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THE SIMPLE EXCLUSION PROCESS AS SEEN FROM A TAGGED PARTICLE[1]

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The simple exclusion process "as seen from a tagged particle" is studied. The set of translation invariant and invariant measures for this process is determined in the translation invariant case on $\mathbb{Z}^d$. The set of all invariant measures is determined in the nearest neighbor asymmetric case on $\mathbb{Z}$. The domains of attraction of the invariant measures are established in the one-dimensional nearest neighbor translation invariant case.

Introduction. The well known ([18], [14]) simple exclusion process describes the behavior of infinitely many particles moving on a countable set S according to the following laws: At the site $x \in S$ there is a particle which, after an exponential time with parameter one, attempts to jump to a site y with probability $p(x, y)$, where $p(x, y)$ is a transition probability function for a Markov chain on S. The jump actually occurs if the site y is empty. No more than one particle is allowed to be at each site. In this paper we study the simple exclusion process "as seen from a tagged particle." We take $S = \mathbb{Z}^d$ (although S could be any countable abelian group) and $p(x, y)$ translation invariant, i.e., $p(x, y) = p(0, y - x)$. For this process the origin is always occupied and, if the site y is empty, after an exponential time of parameter one, with probability $p(0, y)$ a translation of the system occurs in such a way that the new origin will be at site y. The other particles move as before. We call it the "tagged particle process" [14].

The existence of the tagged particle process is known if $p(x, y) = 0$, for $|x - y| > 1$, since in this case there exists a labeled probabilistic version for the simple exclusion process [7]. For general $p(x, y)$ one can construct the process using Liggett's existence criteria or, as we do, by using the fact that the pregenerator of the tagged particle process is a bounded perturbation of the generator of a semigroup. Then we use [5] to prove that it is the generator of a contraction semigroup.

One of the classical problems concerning infinite particle systems is the determination of the set of invariant measures. Since this is a convex and compact set, it suffices to describe the set of extremal invariant measures. For the simple exclusion process Liggett [12] (also [13] and [14]) studied some particular cases. When $S = \mathbb{Z}^d$ and $p(x, y)$ is translation invariant, the set of extremal invariant and translation invariant measures is the one-parameter family

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$\{\nu_\rho: 0 \le \rho \le 1\}$, where ν_ρ is the Bernoulli measure with parameter ρ. In Theorem 2.4 we prove the corresponding result for the tagged particle process. The analogue of translation invariance for measures with mass concentrated on configurations with one particle at the origin is given in Equation (2.1) following Neveu [16]. To prove Theorem 2.4 we use a generalization of a result of Harris [6] and Port and Stone [17], namely we establish the relationship between the measure for the tagged particle process and the measure for the simple exclusion process. This is established in Theorem 2.3.

More interesting is the case $S = \mathbb{Z}$, $p(x, x+1) = p$, $p(x, x-1) = q$, $p + q = 1$. Consider $p > q$. For the simple exclusion process the set of extremal invariant measures is the union of two families. One is the family of Bernoulli measures as in the translation invariant case. The other is a discrete one-parameter family $\{\nu^{(n)}, n \in \mathbb{N}\}$ of "blocking measures" (i.e., measures which give mass one to configurations with a finite number of empty sites to the right of the origin and a finite number of particles to the left of the origin; cf. [12] for a complete description). In Theorem 3.4 we prove that for the tagged particle process we also have the above two classes of invariant measures as seen from one of the particles. We also show that these are not the only ones. Other invariant measures appear. These can only be seen from one of the particles because under these measures the process seen from a fixed site converges to the vacant configuration in distribution. They form a two-parameter family and their mass is concentrated on configurations with a finite number of particles to the left of the tagged particle. This number does not change with time (because of the exclusion condition and the nearest neighbor jump) and it is one of the parameters of the family. To understand the meaning of the other parameter we describe each configuration by labeling the distances between successive particles. So, if ρ $(0 \le \rho < 1)$ is the second parameter, the distance between the mth and the $(m+1)$th particle, counting from the left to the right, has the geometric distribution with parameter

$$\rho_m = \left(\frac{q}{p}\right)^{m+1} + \rho\left[1 - \left(\frac{q}{p}\right)^{m+1}\right].$$

In other words, the number of empty sites between the mth and the $(m+1)$th particle is equal to k with probability $\rho_m^k(1 - \rho_m)$. Furthermore, the distances between different pairs of successive particles are independent. The asymptotic density for positive sites far from the origin is $1 - \rho$. If $\rho = 0$, the measures obtained are analogous to the "blocking measures" found by Liggett (see Theorem 1.4 in [12]), when seen from the nth particle.

The proof of this result is based on a correspondence between the tagged particle process and the zero range process. The zero range process was introduced by Spitzer [18] (see also Liggett [14]) and some of its sets of invariant measures were determined by Andjel [2]. We introduce a modification of this process in order to prove Theorem 3.4. The same correspondence was used by Kipnis [9], De Masi and Ferrari [3], and Ferrari, Presutti, and Vares [4], but the idea of this correspondence was already in Harris [6] and Port and Stone [17].

In Section 4 we determine the domain of attraction of the invariant measures in the one-dimensional nearest neighbor translation invariant case.

1. The tagged particle process. Let $S = \mathbb{Z}^d$ and $p(x, y)$ be a translation invariant probability transition function for a Markov chain on S. Consider the simple exclusion process on $\mathscr{X} = \{0, 1\}^S$ introduced by Spitzer [18]. For $\eta \in \mathscr{X}$ we say that a particle is at x if $\eta(x) = 1$, and that the site x is empty if $\eta(x) = 0$. We also interpret η as the subset (of S) of occupied sites. The existence of the simple exclusion process was proved by Liggett [10] under the condition $\sup_y \Sigma_x p(x, y) < \infty$. This condition is automatically satisfied for translation invariant $p(x, y)$. The generator of the process is the closure in $C(\mathscr{X})$ (with the supremum norm) of the following operator Ω defined for cylindric f by:

$$\Omega f(\eta) = \sum_{x, y:\, \eta(x)=1,\, \eta(y)=0} p(x, y)\left[f(\eta_{xy}) - f(\eta)\right], \tag{1.1a}$$

where

$$\eta_{xy}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(x), & \text{if } z = y, \\ \eta(y), & \text{if } z = x. \end{cases} \tag{1.1b}$$

