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ON CONTINUUM PERCOLATION

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Let $\mathscr{P}$ be a homogeneous Poisson process in $\mathbb{R}^k$. At the points of $\mathscr{P}$, centre k-dimensional spheres whose radii are independent and identically distributed. It is shown that there exists a positive critical intensity for the formation of clumps whose mean size is infinite, if and only if sphere content has finite variance. It is also proved that under a strictly weaker condition than existence of finite variance, there exists a positive critical intensity for the formation of clumps whose size is infinite with positive probability. Therefore these two critical intensities need not be the same. Continuum percolation in the case of general random sets, not just spheres, is studied, and bounds are obtained for a critical intensity.

1. Introduction and summary. The problem of percolation in the continuum may be described as follows. Let $\mathscr{P}$ be a homogeneous Poisson process in k-dimensional Euclidean space, $\mathbb{R}^k$. Let S be a random k-dimensional shape, often a sphere. Centre an independent copy of S at each point of $\mathscr{P}$. We shall say that percolation occurs if, with positive probability, any given random shape is part of an infinite clump of random shapes.

The concept of continuum percolation was introduced by Gilbert [4], albeit in a slightly different form. Gilbert noted similarities between continuum and lattice percolation. One of the results in this paper concerns the extent to which such similarities may be relied upon. It is known that in the case of site or bond percolation on a regular lattice, the critical probability at which percolation takes place is often the same as the probability at which mean cluster size becomes infinite. See for example Kesten [9, pages 52–68]. This property can be particularly useful if critical probability is to be estimated by simulation, since it is relatively easy to estimate the graph of expected cluster size against occupation or passage probability, and determine the asymptote at which the curve diverges. Such a procedure was used by Gilbert [4] and Roberts [14], among others, to estimate critical intensity in the case of percolation in the continuum. We shall show in the present paper that for continuum percolation, in the important case where shapes are random radius spheres, the critical intensities at which clump size and mean clump size become infinite, are not necessarily the same. Indeed, it is possible for one of these quantities to be positive and the other to be zero. We shall prove that the critical intensity at which mean clump size becomes infinite, is strictly positive if and only if sphere content has finite variance.

It is perhaps worth giving an intuitive explanation of our argument. There are two ways of describing the size of a clump on the basis of counting the number of

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shapes. The most common, and that which is most important from a physical point of view, is to define clump size to equal the total number of shapes in the clump. An alternative approach is to define size in terms of the total number of shapes protruding from the clump—that is, the number of shapes comprising the perimeter, or boundary, of the clump. It is likely that as the area covered by a clump increases, the total number of shapes in the clump will increase at a faster rate than the number comprising the perimeter. Therefore there may exist an intensity at which the expected number of shapes in the perimeter is finite, but the expected total number of shapes in the clump is infinite. This is basically the argument which leads us to the results described in the previous paragraph.

In addition to treating the percolation problem for random radius spheres, we shall study continuum percolation for quite general random shapes. Percolation does not require that the k-dimensional shapes have positive Lebesgue measure. For example, percolation in $\mathbb{R}^k$ can occur if the shapes are small sections of hyperplanes. In this case, a necessary and sufficient condition for percolation at sufficiently high intensities is that the orientation of the shapes have a nondegenerate distribution. We shall also provide new, rigorously determined upper and lower bounds to the critical intensity for continuum percolation when the shapes are discs of unit radius placed randomly into the plane. Both bounds improve on the best known previously, and complement estimates which have been obtained via simulation or by extrapolation from the lattice case. Our results will be presented together in Section 3, and their proofs given in Section 4. Section 2 will present notation, and some examples to provide insight into the types of conditions which are necessary to rule out pathological cases.

Continuum percolation lacks much by way of an orderly mathematical structure. Methods based on enumeration, which prove so useful in determining critical probabilities for lattice percolation, lose much of their power in the continuum case. However, continuum and lattice percolation are close in spirit, and so it is worth going into a little more detail about related results in the lattice case. Lattice percolation was introduced by Broadbent and Hammersley [2] as a mathematical model for dispersion of fluid through a random porous medium. Smythe and Wierman [17, Chapter 3] and Kesten [9] have given engaging, rigorous accounts of the general problem of lattice percolation in two dimensions. The case of bond percolation on the square lattice has received the greatest attention in the literature. For this case, let p_H denote the critical probability beyond which infinite clumps start to form, and let p_T be the probability beyond which occur clumps whose expected size is infinite. Clearly, $p_T \le p_H$. Seymour and Welsh [16] showed that $p_H + p_T = 1$, and Kesten [8] proved finally that $p_H = p_T = \frac{1}{2}$. See also Sykes and Essam [18] and Russo [15].

Gilbert [4] introduced continuum percolation as a model for the growth and structure of random networks in communication theory. Today the physical applications of continuum percolation, for example to the modelling of impurity conduction in semiconductors, are of greater importance. In many of these applications the random shapes are taken to be discs in two dimensions, or spheres in three dimensions, whose radii represent some sort of interactive distance. Two intersecting spheres could be said to be bonded in some manner. In conduction models, the critical intensity beyond which infinite clumps form can be interpreted physically as the impurity density beyond which conduction takes place. Pike and Seager [13] have summarised some of the physical applications, and discussed continuum percolation for several types of random shapes. Haan and Zwanzig [5] and Gawlinski and Stanley [3] have each given tabular summaries of various efforts to estimate critical intensities for continuum percolation. Kertész and Vicsek [7] have described simulation results in the case of random radius spheres.

2. Notation and introductory examples. The points of a homogeneous Poisson process $\mathscr{P}$ in $\mathbb{R}^k$, of intensity $\lambda > 0$, may be described by a countable collection of random vectors. We denote these by $\mathbf{X}_1, \mathbf{X}_2, \dots,$ in any systematic order. Let S be a random k-dimensional set, and let $S_1, S_2, \dots$ be independent copies of S, also independent of $\mathscr{P}$. For our purpose, there is no essential loss of generality in assuming that S is a random closed set (RACS). This enables us to employ Matheron's [11] simple definition of a RACS, as a measurable mapping from our probability space into the measure space double consisting of the class of closed subsets of $\mathbb{R}^k$ and its associated σ field. We shall not again make specific reference to the theory of random (closed) sets, except to note here that the fact that S is measurable ensures that many random scalar quantities associated with S, such as the Lebesgue measure of S, are well-defined random variables taking values on the extended real line. This obviates the need to qualify intuitively obvious steps in our arguments by caveats about measurability and "well-definition."

