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ASYMPTOTIC NORMALITY OF TRIMMED SUMS OF Φ-MIXING RANDOM VARIABLES

By M. G. Hahn,[1] J. Kuelbs[2] and J. D. Samur

Tufts University, University of Wisconsin-Madison and Universidad Nacional de La Plata

A Gaussian central limit theorem for trimmed sums of Φ-mixing Hilbert-space-valued random variables is obtained, and implications regarding Ibragimov's conjecture are examined.

1. Introduction. In a recent paper [7], a Gaussian central limit theorem was proved for the partial sums $\{S_n\}$ obtained from an i.i.d. sequence $\{X_j\}$, taking values in a type 2 Banach space provided the partial sums, suitably normalized, were tight, weakly stochastically compact and certain maximal terms of the sample $\{\|X_1\|, \dots, \|X_n\|\}$ were deleted from S_n. The purpose of this paper is to show that a similar result holds in case the sequence $\{X_j\}$ is stationary Φ-mixing with values in a Hilbert space and to examine the implications of this result regarding a conjecture of Ibragimov. We work in the generality of Hilbert space, rather than Banach spaces, because many inequalities resulting from mixing assumptions are known to be valid only if there is an inner product available to give the square of the norm.

The deletion of extreme terms, and then assuming the data are Gaussian or approximately Gaussian, has been a matter of practice in many applied situations for some time. The paper of Stigler [16] provides a theoretical basis for this if one chooses to delete a positive proportion of the sample. What is surprising in the setting of our results is that one need not delete many terms; only $\xi_n r_n$ at stage n, where $\{r_n\}$ and $\{\xi_n\}$ are sequences going to infinity as slowly as one might like [see (1.8), (1.9) and (1.10) for specific details]. A similar result for real-valued i.i.d. sequences $\{X_j\}$ in the domain of attraction of a stable law was obtained in [3] using an entirely different method of proof, depending on a Brownian bridge approximation to the uniform empirical process in weighted supremum norm. It seems unlikely that such a method will apply to the general setting considered here, but a recent preprint of Pruitt [12] also examines this question when the number of terms trimmed goes to infinity arbitrarily slowly using more classical methods. The results of Pruitt and those in [3] differ from ours in the method of trimming as well. Our approach is to trim values only if they are among the largest prescribed group and they are also sufficiently large in magnitude. This makes the procedure insensitive to overtrimming and produces more universal results. However, it is less delicate from a mathematical

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point of view, and we recommend Pruitt's preprint for an interesting discussion of these matters as well as further references and open problems.

Our proof depends in a critical way on [13], and our main result relates to the conjecture of Ibragimov cited in [2] or [6, page 393]. Unfortunately, we do not resolve this conjecture, but we are grateful to R. C. Bradley, Michel Ledoux, Magda Peligrad and Walter Philipp for the encouragement they provided, urging us to investigate the implications of our method on this problem. What we obtain in this area are Theorem 2 and Corollaries 3 and 4.

Throughout, $X, X_1, X_2, \dots$ is a strictly stationary sequence with values in the real separable Hilbert space H. If $M_{a,b}$ denotes the σ-field generated by the random variables $X_a, \dots, X_b$ and $M_{a,\infty}$ is the σ-field generated by $X_a, X_{a+1}, \dots$, the dependence coefficient is defined by

$$\Phi(n) = \sup_k \sup \left\{ \left| \frac{P(E \cap F)}{P(E)} - P(F) \right| : E \in M_{1,k}, \right. \tag{1.1}$$
$$\left. F \in M_{k+n,\infty}, P(E) > 0 \right\}$$

for $n = 1, 2, \dots$. Then $\Phi(1) \le 1$ and $\Phi(n)$ is a nonincreasing sequence as n goes to infinity. The sequence $\{X_j\}$ is called Φ-mixing if $\lim_n \Phi(n) = 0$. There are analogous concepts for triangular arrays and the terminology of [13] is used freely throughout.

The norm on H is denoted by $\|\cdot\|$ and, of course, $\|x\|^2 = \langle x, x\rangle$ for an inner product $\langle \cdot, \cdot \rangle$. H^* denotes the topological dual of H. The law of a random variable X is denoted by $\mathscr{L}(X)$ and $\mathscr{L}(X|A)$ signifies the restriction of the measure to the set A. A sequence of random variables $\{W_n\}$ is said to be tight if for each $\varepsilon > 0$, there is a compact set K_ε such that $\inf_n P(W_n \in K_\varepsilon) > 1 - \varepsilon$. The sequence $\{\mathscr{L}(W_n)\}$ converges weakly to $\mathscr{L}(W)$, and we write

$$\mathscr{L}(W_n) \to_w \mathscr{L}(W),$$

if $\lim_n E(f(W_n)) = E(f(W))$ for all bounded continuous functions f on the range space of W_n. A random variable is called degenerate if its law is concentrated at a single point: otherwise, it is said to be nondegenerate. Finally, a sequence of random variables $\{W_n\}$ is said to be stochastically compact if $\{W_n\}$ is tight and all weak limits of subsequences of $\{W_n\}$ are nondegenerate. The stochastically compact laws on $\mathbb{R}^1$ arising from suitably normalized sums of i.i.d. random variables were studied by Feller [5]. More recently Pruitt [11] characterized the subsequential limit laws which arise in that situation.

To make precise the number of terms to be deleted we define for $n \ge 1$ and $1 \le j \le n$,

$$F_n(j) = \operatorname{card}\{i : \|X_i\| > \|X_j\| \text{ for } 1 \le i \le n \tag{1.2}$$
$$\text{or } \|X_i\| = \|X_j\| \text{ for } 1 \le i \le j\}.$$

If $F_n(j) = k$, set $X_n^{(k)} = X_j$, i.e., $\|X_j\|$ is the kth largest element of $\{\|X_1\|, \dots, \|X_n\|\}$ when $F_n(j) = k$. For any $r > 0$, $n \ge \xi r$, $\tau > 0$, $\xi > 0$ and positive function $d(t)$ defined on the integers define

$$(1.3)\qquad {}^{(\xi r)}_{\tau}S_n = S_n - \sum_{j=1}^{[\xi r]} X_n^{(j)} I\big(\|X_n^{(j)}\| > \tau d([n/r])\big).$$

Here, $[\cdot]$ denotes the greatest integer function and $S_n = X_1 + \cdots + X_n$. Hence, ${}^{(\xi r)}_{\tau}S_n$ denotes the partial sum S_n with the $[\xi r]$ largest terms of the sample $\{\|X_1\|, \dots, \|X_n\|\}$ deleted provided they exceed $\tau d([n/r])$ in norm. Also, define

$$(1.4)\qquad \delta_n(\tau, r) = \sum_{j=1}^{n} E\big(X_j I\big(\|X_j\| \le \tau d([n/r])\big)\big).$$

The main result can now be stated. It is possible to replace the condition $\Phi(1) < 1$ in our results by a suitable alternative, but we have not been able to eliminate the condition $\Phi(1) < 1$. We indicate this alternative in the remarks following Corollary 4. The main implication obtained from this is that Corollary 4 is valid without the assumption $\Phi(1) < 1$.

THEOREM 1. *Let $X, X_1, X_2, \dots$ be a strictly stationary H-valued sequence which is Φ-mixing with $\Phi(1) < 1$. Let $S_n = X_1 + \cdots + X_n$ and assume there are normalizing constants $\{d(n)\}$ such that* $\lim_n d(n) = \infty$,

$$(1.5)\qquad \{S_n/d(n)\} \text{ is tight,}$$

$$(1.6)\qquad \{S_n/d(n)\} \text{ has only nondegenerate limits and}$$

$$(1.7)\qquad \text{if } \{q_n\} \text{ is any sequence of integers such that } \lim_n q_n/n = 0, \quad \text{then } \mathscr{L}\big(S_{q_n}/d(n)\big) \to_w \delta_0.$$

Let $\{r_n\}$ be a sequence of positive integers such that

$$(1.8)\qquad \lim_n r_n = \infty,$$

$$(1.9)\qquad \lim_n n/r_n = \infty$$

and

$$(1.10)\qquad \lim_n r_n \Phi^{1/2}(t_n) = 0,$$

where $t_n = [n/r_n]$. Then, for each $\tau > 0$ and any sequence $\{\xi_n\}$ such that $\lim_n \xi_n = \infty$, *the sequence*

$$(1.11)\qquad \left\{ \frac{{}^{(\xi_n r_n)}_{\tau}S_n - \delta_n(\tau, r_n)}{\sqrt{r_n}\, d(t_n)} \right\}$$

is tight with only centered Gaussian limits. Further, for all $\tau > 0$ sufficiently large, the limits are all nondegenerate.

