*The Annals of Probability*
1987, Vol. 15, No. 2, 690–707

# RECURRENCE AND INVARIANT MEASURES FOR DEGENERATE DIFFUSIONS

BY WOLFGANG KLIEMANN

*Iowa State University*

For a second-order (hypoelliptic) operator $\mathscr{A} = A_0 + \frac{1}{2}\Sigma_{i=1}^m A_i$ on a $d$-dimensional manifold $M^d$, let $x_t$ be the diffusion governed by $\mathscr{A}$ and $\varphi(t)$ its associated deterministic control system. We investigate the relations between transience, recurrence and (finite) invariant measures for $x_t$ using the control theoretic decomposition of $M^d$ with respect to $\varphi(t)$. On the invariant control sets for $\varphi(t)$ we obtain the same classification for $x_t$ as is well known for the nondegenerate case, while outside these sets the diffusion $x_t$ is transient.

**1. Introduction.** Degenerate elliptic operators of second order and their associated diffusion processes have been studied from various points of view, see, e.g., [16] or [26]. In this paper their qualitative behavior, such as transience, (positive) recurrence, existence of invariant measures is classified.

For nondegenerate diffusions this topic was investigated, e.g., in [4], [5] and [15]; some results for strong Feller processes are obtained in [2], [18] and [23]. On the other hand, Azéma, Kaplan-Duflo and Revuz in [2] and [3] and Getoor in [13] considered Hunt processes with their fine topology; see [19] for a survey. This paper intends to fill the gap between these approaches: We study Feller diffusions, where the recurrence notions are with respect to the given topology of a manifold.

This is motivated, e.g., by the study of stochastic systems of the form $\dot{x} = f(x,\xi)$, where the process $\xi$ is governed by some stochastic differential equation. So the pair $(x, \xi)$ is given, e.g., by

$$d\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} f(x,\xi) \\ a(\xi) \end{pmatrix} dt + \begin{pmatrix} 0 \\ A(\xi) \end{pmatrix} dW_t,$$

with suitable initial conditions, giving the pair $(x, \xi)$ as a continuous Feller process, which is not strong Feller in general.

In [1] we introduced weak recurrence notions to take into account the degeneracy of such systems. Here we introduce more smoothness in assuming that the coefficients are $C^\infty$ on a $C^\infty$ manifold. It turns out that the (classical) transience and recurrence notions are sufficient in this case to characterize the qualitative behavior.

We consider diffusion processes whose differential generator is given in Hörmander form, i.e.,

$$dx_t = A_0(x_t)\,dt + \sum_{i=1}^m A_i(x_t) \circ dW_t, \tag{1.1}$$

Received July 1983; revised July 1986.
*AMS* 1980 *subject classification*. 60J60.
*Key words and phrases*. Degenerate diffusions, hypoellipticity, recurrence, invariant measures, geometric control theory.

on a $d$-dimensional $C^\infty$ manifold $M^d$ with $C^\infty$ vector fields $A_0, \dots, A_m$, "$\circ$" denoting the symmetric stochastic integral. Then $x_t$ is a continuous Feller process with generator

$$\mathscr{A} = A_0 + \tfrac{1}{2} \sum_{i=1}^{m} A_i^2; \tag{1.2}$$

compare, e.g., [17], Chapter V, for the setup. Denote by $\mathscr{L} = \mathscr{L}\mathscr{A}\{A_0, \dots, A_m\}$ the Lie algebra generated by $A_0, \dots, A_m$ and by $\mathscr{L}_0$ the ideal in $\mathscr{L}$ generated by $A_1, \dots, A_m$. In order to use geometric control theory we will assume that the distribution $\Delta_{\mathscr{L}}$, defined by $\mathscr{L}$ has the maximal integral manifold property, see [30], which holds, e.g., if $\mathscr{L}$ is locally finitely generated (in particular if everything is analytic) or $\dim \Delta_{\mathscr{L}}(x) = \text{const.}$ for all $x \in M$. Thus $M$ splits into maximal integral manifolds $I_\alpha$ of $\mathscr{L}$. In view of the support theorems (cf. [17], [21], [27] and [28]) we have for the induced probability measures $P_x$ on $C(M)$ that $\operatorname{supp} P_x$ is contained in the closure of the continuous paths on $I_\alpha$, if $x \in I_\alpha$. Hence without loss of generality we can view $I_\alpha$ as the new state space of the diffusion process (1.1). So we assume throughout this paper

$$\text{(H)} \qquad \dim \Delta_{\mathscr{L}}(x) = d, \quad \text{for all } x \in M^d.$$

And since we do not deal with the question of explosion of $x_t$, all vector fields in $\mathscr{L}$ are assumed to be complete, i.e., the systems group is defined for all $t \in \mathbb{R}$.

While $x_t$ is always a Feller diffusion, it is strongly Feller if and only if

$$\text{(J)} \qquad \dim \Delta_{\mathscr{L}_0} = d, \quad \text{for all } x \in M^d,$$

cf. [16]. Hence if $\dim \Delta_{\mathscr{L}_0}(x) = d - 1$ for some $x \in M$, we have to deal with the situation mentioned in the beginning, since then $x_t$ is merely Feller.

In order to characterize the transient and recurrent points of (1.1) precisely for Feller or strong Feller diffusions, we associate to (1.1) a deterministic control system

$$\dot{\varphi}(t) = A_0(\varphi(t)) + \sum_{i=1}^{m} u_i(t) A_i(\varphi(t)), \tag{1.3}$$

where $u$ is taken from the set of admissible controls $\mathscr{U}$, the piecewise continuous functions with values in $\mathbb{R}^m$. Let us denote by $\varphi(t, x, u)$ the solution of (1.3) at time $t$ with initial value $x$ under the control action $u$. Then the positive orbit of $x \in M$ at time $t$ is defined as $\mathscr{O}^+(x, t) = \{y$, there exists $u \in \mathscr{U}$ such that $y = \varphi(t, x, u)\}$ and $\mathscr{O}^+(x) = \bigcup_{t \geq 0} \mathscr{O}^+(x, t)$. In terms of control systems, condition (H) means accessibility, i.e., $\operatorname{int} \mathscr{O}^+(x) \neq \varnothing$ for all $x \in M$, while (J) means strong accessibility, i.e., $\operatorname{int} \mathscr{O}^+(s, t) \neq \varnothing$ for all $t > 0$. Surprisingly enough manifolds on which (H) holds for some $\mathscr{A}$, but not (J), have some global geometric properties; e.g., in the analytic case, if the universal covering space of $M$ is compact, then (H) implies (J), compare [7] for a discussion of related results.

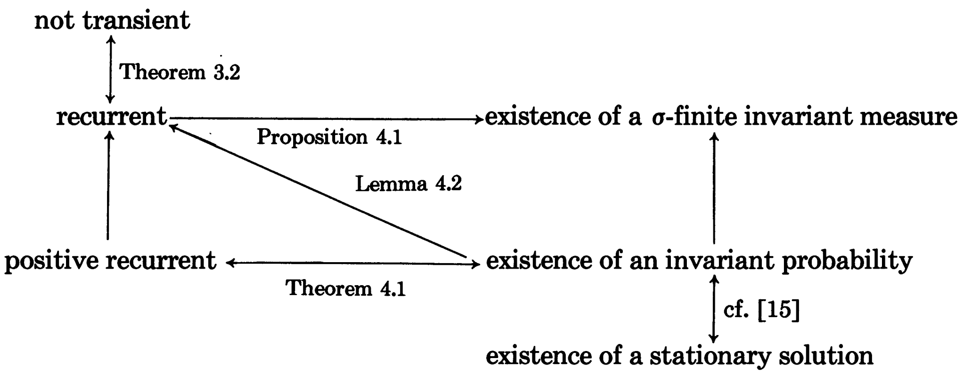
To state the main results of this paper, we need the notion of invariant control sets. To this end, let $A \subset M$ be such that for all $x, y \in A$ we have $y \in \overline{\mathscr{O}^+(x)}$. For any such $A$ there exists a unique maximal set $B \supset A$ with this property, see [1]. Those sets are called control sets for (1.3). [Notice that always

$x \in \mathcal{O}^+(x)$, but we include as control sets only those one point sets, which are steady states, i.e., $A_0(x) = A_i(x) = 0$.]