We shall call the set

$$\mathbf{X}_i + S_i = \left\{\mathbf{x} \in \mathbb{R}^k : \mathbf{x} - \mathbf{X}_i \in S_i\right\},$$

the "random shape S_i centred at $\mathbf{X}_i$." The coverage process $\mathscr{C}$ is the stochastic pattern generated by overlapping random sets $\mathbf{X}_i + S_i$, $i \geq 1$.

Let $\mathscr{A}$ be a bounded, measurable subset of $\mathbb{R}^k$. The expected vacancy within $\mathscr{A}$ due to the coverage process $\mathscr{C}$, equals

$$\begin{aligned} E\{V(\mathscr{A})\} &= \int_{\mathscr{A}} P(\mathbf{x} \text{ is not covered by any random set } \mathbf{X}_i + S_i)\, d\mathbf{x} \\ &= \|\mathscr{A}\| P(\mathbf{0} \text{ is not covered by any random set } \mathbf{X}_i + S_i) \\ &= \|\mathscr{A}\| \exp\{-\lambda E(\|S\|)\}, \end{aligned}$$

where $\|\mathscr{S}\|$ denotes k-dimensional Lebesgue measure of a set $\mathscr{S}$. Therefore

$$P\{V(\mathscr{A}) = 0\} = 1 \quad \text{if and only if } E(\|S\|) = \infty.$$

This implies that k-dimensional Lebesgue measure of the total uncovered area of $\mathbb{R}^k$, equals zero with probability one if and only if $E(\|S\|) = \infty$. In most practical cases, for example where S is a sphere, $E(\|S\|) = \infty$ implies that $\mathbb{R}^k$ is completely covered with probability one. To avoid this type of pathology we shall always assume that $E(\|S\|) < \infty$.

Let $\mathbf{Y}$ be a random vector in $\mathbb{R}^k$. We claim that the mosaic $\mathscr{C}'$ in which random shapes are distributed as $\mathbf{Y} + S$ instead of S, has the same properties as $\mathscr{C}$. To see this, let $\mathbf{Y}_i + S_i$ be independent copies of $\mathbf{Y} + S$, independent also of $\mathscr{P}$. Conditional on $S_1, S_2, \dots,$ the point process $\mathscr{P}' \equiv \{\mathbf{X}_i + \mathbf{Y}_i,\ i \geq 1\}$ is Poisson-distributed with intensity λ. Therefore $\mathscr{C}$ and $\mathscr{C}'$ must have identical properties. This translation invariance will be used on several occasions below, without further comment.

The next two examples illustrate the phenomenon of infinite clumping in the case $k = 1$. In the first example, no infinite clumps can ever form; in the second, each part of each random shape is part of an infinite clump.

EXAMPLE (i). Assume $k = 1$. Suppose the sets S_i are closed intervals of random location and random length. There is no loss of generality in assuming that left hand endpoints of the intervals are points of $\mathscr{P}$. Our restriction that $E(\|S\|) < \infty$ means that the intervals have finite expected lengths, equal to α say. The resulting coverage process may be modelled by any of several classical stochastic processes, such as an $M/G/\infty$ queue. In this way it may be proved that the expected number of intervals making up an arbitrary clump, equals $e^{\alpha\lambda} < \infty$. (This is the same as the expected number of services in an arbitrary busy period of an $M/G/\infty$ queue.) Therefore no clumps containing an infinite number of segments can ever form.

EXAMPLE (ii). Assume $k = 1$. Define the set $\mathscr{S}_n$ by

$$\mathscr{S}_n = \bigcup_{i=-n^2}^{n^2} \left[i/n - (2n^2+1)^{-1}, i/n + (2n^2+1)^{-1} \right], \qquad n \geq 1.$$

The distribution of the random shape S is given by

$$P(S = \mathscr{S}_n) = cn^{-2}, \qquad n \geq 1, \quad \text{where } c = \left(\sum_1^{\infty} n^{-2} \right)^{-1}.$$

Note that $\|\mathscr{S}_n\| = 2$ for each n, so that $\|S\| = 2$. It may be proved that for each $\ell > 0$,

(2.1) the interval $(-\ell, \ell)$ is intersected by an infinite number of the sets $X_i + S_i$, with probability one.

Indeed, if n_0 denotes the smallest integer exceeding $1/2\ell$, then the set $x + \mathscr{S}_n$ will intersect $(-\ell, \ell)$ whenever $|x| \leq n$ and $n \geq n_0$. Therefore (2.1) will follow from

$$P(|X_i| \leq N_i \text{ i.o.}) = 1, \tag{2.2}$$

where $N_1, N_2, \dots$ are independent integer-valued random variables defined by $S_i = \mathscr{S}_{N_i}$. Put

$$E_j = \{\text{for some } i,\ X_i \in (j-1, j) \text{ and } |X_i| \leq N_i\}, \qquad -\infty < j < \infty.$$

The events E_j are independent,

$$P(|X_i| \leq N_i \text{ i.o.}) \geq P(E_j \text{ i.o.}), \tag{2.3}$$

and

$$P(E_j) \geq P\{\text{for some } i,\ X_i \in (j-1, j)\}P(j \leq N_i) = c(1-e^{-\lambda}) \sum_{n \geq j} n^{-2}.$$

Therefore

$$\sum_{j=1}^{\infty} P(E_j) \geq c(1-e^{-\lambda}) \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} n^{-2} = \infty,$$

and so by the Borel–Cantelli lemma, $P(E_j \text{ i.o.}) = 1$. The desired result (2.2), and hence (2.1), now follows from (2.3). Result (2.1) implies that with probability one, each random set $X_i + S_i$ intersects an infinite number of other random sets.

We may avoid the pathological behaviour described in Example (ii) by assuming that with probability one, the shape S is connected. Under this constraint, we can give simple, formal definitions of a clump and an infinite clump. A clump C is a connected union of a collection of random sets $\mathbf{X}_i + S_i$,

$$C = \bigcup_j (\mathbf{X}_{i_j} + S_{i_j}), \tag{2.4}$$

which has empty intersection with each random set not appearing in the union. The clump is infinite if there is an infinite number of distinct subscripts i_j appearing in (2.4). When $k = 1$, the only connected shapes are intervals, and then Example (i) shows that with probability one, no infinite clumps can ever form.