REMARK 1. Condition (1.6) is only used to show that the limits of (1.11) are nondegenerate provided $\tau > 0$ is sufficiently large. Of course, (1.5) and (1.6) combined are the condition that $\{S_n/d(n)\}$ is stochastically compact.

REMARK 2. Since $\lim_n d(n) = \infty$ and $\Phi(n) \downarrow 0$, there are many sequences $\{r_n\}$ satisfying (1.8), (1.9) and (1.10). Indeed, since $\{r_n\}$ can be chosen to go to infinity as slowly as one pleases, condition (1.10) can always be attained.

REMARK 3. If $\mathscr{L}(S_n/d(n)) \to_w \mathscr{L}(Z)$, where Z is nondegenerate, then (1.5) and (1.6) are both satisfied. Further, Theorem 2 in [10] implies that Z is a stable random variable of index $p \in (0, 2]$, and in this case that

$$d(n) = n^{1/p} h(n),$$

where h is a slowly varying function on $[1, \infty)$.

REMARK 4. In case $\{X_j\}$ is an i.i.d. sequence and $\mathscr{L}(S_n/d(n)) \to_w \mathscr{L}(Z)$, where Z is a nondegenerate stable law of index $p \in (0, 2]$, then the sequence (1.11) has a unique nondegenerate centered Gaussian limit G_τ for each $\tau > 0$. This was obtained in [7] for B-valued random variables, and a related result using a different method of trimming was established in [3] for the real-valued case. The sequence (1.11) will also converge to G_τ under suitable mixing conditions in the Hilbert space case. For example, the mixing conditions of Theorem 6.2 of [13] suffice in this regard when we assume $\mathscr{L}(S_n/d(n)) \to_w \mathscr{L}(Z)$ with Z a nondegenerate stable law of index $p \in (0, 2]$. To show there is a unique nondegenerate G_τ for each $\tau > 0$, it is essential to establish that the sequence $\{S_{n,\tau} - ES_{n,\tau}\}$ [see (2.5) for the relevant definitions] is weakly convergent to a nondegenerate law for each $\tau > 0$.

REMARK 5. Condition (1.7) is a special case of property $(*)$ in [13] and is discussed there on pages 395–396. That is, we have the following

DEFINITION. A rowwise stationary triangular array $\{X_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$ is said to satisfy property $(*)$ if for each sequence $\{q_n\}$ with $q_n \le j_n$ and $\lim_n q_n/j_n = 0$, we have

$$\mathscr{L}\left(\sum_{j=1}^{q_n} X_{n,j} \right) \to_w \delta_0.$$

It is easy to see that if the stationary triangular array $\{X_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$ has property $(*)$, then any subarray

$$\left\{ X_{n_l, j}: 1 \le j \le k_{n_l};\ l \ge 1 \right\},$$

where $1 \le k_{n_l} \le j_{n_l}$ also has property $(*)$. Further, if $\{X_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$ is a stationary Φ-mixing triangular array with $\Phi(1) < 1$ satisfying property $(*)$, and such that the random variables also satisfy

$$\sup_{n,j} \|X_{n,j}\| \le C,$$

an application of Proposition 3.5 of [13] easily yields

$$\lim_n E\left\| \sum_{j=1}^{q_n} X_{n,j} \right\| = 0,$$

whenever $\lim_n q_n/j_n = 0$. Hence, the centered triangular array

$$\{X_{n,j} - EX_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$$

also has property (∗).

If the distribution of

$$\sum_{j=1}^{k} X_{n,j}$$

is symmetric for $k = 1, \dots, j_n$, $X_{n,1} \to_{\text{prob}} 0$, and

$$\mathscr{L}\left(\sum_{j=1}^{j_n} X_{n,j}\right)$$

converges weakly, then property (∗) always holds for a stationary Φ-mixing triangular array of the form $\{X_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$. (See [13], Remark 2, page 395.) Further, it is immediate under (1.5) that if

$$\lim_n d(q_n)/d(n) = 0, \tag{1.12}$$

whenever $\lim_n q_n/n = 0$, then (1.7) holds and the triangular array

$$\{X_{n,j} = X_j/d(n): 1 \le j \le n,\ n \ge 1\}$$

satisfies property (∗). The condition (1.12) follows immediately from the representation theorem for slowly varying functions (see, for example, [15]) if $d(n) = n^{\alpha}h(n)$, where $\alpha > 0$ and $h(n)$ is slowly varying on $[1, \infty)$. Hence, by Remark 3 (1.12) and, thus, (1.7) will hold provided $\mathscr{L}(S_n/d(n)) \to_w \mathscr{L}(Z)$, where Z is nondegenerate stable of index $p \in (0, 2]$.

REMARK 6. It is interesting to observe that in the presence of (1.5) and (1.6), condition (1.7) is actually equivalent to (1.12). We have chosen to use (1.7) because in certain situations the asymptotic behavior of $\{d(n)\}$ can be determined by probabilistic considerations (the case of convergence to a stable law and the remark in [13, page 395] amplify this).

Ibragimov conjectured that if $\{X_j\}$ is a strictly stationary Φ-mixing sequence of real-valued random variables with $EX_j = 0$ and $EX_j^2 = 1$, and if $\sigma_n^2 = E(S_n^2) \to \infty$, as $n \to \infty$, then S_n can be normalized to approach a nondegenerate Gaussian limit (see [2] for a nice discussion of this and related matters). Our Theorem 2 shows that once the maximal terms of the sample $\{|X_1|, \dots, |X_n|\}$ are deleted, the only possible nondegenerate limits must be centered Gaussian provided $\Phi(1) < 1$. Corollaries 3 and 4 then apply more directly to Ibragimov's conjecture, but still leave much to be determined.

THEOREM 2. *Let $X, X_1, X_2, \dots$ be a strictly stationary real-valued sequence which is Φ-mixing with $\Phi(1) < 1$. Let $S_n = X_1 + \cdots + X_n$ and assume $E(X_j) = 0$ and $E(X_j^2) = 1$ for $j \ge 1$. Let $\sigma^2(n) = E(S_n^2) \to \infty$ as $n \to \infty$ and let $\{r_n\}$ be a sequence of integers satisfying* (1.8), (1.9) *and* (1.10) *with $t_n = [n/r_n]$. Then, for* *each* $\tau > 0$ *and any sequence* $\{\xi_n\}$ *such that* $\lim \xi_n = \infty$, *the sequence*

$$\left\{ \frac{{}^{(\xi_n r_n)}_{\tau}S_n - nE\big(XI(|X| \le \tau\sigma(t_n))\big)}{\sqrt{r_n}\,\sigma(t_n)} \right\} \tag{1.13}$$

is tight with only centered Gaussian limits. Further, if

$$\mathscr{L}\big(S_{n_k}/\sigma_{n_k}\big) \to_w \lambda, \tag{1.14}$$

where λ *is nondegenerate, then along the same subsequence* $\{n_k\}$ *for all* τ *sufficiently large*

$$\left\{ \frac{{}^{(\xi_{n_k} r_{n_k})}_{\tau}S_{n_k} - n_k E\big(XI\big(|X| \le \tau\sigma\big(t_{n_k}\big)\big)\big)}{\sqrt{r_{n_k}}\,\sigma\big(t_{n_k}\big)} \right\} \tag{1.15}$$

has only nondegenerate Gaussian limits.