In order to define the tagged particle process we consider the set

$$\hat{\mathscr{X}} = \mathscr{X} \cap \{\eta\colon \eta(0) = 1\}.$$

The tagged particle process has state space $\hat{\mathscr{X}}$ and its pregenerator is

$$\hat{\Omega} = \Omega_0 + \Omega_1, \tag{1.2a}$$

where Ω_0 is the operator corresponding to the shifts of the system due to the motion of the tagged particle:

$$\Omega_0 f(\eta) = \sum_{y:\, \eta(y)=0} p(0, y)\left[f(\eta_{0y} - y) - f(\eta)\right] \tag{1.2b}$$

(the configuration $\eta - y$ is defined by $(\eta - y)(x) = \eta(x + y)$) and Ω_1 is the generator of the motion of the other particles:

$$\Omega_1 f(\eta) = \sum_{\substack{x, y \neq 0 \\ \eta(x)=1,\, \eta(y)=0}} p(x, y)\left[f(\eta_{xy}) - f(\eta)\right]. \tag{1.2c}$$

By Liggett's existence criteria [10], Ω_1 is the generator of a contraction semigroup; meanwhile Ω_0 is a bounded operator. Then, by Theorem 2 of Gustafson [5], $\hat{\Omega}$ is the infinitesimal generator of a contraction semigroup, which we call $\hat{S}(t)$. This implies that there exists a unique Markov process on $\hat{\mathscr{X}}$ with generator $\hat{\Omega}$.

REMARK. The arguments above prove the existence of a Markov process on $\hat{\mathscr{X}}$ with generator $\hat{\Omega}$. But, is this the same process that one obtains by tagging a particle in the simple exclusion process, following it, and describing the system as seen from it? The answer is yes and the proof is not difficult: One introduces an auxiliary process (η_t, X_t) where η_t is the simple exclusion process (with generator Ω) and X_t is the position of the particle initially at the origin. Then the process $\eta_t - X_t$ (where $(\eta - x)(z) = \eta(x + z)$) has $\hat{\Omega}$ as generator. An earlier version of this paper contained this construction which is straightforward but tedious. We are indebted to the referee who suggested the perturbation argument used above.

Let $\mathscr{M}(\mathscr{X})[\mathscr{M}(\hat{\mathscr{X}})]$ be the set of probability measures on $\mathscr{X}[\hat{\mathscr{X}}]$. Let $\mathscr{I} = \{\mu \in \mathscr{M}(\mathscr{X}): \mu S(t) = \mu\}$ and $\hat{\mathscr{I}} = \{\mu \in \mathscr{M}(\hat{\mathscr{X}}): \mu \hat{S}(t) = \mu\}$ be the sets of invariant measures for the processes with semigroups $S(t)$ and $\hat{S}(t)$, respectively. These sets are nonempty, convex, and compact in the weak topology. Thus, by the Krein–Millman theorem, in order to describe them it suffices to determine their extreme points $\mathscr{I}_e$ and $\hat{\mathscr{I}}_e$. We determine $\hat{\mathscr{I}}_e$ in some cases in the following sections.

2. The sets of invariant and translation invariant measures. In this section we take S, $p(x, y)$, $S(t)$, and $\hat{S}(t)$ as in Section 1.

Let $\mathscr{S} \subset \mathscr{M}(\mathscr{X})$ be the set of translation invariant measures on $\mathscr{X} = \{0,1\}^S$. Let $\mu \in \mathscr{S}$ be such that $\mu\{\eta(0) = 1\} > 0$. Define $\hat{\mu} \in \mathscr{M}(\hat{\mathscr{X}})$ as the Palm measure of μ by

$$\hat{\mu} = \mu(\cdot|\eta(0) = 1).$$

We can define $\hat{\mu}$ in this simple way because we do not allow multiple occupancy at the sites of S. The general way to define the Palm measure can be found in [16]. An alternative (and equivalent on $\hat{\mathscr{X}}$) way to define $\hat{\mu}$ is, for $\mu \in \mathscr{S}$, $\alpha(\mu) = \mu\{\eta(0) = 1\} > 0$ and all finite $A \subset S$ and f continuous,

$$\int \hat{\mu}(d\eta) f(\eta) = \frac{1}{|A|\alpha(\mu)} \int \mu(d\eta) \sum_{x \in A} f(\eta - x)\eta(x).$$

If μ is the point mass on $\eta \equiv 0$ (the vacant configuration), we define $\hat{\mu}$ as the point mass on the configuration $\eta = \{0\}$ (the configuration in which only the origin is occupied).

Let $\hat{\mathscr{S}}$ be the set

$$\hat{\mathscr{S}} = \Big\{\mu \in \mathscr{M}(\hat{\mathscr{X}}): \int \mu(d\eta) \sum_x \eta(x) f(\eta, x) = \int \mu(d\eta) \sum_x \eta(x) f(\eta - x, -x), \text{ for all nonnegative } f \in C(\hat{\mathscr{X}} \times S)\Big\}. \tag{2.1}$$

In the next proposition, due to Neveu [16], we see that the set $\hat{\mathscr{S}}$ is the set of Palm measures of translation invariant measures.

Proposition 2.2 (Proposition II-11 of [16]). *The transformation $\mu \mapsto \hat{\mu}$ is a function from $\mathscr{S}$ onto $\hat{\mathscr{S}}$.*

Remark. The function of Proposition 2.2 is not one to one. Take for instance $\mu \in \mathscr{S}$ such that $\alpha(\mu) > 0$, and μ_0 as the point mass at $\eta \equiv 0$. Then $\hat{\mu}$ is the Palm measure of $a\mu + (1 - a)\mu_0$ for $0 < a \le 1$.