The situation is rather different in two or more dimensions. There it is possible for the shape S to be connected, and satisfy $E(\|S\|) < \infty$, but still be such that each random set $\mathbf{X}_i + S_i$ is part of an infinite clump with probability one, for all values of the Poisson intensity. One class of sets which exhibits this type of pathological behaviour is that in which S is essentially a sphere with much of its content removed, and such that $E(\|S\|) < \infty$ but the expected content of the smallest sphere containing S is infinite. To eliminate this type of behaviour we shall introduce a little extra notation. Given $\mathscr{S} \subseteq \mathbb{R}^k$, define

$$\bar{s}(\mathscr{S}) \equiv \inf\{\|\mathscr{T}\|: \mathscr{T} \text{ is a } [k] \text{ sphere, and } \mathscr{S} \subseteq \mathscr{T}\}$$

and

$$\underline{s}(\mathscr{S}) \equiv \sup\{\|\mathscr{T}\|: \mathscr{T} \text{ is a } [k] \text{ sphere, and } \mathscr{T} \subseteq \mathscr{S}\}.$$

We shall strengthen the condition $E(\|S\|) < \infty$ to

$$E\{\bar{s}(S)\} < \infty \tag{2.5}$$

3. Main results. We begin by discussing the case where S is a sphere of random radius. Recall from Section 2 that in this situation, $E(\|S\|) = \infty$ is a necessary and sufficient condition for all of $\mathbb{R}^k$ to be covered with probability one. We shall show next that the condition $E(\|S\|^2) = \infty$ is necessary and sufficient for the expected number of spheres in an arbitrary clump to be infinite for all values of the intensity λ. To establish this result it suffices to consider the case where $E(\|S\|) < \infty$, for otherwise *each* sphere is part of an infinite clump with probability one.

THEOREM 1. *Assume that the shape S is a k-dimensional sphere, where $k \geq 2$, and that $E(\|S\|) < \infty$. There exists $\lambda_0 > 0$ such that the expected number of spheres in an arbitrary clump is finite whenever $0 < \lambda < \lambda_0$, if and only if $E(\|S\|^2) < \infty$. Indeed, if $E(\|S\|^2) = \infty$ then the expected number of spheres which are in the same clump as a given sphere and distant no more than one sphere away from that sphere, is infinite for all values of λ.*

It will follow from Corollary 2 that if $E(\|S\|) > 0$, then there exists $\lambda_1 > 0$ such that the expected number of spheres per clump is infinite for all $\lambda > \lambda_1$. Of course, the case $E(\|S\|) = 0$ is trivial, since it implies that with probability one, all spheres are degenerate. Therefore the restriction

$$0 < E\left(\|S\|^2\right) < \infty$$

is *equivalent* to the condition that there exists $0 < \lambda_0 < \infty$ such that the expected number of spheres per clump is finite for all $\lambda < \lambda_0$, and infinite for all $\lambda > \lambda_0$. This type of behaviour is only possible for $k \geq 2$. Example (i) in Section 2 shows that a critical intensity cannot exist in one dimension.

Our next result shows that the restriction $E(\|S\|^2) < \infty$ is stronger than is necessary to ensure that for all sufficiently small λ, all clumps are finite with probability one.

THEOREM 2. *Assume that the shape S is a k-dimensional sphere. If*

$$E\left(\|S\|^{2-(1/k)}\right) < \infty,$$

then for all sufficiently small λ, the number of spheres in each clump is finite with probability one.

If S is a k-dimensional sphere where $k \geq 2$, and if S satisfies

$$E\left(\|S\|^{2-(1/k)}\right) < \infty \quad \text{but} \quad E\left(\|S\|^2\right) = \infty,$$

then the critical intensity for formation of clumps of an infinite number of spheres will be strictly positive, while the critical intensity for formation of clumps with expected number of spheres equal to infinity will be zero.

The following corollary applies to general random shapes, not just spheres. The condition of connectedness is not crucial, but is imposed so that we may use the simple definitions of a clump and an infinite clump given around (2.4). Note the definition of $\bar{s}(S)$ which preceded (2.5).

COROLLARY 1. *Let S be a random set, connected with probability one. If $E\{\bar{s}(S)^2\} < \infty$ then for sufficiently low intensities, the expected number of random shapes making up an arbitrary clump is finite. If $E\{\bar{s}(S)^{2-(1/k)}\} < \infty$ then for sufficiently low intensities, the number of random shapes making up an arbitrary clump is finite with probability one.*

Our next task is to determine conditions which give rise to percolation at high intensities. This requires constraints rather different from those imposed in

Corollary 1. To illustrate the contingencies which can arise, let us consider the case $k = 2$, and examine the distribution of "sticks" (line segments) in the plane. Suppose the sticks are of fixed length, and have their centres at points of a homogeneous Poisson process. If all sticks have the same orientation then no intersections ever occur, and so there is no possibility of even finite clumping. However, Theorem 3 below shows that in all other cases—that is, whenever the distribution of orientation is nondegenerate—percolation occurs for all sufficiently high intensities.

We shall state Theorem 3 in the general k-dimensional case. Place a Cartesian co-ordinate system $(x_1, \ldots, x_k)$ into $\mathbb{R}^k$, and let $\mathscr{S}^{(0)}$ denote the unit $(k-1)$-dimensional sphere lying in the hyperplane perpendicular to the x_k axis and centred at the origin. Given a unit vector $\boldsymbol{\theta}$ from the space Ω of all such vectors, let $\mathscr{S}_{\boldsymbol{\theta}}^{(0)}$ denote the image of $\mathscr{S}^{(0)}$ after rotation to a hyperplane with normal vector $\boldsymbol{\theta}$, still centred at the origin. Let $\boldsymbol{\theta}$ be a random vector distributed on Ω, and define the random shape $S^{(0)}$ to be $\mathscr{S}_{\boldsymbol{\theta}}^{(0)}$. Let $S_1^{(0)}, S_2^{(0)}, \ldots$ be independent copies of $S^{(0)}$, and let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be points of a homogeneous Poisson process $\mathscr{P}$ in $\mathbb{R}^k$, independent of the $S_i^{(0)}$'s. We shall study the coverage process $\mathscr{C}^{(0)}$ generated by the small hyperplanar segments $\mathbf{X}_i + S_i^{(0)}$, $i \geq 1$.