DEFINITION 1.1. A set $B \subset M$ is called invariant for (1.3) if $\mathcal{O}^+(x) \subset \overline{B}$ for all $x \in B$. A control set $C \subset M$ is invariant, if $\overline{\mathcal{O}^+(x)} = \overline{C}$ for all $x \in C$.

Now we can summarize our main results as follows:

Any $x \in M$ not lying in some invariant control set for (1.3) is transient for (1.1) and on invariant control sets we have the following diagram:




Counterexamples concerning the missing implications are furnished by the Wiener process in $\mathbb{R}^2$ or $\mathbb{R}^3$.

This picture is exactly the same as for nondegenerate diffusions. Notice that in the nondegenerate case, where $\dim \Delta_{\mathcal{L}_1}(x) = d$ for all $x \in M$ with $\mathcal{L}_1 = \mathcal{L}\mathcal{A}(A_1, \dots, A_m)$, the whole manifold $M^d$ is the invariant control set for (1.3).

We first study in Section 2 some properties of invariant control sets from the point of view of geometric control theory, making heavy use of the hypoelliptic-ity condition (H). Our method then uses, as is often done in recurrence studies, embedded Markov chains. But as our processes are not strongly controllable (cf., e.g., [22]), we have to pay attention to the time evolution of the diffusion, i.e., we give estimates on upper and lower bounds for exit and entrance times. For the uniform behavior of $x_t$ on invariant control sets continuity properties of exces-sive function are needed. In Section 3, we discuss the transience–recurrence dichotomy and in Section 4, the relations between (positive) recurrence and invariant measures are shown.

Beforehand we fix some notation:

$P_x$ diffusion measure of $x_t$ with initial value $x \in M$;
$E_x$ corresponding expectation;
$P(t, x, \cdot)$ transition probability;
$T_t$ associated semigroup;
$U(x, V) = \int P(t, x, V)\,dt$ for $V$ a Borel set in $M$;
$\sigma_U$ first entrance time of a set $U$;
$\tau_U$ first exit time;
$\theta_t$ shift operator in the space of trajectories.

For a subset $U \subset M$ we use

$\overline{U}, U^c, \partial U$ to denote the closure, the complement, and the boundary of $U$.

Since the qualitative theory for one-dimensional strong Markov processes is completely known, cf. [14], we restrict ourselves to the case $d \geq 2$.

**2. Properties of control sets.** The geometric theory for nonlinear control systems (see, e.g., [8] or [29]) is concerned with accessibility and controllability problems for systems of the form

$$\dot{\varphi}(t) = A_0(\varphi) + \sum_{i=1}^{m} A_i(\varphi) u_i(t) \tag{2.1}$$

on manifolds $M^d$. We assume all data to be $C^\infty$ and suppose (H) to hold true. Then we have the following results for invariant control sets of (2.1):

LEMMA 2.1. *Let $C \subset M$ be an invariant control set for* (2.1). *Then*

(i) *$C$ is closed in $M$;*
(ii) $\operatorname{int}_M C \neq \varnothing$;
(iii) *$C$ is $C$-invariant, i.e., for all $x \in C$, $\mathcal{O}^+(x) \subset C$;*
(iv) *$\mathcal{O}^+(x) = \operatorname{int} C$ for all $x \in \operatorname{int} C$;*
(v) *$C$ is path connected and two invariant control sets $C_1, C_2$ are either identical or $d(C_1, C_2) > 0$, where $d(A, B) = \inf_{x \in A,\ y \in B}\{\rho(x, y)\}$, $\rho$ a Riemannian metric on $M$;*
(vi) *there are at most countably many invariant control sets in $M$.*

PROOF. (i) $C$ invariant means $\overline{\mathcal{O}^+(x)} \subset \overline{C}$ for all $x \in C$. Let $y \in \overline{C}$. Then first of all $\overline{\mathcal{O}^+(y)} \subset \overline{C}$ because: if there exist $t_0 > 0$ and $u_0 \in \mathcal{U}$ such that $y_0 = \varphi(t_0, y, u_0) \notin \overline{C}$, then there is an open neighborhood $U(y_0)$ with $U(y_0) \cap \overline{C} = \varnothing$ and hence an open neighborhood $U(y)$ such that $\varphi(t_0, z, u_0) \in U(y_0)$ for all $z \in U(y)$ and $C$ would not be invariant. Now for all $y \in \overline{C}$ by (H), $\operatorname{int} \mathcal{O}^+(y) \neq \varnothing$ and hence there exists $\tilde{x} \in \operatorname{int} \mathcal{O}^+(y) \cap C$. On the other hand, $\overline{\mathcal{O}^+(\tilde{x})} = \overline{C}$ and thus $y \in C$.

(ii) Since $C$ is invariant $\operatorname{int} \overline{C} \supset \operatorname{int} \overline{\mathcal{O}^+(x)} \neq \varnothing$ for all $x \in C$ and by (i) $C = \overline{C}$.

(iii) For all $x \in C$ we have $\overline{\mathcal{O}^+(x)} \subset \overline{C} = C$.

(iv) First note that for all $x \in C$, $\overline{\mathcal{O}^+(x)} = C$ by invariance and control set property and thus $\operatorname{int} \mathcal{O}^+(x) = \operatorname{int} \overline{\mathcal{O}^+(x)} = \operatorname{int} C$ by [22], Proposition 4.2. To see that $\mathcal{O}^+(x) \subset \operatorname{int} C$ for $x \in \operatorname{int} C$ we use coordinates and the characterization of $\partial C$ in [21], page 42: Each $A_i$, $i = 1, \dots, m$, is tangent to $\partial C$ and $A_0$ is in the inner direction. If $A_0$ is tangential at $y \in \partial C$, then for each exterior normal vector $\nu$ in $x_0$, $A_0(x_0)$ and $\nu$ are orthogonal. Hence there exists an open ball, $B$, centered at $x_1$ with $x_0 \in \partial B(x_1)$ such that all $A_i$, $i = 0, \dots, m$, are tangent to $B$ at $x_0$ and $\mathcal{LA}(A_i, i = 0, \dots, m)(x_0) < d$. So $A_0$ is strictly in the inner direction for all $x \in \partial C$.

(v) $C$ is path connected by (iv). Let $y \in C_1 \cap C_2$, then by (ii) there exist $t_0 > 0$, $u_0 \in \mathscr{U}$, $x \in \operatorname{int} C_1$, such that $x = \varphi(t_0, y, u_0)$, hence by continuous dependence on initial conditions there exist $\tilde{x} \in \operatorname{int} C_2$ with $\varphi(t_0, \tilde{x}, u_0) \in \operatorname{int} C_1$, which is a contradiction to the maximality of control sets. The last assertion follows from (i).

(vi) Since $M$ has a countable dense subset, there are at most countably many invariant control sets by (ii). □

If $M$ is compact, we have furthermore:

LEMMA 2.2. *Let $M$ be compact. Then*
(i) *there exists at least one invariant control set;*
(ii) *there exist at most finitely many invariant control sets.*

PROOF. (i) We show that for each $x_0 \in M$ there is an invariant control set $C_{x_0} \subset \overline{\mathcal{O}^+(x_0)}$: for $x_0 \in M$ consider the collection $\chi := \{\overline{\mathcal{O}^+(x)},\ x \in \overline{\mathcal{O}^+(x_0)}\} \neq \varnothing$. All sets in $\chi$ are invariant, $\chi$ satisfies the assumption in Zorn's lemma w.r.t. set theoretic inclusion "$\subset$." Thus there is a minimal element $C_{x_0} = \overline{\mathcal{O}^+(x_1)}$ for some $x_1 \in \overline{\mathcal{O}^+(x_0)}$. $C_{x_0}$ is invariant and $\overline{\mathcal{O}^+(x)} = C_{x_0}$ for all $x \in C_{x_0}$ because $C_{x_0}$ is minimal, hence $C_{x_0}$ is an invariant control set.

