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ON THE ASYMPTOTIC BEHAVIOUR OF DISCRETE TIME STOCHASTIC GROWTH PROCESSES[1]

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We study the asymptotic behaviour of the solution of the stochastic difference equation $X_{n+1} = X_n + g(X_n)(1 + \xi_{n+1})$, where g is a positive function, (ξ_n) is a 0-mean, square-integrable martingale difference sequence, and the states $X_n < 0$ are assumed to be absorbing. We clarify, under which conditions X_n diverges with positive probability, satisfies a law of large numbers, and, properly normalized, converges in distribution. Controlled Galton–Watson processes furnish examples for the processes under consideration.

1. Introduction. In this paper we investigate the asymptotic behaviour of a recursively defined, discrete time stochastic process

$$X_{n+1} = X_n + g(X_n)(1 + \xi_{n+1}), \qquad X_0 > 0, \quad EX_0^2 < \infty, \tag{1}$$

where $g(t) = o(t)$ is a strictly positive function and (ξ_n) is a zero-mean, square-integrable martingale difference sequence, the conditional second moments of which,

$$\sigma^2(X_n) = E\left[\xi_{n+1}^2 | X_0, \dots, X_n\right], \tag{2}$$

depend only on the present state of the process (X_n) and do not grow too fast. If $X_n < 0$ is absorbing we show that under suitable conditions

$$P\Big(X_n \to \infty \text{ or } \lim_{n\to\infty} X_n \leq 0 \text{ exists}\Big) = 1,$$

$P(X_n \to \infty) > 0$ and investigate the asymptotic behaviour of (X_n) on $\{X_n \to \infty\}$.

The interest in (1) is fueled by Markovian growth models such as controlled Galton–Watson processes which, in contrast to the classical Galton–Watson process, admit a state-dependent reproduction behaviour of the population under consideration [see Küster (1985) and the literature cited therein].

In Keller et al. (1984) we have analyzed the corresponding stochastic differential equation

$$dX_t = g(X_t)(dt + \sigma(X_t)\,dW_t), \qquad X_0 \equiv 1.$$

Our main tool was the transformation $Gt = \int_1^t ds/g(s)$, which applied to X_t yields a process with constant drift 1. This process GX_t admits a rather exact asymptotic description from which information about the process X_t can be regained. In particular one can state conditions on the functions g and σ which

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are equivalent to $X_t \sim \mu_t$ on $\{X_t \to \infty\}$, where $\mu_t = G^{-1}t$ is the solution of the corresponding deterministic equation ($\sigma \equiv 0$).

Although the results for the discrete process defined in (1) turn out to be analogous to those for the process X_t, and although the basic idea of the proof, namely to study the process GX_n first, is the same, too, there are considerable technical differences due to the fact that stochastic calculus is no longer at our disposal. In particular Itô's formula (for the transition from X_t to GX_t) must be replaced by a Taylor-approximation, and this gives rise to error terms which are more difficult to control. In fact, our assumptions are a bit stronger than those made in Keller et al. (1984) and two of our results (Theorems 4 and 5) are weaker than the analogous ones for X_t.

In the next section we formulate and discuss the precise assumptions and derive some simple consequences. In Section 3 we prove a strong law of large numbers for the process GX_n, which is the key to an asymptotic expansion for $(GX_n - Ga_n)$ [a_n is the solution of (1) for $\sigma \equiv 0$] relating it to the martingale $\Sigma_{k=1}^n \xi_k$ (Theorem 2 in Section 4). In Section 5 we come back to the original process X_n and show that if $\sigma(t)$ does not grow too fast then $X_n/a_n \to 1$ in probability on $\{X_n \to \infty\}$ and that in this case $(X_n - a_n)/g(a_n)$ either converges a.s. or, properly normalized, is asymptotically normal (Theorems 3, 4, and 5). If, on the other hand, the growth of $\sigma(t)$ is too rapid, then $X_n/a_n \to 0$ in probability (Theorem 6).

There is a sharp boundary between these two different types of asymptotic behaviour of X_n, which can be described in terms of a joint growth-condition on the functions $\sigma(t)$ and $g(t)$. Exactly "on that boundary" a third type of stochastic behaviour of X_n occurs: X_n, suitably normalized, is asymptotically log-normal on $\{X_n \to \infty\}$. Although the growth-condition is a bit involved, we do not discuss it here but refer the reader to Section 5 in Keller et al. (1984). In Section 6 we give almost sure approximations to X_n, and in Section 7 we present examples for which our results apply and counterexamples violating some of our assumptions which exhibit a different asymptotic behaviour.

Some auxiliary results are proved in the Appendix.

2. Notation and assumptions. We recall the defining equation

$$X_{n+1} = X_n + g(X_n)(1 + \xi_{n+1}), \qquad X_0 > 0, \quad EX_0^2 < \infty \tag{1}$$

of the process (X_n), where $(\xi_n, \mathscr{F}_n)_{n \geq 1}$ is a martingale difference sequence, i.e., $E[\xi_{n+1}|\mathscr{F}_n] = 0$, $\mathscr{F}_0 \subseteq \cdots \subseteq \mathscr{F}_n \subseteq \mathscr{F}_{n+1} \subseteq \cdots$, X_0 is $\mathscr{F}_0$-measurable such that X_n is $\mathscr{F}_n$-measurable. Motivated by the applications to Markovian growth-models we have in mind, we assume a kind of weak Markov property; namely that there exists a measurable function σ^2 such that

$$E\left[\xi_{n+1}^2|\mathscr{F}_n\right] = \sigma^2(X_n) \quad \text{a.e.} \tag{2}$$

In particular $\int_0^\infty P(\xi_{n+1}^2/\sigma^2(X_n) \geq z|\mathscr{F}_n)\,dz < \infty$ a.e. We shall assume a bit more, namely that this integrability is uniform in the following sense: There exists a nonincreasing function F such that

$$P\left(\xi_{n+1}^2/\sigma^2(X_n) \geq z\middle|\mathscr{F}_n\right) \leq F(z) \quad \text{a.e. for all } n \geq 0 \text{ and } z \in \mathbb{R}, \tag{3}$$

and

$$\int_0^\infty F(z)\,dz = -\int_0^\infty z\,dF(z) < \infty.$$

This is implied, e.g., by $E[|\xi_{n+1}|^{2+\delta}|\mathscr{F}_n] = O(\sigma^{2+\delta}(X_n))$ for some $\delta > 0$. Before we make further assumptions on the functions $g(t)$ and $\sigma(t)$ we list all notation used repeatedly in this paper:

$$Z_n = X_n + g(X_n), \tag{4}$$

$$\zeta_{n+1} = \frac{g(X_n)}{g(Z_n)}\xi_{n+1}, \tag{5}$$

$$a_{n+1} = a_n + g(a_n), \qquad a_0 = 1. \tag{6}$$

Note that a_n is the solution of (1) for $\xi_{n+1} \equiv 0$, $X_0 \equiv 1$. Next we introduce some auxiliary functions:

$$Gt = \int_1^t \frac{ds}{g(s)}, \qquad t > 0. \tag{7}$$

If f is a function on the positive axis, $\hat{f}$ denotes the function given by $\hat{f}(t) = f(G^{-1}t)$, G^{-1} being the inverse function of G. Note that $G^{-1}t$ satisfies the differential equation $(d/dt)G^{-1}t = g(G^{-1}t)$. Further for $t > 0$ let

$$\psi(t) = \int_1^t \frac{\sigma^2(s)}{g(s)}\,ds, \quad \text{i.e., } \hat{\psi}(t) = \int_0^t \hat{\sigma}^2(s)\,ds, \tag{8}$$

$$\phi(t) = -\frac{1}{2}\int_1^t \frac{\sigma^2(s)g'(s)}{g(s)}\,ds, \quad \text{i.e., } \hat{\phi}(t) = -\frac{1}{2}\int_0^t \sigma^2(s)\widehat{g'}(s)\,ds, \tag{9}$$

$$r(t) = G(t + g(t)) - Gt - 1. \tag{10}$$

We now formulate our main assumptions:

(A.1) g: $\mathbb{R} \to \mathbb{R}$ *is positive and twice continuously differentiable on* $\{t > 0\}$ *and* $g(t) = 0$ *for* $t < 0$. g *and* g' *are both ultimately concave or convex and*

$$g(t) = o(t).$$

(A.2) σ^2: $\mathbb{R}^+ \to \mathbb{R}^+$ *is positive and continuously differentiable. Additionally* $\int_1^\infty \hat{\sigma}^2(t)t^{-2}\,dt < \infty$ *and* $\hat{\sigma}^2(t)$ *is ultimately concave or convex.*

(A.3) $g'(t)\sigma^2(t) \to 0$, $t \to \infty$.

(A.4) *If* $\psi(\infty) = \infty$ *we suppose that* $|g' \circ \psi^{-1}|$ *is ultimately convex, which is equivalent to* $\lambda(t) := |(g''(t)g(t))/\sigma^2(t)|$ *being ultimately decreasing, because* $\lambda \circ \psi^{-1}(t) = |(g' \circ \psi^{-1})'(t)|$, *whereas if* $\psi(\infty) < \infty$ *we assume that* $|g' \circ G^{-1}|$ *is ultimately convex or, equivalently, that* $|g''(t)g(t)|$ *is ultimately decreasing.*

(A.5) (a) $\xi_{n+1}/\sigma(X_n) \geq -C > -\infty$ *a.e. for all n or*

(b) $t^{-1}g(t)\sigma(t)\sqrt{Gt} \to 0$, $t \to \infty$.

[For decreasing g assumption (A.5b) follows from (A.1) and (A.2); see (11) and (19) below.]

We discuss these assumptions.

The convexity- and differentiability assumptions are not necessary, of course, but convenient. In particular, (A.1) implies:

> Either g is decreasing and then it is convex and $g'(t) \le 0$, $g''(t) \ge 0$ ultimately, or g eventually increases and is concave such that $g'(t) \le g(t)/t$ for large t. In any case
> $$g'(t) \to 0, \qquad t \to \infty.$$

In order to understand the importance of (A.1)–(A.3) for the results we are going to prove consider the following Taylor-expansion of GX_{n+1} at X_n:

$$\begin{aligned} GX_{n+1} &= GX_n + (1 + \xi_{n+1}) - \tfrac{1}{2}g'(X_n)(1 + \xi_{n+1})^2 + \text{remainder} \\ &= GX_n + 1 - \tfrac{1}{2}g'(X_n) + \xi_{n+1}(1 - g'(X_n)) \\ &\quad - \tfrac{1}{2}g'(X_n)\sigma^2(X_n)\big(\xi^2_{n+1}/\sigma^2(X_n)\big) + \text{remainder}. \end{aligned}$$

Suppose for the moment that the remainder is negligible. Then $GX_n/n \to 1$ a.e. on $\{X_n \to \infty\}$ will not hold unless $g'(t) \to 0$ and $g'(t)\sigma^2(t) \to 0$ as $t \to \infty$, i.e., (A.1) and (A.3), and $\sum_{k=0}^{n-1}\xi_{k+1} = o(n)$ a.e. on $\{X_n \to \infty\}$. The best general condition for the latter requirement is $\sum_{k=0}^{\infty}\sigma^2(X_k)/(k+1)^2 < \infty$ a.e. on $\{X_n \to \infty\}$, and in view of $GX_n/n \to 1$ a.e. on $\{X_n \to \infty\}$ this is equivalent to (A.2).

The law of large numbers for $\sum\xi_k$ may hold under slightly weaker assumptions, but it will fail to be true if $\hat{\sigma}^2(t) \sim$ const t. Therefore

$$\hat{\sigma}^2(t) = o(t), \quad \text{i.e., } \sigma^2(t) = o(Gt), \tag{11}$$

which is a consequence of (A.2), is a rather natural requirement, and in fact, this is all we need from (A.2) once we have established that $GX_n/n \to 1$ a.e. on $\{X_n \to \infty\}$ in Theorem 1.

The reason that in Keller et al. (1984) $g(t) = o(t)$ could be replaced by the weaker assumption $G(\infty) = \infty$ is due to the fact that the unperturbed ($\sigma = 0$) solution $\mu_t = G^{-1}t$ of the corresponding differential equation referred to in the introduction satisfies trivially $G\mu_t/t = 1$, whereas the Taylor expansion

$$Ga_{n+1} = Ga_n + 1 - \tfrac{1}{2}g'(a_n) + \text{remainder}$$

shows (negligibility of the remainder is assumed) that $Ga_n/n \to 1$ only if $g'(t) \to 0$, i.e., $g(t) = o(t)$, and this is essential for our approach. Hence $g(t) = o(t)$ [in contrast to $G(\infty) = \infty$] is the price we have to pay for the time-discretization.

In the above considerations we did not care about the remainder terms in the Taylor expansions. A considerable amount of work, however, will be devoted to the control of these terms in later sections, in particular a more complicated expansion will be used in the proofs.

A look at the process $\log X_n$ sheds some more light on condition (A.1): Suppose that the ξ_n are i.i.d. Then

$$\begin{aligned} \log X_{n+1} &= \log X_n + \log\big(1 + g(X_n)/X_n(1 + \xi_{n+1})\big) \\ &= \log X_n + \tilde{g}(\log X_n)(1 + \tilde{\xi}_{n+1}), \end{aligned}$$

for a suitable $\tilde{\xi}_{n+1}$, if $\tilde{g}(\log X_n) = E[\log(1 + g(X_n)/X_n(1 + \xi_{n+1}))|\mathscr{F}_n]$.

Thus we have a recursion for log X_n which is formally identical to that for X_n. If $g(t)/t$ does not tend to 0 as $t \to \infty$, the function $\tilde{g}$ is strongly influenced by the whole distribution of the ξ_n, but if $g(t)/t \to 0$ fast enough, then the influence of the second moments predominates for large X_n: In fact, Theorem 5 will show that, properly normalized, X_n or $\log X_n$ have limiting distributions depending only on g and σ^2 if $g'(t)\psi^{1/2}(t) \to c < \infty$, $t \to \infty$, which in the case of i.i.d. ξ_n means $g'(t)G^{1/2}t \to c/\sigma$, and this is equivalent to $g(t)t^{-1}\log t \to 2c^2/\sigma^2$ [see Section 5.C in Keller et al. (1984)]. We mention without proof that the closer $g(t)$ comes to the identity the higher are the moments determining the limiting distribution of $\log X_n$.

As (A.4) is only a convexity assumption, we turn to (A.5). Note first that (A.5b) and (A.2) imply (A.3) [in view of (11)]. In Section 7 we show that (A.5) is sharp in the following sense: If $g(t), \sigma(t)$ satisfy the regularity assumptions (A.1), (A.2) and $t^{-1}g(t)\sigma(t)\sqrt{Gt} \to \infty$ as $t \to \infty$, then a sequence of i.i.d. random variables η_n can be chosen in such a way that $E[\eta_n] = 0$, $E[\eta_n^2] = 1$, and $\xi_n = \sigma(X_n)\eta_n$ is a martingale difference sequence which gives rise [via (1)] to a process X_n violating Theorem 1.