THEOREM 3. *Assume that $k \geq 2$, and that the distribution of $\boldsymbol{\theta}$ has at least two points of support, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, with $\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_1 \neq -\boldsymbol{\theta}_2$. Then there exists a constant λ_0 depending on the distribution of $\boldsymbol{\theta}$, such that the probability that a random set $\mathbf{X}_i + S_i^{(0)}$ is part of an infinite clump is strictly positive for all $\lambda > \lambda_0$.*

Theorem 3 admits many generalisations. For example, the set $\mathscr{S}^{(0)}$ does not have to be perfectly hyperplanar, and neither does it have to be a $(k-1)$-dimensional sphere. In many applications, the random shape S will have positive k-dimensional Lebesgue measure with positive probability, and in those cases it is often preferable to frame Theorem 3 a little differently. This is done in Corollary 2 below, in which we revert to the notation of Corollary 1, where S is a general random shape.

COROLLARY 2. *Assume $k \geq 2$. Let S be a random set, connected with probability one. If $E\{\underline{s}(S)\} > 0$ then for all sufficiently high intensities, the probability that a given random set is part of an infinite clump is strictly positive.*

Our last theorem gives rigorously determined upper and lower bounds to critical intensity in the two-dimensional case. Both bounds are improvements on the currently best available, $0.151 < \lambda_c < 0.883$, which are due to Kirkwood and Wayne [10]. Gilbert [4] conjectured, but could not completely prove an upper bound of 0.87, which is in excess of the upper bound derived below. [To convert Gilbert's notation into ours, divide his bounds by 4π. Note that there is a numerical error in his display (2).]

THEOREM 4. *Assume $k = 2$, and that the shape S is a disc of unit radius. Let λ_c denote the critical intensity beyond which infinite clumps start to form,* *and* λ'_c *the critical intensity beyond which the mean number of discs in an arbitrary clump is infinite. Then*

$$0.174 < \lambda'_c \le \lambda_c < 0.843.$$

4. Proofs. Throughout these proofs we let v_k denote the content of a k-dimensional sphere of unit radius.

PROOF OF THEOREMS 1 AND 2. Since the Poisson process is homogeneous, there is no loss of generality in assuming that the random sphere S is centred at the origin. In all parts of our argument it will prove expedient to give sphere radius a distribution on the nonnegative integers. Note that Theorem 2 is trivial if $k = 1$; see Example (i) in Section 2.

It is convenient to establish Theorem 2 first, and so we begin by assuming that $E(\|S\|^{2-(1/k)}) < \infty$. Note that this condition is equivalent to $E(R^{2k-1}) < \infty$, where R has the distribution of sphere radius. Let $R' - 1$ equal the integer part of R, and note that $E(R')^{2k-1} < \infty$. If sphere radius is given the distribution of R' instead of R, then the probability of an infinite clump occurring will not decrease. Therefore to establish Theorem 2, it suffices to prove that when the sphere radius takes only positive integer values and satisfies $E(R^{2k-1}) < \infty$, and Poisson intensity is sufficiently small, the number of spheres per clump is finite with probability one. This we do by showing that the expected number of spheres in the *perimeter* of the clump, is finite with probability one.

We shall construct a multitype branching process to bound the number of spheres in the clump perimeter. There will be a countable infinity of types, indexed by positive integers which correspond to sphere radii. Our first step is to determine the distribution of types.

Suppose an initial sphere of radius i is centred at a point z. The number of spheres of radius j in our coverage process which intersect the initial sphere, and which protrude at least partially beyond that sphere, has a Poisson distribution with parameter

$$\mu_{ij} = \lambda v_k \left[(i+j)^k - \{\max(0, i-j)\}^k \right] p_j, \tag{4.1}$$

where $p_j = P(R = j)$. Let this number be N_j. The variables N_j, $j \ge 1$, are stochastically independent. We ignore spheres which are wholly contained within the initial sphere, since they cannot contribute to the perimeter of the clump. If an individual of type i is present in the nth generation, then the vector of numbers of types of his progeny in the $(n+1)$th generation will be given the same distribution as $(N_1, N_2, \dots)$. Here N_j represents the number of children of type j.

Using this type distribution, we construct the branching process as below. The individuals in the process are points in $\mathbb{R}^k$. The individual in the zeroth generation is the centre of some given sphere, which we may take without loss of generality to be the origin. Given individuals $\mathbf{Z}_{n1}, \dots \mathbf{Z}_{nN_n}$ in the nth generation, we define the $(n+1)$th generation as follows. Suppose $\mathbf{Z}_{n\ell}$ is of type i. Let $\mathscr{P}_{n\ell}$ be a Poisson process in $\mathbb{R}^k$ of intensity λ, independent of the previous history of the process and also of $\mathscr{P}_{n\ell'}$ for $\ell' \ne \ell$. Centre spheres at the points of $\mathscr{P}_{n\ell}$, the radii being independent and distributed as R. The progeny of $\mathbf{Z}_{n\ell}$ of type j in the $(n+1)$th generation, are those points of $\mathscr{P}_{n\ell}$ whose associated spheres are of radius j, and which intersect the sphere of radius i centred at $\mathbf{Z}_{n\ell}$, but are not wholly contained within that sphere. The expected number of individuals in all generations of this multitype process, is greater than or equal to the expected number of spheres which protrude from the clump containing the initial sphere in the coverage process. (This observation requires the "lack of memory" property of a Poisson process. Specifically, if $\mathscr{P}$ is homogeneous Poisson and if $\mathbf{x}_1, \dots, \mathbf{x}_m$ are arbitrary fixed points, then the conditional distribution of $\mathscr{P} \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ given that points of $\mathscr{P}$ occur at $\mathbf{x}_1, \dots, \mathbf{x}_m$, is the same as the unconditional distribution of $\mathscr{P}$. This property, and the fact that our construction adds extra points in certain cases, ensure that the conditional distribution of the $(k+1)$st generation of the branching process dominates that of the coverage process.)