(ii) Assume that $\{C_i,\ i \in \mathbb{N}\}$ is a collection of disjoint invariant control sets in $M$. Choose $x_i \in \operatorname{int} C_i$ and denote by $x$ the limit of a subsequence $x_n$. Let $y \in \operatorname{int} C_x$ defined in (i), then there exist $t_0, u_0$ such that $y = \varphi(t_0, x, u_0)$ by Lemma 2.1. For an open neighborhood $U(y) \subset \operatorname{int} C_x$ there exists an open neighborhood $V(x)$ such that $\varphi(t_0, z, u_0) \in U(y)$ for all $z \in V(x)$. Hence for some $N \in \mathbb{N}$ we have $C_n = C_x$ for all $n \geq N$ by Lemma 2.1. □

Noninvariant control sets need not have a nonvoid interior, even if they are compact:

EXAMPLE 2.1. Consider in $\mathbb{R}^2 \sim \{0\}$ the control system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -u^2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Its eigenvalues are $\lambda_{1,2} = -u^2/2 \pm (\frac{1}{4}u^4 - 1)^{1/2}$, hence for $u \equiv 0$ the system moves on circles centered at 0, so any of these circles is contained in a control set. For $u \not\equiv 0$, $\operatorname{Re} \lambda < 0$, hence the circles are the control sets, but they have void interior and none of them is $C$-invariant.

In view of Lemma 2.1 an invariant control set can serve as a new state space of the system (2.1) with lifetime $\infty$. If a trajectory leaves a control set it will never hit it again, so for noninvariant control sets it is sufficient to consider the trajectories until their first exit from $C$, which amounts to a system with finite explosion time:

LEMMA 2.3. (i) *Let $D$ be a noninvariant control set, $x \in D$. Then for all* $y \in \overline{\mathcal{O}^+(x)} \sim \overline{D}$ *we have* $\overline{\mathcal{O}^+(y)} \cap \overline{D} = \varnothing$.

(ii) *Let $x \in M^d$ be in no control set. Then for all $U(x)$ there exists a $V(x)$ such that for all* $y \in \overline{\mathcal{O}^+(x)} \sim U(x)$ *we have* $\overline{\mathcal{O}^+(y)} \cap V(x) = \varnothing$.

PROOF. (i) By contraposition. Assume there is $z \in \overline{\mathcal{O}^+(y)} \cap D$, then for any open neighborhood $U(x)$ there exists an open neighborhood $V(z)$ such that $\mathcal{O}^+(\hat{z}) \cap U(x) \neq \varnothing$ for all $\hat{z} \in V(z)$ since $z$ is in a control set by continuous dependence of the solutions of (2.1) on initial conditions. On the other hand, $z \in \overline{\mathcal{O}^+(y)}$, i.e., there exists $\tilde{z} \in \mathcal{O}^+(y) \cap V(z)$ and hence $x \in \overline{\mathcal{O}^+(y)}$. We also have $y \in \overline{\mathcal{O}^+(x)}$ which would entail $y \in D$ by the maximality of control sets.

(ii) is proven just the same way. □

The proof of Lemma 2.3(i) in particular shows that for all $x \in D$ there is an open neighborhood $U(x)$ such that for all $y \in \overline{O^+(x)} \sim D$ we have $\overline{O^+(y)} \cap U(x) = \varnothing$.

In [20] Kolmogorov introduced the notion of essential states and essential classes for Markov chains, and in [2] Azéma, Kaplan-Duflo and Revuz defined these concepts for Markov processes with respect to the fine topology. In our situation one obtains the following:

For any $x \in M$ we define $E(x) = \{y, P_x\{\sigma_{U(y)} < \infty\} > 0$ for all open neighborhoods $U(y)\}$. Then $x \in M$ is called an essential state if for all $y \in E(x)$ we have $x \in E(y)$. Notice that if $x$ is essential, so are all $y \in E(x)$, and if $x, y, z$ are essential and $y \in E(x)$, $z \in E(y)$, then $z \in E(x)$. Hence for two essential states $x$ and $y$, $E(x)$ and $E(y)$ are either identical or disjoint.

DEFINITION 2.1. $E \subset M$ is an essential class for $x_t$, defined by (1.1), if $E = E(x)$ for some essential state $x \in M$.

In other words, $E$ is an essential class, if for all $x, y \in E$: $y \in E(x)$, and $E$ is maximal with respect to this property. The connection with control sets is given by

PROPOSITION 2.1. *The essential classes of* (1.1) *coincide with the invariant control sets of* (2.1).

PROOF. If $C$ is a control set of (2.1), then for all $x \in C$ and any open neighborhood $U(y)$ of $y \in C$, there exists $t_0 > 0$ such that $P(t_0, x, U(y)) > 0$ by the support theorem, hence $P_x\{\sigma_{U(y)} < \infty\} > 0$. Furthermore, if $C$ is invariant, then it is $C$-invariant by Lemma 2.1, hence $P_x\{x_t \in C \text{ for all } t \geq 0\} = 1$ and $C$ is an essential class.

To see the converse notice that $P_x\{\sigma_{U(y)} < \infty\} > 0$ implies the existence of an $\omega \in \Omega$ such that $x(t_\omega, x, \omega) \in U(y)$, where the trajectory $x(\cdot, x, \omega)$ is continuous. By the support theorem we infer the existence of an $u \in \mathscr{U}$ with $\varphi(t_\varphi, x, u) \in U(y)$, showing that an essential class is a control set. Now $\{y, P_x\{\sigma_{U(y)} < \infty\} > 0 \text{ for all } U(y)\} = \overline{\mathscr{O}^+(x)}$ implies that essential classes are invariant control sets. □

**3. Transience and recurrence.** In this section we establish the transience–recurrence dichotomy for diffusions on invariant control sets and discuss the long-term behavior of diffusions starting in transient points. Our concepts are as follows:

DEFINITION 3.1. A point $x \in M$ is called *recurrent* for $x_t$, if for all open neighborhoods $U(x)$ we have $P_x\{\mathscr{R}_U\} = 1$, where the event $\mathscr{R}_U$ is defined by $\mathscr{R}_U = \{\omega, x_{t_n}(\omega) \in U \text{ for a sequence } t_n \uparrow \infty\}$. A point $x \in M$ is *transient* for $x_t$, if there exists an open neighborhood $V(x)$ such that $P_x\{\mathscr{T}_V\} = 1$ with $\mathscr{T}_V = \{\omega, \text{ there exists } t_0 > 0 \text{ and } x_t(\omega) \notin V \text{ for all } t \geq t_0\}$. The diffusion $x_t$ is recurrent (or transient) on a set $A \subset M$ if all $x \in A$ are recurrent (or transient).

In order to state a transience–recurrence dichotomy for points in $M$ and a uniform behavior on invariant control sets, we need some continuity property of the "first hitting map" $u\colon M \to \mathbb{R}$, $u(x) = P_x\{\sigma_U < \infty\}$. In our context we have the following result (compare [26]):

LEMMA 3.1. *Let $x_t$ be a diffusion defined by* (1.1) *and assume* (H) *holds. Then the excessive functions of $x_t$ are lower semicontinuous* (*lsc*).