The results of this section are the following theorems:

THEOREM 2.3. *If $\mu \in \mathscr{S}$, then for all $t \geq 0$, $\mu S(t) \in \mathscr{S}$ and*

$$\hat{\mu}\hat{S}(t) = [\mu S(t)]\hat{}.$$

THEOREM 2.4. *The following holds*:

$$(\hat{\mathscr{I}} \cap \hat{\mathscr{S}})_e = \{\hat{\nu}_\rho : 0 \leq \rho \leq 1\},$$

where $\hat{\nu}_\rho$ is the Palm measure of ν_ρ, the product of Bernoulli measures with parameter ρ.

REMARK. Theorem 2.3 is an extension (in one way) of Theorem 6.5 of Harris [6] and Theorem 6.5 of Port and Stone [17], for a process constructed without identifying the particles. In [6] and [17] the particles are identified but, in compensation, the result holds for a more general kind of state space. Theorem 2.3 gives a simple way to compute, in the translation invariant case, the measure of the tagged particle process at time t as a function of the measure of the simple exclusion process at the same time. The set of invariant and translation invariant measures has been determined in [12] for the simple exclusion process on $\mathbb{Z}^d$. We use [12] and Theorem 2.3 to obtain an analogous result (Theorem 2.4) for the tagged particle process.

PROOF OF THEOREM 2.3. Define the operator T: $C(\hat{\mathscr{X}}) \to C(\mathscr{X})$ by

$$Tf(\eta) = \eta(0)f(\eta).$$

Then one can see that

$$\int T\hat{\Omega} f \, d\mu = \int \Omega T f \, d\mu \tag{2.5}$$

for all $\mu \in \mathscr{S}$ and f cylindric. We can rewrite the conclusion of Theorem 2.3 as

$$\int T\hat{S}(t) f \, d\mu = \int S(t) T f \, d\mu. \tag{2.6}$$

But (2.5) implies (2.6) as follows from the computation:

$$\begin{aligned} T\hat{S}(t)f - S(t)Tf &= \int_0^t \frac{d}{ds} S(t-s) T\hat{S}(s) f \, ds \\ &= \int_0^t S(t-s)[T\hat{\Omega} - \Omega T]\hat{S}(s) f \, ds \end{aligned}$$

and

$$\begin{aligned} &\int [T\hat{S}(t)f - S(t)Tf] \, d\mu \\ &= \int_0^t \left\{ \int [T\hat{\Omega} - \Omega T]\hat{S}(s) f \, d(\mu S(t-s)) \right\} ds \\ &= 0. \end{aligned}$$

□

PROOF OF THEOREM 2.4. First observe that, since $\nu_\rho \in (\mathscr{T} \cap \mathscr{S})_e$ ([12]) Theorem 2.3 implies that $\hat{\nu}_\rho \in (\hat{\mathscr{T}} \cap \hat{\mathscr{S}})_e$, $0 \le \rho \le 1$. On the other hand, take $\gamma \in (\hat{\mathscr{T}} \cap \hat{\mathscr{S}})_e$. Since $\gamma \in \hat{\mathscr{S}}$, by Proposition 2.2 there exists a measure $\mu \in \mathscr{S}$ such that $\gamma = \hat{\mu}$. Since $\gamma \in \hat{\mathscr{T}}$, $\gamma \hat{S}(t) = \gamma$, i.e., $\hat{\mu}\hat{S}(t) = \hat{\mu}$, and by Theorem 2.3,

$$[\mu S(t)]^{\wedge} = \hat{\mu}. \tag{2.7}$$

Furthermore, since $\mu \in \mathscr{S}$ and $p(x, y) = p(0, y - x)$,

$$\alpha(\mu S(t)) = \alpha(\mu). \tag{2.8}$$

Equations (2.7) and (2.8) and Proposition 2.2 imply that $\mu = \mu S(t)$ and so $\mu \in \mathscr{T}$. Now, if $\mu \in \mathscr{T} \cap \mathscr{S}$ then $\mu = \int \nu_\rho g(d\rho)$, where g is a probability measure on $[0, 1]$ (Liggett [12]). If $\alpha(\mu) = 0$, then $\hat{\mu} = \hat{\nu}_0$. If $\alpha(\mu) > 0$, then

$$\hat{\mu} = \int_{(0,1]} \hat{\nu}_\rho(\rho/\alpha(\mu)) g(d\rho).$$

This implies that $\hat{\mu}$ is a convex combination of $\hat{\nu}_\rho$, $\rho \in (0, 1]$. But $\hat{\mu} = \gamma$ is extremal by assumption. Then $\gamma = \hat{\nu}_\rho$ for some $\rho \in [0, 1]$. □

3. The nearest neighbor one-dimensional case.

3.1. Main theorem. In this section we consider $S = \mathbb{Z}$, $p(x, x+1) = p$, $p(x, x-1) = q$ for all $x \in Z$, $p + q = 1$, $1 > p > q$. Under these conditions the generator $\hat{\Omega}$ of the tagged particle process assumes the form

$$\begin{aligned} \hat{\Omega} f(\eta) = \sum_{x \neq 0} \{&\eta(x)[1 - \eta(x+1)] p[f(\eta_{x,x+1}) - f(\eta)] \\ &+ \eta(x)[1 - \eta(x-1)] q[f(\eta_{x,x-1}) - f(\eta)]\} \\ &+ [1 - \eta(1)] p[f(\eta_{0,1} - 1) - f(\eta)] \\ &+ [1 - \eta(-1) q[f(\eta_{0,-1} + 1) - f(\eta)] \end{aligned} \tag{3.1}$$

for f cylindric on $\hat{\mathscr{X}}$.

Define the following partition of $\hat{\mathscr{X}}$:

$$\hat{\mathscr{X}}_\infty = \left\{\eta \in \hat{\mathscr{X}}: \sum_{x \ge 0} \eta(x) = \infty, \sum_{x \le 0} \eta(x) = \infty\right\},$$

$$\hat{\mathscr{X}}_\infty^+ = \left\{\eta \in \hat{\mathscr{X}}: \sum_{x \ge 0} \eta(x) = \infty, \sum_{x \le 0} \eta(x) < \infty\right\},$$

$$\hat{\mathscr{X}}_\infty^- = \left\{\eta \in \hat{\mathscr{X}}: \sum_{x \ge 0} \eta(x) < \infty, \sum_{x \le 0} \eta(x) = \infty\right\},$$

$$\hat{\mathscr{X}}_0 = \left\{\eta \in \hat{\mathscr{X}}: \sum_{x} \eta(x) < \infty\right\}.$$

Call $\mathscr{M}_\infty[\mathscr{M}_\infty^+, \mathscr{M}_\infty^-, \mathscr{M}_0]$ the family of probability measures with mass concentrated on $\hat{\mathscr{X}}_\infty[\hat{\mathscr{X}}_\infty^+, \hat{\mathscr{X}}_\infty^-, \hat{\mathscr{X}}_0]$. The following lemma allows us to study the set of invariant measures separately for each of these families.