We shall prove that the expected total population size is finite for all sufficiently small λ. The expected number of immediate type j progeny born to a type i individual, equals μ_{ij}. Define $v_i^{(n)}$ to be the expected number of type i individuals in the nth generation, and let $\mathbf{v}^{(n)}$ be the row vector whose ith element is $v_i^{(n)}$. If the initial individual was of type i, then

$$\mathbf{v}^{(n)} = \mathbf{i}\mathbf{M}^n,$$

where $\mathbf{i}$ is the row vector whose ith element is one and has all other elements zero, and where $\mathbf{M} = (\mu_{ij})$ is a matrix with an infinite number of rows and columns. See Athreya and Ney [1, page 184] for the relevant theory of multitype branching processes. Therefore the expected total number of individuals in the nth generation, is

$$\sum_{j=1}^{\infty} v_j^{(n)} = \sum_{j=1}^{\infty} \mu_{ij}^{(n)},$$

where $\mu_{ij}^{(n)}$ is the (i, j)th element of $\mathbf{M}^n$. The expected total number of individuals in all generations, given that the initial individual was of type i, is

$$\mu_i \equiv \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij}^{(n)}. \tag{4.2}$$

In view of formula (4.1), for $i \le j$ we have

$$\mu_{ij} = \lambda v_k (i+j)^k p_j \le 2^k v_k \lambda j^k p_j,$$

while for $i > j$,

$$\mu_{ij} = \lambda v_k \{(i+j)^k - (i-j)^k\} p_j \le \text{const.}\, \lambda i^{k-1} j p_j,$$

where the constant depends on none of i, j or λ. Therefore in general,

$$\mu_{ij} \le c\lambda j \{\max(i, j)\}^{k-1} p_j \le c\lambda i^{k-1} j^k p_j$$

for all i, j and λ, where c is chosen $\ge 2^k v_k$ and depends on none of these parameters. Consequently,

$$\mu_{ij}^{(2)} = \sum_{\ell=1}^{\infty} \mu_{i\ell}\mu_{\ell j} \le (c\lambda)^2 i^{k-1} j^k p_j \sum_{\ell=1}^{\infty} \ell^{2k-1} p_\ell$$
$$= (c\lambda)^2 \mu i^{k-1} j^k p_j \le (c\lambda\mu)^2 i^{k-1} j^k p_j,$$

where $\mu \equiv \sum_\ell \ell^{2k-1} p_\ell < \infty$, the last inequality following from the fact that $E(R^{2k-1}) < \infty$. If $\mu_{ij}^{(n-1)} \le (c\lambda\mu)^{n-1} i^{k-1} j^k p_j$ for all i, j and λ, it follows easily that $\mu_{ij}^{(n)} \le (c\lambda\mu)^n i^{k-1} j^k p_j$, and so the latter formula must be true for all i, j, n and λ, using mathematical induction. Substituting this estimate into (4.2) we see that

$$\mu_i \le i^{k-1} \sum_{n=1}^{\infty} (c\lambda\mu)^n \sum_{j=1}^{\infty} j^k p_j < \infty,$$

provided only that λ is chosen so small that $c\lambda\mu < 1$. In this case, the expected number of spheres which form the perimeter of the clump containing a given sphere of radius i, is finite. Since each of these spheres has finite radius, then the dimensions of the clump are finite with probability one. Therefore the total number of spheres making up the clump must be finite with probability one.

Next we prove a portion of Theorem 1, by showing that if $E(\|S\|^2) < \infty$, or equivalently, if the distribution of R satisfies $E(R^{2k}) < \infty$, then the expected number of spheres in an arbitrary clump is finite provided λ is chosen sufficiently small. This requires a modification of the branching process argument given above. We assume as before that R takes only integer values. On the present occasion we must bound the *total number* of spheres in the clump, not just the number of spheres protruding from the clump.

Suppose a sphere of radius i is centred at a point $\mathbf{z}$. Instead of N_j we consider N_j', equal to the number of spheres of radius j in our coverage process which intersect the initial sphere. The variables N_j' are stochastically independent, and N_j' is Poisson distributed with parameter

$$\mu_{ij}' = \lambda v_k (i+j)^k p_j. \tag{4.3}$$

The branching process is defined as before, except that the type j progeny of the type i individual $\mathbf{Z}_{n\ell}$ in the $(n+1)$th generation, are taken to be those points in $\mathscr{P}_{n\ell}$ whose associated spheres are of radius j and intersect the sphere of radius i centred at $\mathbf{Z}_{n\ell}$. We may derive an analogue of formula (4.2), and so the proof of this part of the theorem will be complete if we show that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij}'^{(n)} < \infty. \tag{4.4}$$

In view of (4.3), we have

$$\mu_{ij}' \le c\lambda \{\max(i, j)\}^k p_j \le c\lambda i^k j^k p_j$$

for all i, j and λ, where $c = 2^k v_k$. It now follows as before that for $n \ge 1$,

$$\mu_{ij}'^{(n)} \le (c\lambda\mu')^n i^k j^k p_j,$$

where $\mu' = \sum_\ell \ell^{2k} p_\ell < \infty$. Therefore (4.4) will hold if λ is chosen so small that $c\lambda\mu' < 1$.

It remains only to show that if $E(\|S\|^2) = \infty$ then the expected number of spheres which are in the same clump as a given sphere and are distant no more than one sphere away from that sphere, is infinite for all values of λ. Let R have the distribution of radius of S, define R'' to be the integer part of R, and note that $E(R'')^{2k} = \infty$. If sphere radius is given the distribution of R'', then the expected number of spheres distant one or more spheres away from a given sphere, will not exceed the expected value in the case where radius has the distribution of R. Therefore we may assume without loss of generality that R takes only nonnegative integer values.

Suppose a sphere $S^{(1)}$ of radius $r \geq 0$ is centred at a point $\mathbf{z}$. The number, N_i, of spheres of radius i which intersect $S^{(1)}$, is Poisson distributed with parameter

$$\mu_i = \lambda v_k (r + i)^k p_i,$$

where $p_i = P(R = i)$. Let M denote the largest value of i for which $N_i > 0$, except that we define $M = -1$ if no spheres intersect $S^{(1)}$. The variables N_i are independent, and so

$$\begin{aligned} P(M = m) &= P(N_m > 0) \prod_{i=m+1}^{\infty} P(N_i = 0) \\ &= \{1 - \exp(-\mu_m)\} \exp\left(- \sum_{i=m+1}^{\infty} \mu_i \right), \qquad m \geq 0, \end{aligned}$$

with

$$P(M = -1) = 1 - \sum_{m=0}^{\infty} P(M = m) = \exp\left(- \sum_{i=0}^{\infty} \mu_i \right).$$