PROOF. We first show that the operator $U^\alpha f(x) = \int_0^\infty e^{-\alpha t} T_t f(x)\,dt$, $\alpha > 0$, maps bounded measurable functions $f$ into continuous ones: By the proof of [6], Theorem 6.1, for each $\alpha > 0$ there exists a Green's function $g_\alpha(x, y)$ such that $U^\alpha f(x) = \int g_\alpha(x, y) f(y)\,dy$ for all $x \in \mathbb{R}^d$, $f \in B(\mathbb{R}^d)$, where $g$ is nonnegative and $C^\infty$ except on the diagonal set $\{(x, x), x \in \mathbb{R}^d\}$. For these points we force $g_\alpha$ to be lsc by defining $g_\alpha(x, x) = \lim_{y \to x,\, z \to x} g_\alpha(y, z)$. It follows that $U^\alpha f(x)$ is lsc, if $f$ is nonnegative. In the same way for any $c \geq |f|$ we have that $U^\alpha(c - f)(x)$ is lsc. Now $c/\alpha = U^\alpha c = U^\alpha f(x) + U^\alpha(c - f)(x)$ proves that $U^\alpha f(x)$ and $U^\alpha(c - f)(x)$ are continuous.

Now if $f$ is excessive for $x_t$, then $\alpha U^\alpha f \leq f$ and $\alpha U^\alpha f \uparrow f$ as $\alpha \uparrow \infty$. Using the above result we see that $f$ is lsc as the monotone limit of lsc functions. □

Using Lemma 3.1 and the arguments in [2] we have the following dichotomy for points:

PROPOSITION 3.1. *Let $x_t$ be defined by* (1.1), *then each point in $M$ is either transient or recurrent. Furthermore,*

*$x$ is recurrent iff $\int P(t, x, U)\,dt = \infty$, for all open neighborhoods $U$ of $x$;*

*$x$ is transient iff $\int P(t, x, V)\,dt < \infty$, for some open neighborhood $V$ of $x$.*

The connection with the notions in [1] is:

COROLLARY 3.1. *For diffusions as in Proposition* 3.1 *we have*

(i) *recurrence and "strong recurrence" are equivalent;*
(ii) *transience and "weak transience" are equivalent.*

The proof just exploits the fact that strongly recurrent points cannot be transient, hence are recurrent by Proposition 3.1.

Our next goal is to show the equivalence of difference notions of recurrence (as compiled, e.g., in [5]) for degenerate diffusions on invariant control sets. We prepare some lemmas:

LEMMA 3.2. *Let* $x, y \in M$ *such that* $y \in \overline{\mathcal{O}^+(x)}$. *Then* $P_x\{\sigma_{U(y)} < \infty\} > 0$ *for all open neighborhoods* $U(y)$. *Furthermore, let* $C$ *be a control set and* $x, y \in C$. *Denote* $\sigma_U(t_0) = \inf\{t > t_0,\ x_t \in U\}$, *then* $P_x\{\sigma_{U(y)}(t_0) < \infty\} > 0$ *for all* $t_0 \geq 0$.

PROOF. The first assertion follows directly from the support theorem. The second one exploits the fact for $x, y \in C$, $x \neq y$, and for any $t_0 > 0$ there exists a control $u_0$ on $[0, t]$ for some $t > t_0$ such that $\varphi(t, x, u_0) \in U(y)$. □

LEMMA 3.3. *Let* $C$ *be an invariant control set for* (2.1) *on* $M$, $U \subset C$ *an open set with compact closure* $\overline{U} \subset \operatorname{int} C$. *Then there exist* $t_0 > 0$, $\alpha_0 > 0$, *such that* $\sup_{y \in \overline{U}} P_y\{\tau_U > t_0\} \leq \alpha_0 < 1$.

PROOF. For all $x \in U$ there exists $t_x > 0$ such that $P_x\{t_U \geq t_x\} < p_x < 1$ by Lemma 3.2. Now Lemma 5.4 in [10] shows that for all $x \in \overline{U}$ there exists an open neighborhood $U(x)$ such that $P_z\{\tau_U \geq t_x\} < p_x < 1$ for all $z \in U(x)$. The assertion follows from the compactness of $\overline{U}$. □

LEMMA 3.4. *Let* $C$ *and* $U$ *be as in Lemma* 3.3. *Then* $\sup_{x \in M} E_x \tau_U < \infty$.

PROOF. Applying Lemma 4.3 in [10] we see the above lemma yields

$$E_x \tau_U \leq t_0/(1 - \alpha_0) < \infty \text{ for all } x \in M. \qquad \square$$

In view of Lemma 3.2 the statements of Lemmas 3.3 and 3.4 remain valid for all $\tau_U(\hat{t}) = \inf\{t > \hat{t},\ x_t \notin U\}$.

LEMMA 3.5. *Let* $C$ *and* $U$ *be as in Lemma* 3.3. *Then there exists a closed set* $F \subset U$ *and* a $t_1 > 0$, $\alpha_1 > 0$, *such that* $\inf_{y \in F} P_y\{\tau_U > t_1\} \geq \alpha_1 > 0$.

PROOF. Since for $U_0 \subset U_1$ we have $\tau_{U_0} \leq \tau_{U_1}$, we may use local charts. Let $x \in U$, $U(x)$ a coordinate neighborhood of $x$ in $U$, with image $V \subset \mathbb{R}^d$ where $x$ is mapped onto 0. Then there exists an open ball $B(0, r) \subset V$. Define $\hat{F} = \overline{B(0, r/4)}$ and $W = \{x,\ r/2 < |x| < 3r/4\}$. For each $z \in \hat{F}$ there exists $u_z, t_z$ such that $|\varphi(t_z, z, u_z)| = 2r/3$, hence an open neighborhood $V(z)$ such that

$\varphi(t_z, y, u_z) \in W$ for all $y \in V(z)$. Thus $P_y\{\tau_{B(0,r)} > t_z\} \geq \alpha_z > 0$. Now compactness of $\hat{F}$ completes the proof. □

LEMMA 3.6. *Let $C$ and $U$ be as in Lemma 3.3, $K \subset C$ compact. Then there exist $t_2 > 0$, $\alpha_2 > 0$, such that* $\inf_{y \in K} P_y\{\sigma_U < t_2\} \geq \alpha_2 > 0$.

PROOF. By Lemma 3.2 we have for all $x \in K$: There exists $t_x$, $\alpha_x$ such that $P_x\{\sigma_u < t_x\} > \alpha_x$. Again using continuous dependence, we see that there exists an open neighborhood $U(x)$ such that $P_z\{\sigma_U < t_x\} > \alpha_x$ for all $z \in U(x)$ and compactness of $K$ finishes the proof. □

With these preparations we are able to prove the equivalence of certain notions of recurrent diffusions on invariant control sets:

THEOREM 3.1. *Let $x_t$ be a diffusion on $M$, defined by* (1.1) *and $C \subset M$ an invariant control set. Then the following statements are equivalent*:

(i) *$x_t$ is recurrent on $C$*;
(ii) $P_x\{\mathscr{R}_U\} = 1$ *for all $x \in C$, all open sets $U \subset C$*;
(iii) $P_x\{x_t \in U$ *for some* $t > 0\} = 1$ *for all $x \in C$, all open sets $U \subset C$*;
(iv) *there exists a compact set $K \subset C$ such that* $P_x\{x_t \in K$ *for some* $t > 0\} = 1$ *for all $x \in C$.*

PROOF. (i) ⇒ (ii) Fix $x \in M$, $y \in \overline{\mathcal{O}^+(x)}$, and let $U(y)$ denote an open neighborhood of $y$, denote $u(x) = P_x\{\sigma_{U(y)} < \infty\}$. By Lemma 3.2 $u(x) > \alpha > 0$ for some $\alpha \in \mathbb{R}$. Now $V = \{z, u(z) > \alpha/2\}$ is open by Lemma 3.1 and $P_x\{\mathscr{T}_V\} = 0$ because of the recurrence of $x$. Hence $\limsup_{t\to\infty} u(x_t(\omega)) > \alpha/2$ $P_x$-a.s. and thus $P_x\{\mathscr{T}_U\} = 0$.

(ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

(iv) ⇒ (iii) Given the open set $U \subset C$ choose $B \subset U$ open and bounded and $U_1 \subset C$ such that $\overline{B} \cup K \subset U_1$. Consider the following embedded chain: Choose $t_0, \alpha_0$ as in Lemma 3.6 with $\inf_{y\in K} P_y\{\sigma_B < t_0\} > \alpha_0$ and define

$$\begin{aligned}
\eta_1 &= \inf\{t, x_t \in K\},\\
\eta_2 &= \inf\{t > \eta_1 + t_0, x_t \in \partial U_1\},\\
&\vdots\\
\eta_{2i} &= \inf\{t > \eta_{2i-1} + t_0, x_t \in \partial U_1\},\\
\eta_{2i+1} &= \inf\{t > \eta_{2i}, x_t \in K\}.
\end{aligned}$$

By the strong Markov property and Lemma 3.4 the $\eta_j$ are a.s. finite. Define $A_i = \{\omega,\ x_t(\omega) \in \overline{B}$ for some $t \in [\eta_{2i-1}, \eta_{2i})\}$ for all $i \in \mathbb{N}$. Then we have $P_x\{A_1\} = P_x\{x_t \in \overline{B}$ for some $t \in [\eta_1, \eta_2)\} > \alpha_0$ for all $x \in C$. Using again the strong Markov property and setting $\tau = \eta_3 + \eta_2$ we obtain

$$\begin{aligned}
P_x\{A_1^c \cap A_2^c\} &= P_x\{A_1^c \cap \theta_\tau A_1^c\}\\
&= \int_{A_1^c} P_{x(\tau)}\{A_1^c\}\, P_x(d\omega) \leq (1-\alpha_0)^2, \quad \text{for all } x \in C.
\end{aligned}$$

Thus by induction $P_x\{\bigcap_{i=1}^n A_i^c\} \le (1-\alpha_0)^n$ and hence

$$P_x\{x_t \in U \text{ for no } t > 0\} \le P_x\{x_t \in \overline{B} \text{ for no } t > 0\}$$
$$\le \lim_{n \to \infty} P_x\left\{\bigcap_{i=1}^n A_i^c\right\} = 0.$$

(iii) $\Rightarrow$ (ii) Let $U \subset C$ be open, $B \subset U$ an open, bounded set with $\overline{B} \subset U$. For $x \in C$ choose an open neighborhood $U(x)$ such that $\overline{U(x)} \cap U = \varnothing$. [For $x \in U$ (ii) is trivial.] Choose $t_0, \alpha_0$ as in Lemma 3.6 with $\inf_{y \in \partial U(x)} P_y\{\sigma_B < t_0\} > \alpha_0$. Consider again an embedded chain

$$\eta_1 = \inf\{t > 0,\ x_t \in \partial U(x)\},$$
$$\vdots$$
$$\eta_{2i} = \inf\{t > \eta_{2i-1} + t_0,\ x_t \in \partial B\},$$
$$\eta_{2i+1} = \inf\{t > \eta_{2i},\ x_t \in U(x)\}.$$

Again the $\eta_j$ are a.s. finite and we have $\lim_{i\to\infty}\eta_{2i} = \infty$ because otherwise the sequences $(x_{\eta_{2i-1}})$ and $(x_{\eta_{2i}})$ have a common limit, which is impossible since $\overline{U(x)} \cap \overline{B} = \varnothing$.

(ii) $\Rightarrow$ (i) is obvious. □

To show the uniform behavior of $x_t$ on invariant control sets, we again premise a lemma:

LEMMA 3.7. *Let $C$ be an invariant control set for* (2.1) *and $x \in C$ transient. Then there exists an open neighborhood $U(x)$ such that all $z \in \overline{U(x)}$ are transient.*

PROOF. $x \in C$ is transient iff there exists an open neighborhood $V(x)$ such that $\int P(t, x, V)\,dt < \infty$. Let $F \subset V(x)$ be compact with $\operatorname{int} F \ne \varnothing$ and denote $f(z) = P(t, z, F)$. Then $\limsup_{y\to x}\int f(y)\,dt \le \int \limsup_{y \to x} f(y)\,dt \le \int f(x)\,dt < \infty$, i.e., there exists an open neighborhood $W(x)$ with $\int f(z)\,dt < \infty$ for all $z \in W(x)$. Define $U(x) = \operatorname{int}(\overline{F} \cap V(x))$, then $\int P(t, z, V)\,dt < \infty$ for all $z \in U(x)$ and the $z$'s are transient. □

THEOREM 3.2. *Let $C$ be an invariant control set for* (2.1). *Then either all points in $C$ are recurrent, or all points in $C$ are transient.*

PROOF. Let $x \in C$ be transient, then there exists a closed, bounded neighborhood $F(x)$ with $\operatorname{int} F(x) \ne \varnothing$ such that $U(z, V) < \infty$ for all $z \in F$, and $F \subset V$, an open set by Lemma 3.7. Let $y \in C$, then

$$U(y, F) = E_y\left(E_{x_{\sigma_F}}\int_0^\infty \mathbb{1}_F(x_t)\,dt\right)$$
$$\le E_y\big(U(x_{\sigma_F}, V)\big) \le U(x, V) + \varepsilon < \infty,$$

and hence for all $y \in C$ we have $U(y, \operatorname{int} F) < \infty$. If $y \in C$ is recurrent, then by the proof of (i) ⇒ (ii) in Theorem 3.1 $P_y\{\mathscr{R}_{U(x)}\} = 1$ for all open neighborhoods of $x$. This implies $U(y, U(x)) = \infty$ by the lemma in [2], page 196. For $U(x) =$ int $F$ we now arrive at a contradiction, and Proposition 3.1 finishes the proof. □

So far we established the recurrence–transience dichotomy on invariant control sets, which generalizes the known results for nondegenerate diffusions, since for these the associated control system is always completely controllable on $M$, i.e., $M$ is the unique invariant control set. For degenerate diffusions all points outside invariant control systems are transient:

PROPOSITION 3.2. *If either*

(i) $x \in M$ *is in no control set for* (2.1) *or*
(ii) $x \in D$, *a noninvariant control set,*

*then $x$ is transient.*

PROOF. (i) If $x$ is in no control set, then $x$ is no absorbing point for $x_t$, hence there exists an open neighborhood $U(x)$ such that $E_x\tau_{U(x)} < \infty$. Now use Lemma 2.3(ii) and the support theorem.

(ii) By assumption there exists a $y \in D$ such that $P_y\{\tau_{\bar{D}} < \infty\} = \alpha > 0$. Hence there is an open neighborhood $U(y)$ such that $P_z\{\tau_{\bar{D}} < \infty\} > \alpha/2$ for all $z \in U(y)$. Now $D$ is a control set, so $P_x\{\sigma_{U(y)} < \infty\} > \beta > 0$ for each fixed $x \in D$. Hence for all $x \in D$ there exists $\gamma > 0$ with $P_x\{\tau_{\bar{D}} < \infty\} > \gamma$. Now Lemma 2.3(i) and the support theorem prove the assertion. □

We conclude this part by mentioning results on the long-term behavior of diffusions starting in transient points:

PROPOSITION 3.3. *Let $x_t$ be a diffusion defined by* (1.1) *and $D \subset M$ the set of all transient points of $x_t$. Then for all compact sets $K \subset D$ we have*

(i) $$\int_0^\infty P(t, x, K)\,dt < \infty, \quad \textit{for all } x \in D,$$

(ii) $$\lim_{t\to\infty} P(t, x, K) = 0, \quad \textit{for all } x \in D.$$

The proof is established the same way as Lemma 3.1 and its corollary in [18] using Lemmas 3.1 and 3.4 and the techniques from Theorem 3.1.