The following corollary is now easy to establish.

COROLLARY 3. *Let* $X, X_1, X_2, \dots$ *be a strictly stationary real-valued sequence which is* Φ*-mixing with* $\Phi(1) < 1$, *and assume* $EX = 0$, $EX^2 = 1$. *Further, assume* $\sigma^2(n) = E(S_n^2) \to \infty$ *as* $n \to \infty$, *where* $S_n = X_1 + \cdots + X_n$. *Then the following are equivalent*:

$$\lim_n nP(|X| > \tau\sigma(n)) = 0 \quad \textit{for some (all) } \tau > 0. \tag{1.16}$$

$$\{S_n/\sigma(n)\} \textit{ is tight with only centered Gaussian limits.} \tag{1.17}$$

REMARK 7. Assuming the conditions of Corollary 3 and (1.16), it is easy to see that the following are equivalent:

$$\mathscr{L}(S_n/\sigma(n)) \to_w N(0,1), \tag{1.18}$$

$$\{S_n^2/\sigma^2(n)\} \text{ is uniformly integrable.} \tag{1.19}$$

The equivalence of (1.18) and (1.19) under only strong mixing [without (1.16)] was proved earlier by Manfred Denker (unpublished).

REMARK 8. If $\sigma(n) \ge c\sqrt{n}$ for some $c \in (0,\infty)$, then the condition (1.16) follows immediately from Markov's inequality for all $\tau > 0$, since $E(X^2) < \infty$. Hence, in this situation, the conditions of Corollary 3 imply (1.17) immediately, and if (1.19) also holds, then

$$\mathscr{L}[S_n/\sigma(n)] \to_w N(0,1)$$

as Ibragimov conjectured. This last fact and some further reaching related results owing to Peligrad are in [9]. Of course, if $E|X|^{2+\delta} < \infty$ for some $\delta > 0$, then (1.16) always holds since $\sigma(n) = \sqrt{nh}(n)$ for h slowly varying.

Since the condition (1.16) is necessary for convergence to a normal distribution (provided $\Phi(1) < 1$), this is assumed in the next corollary.

COROLLARY 4. *Assume the conditions of Corollary 3 together with (1.16) for some $\tau > 0$. Then, either*

(1.20) δ_0 *is a subsequential limit of* $S_n/\sigma(n)$,

or

(1.21) *there is a sequence* $\{\tilde{\sigma}(n)\}$ *such that for some* $c > 1$, $1/c \le \tilde{\sigma}(n)/\sigma(n) \le c$ *for all* $n \ge 1$ *and*

$$\mathscr{L}(S_n/\tilde{\sigma}(n)) \to_w N(0,1).$$

REMARK 9 [An alternative for $\Phi(1) < 1$]. The results in [14] can be used to replace the hypothesis $\Phi(1) < 1$ in Theorems 1 and 2 by weaker, but perhaps less easily verifiable hypotheses. Specifically, let $\rho(n) = d(n)$ in Theorem 1 and $\sigma(n)$ in Theorem 2. Then $\Phi(1) < 1$ in these two theorems can be replaced by the condition:

(H) for some $\tau_0 > 0$, the sequence $\{n\mathscr{L}(X/\rho(n))|B_{\tau_0}^c\}$ is relatively compact, where $B_\tau = \{x \in H\colon \|x\| \le \tau\}$ and B_τ^c is its complement.

The resulting theorems, to be denoted by Theorems 1* and 2*, have the same conclusions with the exception that the tightness implication in (1.11) and (1.13) holds for $\tau \ge \tau_0$ rather than $\tau > 0$. Section 6 contains the necessary details for the proofs.

REMARK 10. Theorem 2* of Remark 9 now allows $\Phi(1) < 1$ to be deleted from a portion of Corollary 3. We denote the result as Corollary 3*. More precisely, assume the notation of Corollary 3, but not necessarily that $\Phi(1) < 1$. Then, Corollary 3* can be stated:

(a) if for some $\tau > 0$,

(1.16*) $$\lim_n nP(|X| > \tau\sigma(n)) = 0,$$

then it follows that $\{S_n/\sigma(n)\}$ is tight, and

(1.17*) $\{S_n/\sigma(n)\}$ has only centered Gaussian limits;

(b) if $\Phi(1) < 1$ and (1.17*) holds, then (1.16*) is satisfied for all $\tau > 0$.

The main implication of Corollary 3* is that Corollary 4 is valid without the assumption $\Phi(1) < 1$.

2. Proof of Theorem 1. Let $B_\tau = \{x \in H\colon \|x\| \le \tau\}$ and B_τ^c its complement. From Theorem 3.4 of [13] we easily see that (1.5) and (1.7) imply that for each $\tau > 0$, the sequence

(2.1) $$\left\{\sum_{j=1}^{n} \mathscr{L}\left(X_j/d(n)|B_\tau^c\right)\right\} \text{ is relatively compact.}$$

Letting

$$S_n^\tau = \sum_{j=1}^{n} X_j I(\|X_j\| > \tau d(n))/d(n), \tag{2.2}$$

Lemma 2.4 of [4] and (2.1) imply that for each $\tau > 0$, the sequence

$$\{S_n^\tau\} \text{ is relatively compact.} \tag{2.3}$$

Now,

$$S_n/d(n) = S_{n,\tau} + S_n^\tau, \tag{2.4}$$

where

$$S_{n,\tau} = \sum_{j=1}^{n} X_j I(\|X_j\| \le \tau d(n))/d(n). \tag{2.5}$$

Since $\{S_n/d(n)\}$ is assumed to be tight, it follows easily that the sequence $\{S_{n,\tau}\}$ is tight for each $\tau > 0$. Further, for each $\tau > 0$, the sequence $\{S_{n,\tau}\}$ also satisfies (1.7), i.e., $\mathscr{L}(S_{q_n,\tau}) \to_w \delta_0$, whenever $\lim_n q_n/n = 0$. This is obvious from (2.4) and the three equations

$$P(\|S_{q_n,\tau}\| > \varepsilon) \le P(\|S_{q_n}/d(n)\| > \varepsilon) + q_n P(\|X\| > \tau d(n)),$$

$$\lim_n P(\|S_{q_n}/d(n)\| > \varepsilon) = 0 \quad (\text{by } (1.7))$$

and

$$\lim_n q_n P(\|X\| > \tau d(n)) = \lim_n \frac{q_n}{n} n P(\|X\| > \tau d(n)) = 0.$$

The latter follows from (2.1) and $\lim_n q_n/n = 0$. In addition, since we know $\{S_{n,\tau}\}$ is tight and (1.7) holds for each $\tau > 0$, Proposition 3.5 of [13] implies

$$\sup_n E(\exp\{\lambda\|S_{n,\tau}\|\}) < \infty \tag{2.6}$$

for some $\lambda > 0$. Given a subsequence of integers, let (n_k) be a subsequence along which $\mathscr{L}(S_{n_k,\tau})$ converges weakly to a limit, say $\mathscr{L}(W_\tau)$. Utilizing (2.6),

$$\lim_k E(S_{n_k,\tau}) = E(W_\tau), \tag{2.7}$$

and as a result

$$\mathscr{L}(S_{n_k,\tau} - ES_{n_k,\tau}) \to_w \mathscr{L}(W_\tau - EW_\tau).$$

Thus, every subsequence of $\{S_{n,\tau} - \mathrm{E}S_{n,\tau}\}$ has a weakly convergent subsequence and, therefore, for each $\tau > 0$, the sequence

$$\{S_{n,\tau} - ES_{n,\tau}\} \text{ is tight.} \tag{2.8}$$

For each integer $n \ge 1$ consider the decomposition

$$S_n = U_n + V_n,$$

where

$$U_n = \sum_{j=1}^{n} u_j, \qquad V_n = \sum_{j=1}^{n} v_j$$

and for $1 \le j \le n$,

$$u_j = u_j(n) = X_j I(\|X_j\| \le \tau d(t_n)),$$
$$v_j = v_j(n) = X_j - u_j.$$

Then $EU_n = \delta_n(\tau, r_n)$ and we have the identity

$$\begin{aligned} {}^{(\xi_n r_n)}_{\tau}S_n - \delta_n(\tau, r_n) &= (U_n - EU_n) + \left(V_n - \sum_{j=1}^{[\xi_n r_n]} X_n^{(j)} I(\|X_n^{(j)}\| > \tau d(t_n))\right) \\ &:= A(\tau, n) + B(\tau, n). \end{aligned} \tag{2.9}$$

With these preliminaries, the proof naturally divides into four steps which separately consider $B(\tau, n)/\sqrt{r_n}\, d(t_n)$, tightness and Gaussian limits of $A(\tau, n)/\sqrt{r_n}\, d(t_n)$ and the nondegeneracy of subsequential limits.