Note that the entire coverage process may be regarded as the superposition of *independent* coverage processes $\mathscr{C}_0, \mathscr{C}_1, \dots,$ where $\mathscr{C}_i$ is generated by spheres of fixed radius i centred at points of a Poisson process of intensity λp_i. The event $\{M = m\}$ is the same as the event $\{N_m > 0;\ N_i = 0 \text{ for } i \geq m + 1\}$, and so is measurable in the σ field generated by the processes $\mathscr{C}_m, \mathscr{C}_{m+1}, \dots$. Thus for any $m > 0$ the events $\{M = m\}$ and $\{M \geq m\}$ are stochastically independent of any event which is measurable in the σ field generated by $\mathscr{C}_0, \mathscr{C}_1, \dots, \mathscr{C}_{m-1}$. Define

$$t = \inf\{n \geq 1\colon p_n > 0\}.$$

Conditional on $M \geq t + 1$, let $S^{(2)}$ be any sphere of radius M intersecting $S^{(1)}$. In view of the preceding discussion, the following is true. Conditional on $M = m$, where $m \geq t + 1$, the number of spheres of radius t which intersect $S^{(2)}$ is Poisson distributed with parameter

$$\nu_m = \lambda v_k (m + t)^k p_t.$$

Therefore the expected number of spheres of radius t distant one sphere or less away from the initial sphere of radius r, is not less than

$$
\begin{aligned}
\rho &\equiv \sum_{m=t+1}^{\infty} P(M=m)\nu_m \\
&\geq \lambda v_k p_t \exp\left(-\sum_{i=0}^{\infty}\mu_i\right)\sum_{m=t+1}^{\infty}\{1-\exp(-\mu_m)\}m^k.
\end{aligned}
\tag{4.5}
$$

Since $E(\|S\|) < \infty$ then

$$\sum_{i=0}^{\infty}\mu_i \leq \text{const.} \sum_{i=1}^{\infty} i^k p_i < \infty,$$

and also,

$$
\begin{aligned}
1-\exp(-\mu_m) \geq \mu_m \exp(-\mu_m) &\geq \lambda v_k m^k p_m \exp(-\mu_m) \\
&\geq \text{const.}\, m^k p_m
\end{aligned}
$$

for all $m \geq 1$, where "const." denotes a generic positive constant not depending on m. Substituting these estimates into (4.5), we see that

$$\rho \geq \text{const.} \sum_{m=t+1}^{\infty} m^{2k} p_m = \infty,$$

since $E(R^{2k}) = \infty$. Therefore $\rho = \infty$, which completes the proof of Theorem 1.

PROOF OF COROLLARY 1. The Corollary is trivial when $k = 1$; note Example (i). Therefore we may confine attention to the case $k \geq 2$. Let $\mathscr{C}$ be a coverage process in which the shapes are distributed as S. Given a random shape S, let T be the closed k-dimensional sphere centred at the origin and such that $\|T\| = \bar{s}(S)$, and let $\mathbf{Y}$ be a random vector in $\mathbb{R}^k$ such that $S \subseteq \mathbf{Y} + T$. Consider the coverage process $\mathscr{C}'$ in which the shapes have the distribution of $\mathbf{Y} + T$, and are centred at points of a Poisson process of the same intensity as that for $\mathscr{C}$. The expected number of shapes per clump, and the probability of a shape being part of an infinite clump, are not greater for $\mathscr{C}$ than for $\mathscr{C}'$. The Corollary follows on applying Theorems 1 and 2 to the process $\mathscr{C}'$.

PROOF OF THEOREM 3. We shall derive a lower bound to the probability of infinite clustering, by comparing our coverage process to a site percolation process on a rectangular lattice in $\mathbb{R}^k$.

Let $\varepsilon > 0$ be so small that the sets

$$\mathscr{T}_i(3\varepsilon) \equiv \{\boldsymbol{\theta} \in \Omega : |\boldsymbol{\theta} - \boldsymbol{\theta}_i| < 3\varepsilon\}, \quad \text{for } i = 1 \text{ and } 2,$$

are disjoint. (The set Ω is the surface of the unit k-dimensional sphere centred at the origin.) Given $\mathbf{z} \in \mathbb{R}^k$ and $\boldsymbol{\theta} \in \Omega$, let $\mathscr{U}(\mathbf{z}, \boldsymbol{\theta})$ denote the open $(k-1)$-dimensional sphere of unit radius centred at $\mathbf{z}$ and whose plane has its normal in the direction of $\boldsymbol{\theta}$. Let $\mathbb{Z}$ denote the set of all integers. The sites in the percolation process will be the points of the lattice $(r_1\mathbb{Z})^k$, for some $r_1 > 0$. Two sites will be said to be adjacent if they are distant exactly r_1 apart.

Choose r_1 so small that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two adjacent sites of the lattice, the two $(k-1)$-dimensional spheres $(1/2)\mathscr{U}(\mathbf{x}_1, \boldsymbol{\phi}_1)$ and $(1/2)\mathscr{U}(\mathbf{x}_2, \boldsymbol{\phi}_2)$ have nonempty intersection whenever $\boldsymbol{\phi}_i \in \mathscr{T}_i(2\varepsilon)$ for $i = 1$ and 2. Next, choose $r_2 \in (0, r_1/2)$ so small that the spheres $\mathscr{U}(\mathbf{y}_1, \boldsymbol{\phi}_1)$ and $\mathscr{U}(\mathbf{y}_2, \boldsymbol{\phi}_2)$ have nonempty intersection whenever $\mathbf{x}_1$ and $\mathbf{x}_2$ are adjacent sites, $|\mathbf{x}_i - \mathbf{y}_i| < r_2$ for $i = 1$ and 2, and $\boldsymbol{\phi}_i \in \mathscr{T}_i(\varepsilon)$ for $i = 1$ and 2.