For compact manifolds we have furthermore:

PROPOSITION 3.4. *Denote $F = \bigcup C_\alpha$, $C_\alpha$ the finitely many invariant control sets of* (2.1). *Then $P_x\{\sigma_F < \infty\} = 1$ for all $x \in M$, $M$ a compact manifold. Furthermore, $E_x\sigma_F < \infty$ for all $x \in M$.*

PROOF. Let $K \subset M \sim F$ be compact, then $P_x\{\tau_K < \infty\} = 1$ by Proposition 3.3. Since $F$ is compact and $\neq \varnothing$ by Lemma 2.2, for all controls $u$ there exists an $\varepsilon > 0$ such that for all $x$ in a $\varepsilon$-neighborhood $F_\varepsilon$ around $F$ there is a $t$ with

$\varphi(t, x, u) \in \operatorname{int} F$. Now choose $K$ such that $\partial K \subset F_\varepsilon$. Then the strong Markov property and the support theorem imply that $P_x\{\sigma_F < \infty\} = 1$ for all $x \in M \sim F$. The last statement follows from compactness with an argument similar to the one used for the proof of Lemmas 3.4 and 3.5. □

On noncompact manifolds transient diffusions may, of course, wander out to $\infty$, even if $M$ is an invariant control set and $x_t$ is nondegenerate, as the example of the Wiener process in $\mathbb{R}^d$, $d \geq 3$, shows.

The long-term behavior of recurrent diffusions is investigated in the next part.

**4. Recurrence and invariant measures.** A measure $\mu$ on $M$ is called invariant for $x_t$, if $\mu(A) = \int P(t, x, A)\, \mu(dx)$ for all $t > 0$, all Borel sets $A \subset M$. An invariant measure is extremal, if it cannot be decomposed into the sum of two different invariant measures, up to constant multiples.

LEMMA 4.1. *Let $\mu$ be an extremal invariant probability measure for $x_t$. Then* $\operatorname{supp} \mu = C$ *for some invariant control set and $\mu$ is the unique invariant probability on $C$.*

PROOF. As in the proof of Proposition 3.2 one shows that $\operatorname{supp} \mu \subset \overline{C}$, $C$ an invariant control set. Now Lemma 2.1 and the support theorem imply $\operatorname{supp} \mu = \overline{C} = C$.

To see the uniqueness of an invariant probability on $C$ observe that $\mu$ has a $C^\infty$-density $\varphi$ with respect to the Riemannian volume, satisfying $\mathscr{A}^*\varphi = 0$, where $\mathscr{A}^*$ is the adjoint operator of $\mathscr{A}$. If there exists another invariant probability $\nu$ on $C$, then $\mu$ and $\nu$ are mutually singular. Hence the density $\psi$ of $\nu$ is zero on a dense set in $C$, hence zero on all of $C$. □

Note that any invariant probability admits a representation as a countable convex combination of the extremal measures, concentrated on the invariant control sets.

LEMMA 4.2. *Let $\mu$ be an* (*extremal*) *invariant probability on some invariant control set $C$. Then all $x \in C$ are recurrent.*

PROOF. By Theorem III, 2.1 in [15] we have

$$\lim_{n \to \infty} \liminf_{T \to \infty} \frac{1}{T} \int_0^T P(t, x, U_n^c)\, dt = 0,$$

$\mu$-a.s. for some exhausting sequence $U_n$ of $C$. Thus for some $x_0 \in C$,

$$\mu_n = \frac{1}{T_n} \int_0^{T_n} P(t, x_0, \cdot)\, dt, \qquad T_n \uparrow \infty,$$

converges towards $\mu$, in particular,

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T P(t, x_0, U)\, dt \geq \mu(U) > 0,$$

for all open neighborhoods of $U$ of $x_0$. By Proposition 3.1 $x_0$ is recurrent and thus all $x \in C$ are recurrent by Theorem 3.2. □

A partial converse of Lemma 4.2 is given by:

PROPOSITION 4.1. *Let the diffusion $x_t$ defined by* (1.1) *be recurrent in the control set $C$. Then $x_t$ admits a $\sigma$-finite invariant measure in $C$.*

PROOF. We use the following chain construction: Choose two open sets $U_0, U_1$ in $C$ with compact closure such that $\overline{U}_0 \subset U_1$, $\overline{U}_1 \subset \operatorname{int} C$ and denote $\partial U_0 = \Gamma_0$, $\partial U_1 = \Gamma_1$. Consider

$$\begin{aligned} \eta_1 &= \inf\{t, x_t \in \Gamma_0\}, \\ \eta_2 &= \inf\{t > \eta_1, x_t \in \Gamma_1\}, \\ &\vdots \\ \eta_{2i} &= \inf\{t > \eta_{2i-1}, x_t \in \Gamma_1\}, \\ \eta_{2i+1} &= \inf\{t > \eta_{2i}, x_t \in \Gamma_0\}. \end{aligned}$$

Then $x_{\eta_{2i-1}}$ is a homogeneous Feller–Markov chain on a compact set $\Gamma_0$ (see [12]), hence has an invariant probability $\tilde{\mu}$. Using the setup in [12], we denote $\tau^A = \int_0^\eta \mathbb{1}_A(x_t)\,dt$, the time spent in a set $A$ during one cycle $\eta$. We first show:

$$\sup_{x \in \Gamma_0} E_x \tau^K < \infty, \quad \text{for compact sets } K \subset C. \tag{4.1}$$

Denote by $\eta'$ the time till reaching $\Gamma_1$, $\eta''$ the time to pass from $\Gamma_1$ to $\Gamma_0$; hence $\eta = \eta' + \eta''$. Then

$$\sup_{x \in \Gamma_0} E_x \eta' \le \sup_{x \in C} E_x \tau_{U_1} < \infty, \quad \text{by Lemma 3.4.} \tag{4.2}$$

On the other hand, if $\tau_{\Gamma_0}^K$ denotes the time spent in $K$ until hitting $\Gamma_0$, then

$$\sup_{x \in \Gamma_0} E_x \tau'' \le \sup_{x \in K} E_x \tau_{\Gamma_0}^K.$$

Define $A_t = \{\tau_{\Gamma_0}^K > t\}$ and let $t_0$ be such that $P_y\{A_{t_0}\} < \alpha < 1$ for all $y \in K$, where $t_0$ and $\alpha$ exist by Lemma 3.3. Let $\sigma(t)$ be the infimum of the solutions of $t + t_0 = \int_0^{\sigma(t)} \mathbb{1}_K(x_s)\,ds$. Then $x_{\sigma(t)} \in K$ and we have

$$\begin{aligned} P_x\{A_{2(t+t_0)}\} &= P_x\{A_{t+t_0} \cap \theta_{\sigma(t)} A_{t+t_0}\} \\ &= \int_{A_{t+t_0}} P_{x_{t+t_0}}\{A_{t+t_0}\} P_x(d\omega) \\ &\le \int_{A_{t+t_0}} P_{x_{t+t_0}}\{A_{t_0}\} P_x(d\omega) \\ &\le \alpha^2. \end{aligned}$$

Hence $\sup_{x \in K} P_x\{A_{n(t+t_0)}\} \leq \alpha^n$ and thus

$$\sup_{x \in K} E_x \tau_{\Gamma_0}^K \leq (t + t_0) \sum_{n=1}^{\infty} n\alpha^{n-1} < \infty. \tag{4.3}$$

(4.2) and (4.3) prove the estimate (4.1).

Now define $\mu(A) = \int_{\Gamma_0} E_x \tau^A \, \tilde{\mu}(dx)$ and the proof of Theorem 2.1 in [18] shows that $\mu$ is a $\sigma$-finite invariant measure for $x_t$. □

COROLLARY 4.1. *The diffusion $x_t$ admits an invariant probability measure in $C$ if and only if $x_t$ is recurrent in $C$ and $\int_{\Gamma_0} E_x \eta \, \tilde{\mu}(dx) < \infty$, where $\eta$ is defined in the above proof.*

For this result mimic the proof of Theorem 3.3 in [18] using Proposition 3.3, Lemma 4.1 and Proposition 4.1.