STEP 1. $B(\tau, n)/\sqrt{r_n}\, d(t_n) \to_{\text{prob}} 0$.

Since $\lim_n \sqrt{r_n}\, d(t_n) = \infty$, this assertion is verified by the next lemma.

LEMMA 2.1. *For any integer sequence* $\{r_n\}$ *satisfying* (1.8), $\tau > 0$, *and sequence* $\{\xi_n\}$ *such that* $\lim_n \xi_n = \infty$, *the quantity*

$$V_n - \sum_{j=1}^{[\xi_n r_n]} X_n^{(j)} I(\|X_n^{(j)}\| > \tau d(t_n)) \to_{\text{prob}} 0. \tag{2.10}$$

PROOF. Let $p_n = P(\|X\| > \tau d(t_n))$. Theorem 3.4 of [13] implies that

$$C(\tau) = \sup_n nP(\|X\| > \tau d(n)) < \infty$$

for each $\tau > 0$. Hence, by stationarity of $\{X_j\}$,

$$\begin{aligned} &P\left(\left\|V_n - \sum_{j=1}^{[\xi_n r_n]} X_n^{(j)} I(\|X_n^{(j)}\| > \tau d(t_n))\right\| > 0\right) \\ &\quad = P\big(\text{at least } ([\xi_n r_n] + 1)\ X_j\text{'s } (1 \le j \le n) \text{ are greater than } \tau d(t_n)\big) \\ &\quad = E\left(I\left(\sum_{j=1}^{n} I(\|X_j\| > \tau d(t_n)) \ge [\xi_n r_n] + 1\right)\right) \\ &\quad \le E\left(\sum_{j=1}^{n} I(\|X_j\| > \tau d(t_n))\right) \Big/ ([\xi_n r_n] + 1) \\ &\quad \le nP(\|X\| > \tau d(t_n))/\xi_n r_n \\ &\quad \le ((t_n + 1)/\xi_n) P(\|X\| > \tau d(t_n)) \\ &\quad \le (C(\tau) + 1)/\xi_n \\ &\quad \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, Lemma 2.1 is proved. □

By Step 1, the first part of the theorem is equivalent to the assertion that for each $\tau > 0$, the sequence

$$\left\{ \frac{U_n - EU_n}{\sqrt{r_n}\, d(t_n)} \right\} \tag{2.11}$$

is tight with only centered Gaussian limits, provided $\lim_n \xi_n = \infty$.

STEP 2. The sequence (2.11) is tight.

Define $l_n = n - r_n t_n$. Then for each $n \geq 1$, we have $0 \leq l_n \leq n - r_n(n/r_n - 1) = r_n$. Next, define

$$I_{j,n} = \{k\colon (j-1)t_n + (j-1) < k \leq jt_n + j\}, \qquad 1 \leq j \leq l_n,$$

and

$$I_{j,n} = \{k\colon (j-1)t_n + l_n < k \leq jt_n + l_n\}, \qquad l_n + 1 \leq j \leq r_n.$$

Now, set

$$Z_{n,j,\tau} = \sum_{k \in I_{j,n}} (u_k(n) - Eu_k(n))/d(t_n), \qquad 1 \leq j \leq r_n.$$

Then,

$$\frac{U_n - EU_n}{\sqrt{r_n}\, d(t_n)} = \sum_{j=1}^{r_n} \frac{Z_{n,j,\tau}}{\sqrt{r_n}},$$

so (2.11) is tight if

$$\left\{ \sum_{j=1}^{r_n} Z_{n,j,\tau}/\sqrt{r_n} \right\} \tag{2.12}$$

is tight. Whenever $l_n \geq 1$, $\{Z_{n,1,\tau}\}$ differs from a subsequence of $\{S_{n,\tau} - ES_{n,\tau}\}$ by one term of the form

$$\frac{u_{t_n+1}(n) - Eu_{t_n+1}(n)}{d(t_n)}.$$

Consequently, $\{Z_{n,1,\tau}\}$ is tight. This implies that for each $\varepsilon > 0$ there is a finite dimensional subspace F of H such that if Q_F is the projection onto the orthogonal complement of F, then

$$\sup_n E\|Q_F Z_{n,1,\tau}\|^2 < \varepsilon. \tag{2.13}$$

If not, there is an $\varepsilon > 0$, an increasing sequence of finite dimensional subspaces $\{F_k\}$ such that $\bigcup_{k \geq 1} F_k$ is dense in H and a sequence $\{n_k\}$ such that

$$E\|Q_{F_k} Z_{n_k,1,t}\|^2 > \varepsilon. \tag{2.14}$$

Choose a subsequence $\{m_k\}$ of $\{n_k\}$ such that $\mathscr{L}(Z_{m_k,1,\tau}) \to_w \mathscr{L}(W)$. Then for every finite dimensional subspace F,

$$\mathscr{L}\big(Q_F Z_{m_k,1,\tau}\big) \to_w \mathscr{L}(Q_F W)$$

and, by Proposition 3.5 of [13],

$$\lim_k E\|Q_F Z_{m_k,1,\tau}\|^2 = E\|Q_F W\|^2 < \infty. \tag{2.15}$$

To apply Proposition 3.5 of [13], the triangular array

$$\left\{\tilde{X}_{k,j} = \left(u_j(m_k) - Eu_j(m_k)\right)/d(t_{m_k}), 1 \le j \le t_{m_k} + 1,\ k = 1,2,\dots\right\} \tag{2.16}$$

must be shown to satisfy property (∗) [see Remark 5]. Since we have shown $\{S_{n,\tau}\}$ satisfies (1.7) for each $\tau > 0$ and the related triangular array

$$\left\{Y_{n,j} = X_j I\left(\|X_j\| \le \tau d(n)\right)/d(n): 1 \le j \le n;\ n \ge 1\right\}$$

is uniformly bounded, the comments of Remark 5 imply that the centered triangular array,

$$\left\{Y_{n,j} - EY_{n,j}: 1 \le j \le n;\ n \ge 1\right\}, \tag{2.17}$$

also satisfies property (∗). Now $\{\tilde{X}_{k,j}\}$, as described in (2.16), is a subarray of (2.17) except for the term $\tilde{X}_{k,t_{m_k}+1}$, which converges to zero in probability in L^1 as $k \to \infty$, and hence (2.16) satisfies property (∗). Hence, (2.15) holds as claimed.