Let $\mathscr{T}(\mathbf{x}, r_2)$ denote the k-dimensional sphere of radius r_2 centred at the point $\mathbf{x}$. Classify each site of the lattice $(r_1\mathbb{Z})^k$ as either "type 1" or "type 2," in such a manner that no type i site is adjacent to another type i site, for $i = 1$ and 2. (Once any given site has been classified, the classification of all other sites is determined, so there are only two different possible classifications.) Distribute the points $\mathbf{X}_j$ of the Poisson process $\mathscr{P}$ throughout $\mathbb{R}^k$. Let $\mathbf{x} \in (r_1\mathbb{Z})^k$ be a site of type i. We shall say that $\mathbf{x}$ is *occupied* if some point of $\mathscr{P}$ lies within the sphere $\mathscr{T}(\mathbf{x}, r_2)$, and is such that the associated random shape $S_j^{(0)}$ (actually, a $(k-1)$-dimensional sphere $\mathscr{U}(\mathbf{0}, \boldsymbol{\theta}_j)$, for some random orientation vector $\boldsymbol{\theta}_j$) has orientation $\boldsymbol{\theta}_j$ lying in the set $\mathscr{T}_i(\varepsilon)$. It follows from our construction that the following properties hold. (i) The probability p_i that a site of type i is occupied, depends only on i and not on other characteristics of the site. (ii) The value of p_i increases to one as Poisson intensity, λ, increases to infinity. (iii) The occupation of a given site $\mathbf{x}$ is stochastically independent of the occupation pattern of any set A of sites for which $\mathbf{x} \notin A$. (iv) If two adjacent sites $\mathbf{x}_1$ and $\mathbf{x}_2$ are occupied, then there exists points $\mathbf{X}_{j_\ell} \in \mathscr{T}(\mathbf{x}_\ell, r_2)$, for $\ell = 1$ and 2, such that the random sets $\mathbf{X}_{j_\ell} + S_{j_\ell}$, $\ell = 1$ and 2, have nonempty intersection.

In view of property (iv), there will exist an infinite clump of random sets $\mathbf{X}_i + S_i$, if there exists an infinite path composed of bonds linking adjacent occupied sites. Consider the simpler site-percolation process on the rectangular lattice $\mathbb{Z}^k$, in which each site is occupied with probability $p = \min(p_1, p_2)$ and vacant with probability $1 - p$, independently of all others. The probability that a given site is part of an infinite clump (or "cluster") for this process, is no less than the probability that a site is part of an infinite cluster for the former two-parameter site-percolation process. It follows from the theory of site percolation that if $p < 1$ is sufficiently large, the probability of an infinite path is strictly positive. (In the case $k = 2$, the critical probability for site percolation on the square lattice satisfies $0.5 < p_c < 1$; see Higuchi [6]. When $k \geq 3$, the critical probability is bounded above by the probability in the case $k = 2$. Thus, $p_c < 1$ for all k.) In view of property (ii) above, this means that the probability of an infinite path is positive for all sufficiently large λ.

PROOF OF COROLLARY 2. Let T denote the open sphere centred at the origin and such that $\|T\| = \underline{s}(S)$. Choose $\mathbf{Y} \in \mathbb{R}^k$ such that $\mathbf{Y} + T \subseteq S$. We claim that there exists a k-dimensional sphere $\mathscr{T}$ with *fixed* centre and radius, such that

$$P(\mathscr{T} \subseteq \mathbf{Y} + T) > 0. \tag{4.6}$$

To prove this, note that since $E\{\underline{s}(S)\} > 0$ we may choose a fixed sphere $\mathscr{T}_1$, of radius $2r > 0$ and centred at the origin, such that $P(\mathscr{T}_1 \subseteq T) > 0$. Let $\mathbf{Y}^*$ have the distribution of $\mathbf{Y}$ conditional on $\mathscr{T}_1 \subseteq T$, and let $\mathbf{y}$ be a continuity point of

$\mathbf{Y}^*$. Then

$$P(\mathbf{y} + (1/2)\mathscr{T}_1 \subseteq \mathbf{Y} + T) \geq P(\mathbf{y} + (1/2)\mathscr{T}_1 \subseteq \mathbf{Y} + \mathscr{T}_1 | \mathscr{T}_1 \subseteq T) \times P(\mathscr{T}_1 \subseteq T)$$
$$\geq P(|\mathbf{Y} - \mathbf{y}| < r) P(\mathscr{T}_1 \subseteq T) > 0.$$

Therefore (4.6) will hold if we take $\mathscr{T} = \mathbf{y} + \mathscr{T}_0$, where $\mathscr{T}_0 = (1/2)\mathscr{T}_1$.

We shall use the sphere $\mathscr{T}$ to construct a new coverage process $\mathscr{C}_1$, as follows. Delete the set $\mathbf{X}_i + S_i$ from the collection $\{\mathbf{X}_i + S_i,\ i \geq 1\}$ if $\mathscr{T} \not\subseteq S_i$, and replace $\mathbf{X}_i + S_i$ by $\mathbf{X}_i + \mathscr{T}$ if $\mathscr{T} \subseteq S_i$. This gives rise to a sequence of random sets $\mathbf{Z}_i + \mathscr{T}$, where $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ are points of a Poisson process in $\mathbb{R}^k$ of intensity

$$\mu = \mu(\lambda) = \lambda P(\mathscr{T} \subseteq S).$$

In view of (4.6), $P(\mathscr{T} \subseteq S) > 0$. Define $\mathscr{C}_1$ to be the coverage process generated by $\mathbf{Z}_i + \mathscr{T}$, $i \geq 1$. If the probability is positive that a given random set in $\mathscr{C}_1$ is part of an infinite clump, then the probability is also positive in the case of $\mathscr{C}$. Therefore the proof will be complete if we show that for some $\mu > 0$, the probability of infinite clumping is positive for $\mathscr{C}_1$. Indeed, since the coverage process $\mathscr{C}_2$ generated by $\mathbf{Z}_i + \mathscr{T}_0$, $i \geq 1$, has the same properties as $\mathscr{C}_1$, we may prove instead that infinite clumps have positive probability of forming in $\mathscr{C}_2$.

We may assume without loss of generality that the sphere $\mathscr{T}_0$ is of unit radius, i.e. $r = 1$, since this situation may always be achieved by rescaling. In this case, the k-dimensional sphere $\mathscr{T}_0$ contains any number of $(k-1)$-dimensional unit spheres centred at the origin, of the type $\mathscr{S}^{(0)}$ considered in Theorem 3. It then follows immediately from Theorem 3 that infinite clumping occurs in $\mathscr{C}_2$ with positive probability if μ is sufficiently large.

PROOF OF THEOREM 4.

Part(i). Lower bound. As in the proof of Theorem 1, our argument relies on a multi-type branching process approximation. However, the distribution of types is quite different in the present case. Types are indexed by the continuum in the interval $(0, 2)$.