For later use we list some further simple consequences of (A.1)–(A.3):

$$t\hat{\sigma}^2(t) = O(\hat{\psi}(t)), \quad \text{from (11)}, \tag{12}$$

$$\hat{\psi}^{1/2}(t) = o(t), \quad \text{from (8) and (11)}, \tag{13}$$

$$\hat{\sigma}^2(t) = o\left(\hat{\psi}(t)^{1/2}\right), \quad \text{from (12) and (13)}, \tag{14}$$

and if t is large enough, (A.2) implies

$$\hat{\sigma}^2(ct) \le c\hat{\sigma}^2(t), \quad \text{for } c \ge 1, \tag{15}$$

$$\hat{\psi}(ct) \le c^2\hat{\psi}(t), \quad \text{for } c \ge 1, \tag{16}$$

$$\hat{\phi}(ct) \le c^2\hat{\phi}(t), \quad \text{for } c \ge 1. \tag{17}$$

As

$$\begin{aligned}\left(\sigma^2(t)g(t)\right)' &= (\hat{\sigma}^2)'(Gt)G'tg(t) + \sigma^2(t)g'(t)\\ &= (\hat{\sigma}^2)'(Gt) + \sigma^2(t)g'(t) \to 0, \qquad t \to \infty,\end{aligned}$$

by (A.2) and (A.3), we infer from l'Hôpital's rule that

$$\sigma^2(t)g(t)/t \to 0, \qquad t \to \infty. \tag{18}$$

These facts will be used freely in the sequel. For ultimately decreasing g we have some further properties:

$$g(t)/t = O(1/Gt) \quad \text{by (A.1)}, \tag{19}$$

and it has been shown in Lemma 1 in Keller et al. (1984) that (A.1) implies for decreasing g

$$\widehat{g'}(t) = O(t^{-1}), \tag{20}$$

which entails in view of (A.3) and (13)

$$g'(t)\sigma(t)\sqrt{Gt} \to 0, \qquad g'(t)\psi(t)^{1/2} \to 0, \qquad t \to \infty. \tag{21}$$

Finally we note that

$$\left|\frac{g(Z_n)}{g(X_n)} - 1\right| = |g'(\tilde{X}_n)| \le \text{const}, \tag{22}$$

for some nonrandom constant, where $X_n \le \tilde{X}_n \le Z_n$.

3. A strong law of large numbers for the transformed process. The main result of this section is

THEOREM 1. *Assume* (A.1), (A.2), (A.3), *and* (A.5). *Then*

(a) $$\varliminf_{n\to\infty} X_n \le 0 \quad or \quad X_n \to \infty,\ n \to \infty \quad a.e.$$

(b) $$P\{X_n \to \infty\} > 0.$$

(c) $$\lim_{n\to\infty} GX_n/n = 1 \quad a.e.\ on\ \{X_n \to \infty\}.$$

The same holds for the process $Z_n = X_n + g(X_n)$.

Stopping rules will play an important role in the proof of the theorem. By a stopping rule [adapted to the filtration $(\mathscr{F}_n)$] we mean a random variable τ taking values in $\{0, 1, 2, \dots, \infty\}$ such that the event $\{\tau = k\}$ is $\mathscr{F}_k$-measurable for all k and $P(\tau < \infty) > 0$. τ gives rise to the restarted process $(\xi_{\tau+n})_{n\ge1}$ which is defined on $\{\tau < \infty\}$. Each $\xi_{\tau+n}$ is measurable with respect to $\mathscr{F}_{\tau+n} = \{B \subseteq \{\tau < \infty\}: B \cap \{\tau = k\} \in \mathscr{F}_k \text{ for all } k\}$. The normalized restriction of P to $\{\tau < \infty\}$ will be denoted by P_τ and expectation with respect to P_τ by $E_\tau[\cdot]$. Observe that (3) implies for $n \ge 0$ and all $z \in \mathbb{R}$:

$$\begin{aligned} &P_\tau\left(\xi^2_{\tau+n+1}/\sigma^2(X_{\tau+n}) \ge z \middle| \mathscr{F}_{\tau+n}\right) \\ &\quad = \sum_{k=0}^{\infty} I(\tau = k) P\left(\xi^2_{k+n+1}/\sigma^2(X_{k+n}) \ge z \middle| \mathscr{F}_{k+n}\right) \\ &\quad \le F(z) \quad \text{a.e.} \end{aligned} \tag{23}$$

In particular, $(\xi^2_{\tau+n+1}/\sigma^2(X_{\tau+n}) - 1)_{n\ge0}$ and $(\xi_{\tau+n+1} I\{X_{\tau+n} \le C_n\})_{n\ge0}$ (C_n any sequence of constants) are integrable martingale difference sequences to which martingale convergence theorems can be applied.

We preface the proof of the theorem with some lemmas, the first of which will be used in later sections, too.

LEMMA 1. *Assume* (A.1), (A.2), (A.3), *and let* $f(t) = 1/Gt$, $f(t) = \psi(t)^{-1/2}$, *or* $f(t) = \sigma^{-1}(t)G(t)^{-1/2}$.

(a) $f(t)\sigma^2(t) \to 0$, $t \to \infty$.

(b) *Let* $c, d > 0$. *Then*

$$P_\tau\left(\sup_{n\ge0} f\left(\tfrac{1}{2}GX_{\tau+n}\right)|\xi_{\tau+n+1}| I\{GX_{\tau+n} \ge c(n+M)\} \ge d\right) \to 0$$

as $M \to \infty$ *uniformly for all stopping rules* τ.

(c) $\hat{f}(\frac{1}{2}GX_n)|\xi_{n+1}|I\{GX_n \geq cn\} \to 0$ *a.e.* $(n \to \infty)$ *for each* $c > 0$.

If g is ultimately decreasing, the same is true for $f(t) = |g'(t)|$ *and* $f(t) = g(t)/t$.

PROOF. (a) follows from (11), (14), (A.3) and (18). From (11), (12), (19) and (20) one deduces that in any case $\hat{f}(t)^2\hat{\sigma}^2(t) \leq a/t$ for some $a > 0$ and all $t \geq 1$. Hence

$$\hat{f}\left(\tfrac{1}{2}GX_n\right)^2\hat{\sigma}^2(GX_n) \leq 2\hat{f}\left(\tfrac{1}{2}GX_n\right)\hat{\sigma}^2\left(\tfrac{1}{2}GX_n\right) \leq 4a/GX_n,$$

and it follows that for $c, d > 0$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_\tau\left(GX_{\tau+n} \geq c(n+M) \text{ and } \hat{f}\left(\frac{1}{2}GX_{\tau+n}\right)|\xi_{\tau+n+1}| \geq d\right)\\
&\quad\leq \sum_{n=0}^{\infty} E_\tau\left[I\{GX_{\tau+n} \geq c(n+M)\}P_\tau\left(\xi^2_{\tau+n+1}/\sigma^2(X_{\tau+n}) \geq \frac{d^2}{4a}GX_{\tau+n}\middle|\mathscr{F}_{\tau+n}\right)\right]\\
&\quad\leq \sum_{n=0}^{\infty} E_\tau\left[I\{GX_{\tau+n} \geq c(n+M)\}F\left(\frac{d^2c}{4a}(n+M)\right)\right] \quad \text{[by (23)]}\\
&\quad\leq \sum_{n=0}^{\infty} F\left(\frac{d^2c}{4a}(n+M)\right)\\
&\quad\leq F\left(\frac{d^2c}{4a}M\right) + \int_M^\infty F\left(\frac{d^2c}{4a}t\right)dt\\
&\quad\to 0 \quad \text{as } M \to \infty.
\end{aligned}$$

This proves (b), and with $M = 0$, $\tau \equiv 0$ the Borel–Cantelli lemma yields (c). □

LEMMA 2. *Suppose* $g(t)$ *is increasing and concave for* $t \geq S_0$. *If* $X_n \geq 4S_0$ *and* $X_{n+1} > 0$, *then*

$$GX_{n+1} - GZ_n - \zeta_{n+1} \geq -2\left(\frac{g(X_n)}{X_n}\right)^2(GX_n + 1)\xi^2_{n+1}.$$

PROOF. Define $H\colon (-\infty, 1] \to \mathbb{R}$ by

$$H(t) = GZ_n - G(S_0 + (1-t)(Z_n - S_0)) - t(Z_n - S_0)/g(Z_n).$$

Then $H(0) = 0$,

$$H'(t) = \frac{Z_n - S_0}{g(S_0 + (1-t)(Z_n - S_0))} - \frac{Z_n - S_0}{g(Z_n)}, \qquad H'(0) = 0,$$

and

$$H''(t) = (Z_n - S_0)^2\frac{g'(S_0 + (1-t)(Z_n - S_0))}{g^2(S_0 + (1-t)(Z_n - S_0))}$$

is positive and monotonically increasing such that $H'(t)$ is convex, and therefore $2t^{-1}H(t) = 2t^{-1}\int_0^t H'(u)\,du \le H'(t)$. If we set $I(t) = t^{-2}H(t)$, this implies $I'(t) \ge 0$.

If $X_{n+1} \ge S_0$ and therefore

$$t_n = -\frac{g(X_n)}{Z_n - S_0}\xi_{n+1} = \frac{Z_n - X_{n+1}}{Z_n - S_0} \le 1,$$

one has

$$\begin{aligned} t_n^2 I(t_n) = H(t_n) &= GZ_n - GX_{n+1} - \frac{Z_n - X_{n+1}}{g(Z_n)} \\ &= GZ_n - GX_{n+1} + \zeta_{n+1}. \end{aligned} \tag{24}$$

On the other hand, if $X_n \ge 4S_0$,

$$\begin{aligned} t_n^2 I(1) = t_n^2 H(1) &\le \left(\frac{g(X_n)}{X_n}\right)^2 \xi_{n+1}^2 \left(\frac{X_n}{Z_n - S_0}\right)^2 GZ_n \\ &\le 2\left(\frac{g(X_n)}{X_n}\right)^2 (GX_n + 1)\xi_{n+1}^2. \end{aligned} \tag{25}$$

From $I'(t) \ge 0$ and $t_n \le 1$ we have $I(t_n) \le I(1)$. Together with (24) and (25) this proves the lemma if $X_{n+1} \ge S_0$. If $X_{n+1} \le S_0$ and $X_n \ge 4S_0$ we finally have

$$2\left(\frac{g(X_n)}{X_n}\right)^2 (GX_n + 1)\xi_{n+1}^2 = 2\left(\frac{X_{n+1} - Z_n}{X_n}\right)^2 (GX_n + 1) \ge \frac{9}{8} GZ_n \ge GZ_n$$

and since in this case $\xi_{n+1} \le 0$, the lemma is true again. □

Before we formulate the next lemma, which constitutes the main step towards the proof of Theorem 1, we give an expansion of GX_{n+1} in terms of GX_n and GZ_n: If $X_{n+1} > 0$, then

$$\begin{aligned} GX_{n+1} &= GZ_n + \frac{g(X_n)}{g(Z_n)}\xi_{n+1} + R_n \\ &= GX_n + 1 + \zeta_{n+1} + r(X_n) + R_n. \end{aligned} \tag{26}$$

[The function r has been defined in (10), while (26) should be considered to be the defining relation for R_n.] If $g(0) > 0$ the same expansion works for $X_{n+1} = 0$.

A Taylor-expansion yields

$$R_n = -\frac{1}{2}\frac{g'(\bar{Z}_n)}{g^2(\bar{Z}_n)} g^2(X_n)\xi_{n+1}^2, \tag{27}$$

with $\bar{Z}_n$ between Z_n ($\ge X_n$) and X_{n+1}.

$$r(X_n) = -\frac{1}{2}\frac{g'(\bar{X}_n)}{g^2(\bar{X}_n)} g^2(X_n), \tag{28}$$

with $\overline{X}_n$ between X_n and Z_n. If $X_{n+1} < 0$ or if $X_{n+1} = 0$ and $g(0) = 0$ (i.e., $X_{n+1} = 0$ is absorbing), we set $R_{n+1} = 0$.

LEMMA 3. *Assume* (A.1), (A.2), (A.3), *and* (A.5) *and let* $c, d > 0$. *Then*

$$p_M = P_\tau\left(\sup_{n\geq 1}(n+M)^{-1}\sum_{k=0}^{n-1}\left(|R_{\tau+k}| + |r(X_{\tau+k})|\right)I\{GX_{\tau+k}\geq c(k+M)\}\geq d\right)$$
$$\to 0$$

as $M \to \infty$, *uniformly for all stopping rules* τ.

PROOF. We consider first the case of ultimately decreasing g. Let

$$B(M,\varepsilon) = \left\{\sup_{n\geq 0} f\left(\tfrac{1}{2}GX_{\tau+n}\right)|\xi_{\tau+n+1}|I\{GX_{\tau+n}\geq c(n+M)\} < \varepsilon\right\},$$

with

$$f(t) = \max\{g(t)/t, 1/Gt, |g'(t)|\}$$

and $\varepsilon \leq \frac{1}{2}$. For all n with $GX_{\tau+n} \geq c(n+M)$, on $B(M,\varepsilon)$

$$\left|\frac{X_{\tau+n+1} - Z_{\tau+n}}{X_{\tau+n}}\right| \leq \frac{g(X_{\tau+n})}{X_{\tau+n}}|\xi_{\tau+n+1}| < \varepsilon \leq \frac{1}{2},$$

so that $X_{\tau+n+1} \geq \frac{1}{2}X_{\tau+n} \geq \frac{1}{2}G^{-1}(c(n+M))$, and the ultimate convexity of Gt implies that for sufficiently large M

$$GX_{\tau+n+1} \geq GX_{\tau+n} - |\xi_{\tau+n+1}| = GX_{\tau+n}(1 - |\xi_{\tau+n+1}|/GX_{\tau+n})$$
$$\geq \tfrac{1}{2}GX_{\tau+n}.$$

Hence

$$\left|\frac{g(X_{\tau+n+1})}{g(X_{\tau+n})} - 1\right| \leq \left|\widehat{g'}\left(\frac{1}{2}GX_{\tau+n}\right)\right| |1 + \xi_{\tau+n+1}|$$
$$< \left|\widehat{g'}\left(\frac{c}{2}(n+M)\right)\right| + \varepsilon$$
$$< 2\varepsilon, \quad \text{for large } M,$$

and similarly

$$\left|\frac{g(Z_{\tau+n})}{g(X_{\tau+n})} - 1\right| < \varepsilon, \quad \text{for large } M \ [\text{cf. (22)}].$$

As $\overline{Z}_n$ lies between Z_n and X_{n+1}, we thus get that for large M and those n with $GX_{\tau+n} \geq c(n+M)$ on $B(M,\varepsilon)$

$$\frac{g(X_{\tau+n})}{g(\overline{Z}_{\tau+n})} \leq (1-2\varepsilon)^{-1}.$$

Set $I(k,M) = I\{GX_{\tau+k} \geq c(k+M)\}$. Fix $\varepsilon > 0$ such that $(1-2\varepsilon)^{-2} \leq 2$. Then we have for large M, say $M \geq M_0$,

$$
\begin{aligned}
&P_\tau\left(\sup_{n\geq 1}(n+M)^{-1}\sum_{k=0}^{n-1}|R_{\tau+k}|I(k,M)\geq \frac{d}{2}\right)\\
(29)\qquad &\leq (1-P_\tau(B(M,\varepsilon)))\\
&\quad +P_\tau\left(\sup_{n\geq 1}(n+M)^{-1}\sum_{k=0}^{n-1}\left|\widehat{g'}\left(\frac{1}{2}GX_{\tau+k}\right)\right|\xi^2_{\tau+k+1}I(k,M_0)\geq \frac{d}{2}\right).
\end{aligned}
$$

By Lemma 1 the first term tends to zero as $M\to\infty$ uniformly in τ. The second one can be estimated by [use (15)]

$$
\begin{aligned}
\tilde{p}_M := P_\tau\Bigg(\sup_{n\geq 1}(n+M)^{-1}\sum_{k=0}^{n-1}&\left|\widehat{g'}\left(\frac{1}{2}GX_{\tau+k}\right)\right|\hat{\sigma}^2\left(\frac{1}{2}GX_{\tau+k}\right)\\
&\times\left(\frac{\xi^2_{\tau+k+1}}{\sigma^2(X_{\tau+k})}-1\right)I(k,M_0)\geq \frac{d}{8}\Bigg)
\end{aligned}
$$

provided M_0 has been chosen so large that $|\widehat{g'}(\frac{1}{2}cM_0)\hat{\sigma}^2(\frac{1}{2}cM_0)| < \frac{1}{8}d$ [cf. (A.3)]. As

$$
\begin{aligned}
&P_\tau\left(\left|\widehat{g'}\left(\frac{1}{2}GX_{\tau+k}\right)\right|\hat{\sigma}^2\left(\frac{1}{2}GX_{\tau+k}\right)I(k,M_0)\frac{\xi^2_{\tau+k+1}}{\sigma^2(X_{\tau+k})}\geq z\,\middle|\,\mathscr{F}_{\tau+k}\right)\\
&\quad\leq P_\tau\left(\left|\widehat{g'}\left(\frac{c}{2}M_0\right)\right|\hat{\sigma}^2\left(\frac{c}{2}M_0\right)\frac{\xi^2_{\tau+k+1}}{\sigma^2(X_{\tau+k})}\geq z\,\middle|\,\mathscr{F}_{\tau+k}\right)\\
&\quad\leq F(z)\quad\text{a.e.,}
\end{aligned}
$$

if $|\widehat{g'}(\frac{1}{2}cM_0)|\hat{\sigma}^2(\frac{1}{2}cM_0)\leq 1$, Theorem A.2(b) of the Appendix shows that $\tilde{p}_M\to 0$ as $M\to\infty$ uniformly in τ. Hence the expression in (29) tends to 0 as $M\to\infty$ uniformly in τ. The proof of a corresponding assertion for the $r(X_{\tau+n})$-terms is similar but simpler, and we leave it to the reader.