The recurrence of $x_t$ alone does not imply the existence of an invariant probability, even if the corresponding operator $\mathscr{A}$ is nondegenerate or the control system is strongly controllable. Counterexamples are, e.g., the Wiener process in $\mathbb{R}^d$, $d \leq 2$, which has the Lebesgue measure as an invariant measure. On the linear system $dx_t = Ax_t \, dt + B \, dW_t$, where $(A, B)$ is a controllable pair of matrices. Here the existence of an invariant probability is equivalent to $A$ being stable, while recurrence is equivalent to $A$ being "of type I," i.e., $A$ has at most one 1- or 2-dimensional eigenspace corresponding to an eigenvalue $\lambda$ with $\operatorname{Re} \lambda = 0$, and all other eigenvalues have negative real part, see [9] and [11]. Remember also that the existence of a $\sigma$-finite invariant measure does not imply the recurrence of $x_t$, as, e.g., the Wiener process in $\mathbb{R}^d$, $d \geq 3$, shows. But just as for nondegenerate diffusions we have in our situation that positive recurrence and the existence of an invariant probability are equivalent.

We call $x_t$ positive recurrent in $C$, if $E_x \sigma_U < \infty$ for all $x \in C$, all open sets $U \subset C$.

LEMMA 4.3. *Consider the diffusion $x_t$ defined by* (1.1) *on an invariant control set $C$. Assume $E_x \sigma_{U_0} < \infty$ for all $x \in C$, some bounded open set $U_0 \subset C$, such that $\overline{U}_0 \subset \operatorname{int} C$. Then $E_x \sigma_U < \infty$ for all open sets $U \subset C$, all $x \in C$.*

PROOF. It suffices to consider open sets $U \subset U_1$, $U_0 \subset U_1$ with $\overline{U} \subset U_1$, $\overline{U}_0 \subset U_1$ and $\overline{U}_1 \subset \operatorname{int} C$. We consider the following chain:

$$\eta_1 = \inf\{t, x_t \in \partial U_0\},$$

$$\vdots$$

$$\eta_{2i} = \inf\{t > \eta_{2i-1} + t_0, x_t \in \partial U_1\},$$

$$\eta_{2i+1} = \inf\{t > \eta_{2i}, x_t \in \partial U_0\},$$

where we choose $t_0$ as in the proof of Theorem 3.1, (iv) $\Rightarrow$ (iii) such that

$\inf_{u \in \overline{U}_0} P_y\{\sigma_U < t_0\} > \alpha > 0$. Then using this proof we have again

$$P_x\left\{\bigcap_{i=1}^{n} A_i^c\right\} \le (1-\alpha)^n, \quad \text{with } A_i = \{x_t \in U \text{ for some } t \in [\eta_{2i-1}, \eta_{2i})\}.$$

Hence $E_x\sigma_U \le E_x\eta_1 + \sum_{i=1}^{\infty}(1-\alpha)^i E_x\eta_{2i}$.

We may assume that $x \in \partial U_1$ and thus

$$\sup_{x \in \partial U_1} E_x\eta_2 \le \sup_{x \in \partial U_1} E_x\sigma_{U_0}(t_0) + \sup_{x \in \partial U_1} \tau_{U_1} \le \beta < \infty.$$

Putting this together yields $E_x\sigma_U \le c + \beta\sum_{i=1}^{\infty} i(1-\alpha)^i < \infty$. □

The assumption of Lemma 4.3 can be verified, e.g., if $x_t$ has some boundedness properties, we refer to [15], Chapter III.7, [24], [25] and [31]. We now prepare a lemma on the smoothness of $E_.\sigma_U$, which is crucial for the following; compare [18], Lemma 5.3.

LEMMA 4.4. *Let $x_t$ be a diffusion defined by* (1.1) *and $\mathscr{A} = A_0 + \sum_{i=1}^{m} A_i^2$ its generator. Let $U$ be an open set with compact closure $\overline{U} \subset \operatorname{int} C$ in some invariant control set. If $E_y\sigma_U < \infty$ for a dense subset of $C$, then $E_.\sigma_U$ is $C^\infty$ in $C \sim U$.*

PROOF. Let $K_n$ be an increasing sequence of compact sets exhausting $C$ and denote by $\tau_n$ the first exit time from $K_n \sim U$. Since $x_t$ is recurrent in $C$ by Theorems 3.1 and 3.2, we have $\tau_n \uparrow \sigma_U$, and $\lim E_x\tau_n = E_x\sigma_U$. Now $\mathscr{A}E_x\tau_n = 1$ in the distributional sense. For any $n_0 \in \mathbb{N}$, $E_x\sigma_U = E_x\tau_{n_0} + \sum_{i=n_0}^{\infty}(E_x\tau_{i+1} - E_x\tau_i)$, where all terms $u_i$ of the infinite sums are positive and fulfill $\mathscr{A}u_i = 0$. Hence by Bony's form of the Harnack inequality (see Theorem 7.1 in [6]), the series converges uniformly in $K_{n_0}$, since it converges on a dense subset by assumption. Now for any $C^\infty$ test function $\varphi$ with compact support, $\operatorname{supp}\varphi \subset K_{n_0}$ for some $n_0$ and since for $n \to \infty$ we have $E_x\tau_n =: u_n \uparrow u := E_x\sigma_U$, one sees that $\int u_n\varphi\,dx \to \int u\varphi\,dx$. Thus in the distributional sense $\mathscr{A}u = -1$ in $C \sim U$. As $\mathscr{A}$ is assumed to be hypoelliptic, $u$ is $C^\infty$ in $C \sim U$. □

THEOREM 4.1. *The diffusion $x_t$ defined by* (1.1) *is positive recurrent in an invariant control set $C$ if and only if there exists a* (*unique*) *invariant probability measure $\mu$ for $x_t$ on $C$.*

PROOF. (i) To show that there exists an invariant probability $\mu$ means in the light of Corollary 4.1: $\int_{\Gamma_0} E_x\eta\,\tilde{\mu}(dx) < \infty$. To see this use the setup of Proposition 4.1 and let $U \subset U_0$ be an open set with $\overline{U} \subset U_0$. Then $E_x\sigma_{U_0} \le E_x\sigma_U$ for all $x$ and $E_x\sigma_U$ is $C^\infty$ in $C \sim U$ by Lemma 4.4. Hence $\sup_{x \in \Gamma_0} E_x\eta \le \sup_{x \in \Gamma_0} E_x\tau_{U_1} + \sup_{x \in \Gamma_1} E_x\sigma_U < \infty$, giving us the "only if" part.

(ii) In a first step we show that there exists a bounded open set $U \subset C$ such that

$$E_x\sigma_U < \infty, \qquad \mu\text{-a.s.} \tag{4.4}$$

By Proposition 4.1 and Corollary 4.1 $\mu$ is of the form

$$\mu(A) = \int_\Gamma E_x\left(\int_0^\eta \mathbb{1}_A(x_t)\,dt\right)\tilde{\mu}(dx),$$

for some open, bounded set $U$ with boundary $\Gamma$. Define $B = \{y,\ E_y\sigma_U = \infty\}$ and assume $\mu(B) > 0$. Then $\int_\Gamma E_x\tau^B\,\tilde{\mu}(dx) > 0$, hence $\tilde{\mu}\{x \in \overline{U},\ P_x\{\eta > \sigma_B\} > 0\} > 0$, since for all $x \in C$ $P_x\{\eta = \sigma_B\} = 0$ because $\overline{U} \cap B = \varnothing$. On the other hand,

$$\begin{aligned} E_x\sigma_U &\geq E_x\left(\mathbb{1}_{(\eta > \sigma_B)}\sigma_U\right) \\ &\geq E_x\left(\left(\sigma_B + \sigma_U \circ \theta_{\sigma_B}\mathbb{1}_{(\eta > \sigma_B)}\right)\right) \\ &\geq E_x\left(\sigma_B\mathbb{1}_{(\eta > \sigma_B)}\right) + E_x\left(\mathbb{1}_{(\eta > \sigma_B)}E_{x_{\sigma_B}}(\sigma_U)\right). \end{aligned}$$

Now $E_{x_{\sigma_B}}(\sigma_U) = \infty$ by the definition of $B$ and by assumption $\{\eta > \sigma_B\}$ has positive $\tilde{\mu}$-measure. Hence there exists a subset $\Gamma'$ of $\Gamma$ such that $\tilde{\mu}(\Gamma') > 0$ and $E_x\eta = \infty$ for all $x \in \Gamma'$, which is a contradiction to $\int_\Gamma E_x\eta\,\tilde{\mu}(dx) < \infty$, thus (4.4) is proved.