Now, choose $F = F_{k_0}$, k_0 sufficiently large, so that

$$E\|Q_F W\|^2 < \varepsilon/2.$$

This contradicts both (2.14) and (2.15). Hence, (2.13) must hold. Using stationarity of the summands involved in $Z_{n,j,\tau}$ and the fact that for $j \ge l_n$ there is a single extra term which converges to zero in probability, (2.13) implies

$$\limsup_n E\|Q_F Z_{n,j,\tau}\|^2 \le \varepsilon \quad \text{for all } j. \tag{2.13'}$$

Now,

$$\begin{aligned}
&E\left\|\sum_{j=1}^{r_n} Q_F(Z_{n,j,\tau})/\sqrt{r_n}\right\|^2 \\
&\quad = 1/r_n\Bigg\{\sum_{j=1}^{r_n} E\left\|Q_F(Z_{n,j,\tau})\right\|^2 \\
&\qquad + 2\sum_{j=2}^{r_n}\left[\sum_{k=1}^{j-2} E\langle Q_F Z_{n,j,\tau}, Q_F Z_{n,k,\tau}\rangle + E\langle Q_F Z_{n,j,\tau}, Q_F Z_{n,j-1,\tau}\rangle\right]\Bigg\} \qquad (2.18)\\
&\quad \le (1/r_n)\sum_{j=1}^{r_n} E\left\|Q_F(Z_{n,j,\tau})\right\|^2 + \delta_n \\
&\qquad + (2/r_n)\sum_{j=2}^{r_n}\left(E\|Q_F Z_{n,j,\tau}\|^2\right)^{1/2}\left(E\|Q_F Z_{n,j-1,\tau}\|^2\right)^{1/2},
\end{aligned}$$

where

$$\delta_n = (2/r_n)\sum_{j=2}^{r_n}\sum_{k=1}^{j-2} E\langle Q_F Z_{n,j,\tau}, Q_F Z_{n,k,\tau}\rangle.$$

Hence, Lemma 1 of [1, page 170], adapted to the Hilbert space inner product, and (2.13) together imply that

$$
\begin{aligned}
\limsup_n |\delta_n| &\le \limsup_n \frac{4}{r_n} \sum_{j=2}^{r_n} \sum_{k=1}^{j-2} \Phi^{1/2}((j-k-1)t_n) \\
&\quad \times \left(E\|Q_F Z_{n,j,\tau}\|^2\right)^{1/2} \left(E\|Q_F Z_{n,k,\tau}\|^2\right)^{1/2} \\
&\le \limsup_n \frac{4}{r_n} \Phi^{1/2}(t_n) \varepsilon \sum_{j=2}^{r_n} (j-2) \\
&= \limsup_n 2 r_n \Phi^{1/2}(t_n) \varepsilon = 0 \quad [\text{by } (1.10)].
\end{aligned} \tag{2.19}
$$

Thus, by (2.13′) and (2.18),

$$
\limsup_n E\left\| \sum_{j=1}^{r_n} Q_F(Z_{n,j,\tau})/\sqrt{r_n} \right\|^2 \le 3\varepsilon. \tag{2.20}
$$

Further, since $\varepsilon > 0$ is arbitrary, this implies (2.12) is tight because the same argument applied to the norm (simply set $F = \{0\}$, so $Q_F(x) = x$) yields that the sequence (2.12) is bounded in probability.

Thus (2.11) is tight, thereby completing Step 2.

STEP 3. The sequence (2.11) has only centered Gaussian subsequential limits.

It suffices to verify that the triangular array

$$
\Delta = \{X_{n,j}: 1 \le j \le n;\ n \ge 1\}, \tag{2.21}
$$

where

$$
X_{n,j} := \left(u_j(n) - Eu_j(n)\right)/\left(\sqrt{r_n}\, d(t_n)\right),
$$

satisfies property (*). Then, since (2.11) is tight with

$$
\|u_j(n) - Eu_j(n)\|/\left(\sqrt{r_n}\, d(t_n)\right) \le 2\tau/r_n \to 0,
$$

Theorem 4.1 of [13] implies that the only possible limits are Gaussian. Furthermore, by Proposition 3.5 of [13],

$$
\sup_n E\|(U_n - EU_n)/\sqrt{r_n}\, d(t_n)\|^2 < \infty. \tag{2.22}
$$

Therefore, the convergence

$$
\mathscr{L}\left(\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\, d(t_{n_k})}\right) \to_w \mathscr{L}(G)
$$

implies

$$
0 = \lim_{k\to\infty} E\left(\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\, d(t_{n_k})}\right) = E(G).
$$

Preliminary to proving that Δ satisfies property ($*$), two lemmas are required. The first lemma concerns arbitrary triangular arrays.

LEMMA 2.2. *Let* $\{X_{n,j}: 1 \le j \le j_n;\ n \ge 1\}$ *be an arbitrary* Φ*-mixing stationary triangular array which satisfies property* ($*$). *If*

$$\left\{\mathscr{L}\left(\sum_{j=1}^{j_n} X_{n,j}\right)\right\}$$

is tight, then the family of probability measures

$$\left\{\mathscr{L}\left(\sum_{j=1}^{l} X_{n,j}\right): 1 \le l \le j_n;\ n \ge 1\right\} \tag{2.23}$$

is also tight.

PROOF. If the family in (2.23) is not tight, there is a subsequence $\{n_k\}$ such that

$$\left\{\mathscr{L}\left(\sum_{j=1}^{l_{n_k}} X_{n_k,j}\right)\right\} \tag{2.24}$$

has no weakly convergent subsequences for some sequence l_{n_k} such that $1 \le l_{n_k} \le j_{n_k}$. Choose a further subsequence, if necessary, so that

$$\mathscr{L}\left(\sum_{j=1}^{j_{n_k}} X_{n_k,j}\right) \to_w \nu. \tag{2.25}$$

Hence, Theorem 3.3 of [13] applied to the stationary Φ-mixing triangular array,

$$\{X_{n_k,j}: 1 \le j \le j_{n_k};\ k \ge 1\},$$

implies that

$$\left\{\mathscr{L}\left(\sum_{j=1}^{l} X_{n_k,j}\right): 1 \le l \le j_{n_k};\ k \ge 1\right\}$$

is tight, and this contradicts the fact that (2.24) has no weakly convergent subsequences. Hence, (2.23) must be tight, and the lemma is proved. □

The second lemma provides specific implications for the terms arising in (2.17).

LEMMA 2.3. *Let* $\{W_{n,j} = Y_{n,j} - EY_{n,j}: 1 \le j \le n;\ n \ge 1\}$ *denote the triangular array in* (2.17). *Then, for each* $\varepsilon > 0$ *and any positive sequence* $\{a_n\}$ *such that* $\lim_n a_n = \infty$,

$$\lim_n \sup_{1 \le l \le n} P\left(\left\|\sum_{j=1}^{l} W_{n,j}\right\| > \varepsilon a_n\right) = 0. \tag{2.26}$$

PROOF. Fix $\varepsilon > 0$ and take $\{a_n\}$ to be a positive sequence such that $\lim_n a_n = \infty$. Using (2.8), (2.17) and the definition of the triangular array $\{W_{n,j}: 1 \le j \le n,\ n \ge 1\}$, we see that it has the following properties:

$$\begin{aligned}&\text{(i)} && \{W_{n,j}: 1 \le j \le n;\ n \ge 1\} \text{ is } \Phi-\text{mixing, stationary,}\\ &\text{(ii)} && \left\{\sum_{j=1}^{n} W_{n,j}: n \ge 1\right\} \text{ is tight, and}\\ &\text{(iii)} && \text{it has property } (*).\end{aligned} \tag{2.27}$$

By Lemma 2.2 it follows that the family of probability measures

$$\left\{\mathscr{L}\left(\sum_{j=1}^{l} W_{n,j}\right): 1 \le l \le n;\ n \ge 1\right\} \tag{2.28}$$

is also tight. Hence, for each $\delta > 0$ there is a compact set K such that

$$\sup_{\substack{1 \le l \le n\\ n \ge 1}} P\left(\sum_{j=1}^{l} W_{n,j} \notin K\right) < \delta,$$

and, thus, since $\{a_n\} \to \infty$, we have

$$\begin{aligned}&\limsup_{n} \sup_{1 \le l \le n} P\left(\left\|\sum_{j=1}^{l} W_{n,j}\right\| > \varepsilon a_n\right)\\ &\quad \le \limsup_{n} \sup_{1 \le l \le n}\left(\sum_{j=1}^{l} W_{n,j} \notin K\right) < \delta.\end{aligned} \tag{2.29}$$

Since $\delta > 0$ is arbitrary, (2.29) implies (2.26) and Lemma 2.3 is proved. □

Finally, the assertion made in (2.21) can be verified.