Let $\mathscr{T}(\mathbf{x})$ denote the disc of radius 2 centred at $\mathbf{x} \in \mathbb{R}^2$. "Individuals" in the branching process are points in $\mathbb{R}^2$. The individual in the zeroth generation is the point $\mathbf{X}_1$. Given points $\mathbf{Z}_{n1}, \dots, \mathbf{Z}_{nN_n}$ in the nth generation, we derive the $(n+1)$th generation as follows. Suppose $\mathbf{Z}_{ni}$ is a child of $\mathbf{Z}_{n-1,j}$ from the $(n-1)$th generation. Let $\mathscr{U}_{ni}$ denote the set of points within the lune

$$\mathscr{T}(\mathbf{Z}_{ni}) \setminus \mathscr{T}(\mathbf{Z}_{n-1,j}) = \mathscr{T}(\mathbf{Z}_{ni}) \cap \mathscr{T}(\mathbf{Z}_{n-1,j})^c,$$

resulting from a Poisson process $\mathscr{P}_{ni}$ of intensity λ, where $\mathscr{P}_{ni}$ is independent of all variables defined previously and also of $\mathscr{P}_{nj}$ for $j \neq i$. The points within $\mathscr{U}_{ni}$ are the progeny of $\mathbf{Z}_{ni}$ in the $(n+1)$th generation.

Let $\mathbf{Z}_{n+1,\ell}$ be any one of the points in $\mathscr{U}_{ni}$. We shall say that $\mathbf{Z}_{n+1,\ell}$ is of type t, where $0 < t < 2$, if $|\mathbf{Z}_{n+1,\ell} - \mathbf{Z}_{ni}| = t$. The distribution of the number of immediate progeny of $\mathbf{Z}_{ni}$ that are of a type lying between t and $t + dt$, depends on previous history of the process only through the type of $\mathbf{Z}_{ni}$. Furthermore, the distribution does not depend on n. These observations follow from the fact that the distribution of the number of immediate children of $\mathbf{Z}_{ni}$ of types between y and $y + dy$, depends only on the area dA of the intersection of the lune $\mathscr{T}(\mathbf{Z}_{ni}) \setminus \mathscr{T}(\mathbf{Z}_{n-1,j})$ with a thin circular shell of radius y:

$$dA = \left\| \left[\mathscr{T}(\mathbf{Z}_{ni}) \setminus \mathscr{T}(\mathbf{Z}_{n-1,j}) \right] \cap \{ \mathbf{z}\colon\ y \leq |\mathbf{z} - \mathbf{Z}_{ni}| \leq y + dy \} \right\|.$$

The value of dA depends on past history only through $x \equiv |\mathbf{Z}_{ni} - \mathbf{Z}_{n-1,j}|$.

The expected number of discs in the clump containing the unit disc centred at $\mathbf{X}_1$, is bounded above by the expected total number of individuals in all generations of the branching process. Our aim is to determine a value λ_1 such that, for all $\lambda < \lambda_1$, the expected total number of individuals in the branching process is finite.

Let $g(y|x)\,dy$ denote the area element dA, given that $|\mathbf{Z}_{ni} - \mathbf{Z}_{n-1,j}| = x$. Then $g(y|x) > 0$ if $2 - x < y < 2$, and $g(y|x) = 0$ if $0 < y < 2 - x$. The expected number of immediate progeny with types between y and $y + dy$, parented by a type x individual, equals $\lambda g(y|x)\,dy$. Let $\lambda^n g_n(y|x)\,dy$ denote the expected number in the nth generation which are of a type between y and $y + dy$, given that the zeroth individual was of type x. Then $g_1 \equiv g$,

$$\begin{aligned} \lambda^{n+1} g_{n+1}(y|x) &= \int_0^2 \lambda g(y|z).\lambda^n g_n(z|x)\,dz \\ &= \lambda^{n+1} \int_0^2 g(y|z) g_n(z|x)\,dz, \end{aligned}$$

and the expected total number of individuals in all generations, given that the zeroth individual was of type x, equals

$$1 + \sum_{n=1}^{\infty} \lambda^n \int_0^2 g_n(y|x)\,dy. \tag{4.7}$$

We shall prove that the series in (4.7) converges whenever $\lambda < \lambda_1 \equiv 0.174$.

Consider the integral operator T, defined by

$$(T\alpha)(x) = \int_0^2 \alpha(y) g(y|x)\,dy.$$

Then

$$\int_0^2 g_n(y|x)\,dy = (T^n 1)(x),$$

where 1 denotes the function which is identically unity on $(0, 2)$. Therefore the desired result will follow if we prove that the maximal eigenvalue, ρ, of the linear operator T, does not exceed λ_1^{-1}. There are numerical procedures for computing maximal eigenvalues, and these show that in the present case, $\rho = 5.718 +$. Note that, after a little trigonometry,

$$g(y|x) = 2y \arccos\left\{ (2xy)^{-1} (4 - x^2 - y^2) \right\}$$

if $2 - x < y < 2$. Furthermore, $g(y|x) = 0$ if $0 < y < 2 - x$.

Part (ii). Upper bound. Our argument is based on site percolation on the regular triangular lattice in $\mathbb{R}^2$. The lattice is composed of equilateral triangles, each of which has unit side length. At each site of the lattice, construct a "curved sided hexagon," in which the arcs forming the sides are portions of circles of unit radius whose centres are the midpoints of the six bonds radiating from the site. We shall define a site to be *occupied* if and only if the curved sided hexagon associated with that site contains a point of the Poisson process of intensity λ in $\mathbb{R}^2$. Thus, the sites are occupied or vacant independently of one another, and for each site the probability of occupation equals

$$p \equiv 1 - e^{-a\lambda},$$

where a denotes the area of the curved sided hexagon.

Each pair of adjacent occupied sites will be thought of as being bonded. If two adjacent sites s_1 and s_2 are bonded, then there is a disc centred within the curved sided hexagon centred on s_1 which intersects a disc centred within the curved sided hexagon centred on s_2. Therefore if site percolation occurs in the triangular lattice—that is, if there is positive probability of any given site being part of an infinite chain of bonded sites—then there is positive probability of any given disc being part of an infinite clump of discs. Site percolation in the triangular lattice occurs if and only if $p > \frac{1}{2}$; see Kesten [9, pages 52–53]. Therefore a sufficient condition for infinite clumping is that

$$\lambda > a^{-1}\log 2.$$

A little trigonometry shows that $a = 0.8227 -$, whence $a^{-1}\log 2 = 0.843 -$. Consequently, $\lambda_c < 0.843$.

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