LEMMA 3.2. *If* $\gamma \in \mathscr{M}_\infty[\mathscr{M}_\infty^+, \mathscr{M}_\infty^-, \mathscr{M}_0]$, *then for all* $t \geq 0$,

$$\gamma S(t) \in \mathscr{M}_\infty[\mathscr{M}_\infty^+, \mathscr{M}_\infty^-, \mathscr{M}_0].$$

PROOF. Let $B_0^- = \{\eta: \Sigma_{x<0}\eta(x) < \infty\}$, $D_x = \{\eta: \eta(y) = 0, y < x\}$. Fix $\varepsilon > 0$, $t, \eta \in D_x$. There exists n_0 $(< x)$ such that for all $n < n_0$, $\delta_\eta \hat{S}(t)(D_n) > 1 - \varepsilon$. Since $B_0^- = \bigcup D_n$, $\delta_\eta \hat{S}(t)(B_0^-) = 1$ for all $\eta \in B_0^-$.

Now let $B_\infty^- = \{\eta: \Sigma_{x<0}\eta(x) = \infty\}$, $B_m^- = \{\eta: \Sigma_{x<0}\eta(x) > m\}$, $D_{x,m} = \{\eta: \Sigma_{x \leq y \leq 0}\eta(y) > m\}$. Fix $\varepsilon, t, \eta \in B_\infty^-$. For each m there exist $x < 0$ and n_0 $(< x)$ such that $\eta \in D_{x,m}$ and for all $n < n_0$, $\delta_\eta \hat{S}(t)(D_{n,m}) > 1 - \varepsilon$. This implies that $\delta_\eta \hat{S}(t)(B_m^-) = 1$ for all m, hence $\delta_\eta \hat{S}(t) B_\infty^- = 1$.

Obtain the lemma by proving the analogous results for $B_0^+ = \{\eta: \Sigma_{x>0}\eta(x) < \infty\}$ and $B_\infty^+ = \{\eta: \Sigma_{x>0}\eta(x) = \infty\}$. □

Let $\hat{\mathscr{I}}$ be the set of invariant measures for the tagged particle process. Define $\hat{\mathscr{I}}_\infty[\hat{\mathscr{I}}_\infty^+, \hat{\mathscr{I}}_\infty^-, \hat{\mathscr{I}}_0] = \hat{\mathscr{I}} \cap \mathscr{M}_\infty[\hat{\mathscr{I}} \cap \mathscr{M}_\infty^+, \hat{\mathscr{I}} \cap \mathscr{M}_\infty^-, \hat{\mathscr{I}} \cap \mathscr{M}_0]$. It follows from Lemma 3.2 that

$$\hat{\mathscr{I}}_e = (\hat{\mathscr{I}}_\infty)_e \,\dot{\cup}\, (\hat{\mathscr{I}}_\infty^+)_e \,\dot{\cup}\, (\hat{\mathscr{I}}_\infty^-)_e \,\dot{\cup}\, (\hat{\mathscr{I}}_0)_e,$$

where $\dot{\cup}$ means disjoint union of sets.

DEFINITION 3.3. For each nonnegative integer n let $B_n = \{\eta \in \hat{\mathscr{X}}_\infty^+: \Sigma_{x<0}\eta(x) = n\}$. If $\eta \in \hat{\mathscr{X}}_\infty^+$, then there exists $r \geq 0$ such that $\eta \in B_r$. In this case we represent η by the sequence $\{x(-r), \dots, x(0), \dots\}$, where $x(0) = 0$ and $x(i) < x(i+1)$ for all $i \geq -r$. In this representation the integer $x(i)$ is the position occupied by the ith particle. Let $\rho \in [0, 1)$. Define $\gamma_{\rho,n}^+ \in \mathscr{M}_\infty^+$ as the measure satisfying:

(a) $\gamma_{\rho,n}^+(B_n) = 1$,
(b) the random variables $[x(m+1) - x(m)]$, $m \geq -n$, are mutually independent, and
(c) $\gamma_{\rho,n}^+\{\eta: x(m+1) - x(m) = k+1\} = \rho_m^k(1 - \rho_m)$, where k is a nonnegative integer, $m \geq -n$, and ρ_m $(m \geq -n)$ are defined as functions of ρ, p, and n in the following way:

$$\rho_m = \left(\frac{q}{p}\right)^{m+n+1} + \rho\left[1 - \left(\frac{q}{p}\right)^{m+n+1}\right].$$

For $\rho \in [0, 1]$, $\hat{\nu}_\rho$ is the Palm measure of the Bernoulli measure with parameter ρ. Recall that for $\rho = 0$, $\hat{\nu}_0$ is the measure with mass concentrated on $\eta = \{0\}$.

The result of this section is:

THEOREM 3.4. *Let* $S = \mathbf{Z}$, $p(x, x+1) = p$, $p(x, x-1) = q$, $p + q = 1$, $q < p < 1$, *and* $p(x, y) = 0$ *if* $|x - y| > 1$. *Then the set of extremal invariant measures* $\hat{\mathscr{I}}_e$ *for the tagged particle process is*

$$\hat{\mathscr{I}}_e = (\hat{\mathscr{I}}_0)_e \cup (\hat{\mathscr{I}}_\infty)_e \cup (\hat{\mathscr{I}}_\infty^+)_e \cup (\hat{\mathscr{I}}_\infty^-)_e, \tag{3.5}$$

where

$$\text{(3.6)}\quad \begin{aligned} &\text{(a)} && (\hat{\mathscr{T}}_0)_e \cup (\hat{\mathscr{T}}_\infty)_e = \{\hat{\nu}_\rho : 0 \le \rho \le 1\},\\ &\text{(b)} && (\hat{\mathscr{T}}^+_\infty)_e = \{\gamma^+_{\rho,n} : 0 \le \rho < 1, n \ge 0\},\\ &\text{(c)} && (\hat{\mathscr{T}}^-_\infty)_e = \varnothing. \end{aligned}$$

REMARK. For $\rho = 0$ the measures $\gamma^+_{\rho,r}$ correspond to the measures $\nu^{(n)}$ of Liggett [12] but as seen from the rth particle.