LEMMA 2.4. *Δ satisfies property* $(*)$.

PROOF. Let (q_n) be a sequence such that $\lim_n(q_n/n) = 0$. Choose $\varepsilon > 0$ and let η_n be an integer such that $0 \le q_n - \eta_n t_n < t_n$. Then

$$\begin{aligned}P\left(\left\|\sum_{j=1}^{q_n} X_{n,j}\right\| > \varepsilon\right) &\le P\left(\left\|\sum_{j=1}^{\eta_n t_n} X_{n,j}\right\| > \varepsilon/2\right)\\ &\quad + P\left(\left\|\sum_{j=\eta_n t_n+1}^{q_n} X_{n,j}\right\| > \varepsilon/2\right).\end{aligned} \tag{2.30}$$

Now, by stationarity,

$$P\left(\left\|\sum_{j=\eta_n t_n+1}^{q_n} X_{n,j}\right\| > \varepsilon/2\right) = P\left(\left\|\sum_{j=1}^{l_n} X_{n,j}\right\| > \varepsilon/2\right), \tag{2.31}$$

where $0 \le l_n = q_n - \eta_n t_n < t_n$. Lemma 2.3, with the accompanying fact that $X_{n,j} = (Y_{t_n,j} - EY_{t_n,j})/\sqrt{r_n}$, yields

$$P\left(\left\|\sum_{j=1}^{l_n} X_{n,j}\right\| > \varepsilon/2\right) = P\left(\left\|\sum_{j=1}^{l_n} \left(Y_{t_n,j} - EY_{t_n,j}\right)\right\| > (\varepsilon/2)\sqrt{r_n}\right) = 0. \tag{2.32}$$

Thus, (2.31) and (2.32) imply that

$$\limsup_n P\left(\left\|\sum_{j=\eta_n t_n+1}^{q_n} X_{n,j}\right\| > \varepsilon/2\right) = 0. \tag{2.33}$$

Now, by arguing as in (2.18) and (2.19), we have

$$\begin{aligned} P\left(\left\|\sum_{j=1}^{\eta_n t_n} X_{n,j}\right\| > \varepsilon/2\right) &\le \frac{4}{\varepsilon^2} E\left\|\sum_{j=1}^{\eta_n t_n} X_{n,j}\right\|^2 \\ &= \frac{4}{\varepsilon^2 r_n}\left\{\sum_{j=1}^{\eta_n} E\|B_j\|^2 + 2\sum_{j=2}^{\eta_n}\sum_{k=1}^{j-2} E\langle B_j, B_k\rangle \right. \\ &\qquad \left. + 2\sum_{j=2}^{\eta_n} E\langle B_j, B_{j-1}\rangle\right\}, \end{aligned} \tag{2.34}$$

where

$$B_l = \sum_{j=(l-1)t_n}^{lt_n} \left(u_j(n) - Eu_j(n)\right)/d(t_n) \quad \text{for } 1 \le l \le \eta_n.$$

Combining stationarity and an application of Proposition 3.5 of [13] [since (2.8) and (2.17) hold],

$$\begin{aligned} \sup_l E\|B_l\|^2 = E\|B_1\|^2 &\le \sup_n E\|S_{n,\tau} - ES_{\eta,\tau}\|^2 \\ &= C < \infty. \end{aligned} \tag{2.35}$$

Hence, by Lemma 1 of [1, page 170], adapted to the Hilbert space inner product,

$$\begin{aligned} \left|\sum_{j=2}^{\eta_n}\sum_{k=1}^{j-2} E\langle B_j, B_k\rangle\right| &\le 2C\sum_{j=2}^{\eta_n}\sum_{k=1}^{j-2} \Phi^{1/2}((j-k-1)t_n) \\ &\le C\Phi^{1/2}(t_n)\eta_n^2. \end{aligned} \tag{2.36}$$

Now, $q_n \ge \eta_n t_n \ge \eta_n(n/r_n - 1)$ and, hence,

$$q_n/n \ge \eta_n/r_n - \eta_n/n.$$

Thus,

$$\eta_n/r_n \le q_n/n + \eta_n/n \le 2q_n/n$$

and, since $\lim_n q_n/n = 0$, we have $\lim_n \eta_n/r_n = 0$. Thus, a combination of (1.10),

$\lim_n \eta_n / r_n = 0$, (2.34), (2.35) and (2.36) yields

$$\lim_n P\left(\left\|\sum_{j=1}^{\eta_n t_n} X_{n,j}\right\| > \varepsilon/2\right) = 0. \tag{2.37}$$

Combining (2.30), (2.31) and (2.37) establishes

$$\lim_n P\left(\left\|\sum_{j=1}^{q_n} X_{n,j}\right\| > \varepsilon\right) = 0,$$

hence, Δ satisfies property $(*)$. □

Step 3 is now complete.

STEP 4. The limit laws are all nondegenerate provided τ is chosen sufficiently large.

This will be accomplished by showing that for each τ sufficiently large there is an $h \in H^*$ such that

$$\liminf_n E\left(h^2\left[(U_n - EU_n)/\left(\sqrt{r_n}\, d(t_n)\right)\right]\right) = \sigma_n^2 > 0. \tag{2.38}$$

Once (2.38) is established for some $\tau > 0$, then by Theorem 4.2 of [13] applied with $\delta > \tau$, the covariance function of any limiting Gaussian law must be nondegenerate and, hence, the Gaussian law itself is nondegenerate. Thus, it suffices to verify (2.38) for each $\tau > 0$ sufficiently large and some $h \in H^*$.

If (2.38) fails, then there is a sequence $\{\tau_i\}$ such that $\lim_i \tau_i = +\infty$ and for each $h \in H^*$, $\tau \in \{\tau_i\}$, we have $\sigma_h^2 = 0$. H separable implies H^* is separable, so choose a countable dense set $\{h_i\colon i \geq 1\}$ in H^*. Since (2.22) holds, failure of (2.38) for all h in $\{h_i\}$ and $\tau \in \{\tau_i\}$, implies failure of (2.38) for all $h \in H^*$ and the same set of τ. Hence, assume (2.38) fails for all h in $\{h_i\}$ and $\tau \in \{\tau_i\}$, where $\lim_i \tau_i = \infty$. Then there is a subsequence $\{n_k\}$ such that for all $\tau \in \{\tau_i\}$,

$$\lim_k E\left(h_i^2\left[\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\, d(t_{n_k})}\right]\right) = 0, \qquad i = 1, 2, \ldots. \tag{2.39}$$

Let $\{\alpha_n\}$ be a sequence of integers such that $\lim_n \alpha_n = \infty$ and $\lim_n d(\alpha_n)/d(t_n) = 0$. Such a sequence is easy to construct since $\lim_n d(n) = \infty$ and $t_n = [n/r_n]$, where $\lim_n t_n = \infty$. Define $\{Z_{n_k}, j, \tau\}$, $1 \leq j \leq r_{n_k}$, as prior to (2.12), and set

$$Z_{n_k, j, \tau} = \tilde{Z}_{n_k, j, \tau} + \tilde{\tilde{Z}}_{n_k, j, \tau},$$

where $\tilde{\tilde{Z}}_{n_k, j, \tau}$ denotes the sum of the last α_{n_k} terms in the block of terms used to define $Z_{n_k, j, \tau}$. Then, for each $h \in H^*$,

$$(2.40)\qquad h\left(\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\,d(t_{n_k})}\right) = h\left(\frac{\sum_{j=1}^{r_{n_k}} Z_{n_i, j, \tau}}{\sqrt{r_{n_k}}}\right) = h\left(\frac{\sum_{j=1}^{r_{n_k}} \tilde{Z}_{n_k, j, \tau}}{\sqrt{r_{n_k}}}\right) + h\left(\frac{\sum_{j=1}^{r_{n_k}} \tilde{\tilde{Z}}_{n_k, j, \tau}}{\sqrt{r_{n_k}}}\right).$$