We now turn to the case of ultimately increasing g and start by proving that

$$
(30)\qquad P_\tau\left(\sup_{n\geq 1}(n+M)^{-1}\sum_{k=0}^{n-1}|R_{\tau+k}|I(k,M_0)\geq \frac{d}{2}\right)\to 0,
$$

as $M\to\infty$ uniformly in τ, if M_0 is big enough. Assume that (A.5a) holds, i.e., $\xi_{\tau+k+1}/\sigma(X_{\tau+k})\geq -C$ a.e. Fix $k\geq 0$. If $\xi_{\tau+k+1}\leq 0$, i.e., $X_{\tau+k+1}\leq Z_{\tau+k}$, we have for large M_0:

$$
|R_{\tau+k}|I(k,M_0)\leq \frac{1}{2}g'(X_{\tau+k+1})\xi^2_{\tau+k+1}\left(\frac{g(X_{\tau+k})}{g(X_{\tau+k+1})}\right)^2.
$$

As $g(t)\sigma(t)/t\to 0$ $(t\to\infty)$ by (A.1) for bounded $\sigma(t)$ or by (18) for unbounded

$\sigma(t)$, we also have if $X_{\tau+k}$ is large enough

$$X_{\tau+k+1} \geq X_{\tau+k}\left(1 - \frac{g(X_{\tau+k})}{X_{\tau+k}} C\sigma(X_{\tau+k})\right) \geq \frac{1}{2} X_{\tau+k}.$$

By the concavity of g one deduces that for large M_0

$$|R_{\tau+k}| I(k, M_0) I\{\xi_{\tau+k+1} \leq 0\} \leq 2g'(\tfrac{1}{2}X_{\tau+k}) C^2 \sigma^2(X_{\tau+k}).$$

If $\xi_{\tau+k+1} \geq 0$, i.e., $X_{\tau+k+1} \geq Z_{\tau+k} \geq X_{\tau+k}$, the ultimate monotonicity of g implies that for large M_0

$$|R_{\tau+k}| I(k, M_0) I\{\xi_{\tau+k+1} \geq 0\} \leq \tfrac{1}{2} g'(X_{\tau+k}) \xi^2_{\tau+k+1}.$$

Putting both estimates together we obtain

$$|R_{\tau+k}| I(k, M_0) \leq \delta(M_0) + \frac{1}{2} g'(X_{\tau+k}) \sigma^2(X_{\tau+k}) \left(\frac{\xi^2_{\tau+k+1}}{\sigma^2(X_{\tau+k})} - 1 \right) I(k, M_0),$$

where

$$\delta(M_0) = \sup_{t \geq G^{-1}(cM_0)} \left\{ \left(2C^2 + \tfrac{1}{2}\right) g'\left(\frac{t}{2}\right) \sigma^2(t) \right\} < \frac{d}{4},$$

for M_0 large enough. [If σ^2 is bounded this follows from (A.1); if σ^2 is unbounded it is a consequence of (A.3) and (15).] Now (30) follows if one can show that

$$P_\tau\left(\sup_{n \geq 1} (n + M)^{-1} \sum_{k=0}^{n-1} \frac{1}{2} g'(X_{\tau+k}) \sigma^2(X_{\tau+k}) \left(\frac{\xi^2_{\tau+k+1}}{\sigma^2(X_{\tau+k})} - 1 \right) I(k, N_0) \geq \frac{d}{4} \right)$$

tends to zero as $M \to \infty$ uniformly in τ. But just as in the case of decreasing g this follows from Theorem A.2(b) of the Appendix provided M_0 is large enough.

If instead of (A.5a) we assume (A.5b), we can deduce from Lemma 2 that for large M_0

$$|R_{\tau+k}| I(k, M_0) \leq 2 \left(\frac{g(X_{\tau+k})}{X_{\tau+k}} \right)^2 (GX_{\tau+k} + 1) \xi^2_{\tau+k+1} I(k, M_0).$$

Observing that (A.5b), (A.1), and (18) imply

$$\left(\frac{g(t)}{t} \right)^2 (Gt + 1) \sigma^2(t) \to 0, \qquad t \to \infty,$$

the proof of (30) is finished as above by an application of Theorem A.2(b).

The assertion for the $r(X_{\tau+k})$-terms which corresponds to (30) is proved again in a similar but simpler way using the fact that $g'(t) \to 0$, $t \to \infty$, and from these two estimates the lemma follows at once. □

COROLLARY. *Under the assumptions of Lemma* 3

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(|R_k| + |r(X_k)| \right) I\{GX_k \geq ck\} = 0 \quad a.e.$$

PROOF. Apply the lemma with $\tau \equiv M$. □

LEMMA 4. *Assume* (A.1), (A.2), (A.3), *and* (A.5) *and let* $\varepsilon > 0$. *Then*

$$P_\tau\big(GX_{\tau+n} \geq (1+\varepsilon)n, \textit{ for all but finitely many } n\big) = 0,$$

for all stopping rules τ.

PROOF. Let $A = \{GX_{\tau+n} \geq (1+\varepsilon)n \text{ for all but finitely many } n\} \cap \{\tau < \infty\}$. By the corollary to the last lemma

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big(|R_{\tau+k}| + |r(X_{\tau+k})|\big) = 0 \quad \text{a.e. on } A,$$

and it follows from (26) that

$$\begin{aligned} GX_{\tau+n} &= n + \sum_{k=0}^{n-1} \zeta_{\tau+k} + o(n) \quad \text{a.e. on } A, \text{ in particular,} \\ GX_n &\leq 2n + \sum_{k=1}^{n} \zeta_k, \quad \text{for sufficiently big } n \text{ a.e. on } A. \end{aligned} \tag{31}$$

So it remains to show that $\sum_{k=0}^{n-1}\zeta_{\tau+k} = o(n)$ a.e. on A or, equivalently, $\sum_{k=1}^{n-1}\zeta_k = o(n)$ a.e. on A. By the martingale convergence theorem [2.18 in Hall and Heyde (1980)] it suffices to prove that

$$\sum_{k=1}^{\infty} k^{-2}\operatorname{var}(\zeta_{k+1}|\mathscr{F}_k) = \sum_{k=1}^{\infty} k^{-2}\left(\frac{g(X_k)}{g(Z_k)}\right)^2 \sigma^2(X_k) < \infty \quad \text{a.e. on } A.$$

This is trivial if σ^2 is bounded. [Observe that $g(X_k)/g(Z_k)$ is bounded by (22).] Otherwise $\hat{\sigma}^2$ is concave, such that

$$\sum_{k=1}^{\infty} k^{-2} E\left[\hat{\sigma}^2\left(2k + \sum_{j=1}^{k} \zeta_k\right)\right] \leq \sum_{k=1}^{\infty} k^{-2}\hat{\sigma}^2(2k) < \infty \quad \text{by (A.2)},$$

and hence

$$\sum_{k=1}^{\infty} k^{-2}\operatorname{var}(\zeta_{k+1}|\mathscr{F}_k) = O\left(1 + \sum_{k=1}^{\infty} k^{-2}\hat{\sigma}^2\left(2k + \sum_{j=1}^{k}\zeta_k\right)\right) < \infty \quad \text{a.e.}$$

on A by (31). □

LEMMA 5. *Assume* (A.1) *and* (A.2) *and let* $c, d > 0$. *Then*

$$q_M := P_\tau\left(\sup_{n\geq 1}(n+M)^{-1}\left|\sum_{k=0}^{n-1}\zeta_{\tau+k+1} I\{GX_{\tau+k} \leq c(k+M)\}\right| \geq d\right) \to 0,$$

as $M \to \infty$ *uniformly for all stopping rules* τ.

PROOF. As $\sum_{k=0}^{n-1}\zeta_{\tau+k+1}I\{GX_{\tau+k} \le c(k+M)\}$ is a square-integrable martingale, it follows from the Hájek–Rényi inequality that

$$\begin{aligned}
q_M &\le d^{-2} \sum_{n=0}^{\infty} (n+1+M)^{-2} E_\tau\left[\zeta_{\tau+n+1}^2 I\{GX_{\tau+n} \le c(n+M)\}\right] \\
&\le d^{-2} \sum_{n=0}^{\infty} (n+1+M)^{-2} E_\tau\left[\left(\frac{g(X_{\tau+n})}{g(Z_{\tau+n})}\right)^2 \hat{\sigma}^2(GX_{\tau+n}) \right. \\
&\qquad\qquad \left. \times I\{GX_{\tau+n} \le c(n+M)\}\right] \\
&\le d^{-2} \operatorname{const} \sum_{n=0}^{\infty} (n+M)^{-2} \hat{\sigma}^2(c(n+M)) \\
&\le d^{-2} c \int_{cM}^{\infty} \frac{\hat{\sigma}^2(t)}{t^2}\, dt \\
&\to 0, \quad \text{as } M \to \infty \text{ uniformly for all } \tau \text{ by (A.2).}
\end{aligned}$$

For the third inequality we have made use of (22). □

Next we investigate some properties of the particular stopping rules

$$\tau = \tau_S = \begin{cases} \min\{k\colon X_k \ge S\}, & \text{if such a } k \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

By $\tau^k = \tau_S^k$ we denote the stopping rule $\inf\{\tau_S, k\}$. The fact that τ is a stopping rule [i.e., that $P(\tau < \infty) > 0$] is proved in

LEMMA 6. *Assume* (A.1). *Then* $P(\tau_S < \infty) > 0$ *for all* $S > 0$ *and* $\lim_{n\to\infty} X_n \le 0$ *exists on* $\{\tau_S = \infty\}$.

PROOF. Fix $S > 0$ and set $Y_n = X_{\tau^n}$. Then $Y_0 = X_0$ and $Y_{n+1} = Y_n + g_S(Y_n)(1 + \xi_{n+1})$, where $g_S(t) = g(t)I\{t < S\}$. Set $A_n = Y_0 + \sum_{k=0}^{n-1} g_S(Y_k)$ and $M_n = Y_n - A_n = \sum_{k=0}^{n-1} g_S(Y_k)\xi_{k+1}$. (M_n) is a square-integrable martingale and $Y_n = A_n + M_n$ is the Doob-decomposition of Y_n. As $g_S(t) = 0$ for $t \ge S$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n^{-2} E\left[(M_{n+1} - M_n)^2 \middle| \mathscr{F}_n\right] &= \sum_{n=0}^{\infty} A_n^{-2} g_S(Y_n)^2 \sigma^2(X_n) \\
&\le \operatorname{const} \sum_{n=0}^{\infty} A_n^{-2}(A_{n+1} - A_n) \\
&\le \operatorname{const} \int_{X_0}^{\infty} t^{-2}\, dt < \infty \quad \text{a.e.,}
\end{aligned}$$

and from two versions of the martingale convergence theorem [e.g., 2.17 and 2.18

in Hall and Heyde (1980)] one can conclude that

$$M_n/A_n \to 0 \quad \text{a.e. on } \{A_n \to \infty\} \text{ and}$$
$$M_n \text{ converges a.e. on } \{\sup A_n < \infty\}.$$

As $Y_n = A_n + M_n$ is pathwise bounded, this implies that A_n and Y_n converge a.e. In particular, $g_S(Y_n) \to 0$ a.e., and on $\{\tau = \infty\}$ $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n \le 0$ a.e. Now suppose for a contradiction that $P\{\tau < \infty\} = 0$, i.e., $\lim_{n\to\infty} Y_n \le 0$ a.e. The contradiction follows from Fatou's lemma applied to $-Y_n \ge -S$:

$$0 \le E\Big[\lim_{n\to\infty}(-Y_n)\Big] \le \liminf_{n\to\infty} E[-Y_n] = \liminf_{n\to\infty} E[-A_n]$$
$$\le E[-Y_0] = -EX_0 < 0. \qquad \square$$

PROOF OF THEOREM 1. Let $S > 0$, $\tau = \tau_S$, $\varepsilon > 0$, define a process V_n on $\{\tau < \infty\}$ by

$$V_0 = GX_\tau,$$
$$V_{n+1} = V_n + 1 + \zeta_{\tau+n+1} I\{GX_{\tau+n} \le (1+3\varepsilon)(n+GS)\}$$
$$+ (R_{\tau+n} + r(X_{\tau+n})) I\{GX_{\tau+n} \ge (1-3\varepsilon)(n+GS)\},$$

and let

$$B_S = \Big\{\sup_{n\ge 0}(n+GS)^{-1}|V_n - V_0 - n| \ge \varepsilon\Big\}.$$

Next let

$$\nu = \nu_S = \begin{cases} \min\{k \ge 0: (1-\varepsilon)(k+GS) \le GX_{\tau+k} \le (1+\varepsilon)(k+GS)\}, \\ \qquad \text{if such a } k \text{ exists,} \\ \infty, \quad \text{otherwise.} \end{cases}$$

By induction on n we show that on $\{\nu < \infty\} \setminus B_S$

$$GX_{\tau+\nu+n} = V_{\nu+n} + GX_{\tau+\nu} - V_\nu \tag{32}$$

and

$$(1-3\varepsilon)(\nu+n+GS) \le GX_{\tau+\nu+n} \le (1+3\varepsilon)(\nu+n+GS), \tag{33}$$

for all $n \ge 0$.

For $n = 0$ assertion (32) is trivial, while (33) is true by definition of ν. Thus suppose that (32) and (33) hold for $n = 0, \dots, N$. Then

$$GX_{\tau+\nu+N+1} = GX_{\tau+\nu+N} + 1 + \zeta_{\tau+\nu+N+1} + R_{\tau+\nu+N} + r(X_{\tau+\nu+N})$$
$$= V_{\nu+N} + 1 + \zeta_{\tau+\nu+N+1} I\{GX_{\tau+\nu+N} \le (1+3\varepsilon)(\nu+N+GS)\}$$
$$+ (R_{\tau+\nu+N} + r(X_{\tau+\nu+N})) I\{GX_{\tau+\nu+N} \ge (1-3\varepsilon)(\nu+N+GS)\}$$
$$+ GX_{\tau+\nu} - V_\nu$$
$$= V_{\nu+N+1} + GX_{\tau+\nu} - V_\nu,$$

i.e., (32) for $n = N+1$. For the second equality (32) and (33) for $n = N$ have been used.

Now

$$\begin{aligned} GX_{\tau+\nu+N+1} &= (V_{\nu+N+1} - V_\nu - (N+1)) + N + 1 + GX_{\tau+\nu} \\ &= (V_{\nu+N+1} - V_0 - (\nu+N+1)) - (V_\nu - V_0 - \nu) + N + 1 + GX_{\tau+\nu} \\ &\le \varepsilon(\nu + N + 1 + GS) + \varepsilon(\nu + GS) + N + 1 + (1+\varepsilon)(\nu + GS) \\ &\le (1+3\varepsilon)(\nu + N + 1 + GS), \end{aligned}$$

by definition of B_S and ν. Similarly

$$\begin{aligned} GX_{\tau+\nu+N+1} &\ge -\varepsilon(\nu + N + 1 + GS) - \varepsilon(\nu + GS) + N + 1 + (1-\varepsilon)(\nu + GS), \\ &\ge (1-3\varepsilon)(\nu + N + 1 + GS), \end{aligned}$$

which proves (33) for $n = N + 1$.