In Section 3.2 we prove part (a) of Theorem 3.4. We prove parts (b) and (c) in Section 3.3.

3.2. Correspondence with the zero range process. The zero range process was introduced by Spitzer [18]. The process was first constructed by Holley [8] for a one-dimensional case (which includes the one studied here) and then by Liggett [11] in a more general context. Sets of invariant measures were studied by Andjel [2]. An intuitive description of the process is the following:

At each site x of a countable set S there are a finite number of particles. Let g be a nonnegative function on the nonnegative integers. At rate $g(k)$ a particle, chosen at random among the k particles at x, jumps to another site y chosen according to a transition probability function $p(x, y)$. The destination site is chosen independently of the number of particles at x and y. We consider the case $S = \mathbb{Z}$, $g(k) \equiv 1$, $p(x, x+1) = q$, $p(x, x-1) = p$, $p + q = 1$, $1 > p > q$. The state space is $\mathscr{Y} = \mathbb{N}^{\mathbb{Z}}$, and the generator of the process is defined for cylindric and bounded $h \in C(\mathscr{Y})$ by

$$\Lambda h(\xi) = \sum_{x \in \mathbb{Z}} 1\{\xi(x) > 0\}\big[q\big(h(\xi_{x,x+1}) - h(\xi)\big) + p\big(h(\xi_{x,x-1}) - h(\xi)\big)\big], \tag{3.7}$$

where

$$\xi_{x,y}(z) = \begin{cases} \xi(z) - 1, & \text{if } z = x,\\ \xi(z) + 1, & \text{if } z = y,\\ \xi(z), & \text{if } z \ne x, y. \end{cases} \tag{3.8}$$

We call $R(t)$ the semigroup generated by Λ. Let $\mathscr{M}(\mathscr{Y})$ be the set of probability measures on $\mathscr{Y}$ and $\mathscr{T}(\Lambda) = \{\beta \in \mathscr{M}(\mathscr{Y}) : \beta R(t) = \beta\}$ be the set of invariant measures for the process.

As a consequence of Theorem 1.11 of Andjel [2] we have that for $\frac{1}{2} < p < 1$,

$$[\mathscr{T}(\Lambda)]_e = \{\beta_\rho : 0 \le \rho < 1\}, \tag{3.9}$$

where β_ρ is the product measure on $\mathscr{Y}$ with marginals

$$\beta_\rho\{\xi : \xi(x) = k\} = \rho^k(1 - \rho), \qquad k \ge 0, x \in \mathbb{Z}.$$

Now we define a bijection between $\hat{\mathscr{X}}_\infty$ and $\mathscr{Y}$. Let $\xi \in \mathscr{Y}$. Define $\eta(\xi) \in \hat{\mathscr{X}}_\infty$ as the configuration satisfying

$$(3.10)\qquad (\eta(\xi))(x) = \begin{cases} 1, & \text{if } \exists\, n \geq 0 \text{ s.t. } \sum_{y=0}^{n} (\xi(y) + 1) = x, \\ 1, & \text{if } \exists\, n < 0 \text{ s.t. } \sum_{y=n}^{-1} (\xi(y) + 1) = -x, \\ 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, let $\eta \in \hat{\mathscr{X}}_\infty$ and $\{x_n\}_{n \in Z}$ be the ordered sequence of sites occupied by η such that $x_0 = 0$:

$$x_i = \begin{cases} \min\{x > x_{i-1}\colon \eta(x) = 1\}, & \text{if } i > 0, \\ \max\{x < x_{i+1}\colon \eta(x) = 1\}, & \text{if } i < 0. \end{cases}$$

Then we define the configuration $\xi(\eta) \in \mathscr{Y}$ as

$$(3.11)\qquad [\xi(\eta)](u) = x_{u+1} - x_u - 1, \qquad u \in \mathbb{Z}.$$

If $f\colon \hat{\mathscr{X}}_\infty \to \mathbb{R}$ is a cylindric function, define $f^\eta\ \mathscr{Y} \to \mathbb{R}$ as

$$(3.12)\qquad f^\eta(\xi) = f(\eta(\xi)).$$

Note that f^η is cylindric and bounded on $\mathscr{Y}$.

The relationship between $\hat{\Omega}$ (equation (3.1)) and Λ (equation (3.7)) is given by the following lemma:

Lemma 3.13. *Let f be a cylindric function on $\hat{\mathscr{X}}_\infty$ and f^η as defined in equation* (3.12). *Then*

$$\hat{\Omega} f(\eta(\xi)) = \Lambda f^\eta(\xi).$$

Proof. Rewrite the generator $\hat{\Omega}$ for the $\{x_n\}_{n \in \mathbb{Z}}$ representation of $\eta \in \hat{\mathscr{X}}_\infty$:

$$\begin{aligned} \hat{\Omega} f(\eta) = \sum_{n \neq 0} \Big\{ & p1\{x_{n+1} - x_n > 1\}\big[f(\eta_{x_n, x_n+1}) - f(\eta)\big] \\ & + q1\{x_n - x_{n-1} > 1\}\big[f(\eta_{x_n, x_n-1}) - f(\eta)\big]\Big\} \\ & + p1\{x_1 > 1\}[f(\eta_{0,1} - 1) - f(\eta)] \\ & + q1\{x_{-1} < -1\}[f(\eta_{0,-1} + 1) - f(\eta)] \end{aligned}$$

and, if $\eta = \eta(\xi)$,

$$\begin{aligned} \hat{\Omega} f(\eta(\xi)) = \sum_{n \neq 0} \{ & p1\{\xi(n) > 0\}[f(\eta(\xi_{n,n-1})) - f(\eta(\xi))] \\ & + q1\{\xi(n-1) > 0\}[f(\eta(\xi_{n-1,n})) - f(\eta(\xi))]\} \\ & + p1\{\xi(0) > 0\}[f(\eta(\xi_{0,-1})) - f(\eta(\xi))] \\ & + q1\{\xi(-1) > 0\}[f(\eta(\xi_{-1,0})) - f(\eta(\xi))] \\ = {} & \Lambda f^\eta(\xi). \end{aligned}$$

