\,dy\,| \le 2Ma_n.$$

Now there are two cases. If

$$G_+(\rho\eta a_n) \le Mn^{-1}, \tag{3.16}$$

then we have by (2.4) since $\rho\eta < 1$

$$n\int_{\rho\eta a_n}^{a_n} G_+(y)\,dy \le n a_n G_+(\rho\eta a_n) \le Ma_n$$

and so

$$nEX \le n\mu_n + 3Ma_n.$$

In conjunction with (3.15), this gives the required lower bound. If (3.16) fails, then (3.4) holds. Since we have

$$nEX \le n\mu_n + C_1 a_n$$

by (3.14) we again obtain the lower bound by taking for C in Lemma 2 the sum of C_1 and M. The other lower bound follows on replacing X by $-X$. It is now clear that nEX may be used for δ_n. Finally, to see that the given value works for β_n, truncate at $(4\varepsilon^{-1})^{1/\lambda}a_n$, use Chebyshev, and estimate the difference between EX and the truncated mean as above.

4. Upper estimate for differences. In the classical case of finite variance this bound may be deduced by combining the local limit theorem with a version due to Smith which gives a better error term for large x (see [10, page 79]).

THEOREM 4. *Assume* (A). *Then there is a C such that*

$$|\,P\{S_n = x\} - P\{S_n = y\}\,| \le \frac{C\,|\,y - x\,|}{a_n^2}$$

for all n and all x, y such that $y - x \in p\mathbb{Z}$.

PROOF. By the inversion formula and (2.15)

$$P\{S_n = x\} - P\{S_n = y\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-ixu} - e^{-iyu})\varphi^n(u)\, du,$$

$$\begin{aligned} |P\{S_n = x\} - P\{S_n = y\}| &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - e^{-i(y-x)u}|\, |\varphi(u)|^n\, du \\ &= \frac{p}{\pi} \int_0^{\pi/p} |1 - e^{-i(y-x)u}|\, |\varphi(u)|^n\, du \\ &\le \frac{p}{\pi} |y - x| \int_0^{\pi/p} u \exp\{-cnQ(u^{-1})\}\, du \\ &= \frac{p}{\pi} |y - x| \int_p^{\infty} \exp\{-cnQ(v)\} \frac{dv}{v^3}. \end{aligned}$$

To estimate the integral, we break the range of integration at a_n with the integral over $[a_n, \infty)$ clearly being at most $1/2a_n^2$. By (2.3),

$$\begin{aligned} \int_{p/\pi}^{a_n} \exp\{-cnQ(v)\} \frac{dv}{v^3} &\le \int_{p/\pi}^{a_n} \exp\{-c_1 a_n^{\lambda} v^{-\lambda}\} \frac{dv}{v^3} \\ &\le a_n^{-2} \int_0^1 \exp\{-\, c_1 w^{-\lambda}\} \frac{dw}{w^3} = C_1 a_n^{-2}. \end{aligned}$$

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