From (32) and the definition of B_S we conclude that on $\{\nu_S < \infty\} \setminus B_S$

$$(1-\varepsilon) \le \liminf_{n\to\infty} GX_n/n \le \limsup_{n\to\infty} GX_n/n \le 1 + \varepsilon. \tag{34}$$

In particular,

$$\{\nu_S < \infty\} \setminus B_S \subseteq \bigcap_{M>0} \{\tau_M < \infty\}.$$

Now suppose for the moment that

$$P_{\tau_S}(\{\nu_S < \infty\} \setminus B_S) \to 1 \quad \text{as } S \to \infty. \tag{35}$$

Then $P(\bigcap_{M>0}\{\tau_M < \infty\}) > 0$, and we have:

$$(34) \text{ holds a.e. on } \bigcap_{M>0} \{\tau_M < \infty\},$$

$$\lim_{n\to\infty} X_n \le 0 \quad \text{a.e. on} \left[\bigcap_{M>0} \{\tau_M < \infty\}\right]^c = \bigcup_{M>0} \{\tau_M = \infty\}, \quad \text{by Lemma 6.}$$

As $\varepsilon > 0$ was arbitrary, this will prove the theorem after (35) is established.

Applying Lemma 3 with $c = 1 - 2\varepsilon$, $d = \varepsilon/2$, and Lemma 5 with $c = 1 + 3\varepsilon$, $d = \varepsilon/2$ we get

$$P_{\tau_S}(B_S) \le p_S + q_S \to 0 \quad \text{as } S \to \infty,$$

and we only have to show that $P_{\tau_S}(\nu_S = \infty) \to 0$ as $S \to \infty$. To this end let

$$\mu = \mu_S = \begin{cases} \min\{k \ge 1\colon GX_{\tau+k} \le (1+\varepsilon)(k + GS)\}, & \text{if such a } k \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

By Lemma 4, $P_{\tau_S}(\mu_S = \infty) = 0$ for all S, and it suffices to show that

$$P_{\tau_S}(\{\nu_S = \infty\} \cap \{\mu_S < \infty\}) \to 0 \quad \text{as } S \to \infty.$$

But on $\{\tau_S < \infty\} \cap \{\mu_S < \infty\} \cap \{\nu_S = \infty\}$

$$\begin{aligned} GX_{\tau+\mu-1} &> (1+\varepsilon)(\mu - 1 + GS), \\ GX_{\tau+\mu} &< (1-\varepsilon)(1 + GS), \end{aligned} \tag{36}$$

i.e.,

$$GX_{\tau+\mu-1} > (1-\varepsilon)^{-1} GX_{\tau+\mu},$$

if S is so large that $(1+\varepsilon)(GS-1) \geq GS$. Hence

$$\begin{aligned}\varepsilon GX_{\tau+\mu-1} &< -(GX_{\tau+\mu} - GX_{\tau+\mu-1}) \\ &= -(1 + \zeta_{\tau+\mu} + R_{\tau+\mu-1} + r(X_{\tau+\mu-1})).\end{aligned}$$

Therefore

$$\begin{aligned}&P_{\tau_S}(\{\nu_S = \infty\} \cap \{\mu_S < \infty\}) \\ &\quad \leq P_\tau\left(\left\{|1 + \zeta_{\tau+\mu}|/GX_{\tau+\mu-1} > \frac{\varepsilon}{2}\right\} \cap \{\mu < \infty\}\right) \\ &\quad + P_\tau\left(\left\{\left(|R_{\tau+\mu-1}| + |r(X_{\tau+\mu-1})|\right)/GX_{\tau+\mu-1} > \frac{\varepsilon}{2}\right\} \cap \{\mu < \infty\}\right) \\ &\quad \to 0 \quad \text{as } S \to \infty.\end{aligned}$$

[The first term in the sum tends to 0 because of (36), (22), and Lemma 1. The convergence of the second term follows (36) and Lemma 3.] □

4. An asymptotic expansion for the process GX_n. The main result of this section is

THEOREM 2. *Suppose* (A.1)–(A.5) *hold.*

(a) *If $\hat{\psi}(\infty) < \infty$, then $GX_n - Ga_n$ converges a.s. on $\{X_n \to \infty\}$.*
(b) *If $\hat{\psi}(\infty) = \infty$, then*

$$GX_n = Ga_n + \sum_{k=1}^{n} \xi_k + \hat{\phi}(n)(1 + o(1)) + o(\hat{\psi}^{1/2}(n)) \quad a.s. \text{ on } \{X_n \to \infty\}.$$

The asymptotic behaviour of the martingale $\sum_{k=1}^{n} \xi_k$ occurring in part (b) of the theorem is controlled by

PROPOSITION 1. *Suppose* (A.1)–(A.5). *If $\hat{\psi}(\infty) = \infty$, then the conditional distribution of $\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k$ on $\{X_n \to \infty\}$ converges weakly to a standard normal distribution $\mathcal{N}(0,1)$.*

PROPOSITION 2. *Assume* (A.1)–(A.5) *and additionally that $E[|\xi_{n+1}|^{2+\delta} \mathscr{F}_n] \leq C\sigma^{2+\delta}(X_n)$ for some $0 < \delta \leq 2$, $C > 0$. If $\hat{\psi}(\infty) = \infty$, then one can redefine the process ξ_n on a richer probability space together with a standard Brownian motion $B(t)$ in such a way that*

$$\sum_{k=1}^{n} \xi_k = B(\hat{\psi}(n) + o(\hat{\psi}(n))) \quad a.e. \text{ on } \{X_n \to \infty\}.$$

Set $S_n = \Sigma_{k=0}^{n-1} R_k$, and denote by D the event $\{X_n \to \infty\}$. We start the proof of the above result with a lemma:

LEMMA 7. *Suppose* (A.1), (A.2), (A.3), *and* (A.5) *hold.*

(a) *If* $|\hat{\phi}(\infty)| < \infty$, *then* S_n *converges a.s. on* D
(b) *If* $|\hat{\phi}(\infty)| = \infty$, *then* $S_n = \hat{\phi}(n)(1 + o(1))$ *a.s. on* D.

PROOF. In (26) R_k has been defined as $R_k = GX_{k+1} - GZ_k - (g(X_k)/g(Z_k))\xi_{k+1}$. A Taylor-expansion of GX_{k+1} at Z_k yields [cf. (27)]

$$R_k = -\frac{1}{2}\frac{g'(\bar{Z}_k)}{g^2(\bar{Z}_k)} g^2(Z_k)\xi_{k+1}^2,$$

with $\bar{Z}_k$ between Z_k and $X_{k+1} = Z_k + g(X_k)\xi_{k+1}$.

Fix $c < 1$. By Theorem 1 for k large enough

$$\left|\widehat{g'}\left(\frac{k}{c}\right)\right|\xi_{k+1}^2 \le |g'(\bar{Z}_k)|\xi_{k+1}^2 \le |\widehat{g'}(ck)|\xi_{k+1}^2 \tag{37}$$

holds a.s. on D.

For $L \ge 0$ set $I_{k,L} = I\{\xi_{k+1}^2 \ge L\sigma^2(X_k)\}$. From (15) and Theorem 1 one deduces that a.s. on D

$$\sum_{k=0}^{n-1} |\widehat{g'}(ck)|\xi_{k+1}^2 I_{k,L} \le c^{-2} \sum_{k=0}^{n-1} |\widehat{g'}(ck)|\hat{\sigma}^2(ck)\frac{\xi_{k+1}^2}{\sigma^2(X_k)} I_{k,L} + O(1). \tag{38}$$

If $|\hat{\phi}(\infty)| = |\frac{1}{2}\int_0^\infty \widehat{g'}(s)\hat{\sigma}^2(s)\,ds| < \infty$, then the regularity assumptions on g and σ^2 imply $\Sigma_{k=0}^\infty |\widehat{g'}(ck)|\hat{\sigma}^2(ck) < \infty$. Thus the right-hand side of (38) converges a.e. on D, and together with (37) one concludes ($L = 0$)

$$\sum_{k=0}^{\infty} |g'(\bar{Z}_k)|\xi_{k+1}^2 < \infty \quad \text{a.e. on } D. \tag{39}$$

If $|\hat{\phi}(\infty)| = \infty$, the regularity assumptions on g and σ^2 imply that

$$\sum_{k=0}^{n-1} |\widehat{g'}(ck)|\hat{\sigma}^2(ck) \sim 2c^{-1}|\hat{\phi}(cn)| \to \infty \quad \text{as } n \to \infty,$$

and from (38), (3), and Theorem A.2(a) of the Appendix one deduces that a.s. on D

$$\sum_{k=0}^{n-1} |\widehat{g'}(ck)|\xi_{k+1}^2 I_{k,L} \le 2c^{-3}\hat{\phi}(cn)\left(\sup_k E\left[\frac{\xi_{k+1}^2}{\sigma^2(X_k)} I_{k,L}\middle|\mathscr{F}_k\right] + o(1)\right). \tag{40}$$

Taking $L = 0$ this and (37) imply that a.s. on D

$$\sum_{k=0}^{n-1} |g'(\bar{Z}_k)|\xi_{k+1}^2 \leq 2c^{-3}\hat{\phi}(cn)(1 + o(1)) \leq 2c^{-3}\hat{\phi}(n)(1 + o(1)),$$

and as $c < 1$ was arbitrary,

$$\sum_{k=0}^{n-1} |g'(\bar{Z}_k)|\xi_{k+1}^2 \leq 2|\hat{\phi}(n)|(1 + o(1)) \quad \text{a.s. on } D. \tag{41}$$

Similarly one shows that

$$\sum_{k=0}^{n-1} |g'(\bar{Z}_k)|\xi_{k+1}^2 \geq 2|\hat{\phi}(n)|(1 + o(1)) \quad \text{a.s. on } D. \tag{42}$$

We still have to show that one may introduce the factors $(g(Z_k)/g(\bar{Z}_k))^2$ into the sums (39), (41), and (42) without changing their asymptotic behaviour. It follows from Theorem 1 and the fact that $\bar{Z}_k$ is between Z_k and $Z_k + g(X_k)\xi_{k+1}$ that for large k

$$\begin{aligned}\left|\frac{g(\bar{Z}_k)}{g(Z_k)} - 1\right| &\leq |\widehat{g'}(ck)|\,|\xi_{k+1}|\frac{g(X_k)}{g(Z_k)} \\ &\leq \text{const}|\widehat{g'}(ck)|\,|\xi_{k+1}|\end{aligned} \tag{43}$$

a.s. on D (observe again that $g(X_k)/g(Z_k)$ is bounded).

If $|\hat{\phi}(\infty)| < \infty$, then $|\widehat{g'}(ck)|\xi_{k+1}^2 \to 0$, $k \to \infty$, a.e. on D by (39), and as $|g'(t)|$is bounded, it follows that the expression in (43) tends to 0 a.e. on D. Hence we are done.

We turn to the case $|\hat{\phi}(\infty)| = \infty$. If g ultimately decreases or if g ultimately increases and (A.5b) holds, whence $|g'(t)| \leq t^{-1}g(t) = o(\sigma^{-1}(t)G(t)^{-1/2})$, the expression in (43) tends to 0 a.e. on D by Lemma 1. Otherwise g ultimately increases and (A.5a) holds, i.e., $\xi_{k+1}/\sigma(X_k) \geq -C$ a.e. We combine (41) and (42) and reformulate it:

$$\begin{aligned} S_n = \hat{\phi}(n)(1 + o(1)) &- \frac{1}{2}\sum_{k=0}^{n-1} g'(\bar{Z}_k)\xi_{k+1}^2\left(\frac{g^2(Z_k)}{g^2(\bar{Z}_k)} - 1\right)I_{k,L} \\ &- \frac{1}{2}\sum_{k=0}^{n-1} g'(\bar{Z}_k)\xi_{k+1}^2\left(\frac{g^2(Z_k)}{g^2(\bar{Z}_k)} - 1\right)(1 - I_{k,L}).\end{aligned} \tag{44}$$

From (43), (15), and Theorem 1 we conclude that a.s. on D for large k

$$\left|\frac{g(\bar{Z}_k)}{g(Z_k)} - 1\right|(1 - I_{k,L}) \leq \text{const}|\widehat{g'}(ck)|L^{1/2}c^{-2}\hat{\sigma}(ck),$$

and this tends to 0, because $g'(t)\sigma(t) \to 0$, $t \to \infty$, by (A.1) and (A.3). Hence the second sum in (44) is a $o(\hat{\phi}(n))$ by (41). To the first sum in (44) only those k contribute for which $\xi_{k+1}^2 \geq L\sigma^2(X_k)$. If $L > \sqrt{C}$, this is possible only if $\xi_{k+1} \geq 0$, i.e., $X_k \le Z_k \le \bar{Z}_k \le X_{k+1}$. Therefore, if a k of this type is big enough, we have $|g'(\bar{Z}_k)| \le |g'(X_k)|$ and $g(Z_k) \le g(\bar{Z}_k)$, and the first sum in (44) is bounded in absolute value a.s. on D by

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1} |g'(X_k)|\xi_{k+1}^2 I_{k,L} + O(1) \\
&\quad \le c^{-3}\hat{\phi}(n)\left(\sup_k E\left[\frac{\xi_{k+1}^2}{\sigma^2(X_k)} I_{k,L}\middle|\mathscr{F}_k\right] + o(1)\right) + O(1) \quad [\text{by } (40)] \\
&\quad \le c^{-3}\hat{\phi}(n)\left(\int_L^\infty F(z)\,dz + o(1)\right) \quad [\text{by } (3)].
\end{aligned}$$

Since this is true for all $L > \sqrt{C}$, one can conclude from (44) that

$$S_n = \hat{\phi}(n)(1 + o(1)). \qquad \square$$

Proof of Theorem 2. We have defined $r(t) = G(t + g(t)) - Gt - 1$ [see (10)]. Hence

$$\begin{aligned}
r'(t) &= \frac{1 + g'(t)}{g(t + g(t))} - \frac{1}{g(t)} \\
&= \frac{1 + g'(t)}{g(t) + g(t)g'(t + \theta g(t))} - \frac{1}{g(t)},
\end{aligned}$$

for some $0 \le \theta \le 1$, and because of (A.1) for large t

$$|r'(t)| = \frac{1}{g(t)}\left|\frac{g'(t) - g'(t + \theta g(t))}{1 + g'(t + \theta g(t))}\right| \le 2|g''(t)|,$$

i.e.,

$$\begin{aligned}
|\hat{r}'(t)| &= |r'(G^{-1}t)\hat{g}(t)| \\
&\le 2|\widehat{g''}(t)\hat{g}(t)| = 2|\hat{\lambda}(t)\hat{\sigma}^2(t)|.
\end{aligned} \tag{45}$$

In view of (26) there is U_n between X_n and a_n such that

$$\begin{aligned}
GX_{n+1} - Ga_{n+1} &= GX_n - Ga_n + r(X_n) - r(a_n) + \zeta_{n+1} + R_n \\
&= (GX_n - Ga_n)(1 + \hat{r}'(GU_n)) + (\zeta_{n+1} + R_n).
\end{aligned}$$

By induction this leads to

$$\begin{aligned}
GX_n - Ga_n &= (GX_0 - Ga_0)\prod_{i=0}^{n-1}(1 + \hat{r}'(GU_i)) + \sum_{k=0}^{n-1}(\zeta_{k+1} + R_k) \\
&\quad + \sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1}\zeta_{k+1} + R_k\right)\hat{r}'(GU_j)\prod_{i=j+1}^{n-1}(1 + \hat{r}'(GU_i)).
\end{aligned}$$

Using the notation $M_n = \sum_{k=0}^{n-1}\zeta_{k+1}$ and $\Pi_j^n = \prod_{i=j}^{n-1}(1 + \hat{r}'(GU_i))$, this can be rewritten as

$$(46)\qquad \begin{aligned}&(GX_n - Ga_n) - (M_n + S_n)\\ &\quad = (GX_0 - Ga_0)\prod_0^n + \sum_{j=1}^{n-1} (M_j + S_j)\hat{r}'(GU_j)\prod_{j+1}^n .\end{aligned}$$

Next we show that a.e. on D

$$(47)\qquad \prod_k^{k+l} \to 1 \quad \text{as } k \to \infty \text{ uniformly in } l.$$

As $\prod_k^{k+l} = \exp(\sum_{i=k}^{k+l-1}\log(1 + \hat{r}'(GU_i)))$, it suffices to show that $\sum_{i=k}^{\infty}|\hat{r}'(GU_i)| \to 0$ as $k \to \infty$ a.e. on D.