Arguing as in (2.18) and (2.19) for each $h \in H^*$ and $\tau > 0$,

$$\begin{aligned}\lim_k r_{n_k}^{-1} E\left(h^2\left(\sum_{j=1}^{r_{n_k}} \tilde{\tilde{Z}}_{n_k, j, \tau}\right)\right) &= \lim_k r_{n_k}^{-1}\Bigg\{\sum_{j=1}^{r_{n_k}} Eh^2\left(\tilde{\tilde{Z}}_{n_k, j, \tau}\right) + 2\sum_{j=2}^{r_{n_k}} E\left(h\left(\tilde{\tilde{Z}}_{n_k, j, \tau}\right)h\left(\tilde{\tilde{Z}}_{n_k, j-1, \tau}\right)\right)\\ &\qquad + 2\sum_{j=2}^{r_{n_k}}\sum_{l=1}^{j-2} E\left(h\left(\tilde{\tilde{Z}}_{n_k, j, \tau}\right)h\left(\tilde{\tilde{Z}}_{n_k, l, \tau}\right)\right)\Bigg\}\\ &= 0,\end{aligned}$$

since

(i) $\lim_k d(\alpha_{n_k})/d(t_{n_k}) = 0$,

(ii) $\lim_k r_{n_k}\Phi^{1/2}(t_{n_k}) = 0$ by (1.10), and

(iii) $\lim_k \sup_{1 \le j \le r_{n_k}} Eh^2(\tilde{\tilde{Z}}_{n_k, j, \tau}) d^2(t_{n_k})/d^2(\alpha_{n_k}) < \infty$,

by stationarity and (2.13′) applied when Q_F is the identity map. Hence, from (2.40) for all h_i, $i = 1, 2, \ldots$, and $\tau \in \{\tau_i\}$

$$0 = \lim_k E\left(h_i^2\left(\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\,d(t_{n_k})}\right)\right) = \lim_k E\frac{h_i^2\left(\sum_{j=1}^{r_{n_k}} \tilde{Z}_{n_k, j, \tau}\right)}{r_{n_k}}$$

and, again by the ideas in (2.18) and (2.19) along with $\lim_n \Phi^{1/2}(\alpha_n) = 0$, it follows that

$$\lim_k E\left(h_i^2\left(\sum_{j=1}^{r_{n_k}} \tilde{Z}_{n_k, j, \tau}\right)\right)\Big/ r_{n_k} = \lim_k \sum_{j=1}^{r_{n_k}} Eh_i^2\left(\tilde{Z}_{n_i, j, \tau}\right)/r_{n_k}.$$

Since

$$\lim_k \sup_{1 \le j \le r_{n_k}} Eh^2\left(\tilde{\tilde{Z}}_{n_k, j, \tau}\right) = 0$$

for all $h \in H^*$, the previous three equations yield that for all h_i, $i = 1, 2, \ldots,$

and $\tau \in \{\tau_i\}$

$$(2.41)\qquad \begin{aligned} 0 &= \lim_k E\left(h_i^2\left[\frac{U_{n_k} - EU_{n_k}}{\sqrt{r_{n_k}}\, d(t_{n_k})} \right] \right) \\ &= \lim_k \sum_{j=1}^{r_{n_k}} E \frac{h_i^2(Z_{n_k, j, \tau})}{r_{n_k}}. \end{aligned}$$

Now (2.41), stationarity and our definition of l_{n_k} prior to (2.12) implies for all h_i $(i \geq 1)$ and $\tau \in \{\tau_i\}$ that

$$(2.42)\qquad 0 = \lim_k \left\{ l_{n_k} E\left(h_i^2(Z_{n_k, 1, \tau})\right) + (r_{n_k} - l_{n_k}) E\left(h_i^2(Z_{n_k, r_{n_k}, \tau})\right) \right\} / r_{n_k}.$$

Now $Z_{n_k, 1, \tau}$ differs from $Z_{n_k, r_{n_k}, \tau}$ be at most one term of the form

$$(u_j(n_k) - Eu_j(n_k))/d(t_{n_k})$$

and, since

$$(2.43)\qquad \lim_k E\|u_1(n_k) - Eu_1(n_k)\|^2/d^2(t_{n_k}) = 0,$$

stationarity implies that for each $h \in H^*$,

$$(2.44)\qquad \lim_k \left| E\left(h^2(Z_{n_k, 1, \tau})\right) - E\left(h^2(Z_{n_i, r_{n_k}, \tau})\right) \right| = 0.$$

Combining (2.42) and (2.44) yields for each h_i $(i \geq 1)$ and $\tau \in \{\tau_i\}$ that

$$(2.45)\qquad \lim_k E h_i^2(Z_{n_k, 1, \tau}) = 0.$$

The sequence $\{Z_{n, 1, \tau}\}$ was previously shown to be tight with (2.13) bounded when $F = \{0\}$. Therefore, (2.45) and denseness of $\{h_i\}$ in H^* imply that

$$(2.46)\qquad \mathscr{L}(Z_{n_k, 1, \tau}) \to_w \delta_0.$$

Now, extract a further subsequence $\{m_k\}$ of $\{n_k\}$, such that both

$$(2.47)\qquad \mathscr{L}(S_{t_{m_k}, \tau}) \to_w \mathscr{L}(W_\tau), \quad \text{for } \tau \in \{\tau_i\}$$

and

$$(2.48)\qquad \mathscr{L}(S_{t_{m_k}}/d(t_{m_k})) \to_w \mathscr{L}(Z),$$

with Z nondegenerate. This latter is possible by assumption (1.6). Of course, by previous remarks,

$$(2.49)\qquad \lim_k E(S_{t_{m_k}, \tau}) = EW_\tau.$$

Since (2.43) holds, the definition of $Z_{n_k, 1, \tau}$ and stationarity, together with (2.47) and (2.49), imply that for each $\tau \in \{\tau_i\}$, both

$$(2.50)\qquad \mathscr{L}(S_{t_{m_k}, \tau} - ES_{t_{m_k}, \tau}) \to_w \delta_0$$

and

$$(2.51)\qquad \mathscr{L}(S_{t_{m_k}, \tau} - ES_{t_{m_k}, \tau}) \to_w \mathscr{L}(W_\tau - EW_\tau).$$

Combining (2.50) and (2.51) yields

$$\mathscr{L}(W_\tau - EW_\tau) = \delta_0,$$

so $\mathscr{L}(W_\tau) = \delta_{EW_\tau}$ for each $\tau \in \{\tau_i\}$.

We conclude with two lemmas. The first, in conjunction with (2.47) and (2.52) will establish that for any $\varepsilon > 0$, there exists $\tau(\varepsilon)$ so that for $\tau \geq \tau(\varepsilon)$, $\tau \in \{\tau_i\}$, the Prohorov distance between δ_{EW_τ} and $\mathscr{L}(Z)$ is less than ε. The final lemma will then imply that $\mathscr{L}(Z)$ must be degenerate, contradicting (2.48). Hence, the assumption that (2.38) fails leads to a contradiction, thereby, verifying Step 4. Thus, the proof of Theorem 1 is complete modulo the proofs of the final two lemmas.

Recall the following definitions which are required for the lemmas. If (S, d) is a metric space, for $x \in S$ and any set $A \subset S$, define

$$d(x, A) = \inf_{y \in A} d(x, y),$$

and

$$A^\varepsilon = \{x: d(A, x) < \varepsilon\}, \quad \text{the } \varepsilon \text{ ball about } A.$$

For any Borel probability measures μ and ν on (S, d), the Prohorov metric ρ is defined by

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0: \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for all closed sets } A\}.$$

LEMMA 2.5. *For each $\varepsilon > 0$, there is a $\tau(\varepsilon) > 0$ such that for $\tau \geq \tau(\varepsilon)$ and $\tau \in \{\tau_i\}$,*

$$\limsup_k \rho\Big(\mathscr{L}\big(S_{t_{m_k}, \tau}\big), \mathscr{L}(Z)\Big) < \varepsilon, \tag{2.52}$$

where $\mathscr{L}(Z)$ is the nondegenerate law of (2.48).