□

REMARK. If in the zero range process the site u is not empty, then particles u and $u+1$ of the tagged particle process are not neighbors (i.e., $x_{u+1} - x_u > 1$). With rate p a zero range particle jumps from site u to the left, which is equivalent to a jump to the right of the $(u+1)$th exclusion particle (occupying site x_{u+1}). With rate q a zero range particle jumps to the right which is equivalent to the left jump of the uth exclusion particle.

The corresponding relation between $\mathscr{M}(\hat{\mathscr{X}}_\infty)$ and $\mathscr{M}(\mathscr{Y})$ is given in the following lemma. For $\mu \in \mathscr{M}(\hat{\mathscr{X}}_\infty)$ define $\lambda_\mu \in \mathscr{M}(\mathscr{Y})$ as the measure satisfying

$$\lambda_\mu f^\eta = \mu f \quad \text{for all } f \in C(\hat{\mathscr{X}}_\infty). \tag{3.14}$$

LEMMA 3.15. *If $\mu \in \hat{\mathscr{I}}_\infty$ then $\lambda_\mu \in \mathscr{I}(\Lambda)$.*

PROOF. We have that (see [14])

$$\hat{\mathscr{I}}_\infty = \{\mu \in \mathscr{M}(\hat{\mathscr{X}}_\infty) : \mu\hat{\Omega} f = 0, \text{ for all cylindric } f\},$$
$$\mathscr{I}(\Lambda) = \{\lambda \in \mathscr{M}(\mathscr{Y}) : \lambda\Lambda h = 0, \text{ for all cylindric and bounded } h\}.$$

The lemma follows from Lemma 3.13 and equation (3.14). □

PROOF OF THEOREM 3.4(a). It is easy to check that $\hat{\mathscr{I}}_0 = \{\hat{\nu}_0\}$. Suppose then that $\rho > 0$, thus $\hat{\nu}_\rho \in \mathscr{M}(\hat{\mathscr{X}}_\infty)$. Since ν_ρ are invariant for the simple exclusion process ([12]), Theorem 2.3 implies that $\hat{\nu}_\rho$ are invariant for the tagged particle process. Use equation (3.9) and Lemma 3.15 to complete the proof. □

3.3. The semiinfinite case. In this subsection we introduce a modified zero range process on $\mathscr{Y}^+ = \mathbb{N}^{\mathbb{N}}$. We assume that at the "site" -1 there are infinitely many particles which have the same behavior as the other particles, i.e., particles jump from and to "site" -1 with the same rates they do at the other sites. The generator Λ^+ of this process is defined on bounded cylindric functions by

$$\begin{aligned}\Lambda^+ h(\xi) = \sum_{u=0}^{\infty} 1\{\xi(u) > 0\}\{&p[h(\xi_{u,u-1}) - h(\xi)] \\ &+ q[h(\xi_{u,u+1}) - h(\xi)]\} \\ + q[h(\xi_{-1,0}) - h(\xi)],&\end{aligned} \tag{3.16}$$

where, for u, $u \pm 1 \geq 0$, $\xi_{u,u\pm1}$ was defined in equation (3.8) and

$$\begin{aligned}\xi_{0,-1}(u) &= \begin{cases} \xi(u), & \text{if } u \geq 1, \\ \xi(u) - 1, & \text{if } u = 0, \end{cases} \\ \xi_{-1,0}(u) &= \begin{cases} \xi(u), & \text{if } u \geq 1, \\ \xi(u) + 1, & \text{if } u = 0. \end{cases}\end{aligned} \tag{3.17}$$

REMARK. Since $g(k) = 1$ is bounded and the birth rate at the origin (i.e., the rate of jump from -1) is bounded by $q < 1$, one can use Holley's techniques [8] to prove the existence of a semigroup $R^+(t)$ generated by Λ^+.

Now we establish relations between the sets $\mathscr{Y}^+$ and $B_0 = \{\eta \in \hat{\mathscr{X}}_\infty^+ : \Sigma_{x<0}\eta(x) = 0\}$. Represent $\eta \in B_0$ by the sequence $\{x_0, x_1, \dots\} \subset \mathbb{N}$, where $x_0 = 0$ and $x_i > x_{i-1}$. Let ξ belong to $\mathscr{Y}^+$. Define $\eta(\xi)$ as the sequence $\{0, x_1, x_2, \dots\}$ such that $x_i = \Sigma_{u=0}^{i-1}(\xi(u) + 1)$. If $f \in C(B_0)$ is a cylindric function, define $f^\eta \in C(\mathscr{Y}^+)$ (which is cylindric too) as in equation (3.12). If $\mu \in \mathscr{M}(B_0)$, define $\lambda_\mu \in \mathscr{M}(\mathscr{Y}^+)$ as in equation (3.14). Now, if $\mathscr{T}(\Lambda^+)$ is the set of invariant measures for the process with generator Λ^+, we have the analogues to Lemmas 3.13 and 3.15:

LEMMA 3.18. *Let $f \in C(B_0)$ be a cylindric function. Then*

(a) $$\hat{\Omega} f(\eta(\xi)) = \Lambda^+ f^\eta(\xi);$$

(b) $$\text{if} \quad \mu \in \hat{\mathscr{T}}_\infty^+ \cap \mathscr{M}(B_0), \quad \text{then } \lambda_\mu \in \mathscr{T}(\Lambda^+).$$

PROOF. The proof is similar to those of Lemmas 3.13 and 3.15. □

We characterize $\mathscr{T}(\Lambda^+)$ in the following theorem:

THEOREM 3.19. *Let $\mathscr{T}(\Lambda^+)$ be the set of invariant measures for the semiinfinite zero range process with generator Λ^+ defined in equation* (3.16). *Then*:

(a) $$\text{If} \quad p \le q, \quad \text{then } \mathscr{T}(\Lambda^+) = \varnothing.$$

(b) $$\text{If} \quad p > q, \quad \text{then } [\mathscr{T}(\Lambda^+)]_e = \{\lambda_\rho : 0 \le \rho < 1\},$$

where λ_ρ is the product measure with marginals

(3.20) $$\lambda_\rho\{\xi(m) = k\} = \rho_m^k(1 - \rho_m), \quad k \ge 0,$$

and ρ_m is the following function of ρ, p, and m:

$$\rho_m = \rho\left(1 - \left(\frac{q}{p}\right)^{m+1}\right) + \left(\frac{q}{p}\right)^{m+1}.$$

REMARK. Part (a) of Theorem 3.19 says that for $p \le \frac{1}{2}$ there are no invariant measures, for the tagged particle process, which concentrate on configurations with only a finite number of particles to the left of the tagged particle.

PROOF. (a) Suppose that $p \le q$ and that there exists a measure $\lambda \in \mathscr{T}(\Lambda^+)$. For each $x \in \mathbb{N}$ consider the cylindric function $h_x(\xi) = \xi(x)$. Since λ is invariant, $\int \Lambda^+ h_x \, d\lambda = 0$ for all $x \in \mathbb{N}$. Rewrite this expression to obtain the equations

$$a_x = pa_{x+1} + qa_{x-1}, \qquad x \ge 0,$$

where $a_x = \lambda\{\xi : \xi(x) > 0\}$ for $x \ge 0$ and $a_{-1} = 1$.

Hence

$$a_0 = q + pa_1,$$

$$a_x = a_0 \sum_{k=0}^{x} \left(\frac{q}{p}\right)^k - \sum_{k=1}^{x} \left(\frac{q}{p}\right)^k, \qquad x \ge 1,$$

and, since $0 \le a_x \le 1$ for all $x \in Z^+$,

$$\frac{\Sigma_{k=1}^{x}(q/p)^k}{\Sigma_{k=0}^{x}(q/p)^k} \le a_0 \le \frac{1}{\Sigma_{k=0}^{x}(q/p)^k} + \frac{\Sigma_{k=1}^{x}(q/p)^k}{\Sigma_{k=1}^{x}(q/p)^k}.$$

But $q \ge p$ implies $a_0 = 1$, thus $a_x = 1$ for all x and $\lambda\{\eta\colon\ \eta(x) = 0\} = 0$. Hence $\lambda\{\eta\colon\ \eta(x) = k\} = 0$ for all x, all k. Then λ cannot be a probability measure on $\mathscr{Y}^+$.

(b) Consider $p > q$. To prove the invariance of the measures λ_ρ one first observes that for all $x \ge 0$:

$$\rho_{x-1}q - \rho_x p = \rho(q-p), \tag{3.21a}$$

$$p(\rho_{x+1}/\rho_x) + q(\rho_{x-1}/\rho_x) = 1 \tag{3.21b}$$

(with the convention $\rho_{-1} = 1$).

Now consider the family of cylindric functions $h(\xi) = 1\{\xi(x) = k(x),\ 0 \le x \le n\}$, $k(x) \ge 0$, $n \ge 0$. This family generates a dense subset of $C(\mathscr{X})$. Then

$$\lambda_\rho \Lambda^+ h = \prod_{x=0}^{n} \rho_x^{k(x)}(1-\rho_x) \times \left[\sum_{x=0}^{n} 1\{k(x) > 0\}\left(p\frac{\rho_{x+1}}{\rho_x} + q\frac{\rho_{x-1}}{\rho_x} - 1\right) + p\rho_0 + q\rho_n - q - p\rho_{n+1}\right] = 0$$

by equations (3.21). Thus $\lambda_\rho \Lambda^+ f = 0$ for all bounded cylindric f, which implies (see [14]) that λ_ρ are invariant for the process with generator Λ^+.

The proof that $\{\lambda_\rho\colon 0 \le \rho < 1\}$ are the unique extremal invariant measures is analogous to that of Theorem 1.11 of Andjel [2], and so we only sketch it. As in [2] we define a coupling on the space $\mathscr{Y}^+ \times \mathscr{Y}^+$ in such a way that each marginal has $R^+(t)$ as semigroup. In this coupling, particles occupying the same site in different marginals jump together. The aim of this coupling is to prove that for each invariant measure $\lambda \in [\mathscr{I}(\Lambda^+)]_e$ and $\rho \in [0,1)$, there exists a measure $\tilde{\lambda}_\rho$ on $\mathscr{Y}^+ \times \mathscr{Y}^+$, invariant for the coupled process, with marginals λ and λ_ρ such that $\tilde{\lambda}_\rho\{\zeta \ge \xi \text{ or } \xi \ge \zeta\} = 1$. Then, the characterization of λ as a λ_ρ measure for some $\rho = \rho_0$ follows. The generator of the coupled process is

$$\begin{aligned}\tilde{\Lambda}^+ f(\zeta, \xi) = \sum_{u=0}^{\infty} \{&1\{\zeta(u) > 0, \xi(u) > 0\}[p[f(\zeta_{u,u-1}, \xi_{u,u-1}) - f(\zeta,\xi)] \\ &+ q[f(\zeta_{u,u+1}, \xi_{u,u+1}) - f(\zeta,\xi)]] + 1\{\zeta(u) > 0, \xi(u) = 0\} \\ &\times [p[f(\zeta_{u,u-1}, \xi) - f(\zeta,\xi)] + q[f(\zeta_{u,u+1}, \xi) - f(\zeta,\xi)]] \\ &+ 1\{\zeta(u) = 0, \xi(u) > 0\}[p[f(\zeta, \xi_{u,u-1}) - f(\zeta,\xi)] \\ &+ q[f(\zeta, \xi_{u,u+1}) - f(\zeta,\xi)]]\} + q[f(\zeta_{-1,0}, \xi_{-1,0}) - f(\zeta,\xi)],\end{aligned}$$