By Lemma 4.1 (4.2) is true for a dense subset of $C$ and so by Lemma 4.4 for all $x \in C$. Now use Lemma 4.3 to conclude the "if" part of the theorem. □

Theorem 3.2, Proposition 4.1 and Theorem 4.1 now provide the diagram given in the introduction. Counterexamples for the missing implications can be obtained by considering the Wiener process in $\mathbb{R}^2$ or $\mathbb{R}^3$.

On compact manifolds all invariant control sets are compact, hence each of those is the support of an extremal invariant probability measure.

For recurrent diffusions one obviously has a strong law of large numbers $\mu$-a.s. The construction above even guarantees a law for all points in invariant control sets: If $x_t$ is a recurrent diffusion on an invariant control set $C$ with invariant probability $\mu$, then for any $f \in L^1(\mu)$ and all $x \in C$

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x_t)\,dt = \int_C f(x)\,d\mu\right\} = 1.$$

This is Theorem 5.1 in [23]. Furthermore, if $M$ is compact with invariant control sets $C_\alpha$, $\alpha = 1,\ldots,l$, define for $x \in M$, $P_x\{\sigma_{C_\alpha} < \infty\} = p_\alpha$. By Proposition 3.4 we have $\Sigma p_\alpha = 1$ and if $\mu_\alpha$ is the invariant probability on $C_\alpha$, then for all $x \in M$, $\mu_x = \Sigma p_\alpha\mu_\alpha$ is an invariant probability on $\bigcup C_\alpha$. Now the above strong law of large numbers for compact manifolds reads

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x_t)\,dt = \int_M f(x)\,d\mu_x\right\} = 1, \quad \text{for all } x \in M.$$

**Acknowledgment.** The author is very thankful for the comments of a helpful referee, in particular for pointing out this elegant proof of Lemma 3.1.

## REFERENCES


[1] ARNOLD, L. and KLIEMANN, W. (1983). Qualitative theory of stochastic systems. In *Probabilistic Analysis and Related Topics* (A. T. Bharucha-Reid, ed.) **3** 1–79. Academic, New York.

[2] AZÉMA, J., KAPLAN-DUFLO, M. and REVUZ, D. (1966). Récurrence fine des processus de Markov. *Ann. Inst. H. Poincaré Sect. B* **2** 185–220.

[3] AZÉMA, J., KAPLAN-DUFLO, M. and REVUZ, D. (1967). Mesure invariante sur les classes récurrentes des processus de Markov. *Z. Wahrsch. verw. Gebiete* **8** 157–181.

[4] AZENCOTT, R. (1974). Behavior of diffusion semi-groups at infinity. *Bull. Soc. Math. France* **102** 193–240.

[5] BHATTACHARYA, R. N. (1978). Criteria for recurrence and existence of invariant measures for multidimensional diffusions. *Ann. Probab.* **6** 541–553. Correction in **8** 1194–1195 (1980).

[6] BONY, J.-M. (1969). Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptique dégénérés. *Ann. Inst. Fourier (Grenoble)* **19** 277–304.

[7] BOOTHBY, W. M. (1977). Transversely complete *e*-foliations of codimension one and assessibility properties of non-linear systems. *Proc. 1976 Ames Research Center (NASA) Conference on Geometric Control Theory* 361–385. Math. Sci. Press, Brookline, Mass.

[8] BROCKETT, R. W. (1973). Lie algebras and Lie groups in control theory. In *Geometric Methods in Systems Theory* (D. Q. Mayne and R. W. Brockett, eds.) 43–82. Reidel, Dordrecht.

[9] DYM, H. (1966). Stationary measures for the flow of a linear differential equation driven by white noise. *Trans. Amer. Math. Soc.* **123** 130–164.

[10] DYNKIN, E. B. (1965). *Markov Processes* **1**, **2**. Springer, New York.

[11] ERICKSON, R. V. (1971). Constant coefficient linear differential equations driven by white noise. *Ann. Math. Statist.* **42** 820–823.

[12] EZHOV, I. I. and SHURENKOV, V. M. (1976). Ergodic theorems connected with the Markov property of random processes. *Theory Probab. Appl.* **21** 620–624.

[13] GETOOR, R. K. (1980). Transience and recurrence of Markov processes. *Lecture Notes in Math.* **784** 397–409. Springer, New York.

[14] GIHMAN, I. I. and SKOROHOD, A. V. (1979). *The Theory of Stochastic Processes* **3**. Springer, New York.

[15] HAS'MINSKII, R. Z. (1980). *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Alphen aan den Rijn.

[16] ICHIHARA, K. and KUNITA, H. (1974). A classification of the second order degenerate elliptic operators and its probabilistic characterization. *Z. Wahrsch. verw. Gebiete* **30** 235–254. Correction in **39** 81–84 (1977).

[17] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam.

[18] KHAS'MINSKII, R. Z. (1960). Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations. *Theory Probab. Appl.* **5** 179–195.

[19] KLIEMANN, W. (1983). Transience, recurrence and invariant measures for diffusions. In *Nonlinear Stochastic Problems* (R. S. Bucy and J. M. F. Moura, eds.) 437–454. Reidel, Dordrecht.

[20] KOLMOGOROV, A. N. (1936). Anfangsgründe der Theorie der Markoffschen Ketten mit unendlich vielen möglichen Zuständen. *Mat. Sb. (N. S.)* 607–610.

[21] KUNITA, H. (1974). Diffusion processes and control systems. Course at the University of Paris VI.

[22] KUNITA, H. (1976). Supports of diffusion processes and controllability problems. In *Proc. of the International Symposium on Stochastic Differential Equations* (K. Itô, ed.) 163–185. Kino Kuniya Book-Store, Tokyo.

[23] MARUYAMA, G. and TANAKA, H. (1959). Ergodic property of *N*-dimensional recurrent Markov processes. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **13** 157–172.

[24] MIYAHARA, Y. (1972). Ultimate boundedness of the systems governed by stochastic differential equations. *Nagoya Math. J.* **47** 111–144.

[25] MIYAHARA, Y. (1973). Invariant measures of ultimately bounded stochastic processes. *Nagoya Math. J.* **49** 149–153.

[26] ROYNETTE, B. (1975). Sur les processus de diffusion de dimension 2. *Z. Wahrsch. verw. Gebiete* **32** 95–110.

[27] STROOCK, D. W. and VARADHAN, S. R. S. (1972). On the support of diffusion processes with applications to the strong maximum principle. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **3** 333–359. Univ. California Press.

[28] STROOCK, D. W. and VARADHAN, S. R. S. (1972). On degenerate elliptic–parabolic operators of second order and their associated diffusions. *Comm. Pure Appl. Math.* **25** 651–713.

[29] SUSSMANN, H. J. and JURDJEVIC, V. (1972). Controllability of nonlinear systems. *J. Differential Equations* **12** 95–116.

[30] SUSSMANN, H. J. (1973). Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.* **180** 171–188.

[31] WONHAM, W. M. (1966). Lyapunov criteria for weak stochastic stability. *J. Differential Equations* **2** 365–377.

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011