Now U_i is between a_i and X_i, so Theorem 1 implies $GU_i/i \to 1$ a.e. on D [note that (a_n) is a special case of a process X_n tending everywhere to infinity]. Hence by (45), (A.4) and (15)

$$(48)\qquad |\hat{r}'(GU_i)| \le 2\left|\hat{\lambda}\left(\frac{i}{2}\right)\right| 4\hat{\sigma}^2\left(\frac{i}{2}\right) = 8\left|\widehat{g''}\left(\frac{i}{2}\right)\hat{g}\left(\frac{i}{2}\right)\right|$$

a.e. on D for i big enough and

$$(49)\qquad \begin{aligned}\sum_{i=k}^{\infty} |\hat{r}'(GU_i)| &\le 8\sum_{i=k}^{\infty}\left|g''\left(G^{-1}\frac{i}{2}\right)g\left(G^{-1}\frac{i}{2}\right)\right|\\ &= 8\sum_{i=k}^{\infty}\left|(g'\circ G^{-1})'\left(\frac{i}{2}\right)\right|\\ &\le \left|18\int_{(k-1)/2}^{\infty}(g'\circ G^{-1})'(t)\,dt\right|\\ &= \left|18g'\left(G^{-1}\frac{k-1}{2}\right)\right|\\ &\to 0, \qquad k \to \infty \text{ a.e. on } D.\end{aligned}$$

A first consequence of (47) is that

$$(50)\qquad (GX_0 - Ga_0)\prod_0^n \text{ converges a.e. on } D.$$

Assume now that $\hat{\psi}(\infty) < \infty$. Define the martingale

$$\tilde{M}_j = \sum_{k=0}^{j-1} \zeta_{k+1} I\{k/2 \le GX_k \le 2k\}.$$

Then

$$\begin{aligned}E\left[\tilde{M}_j^2\right] &= E\left[\sum_{k=0}^{j-1}\hat{\sigma}^2(GX_k)\left(\frac{g(X_k)}{g(Z_k)}\right)^2 I\{k/2 \le GX_k \le 2k\}\right]\\ &\le \text{const}\,\hat{\psi}(j) \quad \text{as } g(X_k)/g(Z_k) \text{ is bounded}\\ &\le \text{const}\,\hat{\psi}(\infty) < \infty,\end{aligned}$$

such that $\tilde{M}_j$ converges a.e. by the martingale convergence theorem. Now Theorem 1 implies that M_j converges a.e. on D. If $|\hat{\phi}(\infty)| < \infty$, which is always the case if $\hat{\psi}(\infty) < \infty$, then S_n converges a.e. on D by Lemma 7, and it follows from (46), (47), (49) and (50) that $GX_n - Ga_n$ converges a.e. on D.

Suppose now that $\hat{\psi}(\infty) = \infty$. From (46), (47), (48) and (50) we see that a.e. on D

$$(51)\qquad |GX_n - Ga_n - M_n - S_n| = O\left(\sum_{j=1}^{n-1} |M_j + S_j|\hat{\lambda}\left(\frac{j}{2}\right)\hat{\sigma}^2\left(\frac{j}{2}\right)\right).$$

We show that

$$(52)\qquad \sum_{j=1}^{n-1} |M_j|\hat{\lambda}\left(\frac{j}{2}\right)\hat{\sigma}^2\left(\frac{j}{2}\right) = o\left(\hat{\psi}(n)^{1/2}\right) \quad \text{a.e. on } D.$$

As $GX_n/n \to 1$ a.e. on D, it suffices to prove the same for the martingale $\tilde{M}_j = \sum_{k=0}^{j-1}\zeta_{k+1}I\{k/2 \le GX_k \le 2k\}$, and as

$$\sum_{j=0}^{n-1} \frac{\hat{\sigma}^2(j/2)}{\hat{\psi}(j/2)^{1/2}} \sim 2\int_0^{n/2} \frac{\hat{\sigma}^2(t)}{\hat{\psi}(t)^{1/2}}\,dt$$
$$\le 4\int_0^{n}\left(\hat{\psi}(t)^{1/2}\right)'\,dt = 4\hat{\psi}(n)^{1/2},$$

it is enough to show that for each $d > 0$

$$P\left\{|\tilde{M}_j|\hat{\lambda}\left(\frac{j}{2}\right)\hat{\psi}\left(\frac{j}{2}\right)^{1/2} \ge d \text{ for some } j \ge N\right\} \to 0 \quad \text{as } N \to \infty.$$

As $\lambda \circ \psi^{-1} = |(g' \circ \psi^{-1})'|$ is ultimately decreasing and integrable by (A.4) and (A.1), it follows that $t\lambda(\psi^{-1}t)$ and hence $\hat{\psi}(t)\hat{\lambda}(t) \to 0$ as $t \to \infty$.

Therefore it suffices to show

$$(53)\qquad P\left\{|\tilde{M}_j|\hat{\lambda}\left(\frac{j}{2}\right)^{1/2} \ge d \text{ for some } j \ge N\right\} \to 0 \quad \text{as } N \to \infty.$$

Observing (15), (22) and (A.4) the Hájek–Rényi inequality yields the following estimate for the probability in (53):

$$\le d^{-2}\hat{\lambda}\left(\frac{N}{2}\right)\sum_{j=0}^{N-1} E\left[\zeta_{j+1}^2 I\left\{\frac{j}{2} \le GX_j \le 2j\right\}\right]$$
$$+d^{-2}\sum_{j=N}^{\infty}\hat{\lambda}\left(\frac{j}{2}\right)E\left[\zeta_{j+1}^2 I\left\{\frac{j}{2} \le GX_j \le 2j\right\}\right]$$
$$\le \text{const}\left(\hat{\lambda}\left(\frac{N}{2}\right)\int_0^{N/2}\hat{\sigma}^2(t)\,dt + \int_{N/2}^{\infty}\hat{\lambda}(t)\hat{\sigma}^2(t)\,dt\right)$$
$$= \text{const}\left(\hat{\lambda}\left(\frac{N}{2}\right)\int_0^{N/2}\hat{\sigma}^2(t)\,dt + \int_{N/2}^{\infty}\left(\lambda \circ \psi^{-1} \circ \hat{\psi}(t)\right)\hat{\psi}'(t)\,dt\right)$$
$$= \text{const}\left(\hat{\lambda}\left(\frac{N}{2}\right)\hat{\psi}\left(\frac{N}{2}\right) + \int_{\hat{\psi}(N/2)}^{\infty}\left|(g' \circ \psi^{-1})'(t)\right|dt\right)$$
$$\to 0 \quad \text{as } N \to \infty,$$

by the above considerations on $\lambda \circ \psi^{-1}$. This proves (53) and hence (52).

Next we estimate $\sum_{j=1}^{n-1}|S_j|\hat{\lambda}(j/2)\hat{\sigma}^2(j/2)$. Because of Lemma 7, a.e. on D

$$\begin{aligned}\sum_{j=1}^{n-1}|S_j|\hat{\lambda}\left(\frac{j}{2}\right)\hat{\sigma}^2\left(\frac{j}{2}\right) &= O\left(\sum_{j=1}^{n-1}|\hat{\phi}(j)|\hat{\lambda}\left(\frac{j}{2}\right)\hat{\sigma}^2\left(\frac{j}{2}\right)\right)\\ &= O\left(\int_0^n|\hat{\phi}(t)|\hat{\lambda}(t)\hat{\sigma}^2(t)\,dt\right) \quad \text{[by (17)]}\\ &= O\left(\int_0^n|\hat{\phi}(t)|\,|(g'\circ G^{-1})'(t)|\,dt\right)\\ &= O\left(|\hat{\phi}(n)|\,|\widehat{g'}(n)| + \int_0^n|\hat{\phi}'(t)|\,|\widehat{g'}(t)|\,dt\right).\end{aligned}$$

If $|\hat{\phi}(\infty)| < \infty$ this means that the above sum converges a.e. on D such that it is $o(\hat{\psi}^{1/2}(n))$, whereas in the case of $|\hat{\phi}(\infty)| = \infty$ it implies that the sum is $o(\hat{\phi}(n))$ a.e. on D because $|g'(t)| \to 0$, $t \to \infty$.

Together with (51), (52) and Lemma 7 this proves

$$GX_n - Ga_n = M_n + \hat{\phi}(n)(1 + o(1)) + o\left(\hat{\psi}(n)^{1/2}\right) \quad \text{a.e. on } D,$$

and we can finish the proof of (b) of Theorem 2 by showing that

$$\left|M_n - \sum_{j=0}^{n-1}\xi_{j+1}\right| = o(|\hat{\phi}(n)|) + O(1) \quad \text{a.e. on } D.$$

From (22) we know that for large X_j $|g(Z_j)/g(X_j) - 1| \le |g'(X_j)|$, such that a.e. on D

$$\begin{aligned}&\sum_{j=1}^{\infty}\hat{\phi}(j)^{-2}\left(\frac{g(X_j)}{g(Z_j)} - 1\right)^2\sigma^2(X_j)\\ &\quad = O\left(\sum_{j=1}^{\infty}\hat{\phi}(j)^{-2}\widehat{g'}\left(\frac{j}{2}\right)^2\hat{\sigma}^2\left(\frac{j}{2}\right)\right)\\ &\quad = O\left(\int_1^{\infty}\frac{\widehat{g'}(t)^2\hat{\sigma}^2(t)}{\hat{\phi}(t)L^2}\,dt\right) = O\left(\int_1^{\infty}\widehat{g'}(t)\left(\frac{-1}{\hat{\phi}(t)}\right)'dt\right)\\ &\quad = O\left(\left|\int_1^{\infty}\left(\frac{1}{\hat{\phi}l(t)}\right)'dt\right|\right) = O\left(\frac{1}{\hat{\phi}(1)} - \frac{1}{\hat{\phi}(\infty)}\right) < \infty,\end{aligned}$$

and we can conclude from the martingale convergence theorem [and the Kronecker lemma, if $\hat{\phi}(\infty) = \infty$] that

$$\left|\sum_{j=0}^{n-1}\xi_{j+1}\left(\frac{g(X_j)}{g(Z_j)} - 1\right)\right| = o(\hat{\phi}(n)) + O(1) \quad \text{a.e. on } D. \qquad \square$$

Proof of Proposition 1. As $GX_n/n \to 1$ a.s. on D, it suffices to prove that for $\overline{M}_n = \sum_{k=0}^{n-1}\xi_{k+1}I\{k/2 \le GX_k \le 2k\}$

$$\hat{\psi}(n)^{-1/2}\overline{M}_n \to 0 \quad \text{a.e. on } D^c, \tag{54}$$

$$\hat{\psi}(n)^{-1/2}\overline{M}_n \to_{\mathscr{D}} \eta\mathscr{N}(0,1), \tag{55}$$

where η is independent of $\mathscr{N}(0,1)$, $P\{\eta = 1\} = P(D)$ and $P(\eta = 0) = 1 - P(D)$. While (54) follows immediately from Theorem 1, (55) is nothing but Corollary 3.1 in Hall and Heyde (1980) together with the remark thereafter. We check the assumptions of this corollary.

First we need a conditional Lindeberg condition. For any $\varepsilon > 0$

$$\begin{aligned}
&\sum_{k=0}^{n-1} E\left[\frac{\xi_{k+1}^2}{\hat{\psi}(n)} I\left\{\frac{k}{2} \le GX_k \le 2k,\ \frac{\xi_{k+1}^2}{\hat{\psi}(n)} > \varepsilon\right\}\middle|\mathscr{F}_k\right] \\
&\quad = \hat{\psi}(n)^{-1}\sum_{k=0}^{n-1}\hat{\sigma}^2(GX_k)I\left\{\frac{k}{2} \le GX_k \le 2k\right\} \\
&\qquad \times E\left[\frac{\xi_{k+1}^2}{\sigma^2(X_k)} I\left\{\frac{\xi_{k+1}^2}{\sigma^2(X_k)} > \varepsilon\frac{\hat{\psi}(n)}{\sigma^2(X_k)}\right\}\middle|\mathscr{F}_k\right] \\
&\quad \le 4\hat{\psi}(n)^{-1}\sum_{k=1}^{n-1}\hat{\sigma}^2(k/2)\int_{(\varepsilon/4)\hat{\psi}(n)/\hat{\sigma}^2(k/2)}^{\infty} F(x)\,dx \qquad \text{[by (8) and (15)]} \\
&\quad = o\left(\hat{\psi}(n)^{-1}\sum_{k=1}^{n-1}\hat{\sigma}^2\left(\frac{k}{2}\right)\right) \\
&\quad = o(1),
\end{aligned}$$

since $\hat{\psi}(n)/\hat{\sigma}^2(k/2) \ge \text{const}\ \hat{\psi}(n) \to \infty$ for decreasing σ^2 and $\hat{\psi}(n)/\hat{\sigma}^2(k/2) \ge \hat{\psi}(n)/\hat{\sigma}^2(n) \ge n/2 \to \infty$ by (12) for increasing σ^2. The second hypothesis concerns the conditional variances:

$$\begin{aligned}
&\sum_{k=0}^{n-1} E\left[\frac{\xi_{k+1}^2}{\hat{\psi}(n)} I\left\{\frac{k}{2} \le GX_k \le 2k\right\}\middle|\mathscr{F}_k\right] \\
&\quad = \hat{\psi}(n)^{-1}\sum_{k=0}^{n-1}\hat{\sigma}^2(GX_k)I\left\{\frac{k}{2} \le GX_k \le 2k\right\} \\
&\quad \to 1_D, \qquad n \to \infty \text{ a.e. by (8) and Theorem 1.} \qquad \square
\end{aligned}$$

Proof of Proposition 2. Let $Y_n = \sum_{k=1}^{n}\xi_k$. As $(Y_n, \mathscr{F}_n)$ is a square-integrable, zero-mean martingale, there is a Skorokhod representation for Y_n [see, e.g., Theorem A.1 in Hall and Heyde (1980)], i.e., by extending our probability space if necessary, we may suppose that there is a standard Brownian motion $B(t)$ and random variables $0 = T_0 \le T_1 \le T_2 \le \cdots$ such that $Y_n = B(T_n)$. Furthermore, if $\tau_n = T_n - T_{n-1}$, $n \geq 1$, and $\mathscr{G}_n$ is the σ-algebra generated by $\xi_1, \dots, \xi_n$ and $(B(t): 0 \leq t \leq T_n)$, then

$$E[\tau_{n+1}|\mathscr{G}_n] = E[\xi_{n+1}^2|\mathscr{G}_n] = E[\xi_{n+1}^2|\mathscr{F}_n] = \sigma^2(X_n),$$

$$E[\tau_{n+1}^{1+\delta/2}|\mathscr{G}_n] \leq C'E[\xi_{n+1}^{2+\delta}|\mathscr{G}_n] = C'E[\xi_{n+1}^{2+\delta}|\mathscr{F}_n] \leq CC'\sigma^{2+\delta}(X_n),$$

by the assumption on the $(2+\delta)$-moments of (ξ_n).