where $\zeta_{u,u\pm1}$ and $\xi_{u,u\pm1}$ are defined in equations (3.8) and (3.17). The existence of the measure $\tilde{\lambda}_\rho$ on $\mathscr{Y}^+ \times \mathscr{Y}^+$, invariant for the process with generator $\tilde{\Lambda}^+$ and marginals λ and λ_ρ is easily proven following [2]. To prove that $\tilde{\lambda}_\rho\{\zeta \geq \xi$ or $\zeta \leq \xi\} = 1$, one considers the function $f_n(\zeta, \xi) =:$ number of changes of sign of $\zeta(0) - \xi(0), \zeta(1) - \xi(1), \dots, \zeta(n) - \xi(n)$, $n = 1, 2, \dots$, and the sequence $A_n = \int(f_{n+1} - f_n)\, d\tilde{\lambda}_\rho$. As in Lemma 7.1 of [2] one proves that there exists a subsequence of A_n converging to 0. Now we use this fact to prove that the probability that there is any change of sign is zero. We argue by contradiction:

Suppose that there exist $a < b \in Z^+$ such that

$$\tilde{\lambda}_\rho\{\zeta(a) > \xi(a), \zeta(b) < \xi(b), \zeta(x) = \xi(x) \quad \text{for } 0 \leq x < a, a < x < b\} > 0;$$

then an inductive argument shows that

$$\tilde{\lambda}_\rho\{1 = \zeta(0) > \xi(0), \zeta(x) = \xi(x), 1 \leq x < b, \zeta(b) < \xi(b)\} > 0. \tag{3.22}$$

Now take $A_{n_l} \to 0$. Since $\tilde{\lambda}_\rho$ is invariant, $\int \tilde{\Lambda} f_{n_l}\, d\tilde{\lambda}_\rho = 0$. The positive terms of this integral are bounded by pA_{n_l} and therefore go to 0 as l goes to ∞. But (3.22) implies that the absolute value of the negative terms is bounded below by a positive constant for $n > b$. This contradiction proves that

$$\tilde{\lambda}_\rho\{\zeta \geq \xi \text{ or } \xi \geq \zeta\} = 1. \tag{3.23}$$

□

Notice that we are using that (a) new changes of signs cannot be generated by the creation of particles at the origin and (b) the destruction of particles at the origin can only decrease the number of changes of sign. This allows us to give a proof of equation (3.23) which is simpler than the one given by Andjel for the doubly infinite case.

PROOF OF THEOREM 3.4(b) AND (c). First, it follows from Theorem 3.19(a) and Lemma 3.18 that $\hat{\mathscr{I}}_\infty^- = \varnothing$. This proves (c). To prove (b) write $\mathscr{M}_\infty^+ = \bigcup_{n \geq 0} \mathscr{B}_n$, where $\mathscr{B}_n$ is the set of measures concentrating mass on the set $B_n = \{\eta \in \hat{\mathscr{X}}_\infty^+ : \Sigma_{x<0}\eta(x) = n\}$. From the proof of Lemma 3.2, we have that if $\gamma \in \mathscr{B}_n$ then $\gamma\hat{S}(t) \in \mathscr{B}_n$ for each $t \geq 0$. Then $(\hat{\mathscr{I}}_\infty^+)_e = \bigcup_{n \geq 0}(\hat{\mathscr{I}}_\infty^+ \cap \mathscr{B}_n)_e$. From Theorem 3.19 and Lemma 3.18 we know that $(\hat{\mathscr{I}}_\infty^+ \cap \mathscr{B}_0)_e = \{\gamma_{\rho,0}^+, 0 \leq \rho < 1\}$. To complete the proof of part (b) it suffices to change the origin of coordinates for the process $R^+(t)$ in Theorem 3.19 in such a way that birth and death of particles occur at the site $-n$ (instead of the origin) and so one obtains that $(\hat{\mathscr{I}}_\infty^+ \cap \mathscr{B}_n)_e = \{\gamma_{\rho,n}^+ : 0 \leq \rho < 1\}$ for $n \geq 0$. □

4. Domains of attraction. In this section $S = \mathbb{Z}$, $p(x, y) = p$ and q for $y = x + 1$ and $x - 1$, respectively. Let $\mathscr{S}$ be the set of translation invariant measures and $\hat{\mathscr{S}}$ be as defined in Section 2. If $\mu \in \mathscr{S}$, the intensity of μ is $\alpha(\mu) = \int \eta(0)\, d\mu$. As in the previous sections, $S(t)$ is the semigroup for the simple exclusion process and $\hat{S}(t)$ the one for the tagged particle process.

THEOREM 4.1. *Let μ be an ergodic translation invariant measure on $\{0,1\}^{\mathbf{Z}}$ with intensity $\alpha(\mu) = \rho$; then $\hat{\mu}\hat{S}(t) \Rightarrow \hat{\nu}_\rho$, i.e., $\hat{\mu}\hat{S}(t)$ converges weakly to the Palm measure of ν_ρ (the Bernoulli measure of parameter ρ).*

PROOF. The case $\rho = 0$ is trivial; we then assume $\rho > 0$. It is proven by Andjel [1] that $\mu S(t) \Rightarrow \nu_\rho$. Let $\mathscr{A} = \{\mu: \mu\{\eta(x) = 0 \ \forall\ x\} = 0\}$. Then the map from $\mathscr{S} \cap \mathscr{A}$ to $\hat{\mathscr{S}} \cap \mathscr{A}$ defined by $\mu \mapsto \hat{\mu}$ is a homeomorphism of $\mathscr{S} \cap \mathscr{A}$ onto $\hat{\mathscr{S}} \cap \mathscr{A}$ in the weak topology (Theorem 7.6 of [17]). Since by ergodicity $\mu \in \mathscr{A}$, $[\mu S(t)]\hat{} \Rightarrow \hat{\nu}_\rho$. The proof follows from Theorem 2.3: $[\mu S(t)]\hat{} = \hat{\mu}\hat{S}(t)$. □

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