So all we have to show is that $T_n/\hat{\psi}(n) \to 1$ a.e. on D. Let $V_n = \sum_{k=0}^{n-1}\sigma^2(X_k)$. By (15) and Theorem 1 we have

$$V_n \sim \sum_{k=0}^{n} \hat{\sigma}^2(k) \sim \hat{\psi}(n) \quad \text{a.e. on } D,$$

and we can finish the proof by showing that

$$T_n - V_n = \sum_{k=1}^{n} (\tau_k - E[\tau_k|\mathscr{G}_{k-1}]) = o(\hat{\psi}(n)) \quad \text{a.e. on } D.$$

But this follows from the martingale convergence theorem [2.18 in Hall and Heyde (1980)], since a.s. on D

$$\begin{aligned}
&\sum_{n=1}^{\infty} \hat{\psi}(n)^{-1-\delta/2} E[\tau_k^{1+\delta/2}|\mathscr{G}_{k-1}] \\
&\quad \leq \text{const} \sum_{n=1}^{\infty} \hat{\psi}(n)^{-1-\delta/2}\hat{\sigma}^{2+\delta}(n) \quad [\text{by (15) and Theorem 1}] \\
&\quad \leq \text{const} \int_1^{\infty} \hat{\sigma}^{2+\delta}(t)\hat{\psi}(t)^{-1-\delta/2}\,dt \\
&\quad < \infty \quad [\text{by (12)}].
\end{aligned}$$

□

5. Asymptotic results for the process X_n. If, given the growth rate g, σ^2 does not grow too fast, the asymptotic behaviour of X_n is described by

THEOREM 3. *Assume* (A.1), (A.2), (A.4), *and that* $g'(t)\psi(t)^{1/2} \to 0$ *as* $t \to \infty$.

(a) *If* $\hat{\psi}(\infty) < \infty$, *then* $X_n/a_n \to 1$ *a.e. on* $\{X_n \to \infty\}$, *and* $(X_n - a_n)/g(a_n)$ *converges a.e. on* $\{X_n \to \infty\}$.

(b) *If* $\hat{\psi}(\infty) = \infty$, *then* $X_n/a_n \to 1$ *in probability on* $\{X_n \to \infty\}$ *and* $(X_n - a_n)/g(a_n) = \sum_{k=1}^n \xi_k + o(\hat{\psi}(n)^{1/2})$ *in probability on* $\{X_n \to \infty\}$ [*in this case* $\mathscr{D}(\hat{\psi}(n)^{-1/2}\sum_{k=1}^n \xi_k|\{X_n \to \infty\}) \to \mathscr{N}(0,1)$, cf. *Proposition* 1].

The next theorems show that if (A.1)–(A.5) are assumed, then the behaviour described in Theorem 3 is equivalent to $g'(t)\psi(t)^{1/2} \to 0$.

THEOREM 4. *Assume* (A.1)–(A.5). *The following are equivalent*:

(i) $g'(t)\psi^{1/2}(t) \to 0$, $t \to \infty$.

(ii) $X_n/a_n \to 1$ *in probability on* $\{X_n \to \infty\}$.

THEOREM 5. *Assume* (A.1)–(A.5). *The following are equivalent*:

(i) $g'(t)\psi^{1/2}(t) \to c,\ t \to \infty,\ 0 \le c < \infty$.

(ii) *There is an increasing sequence of constants* b_n *such that the conditional distribution of* $(X_n - a_n)/b_n$ *on* $\{X_n \to \infty\}$ *converges weakly to a nondegenerate distribution.*

If (i) *and* (ii) *are true, then*

(a) *for* $c = 0$ *Theorem* 3 *applies,*

(b) *for* $0 < c < \infty$, $\log(X_n/a_n) = c\hat{\psi}(n)^{-1/2}\sum_{k=1}^{n}\xi_k - c^2 + o(1)$ *in probability on* $\{X_n \to \infty\}$.

[*Observe that* $c > 0$ *implies* $\hat{\psi}(\infty) = \infty$, *hence Proposition* 1 *applies such that* $\mathscr{D}(c^{-1}\log(X_n/a_n)|\{X_n \to \infty\}) \to \mathscr{N}(-c, 1)$.]

If $0 < c < \infty$ in Theorem 5, the limiting distribution is no longer centered at the origin, namely $P(X_n \le a_n|D) \to H(c)$, where $H(z)$ denotes the standard normal distribution function. This tendency of X_n to be smaller than a_n becomes even stronger, if $c = \infty$:

THEOREM 6. *Assume* (A.1)–(A.5). *If* $g'(t)\psi(t)^{1/2} \to \infty$ *as* $t \to \infty$, *then* $X_n/a_n \to 0$ *in probability.*

For a further discussion of these results (in the setting of stochastic differential equations) see Keller et al. (1984). The following identity is basic for the proofs of the theorems:

$$X_n - a_n = (GX_n - Ga_n)g(U_n), \tag{56}$$

for some U_n between X_n and a_n, such that a.e. on D for big n, $n/2 \le GU_n \le 2n$.

PROOF OF THEOREM 3. [The proof is analogous to the corresponding one in Keller et al. (1984).] As $Gt\sigma^2(t) = O(\psi(t))$ follows from (A.2) alone [cf. (12)], the assumptions of the theorem and Lemma A.1 of the Appendix imply that $t^{-1}g(t)\sigma(t)\sqrt{Gt} = o(1)$, i.e., (A.5b) and hence (A.3) (cf. Section 2). Furthermore,

$$\begin{aligned} \frac{|g(a_n) - g(U_n)|}{\max\{g(a_n), g(U_n)\}} &\le \widehat{g'}\left(\frac{n}{2}\right)|Ga_n - GU_n| \\ &\le \widehat{g'}\left(\frac{n}{2}\right)|GX_n - Ga_n| \\ &= o\left(\hat{\psi}(n)^{-1/2}|GX_n - Ga_n|\right) \quad [\text{observe } (16)]. \end{aligned} \tag{57}$$

Hence $g(U_n)/g(a_n) \to 1$ on D in the almost sure sense if $\hat{\psi}(\infty) < \infty$ (see Theorem 2a) and in probability if $\hat{\psi}(\infty) = \infty$ (see Theorem 2b, Proposition 1, and Lemma A.1 of the Appendix). Now part (a) of the theorem follows from (56), Theorem 2, and (A.1), and if one observes additionally Proposition 1 and Lemma A.1, then part (b) also follows. □

PROOF OF THEOREM 4. In view of Theorem 3 we only must show (ii) $\Rightarrow$ (i), and (i) is trivial for $\hat{\psi}(\infty) < \infty$. Because of (21) we may assume that g is ultimately increasing such that $\hat{\phi}(t) \le \gamma$ for some constant γ. Let $0 < c < 1$. Theorem 2 implies

$$\begin{aligned} P\left(Ga_n + \sum_{k=1}^{n} \xi_k + o(\hat{\psi}^{1/2}(n)) + \gamma \le G(ca_n) \right) \\ \le P(X_n \le ca_n) \\ \to 0 \quad \text{as } n \to \infty \text{ by assumption,} \end{aligned}$$

so by Proposition 1 for all $d > 0$,

$$G(a_n) - G(ca_n) > d\hat{\psi}^{1/2}(n) \quad \text{ultimately.} \tag{58}$$

Since for ultimately increasing concave $g(t)$

$$\left(\frac{t}{g(t)} \right)' = \frac{g(t) - tg'(t)}{(g(t))^2} \ge 0, \quad \text{for large } t,$$

we have

$$\begin{aligned} G(a_n) - G(ca_n) &= (1-c)a_n/g(\delta a_n), \quad \text{for some } c \le \delta \le 1 \\ &= \frac{1-c}{\delta} \frac{\delta a_n}{g(\delta a_n)} \\ &\le \frac{1-c}{c} \frac{a_n}{g(a_n)}, \end{aligned} \tag{59}$$

and together with (58) this implies in view of (16) and Theorem 1,

$$\frac{g(t)}{t} \psi^{1/2}(t) \to 0, \qquad t \to \infty.$$

Assertion (i) follows now from Lemma A.1 of the Appendix. □

PROOF OF THEOREM 5. Suppose first that (i) is true.

If $c = 0$ the (ii) and (a) follow from Theorem 3.

If $c > 0$ and hence necessarily $\hat{\psi}(\infty) = \infty$, we apply the mean-value theorem to $\log G^{-1}t$, and we see that there is V_n between X_n and a_n such that

$$\log X_n - \log a_n = (GX_n - Ga_n) \frac{g(V_n)}{V_n}.$$

Now $g(V_n)/V_n \sim c\psi(V_n)^{-1/2} = c\hat{\psi}(GV_n)^{-1/2} \sim c\hat{\psi}^{-1/2}(n)$ a.s. on D by (16) and Theorem 1, and $\phi(t)\psi(t)^{-1/2} \to -c$ by Lemma A.1 of the Appendix. Hence Theorem 2 and Proposition 1 imply

$$\log(X_n/a_n) = c\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k - c^2 + o(1) \quad \text{in probability on } D.$$

This is assertion (b) of the theorem and (ii) follows with $b_n = a_n$.

Suppose now that (ii) holds. We show (i). Without loss we may assume that $\psi(\infty) = \infty$ and that g is ultimately increasing [see (21)]. As $a_{n+1}/a_n = 1 + g(a_n)/a_n \to 1$ by (A.1), it follows from Lemma A.1, (16), Theorem 1, and from the concavity of g that (i) is equivalent to

(60) $$g(a_n)\hat{\psi}(n)^{1/2}/a_n \to c, \qquad n \to \infty, 0 \le c < \infty.$$

Denote the limiting distribution of $(X_n - a_n)/b_n$ on D by H. If H is continuous at z, then by Theorem 2

$$H(z) = \lim_{n\to\infty} P(X_n \le a_n + b_n z | D)$$

(61) $$= \lim_{n\to\infty} P\left(\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k \le \hat{\psi}(n)^{-1/2} \times (G(a_n + b_n z) - Ga_n - \hat{\phi}(n)(1 + o(1))) + o(1) \middle| D\right).$$

We show that $|\hat{\phi}(n)| = O(\hat{\psi}(n)^{1/2})$:

Assume for a contradiction that there is a subsequence n' such that $-\hat{\phi}(n')/\hat{\psi}(n')^{1/2} \to \infty$. Let $z_0 = \inf\{z|H(z) > 0\}$, $z_1 = \sup\{z|H(z) < 1\}$. As H is nondegenerate, $z_0 < z_1$, and if H is continuous at z, $z_0 < z < z_1$, then because of (61)

$$f_{n'}(z) := (G(a_{n'} + b_{n'}z) - G(a_{n'}))/\hat{\phi}(a_{n'}) \to 1, \qquad n' \to \infty.$$

Observe that $a_n + b_n z > 0$ if $z > z_0$ and n is big enough. Hence

$$f(z) = \limsup_{n'\to\infty} f_{n'}(z) \quad [\text{possibly } f(z) = \pm\infty]$$

is well defined for $z \ge 0$ and for $z > z_0$, f is decreasing and f is convex on $\{|f| < \infty\}$ since the $f_{n'}$ are ($\hat{\phi}(a_{n'}) \le 0$!), $f(0) = 0$, and $f(z) = 1$ for $z_0 < z < z_1$. This is a contradiction.

Now (61) implies that if H is continuous at z

$$H(z) = \lim_{n\to\infty} P\left(\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k \le \hat{\psi}(n)^{-1/2}(G(a_n + b_n z) - G(a_n) - \hat{\phi}(n)) \middle| D\right)$$

and Theorem 2 and Proposition 1 imply together with Proposition 1 in Keller et al. (1984) that $g(r_n)s_n/r_n \to c$ for some $0 \le c < \infty$, where $r_n = G^{-1}(Ga_n + \hat{\phi}(n))$, $s_n = \hat{\psi}(n)^{1/2}$. If $c = 0$ this proves (60) because $r_n \le a_n$ for large n. If $c > 0$, the proof of Proposition 1 in Keller et al. (1984) also tells us that $g(b_n)s_n/b_n \to c$, that $g(t)/t$ is a slowly varying function and that we may assume without loss $a_n = \text{const}\ b_n$. Combining these three facts we obtain (60) again. □

PROOF OF THEOREM 6. If $g'(t)\psi(t)^{1/2} \to \infty$ then g is ultimately increasing by (21), so by (59) and Lemma A.1 for each $0 < \varepsilon < 1$

$$Ga_n - G(\varepsilon a_n) = o(\phi(a_n)) = o(\hat{\phi}(n)).$$

Since $Ga_{n+1} - Ga_n = 1 + r(a_n)$ and $r(t) \leq 0$ for large t,

$$a_n \leq G^{-1}(n + \text{const.})$$

and

$$|\hat{\phi}(n + \text{const}) - \hat{\phi}(n)| \leq \text{const}\, \hat{\sigma}^2(n + \text{const}')\widehat{g'}(n + \text{const}') \leq \text{const}''.$$

Now Theorem 2 implies

$$\begin{aligned} &P(X_n \leq \varepsilon a_n | D) \\ &\quad = P\big(GX_n \leq G(\varepsilon a_n)|D\big) \\ &\quad = P\Bigg(\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k \leq \hat{\psi}(n)^{-1/2} \\ &\qquad\qquad \times (G(\varepsilon a_n) - Ga_n - \hat{\phi}(n)(1 + o(1))) + o(1) \Bigg| D\Bigg) \\ &\quad = P\Bigg(\hat{\psi}(n)^{-1/2} \sum_{k=1}^{n} \xi_k \leq -\hat{\phi}(n)\hat{\psi}(n)^{-1/2}(1 + o(1)) + o(1) \Bigg| D\Bigg), \end{aligned}$$

which converges to 1 as $n \to \infty$ by Lemma A.1 of the Appendix and by Proposition 1. As X_n converges a.e. on D^c by Theorem 1, this proves Theorem 6. □

6. Almost sure approximations for the process X_n. The law of large numbers and the normal approximation for X_n which are stated in Theorem (3b) are only approximations in probability. They have been characterized to be equivalent to $g'(t)\psi(t)^{1/2} \to 0$, $t \to \infty$, in Theorems 4 and 5. For the corresponding almost sure approximations we find similar characterizations, if we assume that

(A.6) *there are* $C > 0$, $0 < \delta \leq 2$, *such that*

$$E\left[\, |\xi_{n+1}|^{2+\delta} | \mathscr{F}_n \right] \leq C\sigma^{2+\delta}(X_n) \quad a.e.$$

THEOREM 7. *Assume* (A.1)–(A.6) *and* $\hat{\psi}(\infty) = \infty$. *Then*

(i) $$t^{-1}g(t)\psi(t)^{1/2}(\log\log \psi(t))^{1/2} \to 0, \qquad t \to \infty,$$

implies

(ii) $$X_n/a_n \to 1 \quad a.e.\ on\ \{X_n \to \infty\}.$$

If the left-hand side of (i) *is ultimately monotone, then* (ii) *implies* (i).

THEOREM 8. *Assume* (A.1)–(A.6) *and* $\hat{\psi}(\infty) = \infty$. *Then*

(iii) $$g'(t)\psi(t)^{1/2}\log\log\psi(t) \to 0, \qquad t \to \infty$$

implies

(iv) $$(X_n - a_n)/g(a_n) = \sum_{k=1}^{n} \xi_k + o\big(\hat{\psi}(n)^{1/2}\big) \quad a.e.\ on\ \{X_n \to \infty\}.$$

If the left-hand side of (iii) *is ultimately monotone, then* (iv) *implies* (iii). [*Observe that* $\sum_{k=1}^{n}\xi_k = B(\hat{\psi}(n) + o(\hat{\psi}(n))$ *a.e. on* $\{X_n \to \infty\}$.]

REMARK. Any of the conditions (i)–(iv) implies $g'(t)\psi(t)^{1/2} \to 0$, $t \to \infty$. For condition (i) this follows from Lemma A.1, for (ii) from Theorem 4, and for (iv) from Theorem 5. Hence we may assume for the proof of both theorems that $\phi(t) = o(\psi(t)^{1/2})$ (see Lemma A.1).

PROOF OF THEOREM 7. In view of the preceding remark it follows from (56), Theorem 2 and (16) that $X_n/a_n \to 1$ a.e. on D if and only if

(62) $$\frac{g(U_n)}{g(a_n)}\left(\sum_{k=1}^{n} \xi_k/(\hat{\psi}(n)\log\log\hat{\psi}(n))^{1/2} + o\big((\log\log\hat{\psi}(n))^{-1/2}\big)\right) \times \frac{g(a_n)}{a_n}(\psi(a_n)\log\log\psi(a_n))^{1/2} \to 0 \quad \text{a.e. on } D.$$

Hence (i) ⇒ (ii) follows from Proposition 2 and the law of the iterated logarithm (LIL), if we can show that $g(U_n)/g(a_n) \to 1$ a.e. on D, or, because of (57), that $g'(n/2)|GX_n - Ga_n| \to 0$ a.e. on D. This is a consequence of Theorem 2, Proposition 2, and the LIL provided that $g'(t)(\psi(t)\log\log\psi(t))^{12} \to 0$. But for ultimately increasing (and concave!) g this follows from (i).

For decreasing g this follows from (20) and from

$$\hat{\psi}(t) = O\left(\frac{t^2}{\log t}\right).$$

This statement is immediate in view of (8), if $\hat{\sigma}(t)^2$ is ultimately decreasing. Let $\hat{\sigma}(t)$ be increasing and concave. Then $\hat{\sigma}^2(s) \geq (s/t)\hat{\sigma}^2(t)$, if $1 \leq s \leq t$ and t is large enough. From (A.2)

$$\frac{\log t}{t}\hat{\sigma}(t)^2 \leq \int_1^t \frac{\hat{\sigma}(s)^2}{s^2}\,ds \leq C < \infty,$$

and from (8) our claim follows again.

For the reverse implication note that Theorem 2, Proposition 2 and the LIL imply that a.e. on D for infinitely many n

$$\sum_{k=1}^{n} \xi_k \geq (\hat{\psi}(n)\log\log\hat{\psi}(n))^{1/2}$$

and

$$GX_n \geq GU_n \geq Ga_n \quad (g \text{ increasing})$$

or

$$GX_n \leq GU_n \leq Ga_n \quad (g \text{ decreasing}).$$

Now (62) implies (i) if $t^{-1}g(t)(\psi(t)\log\log\psi(t))^{1/2}$ is ultimately monotone. □

PROOF OF THEOREM 8. A Taylor expansion of $G^{-1}t$ at Ga_n gives

$$X_n = a_n + (GX_n - Ga_n)g(a_n) + \tfrac{1}{2}(GX_n - Ga_n)^2 g'(W_n)g(W_n),$$

with W_n between a_n and X_n. Hence, in view of Theorem 2 and the remark after Theorem 8, (iv) is equivalent to

$$(GX_n - Ga_n)^2 g'(W_n)\hat{\psi}(n)^{-1/2}\frac{g(W_n)}{g(a_n)} \to 0 \quad \text{a.e. on } D,$$

and the proof is finished along the same lines as that of Theorem 7. □

REMARK. One of the referees suggested a nice approach to the a.s. convergence of X_n/a_n, which we would like to repeat here, although it needs a slightly stronger assumption than condition (i) of Theorem 7, namely

$$(*) \qquad \int_1^\infty \frac{\sigma^2(u)g(u)}{u^2}\,du = \int_1^\infty \psi'(u)\frac{g(u)^2}{u^2}\,du < \infty.$$

Let $R(x) = (x + g(x))/x$. Then $X_{n+1} = R(X_n)X_n + g(X_n)\xi_{n+1}$; thus

$$M_n = \frac{X_n}{\prod_{k=0}^{n-1}R(X_k)} = X_0\prod_{k=0}^{n-1}\left(1 + \frac{g(X_k)}{X_kR(X_k)}\xi_{k+1}\right).$$

Now $\lim_{n\to\infty}M_n > 0$, if $\sum_{k=0}^{\infty}(g(X_k)/X_kR(X_k))\xi_{k+1}$ converges, which is implied by

$$\sum_{k=0}^{\infty}\frac{g(X_k)^2\sigma(X_k)^2}{X_k^2R(X_k)^2} < \infty.$$

This latter statement follows from $(*)$ and (A.1), (A.2) if only $GX_n/n \to 1$. In view of Theorem 1 a.s. $\lim_{n\to\infty}M_n > 0$ on $\{X_n \to \infty\}$. Now

$$\frac{X_n}{a_n} = \frac{X_0}{a_0}M_n\prod_{k=0}^{n-1}\frac{R(X_k)}{R(a_k)}.$$

$R(x)$ is ultimately nonincreasing, therefore $X_k/a_k > 1$ implies $R(X_k)/R(a_k) \leq 1$, while $X_k/a_k < 1$ implies $R(X_k)/R(a_k) \geq 1$.

From this observation it is not difficult to deduce the a.s. convergence of X_n/a_n to a positive limit on $\{X_n \to \infty\}$. Now if $t_0 \le t$ and t_0 is sufficiently large,

$$\frac{g(t)^2}{t^2}(\psi(t) - \psi(t_0)) \le \int_{t_0}^{t} \psi'(s)\frac{g(s)^2}{s^2}\,ds.$$

Therefore $(*)$ entails $(g(t)/t)\psi(t)^{1/2} \to 0$. From Lemma A.1c (Appendix) it is seen that $(*)$ implies (i) of Theorem 4. If we assume $g(t)/t \sim \psi(t)^{-1/2}(\log \psi(t))^{-\alpha/2}$, as $t \to \infty$, $(*)$ is satisfied, if and only if $\alpha > 1$. From this example it is seen that essentially $(*)$ is a stronger requirement than condition (i) of Theorem 7.

7. Examples.

A. *Controlled Galton–Watson processes.* As pointed out in the introduction, growth models like (1) are well suited to describe controlled Galton–Watson processes. These are integer-valued Markovian branching processes satisfying the recursive relation

$$X_{n+1} = \sum_{i=1}^{X_n} \eta_{i,n}(X_n), \qquad X_0 > 0,$$

where $\{\eta_{i,n}(k): i, n, k \in \mathbb{N}_0\}$ is a family of nonnegative, integer-valued, independent random variables which are identically distributed for each fixed k. In Küster (1985) such processes (and more general ones) have been investigated using a general growth model like (1), and details as well as further references can be found there.

Using our notation one main result of Küster (1985) can be roughly stated as follows: Assuming some rather weak regularity properties for g (none for σ^2), $g(t) = O(t)$ and a rather involved joint growth condition on g and σ^2, which comes close to $\sigma^2(t)g(t)/t = O(t^{-\delta})$ for some $\delta > 0$ and hence is stronger than that of Theorem 7, a strong law of large numbers for X_n/a_n holds on $\{X_n \to \infty\}$, where the limit is not necessarily 1 but may be some positive random variable. Nothing is said about $P(X_n \to \infty) >$ or $= 0$. [For controlled Galton–Watson processes with particular choices of g and σ^2 this problem has been treated in Fujimagari (1976) and Lévy (1979).]

B. *Special choices of g and σ^2 and counterexamples.* For two special, "natural" families of functions g and σ^2 we shall give simple criteria for assumptions (A.2), (A.3), (A.5b), and $g'(t)\psi(t)^{1/2} \to 0$. This will show that these conditions—although very closely related—are really different. Afterwards we give some examples showing that, as soon as these conditions fail to be true, a completely different asymptotic behaviour may be observed.

Let

$$g(t) = At^{a}(\log t)^{b}(\log\log t)^{c}, \qquad A > 0,$$
$$\sigma^2(t) = Bt^{\alpha}(\log t)^{\beta}(\log\log t)^{\gamma}, \qquad B > 0.$$

Using the notation $(r, s, t) < (u, v, w)$ if $r < u$ or $r = u,\ s < v$ or $r = u,\ s = v,\ t < w$, some elementary calculations lead to the following criteria:

$$(\text{A.1}) \Leftrightarrow (a, b, c) < (1,0,0),$$

$$(\text{A.2}) \Leftrightarrow (a + \alpha, b + \beta, c + \gamma) < \begin{cases} (1,-1,-1), & \text{if } a < 1, \\ (1,1,-1), & \text{if } a = 1, \end{cases}$$

$$(\text{A.3}) \Leftrightarrow (a + \alpha, b + \beta, c + \gamma) < \begin{cases} (1,0,0), & \text{if } a \neq 0, \\ (1,1,0), & \text{if } a = 0,\ b \neq 0, \\ (1,1,1), & \text{if } a = b = 0,\ c \neq 0, \\ (\infty,\infty,\infty), & \text{if } a = b = c = 0, \end{cases}$$

$$(\text{A.5b}) \Leftrightarrow (a + \alpha, b + \beta, c + \gamma) < \begin{cases} (1,0,0), & \text{if } a < 1, \\ (1,-1,0), & \text{if } a = 1. \end{cases}$$

We see that (A.2), (A.3) and (A.5b), although they coincide if $b = c = \beta = \gamma = 0$, are nonequivalent. Furthermore,

$$(\text{A.2}) \wedge (\text{A.3}) \Leftrightarrow (a + \alpha, b + \beta, c + \gamma) < \begin{cases} (1,-1,-1), & \text{if } a < 1, \\ (1,0,0), & \text{if } a = 1, \end{cases}$$

and (A.2) ∧ (A.3) ∧ (A.5b) is really stronger than (A.2) ∧ (A.3), namely,

$$(\text{A.2}) \wedge (\text{A.3}) \wedge (\text{A.5b}) \Leftrightarrow (a + \alpha, b + \beta, c + \gamma) < \begin{cases} (1,-1,-1), & \text{if } a < 1, \\ (1,-1,0), & \text{if } a = 1. \end{cases}$$

In Theorems 3–6 the different types of asymptotic behaviour of X_n were characterized by $\hat{\psi}(\infty) <$ or $= \infty$ and $g'(t)\psi(t)^{1/2} \to c$ for $0 \le c \le \infty$. It is easily checked that

$$\hat{\psi}(\infty) < \infty \Leftrightarrow (\alpha, \beta, \gamma) < (a - 1, b - 1, c - 1),$$

and that, if (A.1)–(A.3) and (A.5b) are assumed, then $g'(t)\psi(t)^{1/2} \to 0$ imposes the additional requirement:

$$\text{If } a = 1,\ \alpha = 0,\ b = 0,\ \beta = -1, \text{ then } \gamma + c < -1.$$

In particular, we see that if (A.1)–(A.3) are assumed, $g'(t)\psi(t)^{1/2} \to 0$ is strictly stronger than (A.5b) (cf. the proof of Theorem 3).

Asymptotic log-normality of X_n/a_n as described in Theorem 5 occurs in the following cases

$$a = 1, \quad \alpha = 0, \quad b < 0, \quad b + \beta = -1, \quad c + \gamma = 0,$$

$$a = 1, \quad \alpha = 0, \quad b = 0, \quad \beta = -1, \quad c + \gamma = -1.$$

If we neglect the logarithmic terms, Figure 1 represents some of these results.

We now give some examples where (A.1)–(A.5) just fail to be true showing a completely different asymptotic behaviour: (We always assume $b = c = \beta = \gamma = 0$). (Their location is indicated in Figure 1.)

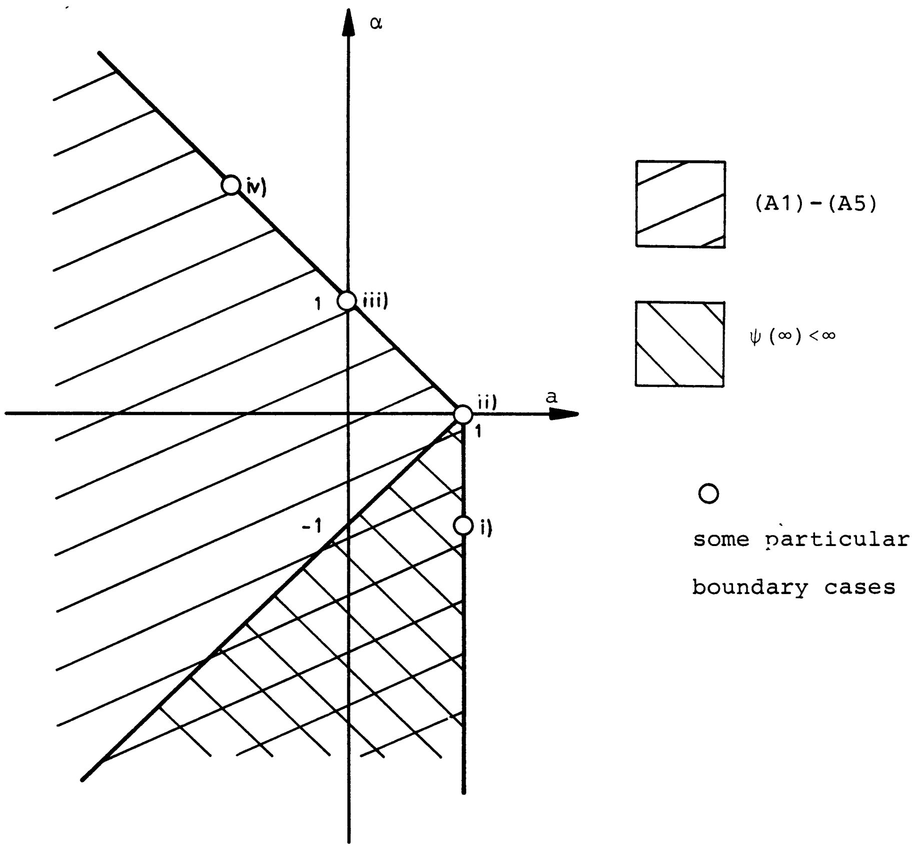

FIG. 1.

(i) $a = 1$, $\alpha = -1$. An example for this case is the classical supercritical Galton–Watson process, for which it is well known that $X_n/(1 + A)^n$ converges a.s. on $\{X_n \to \infty\}$ to a nondegenerate random variable [see, e.g., Harris (1963)]. Here the constant A comes from $g(t) = At$.

(ii) $a = 1$, $\alpha = 0$. Let $\xi_i \geq -2 + \varepsilon$, $\varepsilon > 0$, be i.i.d. random variables with $E\xi_i = 0$ and $\mathrm{var}(\xi_i) = \sigma^2$, and define X_n by $X_0 \equiv 1$, $X_{n+1} = X_n(2 + \xi_{n+1})$. Then $\log X_n = \Sigma_{i=1}^n \log(2 + \xi_i)$ is a sum of square-integrable i.i.d. random variables and

$$n^{-1/2}(\log X_n - nE[\log(2 + \xi_1)])$$

is asymptotically normal. It should be observed that $a = 1$, $\alpha = 0$ is the case where for suitable choices of b, c, β, γ asymptotic log-normality of X_n/a_n (Theorem 5) or $X_n/a_n \to 0$ in probability (Theorem 6) may happen. As in the present example $a_{n+1} = 2a_n$, $a_0 = 1$, i.e., $E \log(2 + \xi_1)^n = o(2^n) = o(a_n)$, we have in particular that $X_n/a_n \to 0$ in probability thus extending Theorem 6. Although we did not give results on the log-normality of processes covered by Theorem 6, this example suggests that under suitable regularity assumptions on g and σ^2 and moment assumptions on ξ_n there are norming constants μ_n, ν_n such that $\mu_n \log X_n/\nu_n$ is asymptotically normal. For the corresponding stochastic differential equation this has been shown in Keller et al. (1984), Theorem 5.

(iii) $a = 0$, $\alpha = 1$. We give two examples for this situation.

(1) Let Z_n be a symmetric random-walk on $\mathbb{Z}^d$ and $X_n = \|Z_n\|^2$, where $\|\cdot\|$ is the Euclidean norm. Simple computations show that

$$g(X_n) = E[X_{n+1}|X_n] - X_n = 1$$

and

$$\sigma^2(X_n) = \operatorname{var}[X_{n+1} - X_n|X_n] = \frac{4}{d}X_n.$$

It is a classical result of Pólya (1921) that Z_n (and hence X_n) is recurrent if $d \leq 2$, whereas for $d \geq 3$ it is transient. In any case $Z_n/\sqrt{n}$ is asymptotically normal with mean 0 and covariance-matrix $d^{-1}Id$, such that X_n/n is asymptotically gamma-distributed with parameters $d/2$, $d/2$.

(2) A controlled Galton–Watson process with

$$E[\eta_{i,n}(k)] = 1 + A/k \quad \text{and} \quad \operatorname{var}[\eta_{i,n}(k)] = B > 0$$

(for the notation see part A of this section) is another example for the case $a = 0$, $\alpha = 1$. Such processes have been studied in Höpfner (1983), where (among others) the following result can be found: Suppose $E[|\eta_{i,n}(k)|^{2+\delta}] \leq \text{const}$ for some $\delta > 0$. Then $P(X_n \to \infty) > 0$ if and only if $B < 2A$, and in this case X_n/n is asymptotically gamma-distributed with parameters $2/B$ and $2A/B$. [Compare also Klebaner (1984).]

(iv) $a = -1$, $\alpha = 2$. Here is an example for this situation, which can be found in Guivarc'h et al. (1977): X_n is a random walk on $\{0, 1, 2, \dots\}$ with transition probabilities

$$p(0,1) = 1,\ p(i, i+1) = p_i,\ p(i, i-1) = 1 - p_i, \qquad i \geq 1,$$

where $p_i = \frac{1}{2}(1 + (\lambda/(i+\lambda))$ for some $\lambda > -\frac{1}{2}$.

Simple computations show that

$$g(X_n) = E[X_{n+1}|X_n] - X_n = \lambda/(X_n + \lambda),$$

$$\sigma^2(X_n) = g(X_n)^{-2}\operatorname{var}[X_{n+1} - X_n|X_n] = X_n(X_n + 2\lambda)/\lambda^2,$$

and $\xi_{n+1} = (X_{n+1} - X_n)/g(X_n) - 1$ is a martingale difference sequence. It is known that $X_n/\sqrt{n}$ converges weakly to a distribution on $\mathbb{R}^+$ with density $(2^{\lambda-1/2}\Gamma(\lambda + \frac{1}{2}))^{-1}x^{2\lambda}\exp(-x^2/2)$ [Chapter VI, Theorem 42 in Guivarc'h et al. (1977)] and that X_n is transient if and only if $\lambda > \frac{1}{2}$ [Chapter VI, Corollary 39 in Guivarc'h et al. (1977)].

In all of the above examples it is easy to obtain results on GX_n from those on X_n:

(i) GX_n/n converges a.e. on $\{X_n \to \infty\}$ to a nondegenerate random variable.
(ii) GX_n/n converges a.e. on $\{X_n \to \infty\}$ to a constant.
(iii) and (iv) GX_n/n is asymptotically gamma-distributed.

C. *A class of examples showing that* (A.5b) *is sharp.* As announced in Section 2 we now show that if $g(t)$, $\sigma(t)$ are functions satisfying the regularity assumptions (A.1), (A.2), and $t^{-1}g(t)\sigma(t)\sqrt{Gt} \to \infty$ as $t \to \infty$, then there is a sequence η_n of i.i.d. random variables with $E[\eta_n] = 0$, $E[\eta_n^2] = 1$ such that the process X_n defined by $X_0 = 1$, $X_{n+1} = X_n + g(X_n)(1 + \sigma(X_n)\eta_{n+1})$ violates the conclusion of Theorem 1.

To this end let $h(t) := t^{-1}g(t)\sigma(t)\sqrt{Gt} \to \infty$, $t \to \infty$. If Theorem 1 holds, then $GX_n/n \to 1$ a.e. on $\{X_n \to \infty\}$ and $P(X_n \to \infty) > 0$, and in view of the regularity assumptions on g and σ we get

$$X_{n+1} = X_n\left(1 + \frac{g(X_n)}{X_n}(1 + \sigma(X_n)\eta_{n+1})\right)$$
$$= X_n\left(1 + \frac{\hat{h}(n)}{\sqrt{n}}\eta_{n+1}\right)(1 + o(1)) \quad \text{a.e. on } \{X_n \to \infty\}.$$

As $h(t) \to \infty$, $t \to \infty$, one can find a symmetric distribution for the η_n such that $E[\eta_n] = 0$, $E[\eta_n^2] = 1$, and $\sum_{n=1}^{\infty} P(\frac{1}{2}\hat{h}(n)^2\eta_{n+1}^2 \geq n) = \infty$. Hence for infinitely many n almost surely $\hat{h}(n)n^{-1/2}\eta_{n+1} \leq -2$ and therefore $X_{n+1} < 0$ for some n almost surely, which contradicts $P(X_n \to \infty) > 0$, as the negative states are assumed to be absorbing.

APPENDIX

We start with a purely analytical lemma.

LEMMA A.1. *Assume* (A.1), (A.2) *and* (A.3), *and fix some* $0 \leq c \leq \infty$. *Consider the following five statements*:

(i) $$g'(t)\psi(t)^{1/2} \to c;$$
(ii) $$t^{-1}g(t)\psi(t)^{1/2} \to c;$$
(iii) $$g'(t)\phi(t) \to -c^2;$$
(iv) $$t^{-1}g(t)\phi(t) \to -c^2;$$
(v) $$\phi(t)\psi(t)^{-1/2} \to -c.$$

If $\psi(\infty) = \infty$ *then*

(a) (i) *implies all the other statements, and* (iii) *implies* (iv) *and* (v).
(b) *If* $c < \infty$, *then* (i) $\Leftrightarrow$ (ii) *and* (iii) $\Leftrightarrow$ (iv).
(c) *If* $c = 0$, *then all statements are equivalent.*

PROOF. We first show (a).

(i) ⇒ (v) $$\phi(t) = -\tfrac{1}{2}\int_1^t g'(s)\psi'(s)\,ds \sim -\tfrac{1}{2}c\int_1^t \psi(s)^{-1/2}\psi'(s)\,ds = -c\psi(t)^{1/2}.$$

(i) ⇒ (iii) $$g'(t)\phi(t) = g'(t)\psi(t)^{1/2}\phi(t)\psi(t)^{-1/2} \sim -c^2.$$

(i) ⇒ (ii) $$g(t)\psi(t)^{1/2} = \int_1^t g'(s)\psi(s)^{1/2}\,ds + \tfrac{1}{2}\int_1^t \sigma^2(s)\psi(s)^{-1/2}\,ds \sim ct + o(t) \quad \text{by (14)}.$$

(iii) ⇒ (iv) $$g(t)\phi(t) = \int_1^t g'(s)\phi(s)\,ds - \tfrac{1}{2}\int_1^t \sigma^2(s)g'(s)\,ds \sim -c^2t + o(t) \quad \text{by (A.3)}.$$

(iii) ⇒ (v) $$\phi(t)^2 = -\int_1^t g'(s)\psi'(s)\phi(s)\,ds \sim c^2\int_1^t \psi'(s)\,ds = c^2\psi(t).$$

This proves (v) up to a factor ± 1. If g is eventually increasing, the limit in (v) must be ≤ 0, i.e., $-c$, whereas for decreasing g assertion (i), and hence (iii), is always true with $c = 0$ [cf. (21)].

(b) Implication (ii) ⇒ (i) was shown in Lemma 3 of Keller et al. (1984) If one replaces each $\hat{\psi}(t)^{1/2}$ occurring in that proof by $\hat{\phi}(t)$, one gets a proof of (iv) ⇒ (iii).

(c) We show that (v) ⇒ (i) if $c = 0$:

$$\left|g'(t)\psi(t)^{1/2}\right| = \left|g'(t)\frac{\psi(t)}{\phi(t)}\right| \left|\phi(t)\psi(t)^{-1/2}\right| = O\left(\left|\phi(t)\psi(t)^{-1/2}\right|\right) \to 0$$

by the monotonicity of $g'(t)$ for large t. □

Next we prove two inequalities of the Hájek–Rényi type, which imply as corollaries strong laws of large numbers. Similar results can be found in the literature, cf., e.g., Theorem 2.19 in Hall and Heyde (1980).

THEOREM A.2. *Let $\mathscr{F}_n$, $n \geq 0$, be an increasing sequence of σ-algebras and let Y_n be $\mathscr{F}_n$-measurable random variables satisfying $P(|Y_{n+1}| \geq x|\mathscr{F}_n) \leq F(x)$ a.e. for all $x \geq 0$, where $F(x)$ is a decreasing function with $\int_0^\infty F(x)\,dx = -\int_0^\infty x\,dF(x) < \infty$. Suppose further that h: $\mathbb{R}^+ \to \mathbb{R}^+$ is a bounded, ultimately decreasing function with $H(t) := \int_0^t h(s)\,ds \to \infty$ as $t \to \infty$. Let τ be a stopping rule adapted to $(\mathscr{F}_n)$ and set*

$$S_n = \sum_{i=1}^n h(i)\left(Y_{\tau+i} - E\left[Y_{\tau+i}|\mathscr{F}_{\tau+i-1}\right]\right).$$

(a) *For each* $\varepsilon > 0$

$$P\left(\max_{n\geq N} H(n)^{-1}|S_n| \geq \varepsilon \middle| \tau < \infty\right) \to 0$$

as $N \to \infty$ *uniformly in* τ.

(b) *If* $h(t) \equiv 1$, *then for each* $\varepsilon > 0$

$$P\left(\max_{n\geq 1} (n + 2C)^{-1}|S_n| \geq \varepsilon \middle| \tau < \infty\right) \to 0$$

as $C \to \infty$ *uniformly in* τ.

PROOF. Let $U_n = Y_n I\{|Y_n| \leq n - \tau + C\}$, $\bar{S}_0 = 0$,

$$\bar{S}_n = \sum_{i=1}^{n} h(i)(U_{\tau+i} - E[U_{\tau+i}|\mathscr{F}_{\tau+i-1}]), \quad \text{for } n \geq 1.$$

$(\bar{S}_n, \mathscr{F}_{\tau+n})$ is a martingale with respect to the probability $P_\tau = P(\cdot|\tau < \infty)$, and

$$P_\tau(|U_{\tau+n}| \geq x|\mathscr{F}_{\tau+n-1}) \leq F(x)I\{x \leq n + C\} \quad \text{a.e.}$$

We start by proving the theorem for $\bar{S}_n$ (in place of S_n) and have to show that for each $\varepsilon > 0$,

$$p := P_\tau\left(\sup_{n\geq N} (H(n) + 2Ch(n))^{-1}|\bar{S}_n| \geq \frac{\varepsilon}{2}\right)$$

tends to 0 uniformly in τ in each of the following two cases:

(a′) $N = 1$, $h(t) \equiv 1$, $C \to \infty$.

(b′) $C = 0$, $t_0 \leq N \to \infty$, where t_0 has been chosen such that $h(t)$ is decreasing for $t \geq t_0$.

Observe that in both cases $H(t) + Ch(t)$ is monotonically increasing for $t \geq 0$, and we can apply the Hájek–Rényi inequality in order to estimate p,

$$\begin{aligned}\frac{\varepsilon^2}{4}p &\leq (H(N) + 2CH(n))^{-2}E_\tau[\bar{S}_N^2] \\ &\quad + \sum_{n=N+1}^{\infty} (n - t_0 + 2C)^{-2}E_\tau[U_{\tau+n}^2] \\ &=: \gamma_1 + \gamma_2,\end{aligned}$$

and

$$\begin{aligned}\gamma_2 &\leq 2\sum_{n=N+1}^{\infty} (n - t_0 + 2C)^{-2}\int_0^{n+C} xF(x)\,dx \\ &\leq 2\sum_{n=N+1}^{\infty} \sum_{1\leq i<n+C+1} (n - t_0 + 2C)^{-2} i\int_{i-1}^{i} F(x)\,dx \\ &= 2\sum_{i=1}^{\infty}\left(\sum_{n>i-C-1,\, n>N} (n - t_0 + 2C)^{-2}\right) i\int_{i-1}^{i} F(x)\,dx \\ &\leq \text{const.}\sum_{i=1}^{\infty} \frac{i}{\max\{i, N\} + C - t_0}\int_{i-1}^{i} F(x)\,dx.\end{aligned}$$

Observe now that

$$\sum_{i=1}^{\infty} \int_{i-1}^{i} F(x)\,dx = \int_{0}^{\infty} F(x)\,dx < \infty.$$

In case (a′) ($N = 1$), $\gamma_2 \to 0$, $C \to \infty$, and $\gamma_1 \to 0$, $C \to \infty$. In case (b′) ($C = 0$), $\gamma_2 \to 0$, $N \to \infty$ and

$$\begin{aligned}
\gamma_1 &= H(N)^{-2} \sum_{n=1}^{N} h(n)^2 E_\tau\left[U_{\tau+n}^2\right] \\
&\le 2H(N)^{-2} \sum_{n=1}^{N} h(n)^2 \sum_{i=1}^{n} i \int_{i-1}^{i} F(x)\,dx \quad \text{(as above)} \\
&= 2H(N)^{-2} \sum_{i=1}^{N} \left(\sum_{n=i}^{N} h(n)^2 \right) i \int_{i-1}^{i} F(x)\,dx \\
&\le 2H(N)^{-1} \left(\sum_{i=1}^{N} i h(i) \int_{i-1}^{i} F(x)\,dx + \text{const} \right) \\
&\le 2 \sum_{i=1}^{N} \frac{H(i) + \text{const}}{H(N)} \int_{i-1}^{i} F(x)\,dx + \text{const}/H(N) \\
&\to 0, \qquad N \to \infty \text{ uniformly in } \tau.
\end{aligned}$$

Here we have used that $h(n) \le h(i)$ for $n \ge i \ge t_0$, $\Sigma_{n=i}^{N} h(i) \le H(N)$ for $i \ge t_0$ and $ih(i) \le H(i) + \text{const}$.

Hence in both cases $\underline{p} \to 0$ as $C \to \infty$ or $N \to \infty$, respectively, uniformly in τ. In order to pass from $\overline{S}_n$ to S_n we must compare the $Y_{\tau+i}$ to the $U_{\tau+i}$ and the $E[U_{\tau+i} | \mathscr{F}_{\tau+i-1}]$ to the $E[Y_{\tau+i} | \mathscr{F}_{\tau+i-1}]$:

$$\begin{aligned}
P_\tau\{\exists i \ge N \colon Y_{\tau+i} \ne U_{\tau+i}\} &\le \sum_{i=N}^{\infty} P_\tau\{|Y_{\tau+i}| > i + C\} \\
&\le \sum_{i=N}^{\infty} F(i + C) \le \int_{C+N-1}^{\infty} F(x)\,dx \to 0
\end{aligned} \tag{63}$$

as $C \to \infty$ or $N \to \infty$ uniformly in τ.

Finally, for each n one has

$$\begin{aligned}
&H(n)^{-1} \sum_{i=1}^{n} h(i) \left| E_\tau[Y_{\tau+i} - U_{\tau+i} | \mathscr{F}_{\tau+i-1}] \right| \\
&\quad \le H(n)^{-1} \sum_{i=1}^{n} h(i) E_\tau\left[\, |Y_{\tau+i}| I\{|Y_{\tau+i}| > i + C\} | \mathscr{F}_{\tau+i-1}\right] \\
&\quad = H(n)^{-1} \sum_{i=1}^{n} h(i) \int_{i+C}^{\infty} P_\tau\left(|Y_{\tau+i}| \ge x | \mathscr{F}_{\tau+i-1}\right) dx \\
&\quad \le H(n)^{-1} \sum_{i=1}^{n} h(i) \int_{i+C}^{\infty} F(x)\,dx \\
&\quad < \frac{\varepsilon}{2},
\end{aligned}$$

for all $n \geq 1$ and τ if C is big enough (case a′) and for $C = 0$ and all τ if $n \geq N$ is big enough (case b′). As $p \to 0$ ($C \to \infty$ or $N \to \infty$), the theorem follows from (63) and the last estimate. □

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