PROOF. For any closed set F in H we have

$$\begin{aligned} P\big(S_{t_{m_k}, \tau} \in F\big) &\leq P\big(S_{t_{m_k}} \in F, S^\tau_{t_{m_k}} = 0\big) + P\big(S^\tau_{t_{m_k}} \neq 0\big) \\ &\leq P\big(S_{t_{m_k}}/d(t_{m_k}) \in F\big) + \sum_{j=1}^{t_{m_k}} P\big(\|X_j\| > \tau d(t_{m_k})\big) \\ &\leq P(Z \in F^{\varepsilon_k}) + \varepsilon_k + t_{m_k} P\big(\|X\| > \tau d(t_{m_k})\big), \end{aligned} \tag{2.53}$$

where

$$\varepsilon_k > \rho\Big(\mathscr{L}\big(S_{t_{m_k}}/d(t_{m_k})\big), \mathscr{L}(Z)\Big)$$

and, hence by (2.48), it is possible to have $\lim_k \varepsilon_k = 0$. Further, by Theorem 3.4 of [13], the sequence of measures

$$\{n\mathscr{L}(X_1/d(n)|B_1^c)\}$$

is tight, so there is a $\tau_0(\varepsilon)$ sufficiently large so that for $\tau \geq \tau_0(\varepsilon)$,

$$\sup_n nP(\|X\| > \tau d(n)) < \varepsilon/2.$$

Thus, for $\tau \geq \tau_0(\varepsilon)$ and τ in $\{\tau_i\}$, (2.53) implies that

$$\rho\Big(\mathscr{L}\big(S_{t_{m_k}, \tau}\big), \mathscr{L}(Z)\Big) \leq \varepsilon_k + \varepsilon/2.$$

Taking limits completes the proof of the lemma. □

LEMMA 2.6. *Let (S, d) be a complete separable metric space. Let μ be a nondegenerate Borel probability on S. Then*

$$\inf_{x \in S} \rho(\delta_x, \mu) > 0.$$

PROOF. This is an elementary consequence of Lemma 6.1 in [8, page 42] and that (S, d) is a complete metric space. □

3. Proof of Theorem 2. Let $d(n) = \sigma(n)$. That (1.13) has only mean zero Gaussian limits now follows immediately from Theorem 1 and its proof. That is, it is trivial to show

$$\{S_n/\sigma(n)\} \text{ is tight} \tag{3.1}$$

and, by [6, Theorem 18.2.3],

$$\sigma(n) = n^{1/2}h(n), \tag{3.2}$$

where $h(t)$ is slowly varying on $[1, \infty)$. Therefore, by Remark 5, condition (1.12) is satisfied, so (1.7) holds for $\{S_n/\sigma(n)\}$. Since (1.6) is used only to prove the second part of Theorem 1, we can conclude that the only limits of (1.13) are centered Gaussian.

Condition (1.14) provides the analogue of (1.6) on the subsequence $\{n_k\}$, so by using the argument in the second half of the proof of Theorem 1, with the subsequence $\{n_k\}$ replacing $\{n\}$, it is evident that (1.15) has only nondegenerate limits for all τ sufficiently large. Thus, Theorem 2 holds as claimed. □

4. Proof of Corollary 3. Let $d(n) = \sigma(n)$ and assume (1.16) holds for some $\tau > 0$. In view of Theorem 2,

$$\left\{ \frac{S_n - nE(XI(|X| \leq \tau\sigma(t_n)))}{\sigma(n)} \right\} \tag{4.1}$$

is tight with only centered Gaussian limits, if for *some* $\{\xi_n\}$ such that $\lim_n \xi_n = \infty$ and *some* sequence $\{r_n\}$ satisfying (1.8), (1.9) and (1.10) with ($t_n = [n/r_n]$), both

$$\left({}^{(\xi_n r_n)}_{\tau}S_n - S_n\right)/\left(\sqrt{r_n}\,\sigma(t_n)\right) \to_{\text{prob}} 0, \tag{4.2}$$

and

$$\lim_n \sigma(n)/\left(\sqrt{r_n}\,\sigma(t_n)\right) = 1. \tag{4.3}$$

To prove (4.2) choose a $\tau > 0$ satisfying (1.16). Hence, for each $\varepsilon > 0$,

$$\begin{aligned}
&P\left(\left|{}^{(\xi_n r_n)}_{\tau}S_n - S_n\right| > \varepsilon\right) \\
&= P\left(\left|\sum_{j=1}^{[\xi_n r_n]} X_n^{(j)} I(|X_n^{(j)}| > \tau\sigma(t_n))\right| > \varepsilon\right) \\
&\leq \sum_{j=1}^{[\xi_n r_n]} P\left(|X_n^{(j)} I(|X_n^{(j)}| > \tau\sigma(t_n))| > 0\right) \\
&\leq \xi_n r_n n P(|X| > \tau\sigma(t_n)) \\
&\leq \xi_n r_n^2 (t_n + 1) P(|X| > \tau\sigma(t_n)).
\end{aligned} \tag{4.4}$$

Now, (1.16) implies for our $\tau > 0$ (fixed) that

$$\max_{k \geq n} kP(|X| > \tau\sigma(k)) = \varepsilon(n), \tag{4.5}$$

where $\lim_n \varepsilon(n) = 0$ and, hence, (4.4) yields (4.2) if

$$\lim_n \xi_n r_n^2 \varepsilon(t_n) = 0. \tag{4.6}$$

Choose $\{\xi_n\}$ such that $\lim_n \xi_n = \infty$ and (r_n) satisfies (1.8), (1.9) and (1.10) with $t_n = [n/r_n] \geq n/\log n$, and

$$\lim_n \xi_n r_n^2 \varepsilon(n/\log n) = 0. \tag{4.7}$$

Since $\{\varepsilon(n)\}$ as defined in (4.5) is decreasing, this choice of $\{\xi_n\}$ and $\{r_n\}$ yields (4.6) and, hence, (4.2) is verified.

To prove (4.3), recall $\sigma(n) = n^{1/2}h(n)$, where $h(t)$ is slowly varying on $[1, \infty)$ [6, Theorem 18.2.3]. Hence, by the representation theorem for slowly varying functions [15, page 2]

$$h(t) = \exp\left\{\eta(t) + \int_1^t \frac{\varepsilon(s)}{s}\, ds\right\}, \tag{4.8}$$

where $\lim_{t \to \infty} \eta(t) = c$ with $|c| < \infty$ and $\varepsilon(s)$ is continuous with $\lim_{s \to \infty} \varepsilon(s) = 0$. Hence, for $\{r_n\}$ satisfying (1.8), (1.9) and (1.10),

$$\begin{aligned} \sqrt{r_n}\,\sigma(t_n)/\sigma(n) &\sim \frac{h(t_n)}{h(n)} \\ &> \exp\left\{-\int_{n/r_n}^{n} \frac{\varepsilon(s)}{s}\, ds\right\} \\ &\leq \exp\left\{\sup_{n/r_n \leq s \leq n} |\varepsilon(s)| \log r_n\right\}. \end{aligned} \tag{4.9}$$

Choosing $\{r_n\}$ to satisfy (1.8), (1.9), (1.0), (4.6) and also the conditions

$$t_n = [n/r_n] \geq n/\log n, \tag{4.10}$$

and

$$\lim_n \sup_{n/\log n \leq s \leq n} \varepsilon(s) \log r_n = 0, \tag{4.11}$$

it follows from (4.9) that (4.3) holds.

Since (4.2) and (4.3) hold, Theorem 2 implies (4.1) is tight with only centered Gaussian limits. Further, since $\{S_n/\sigma(n)\}$ is tight with $E(S_n^2/\sigma^2(n)) = 1$, it follows that all limits of $S_n/\sigma(n)$ are also mean zero limits. Subtracting, we easily see the sequence $\{nE(XI(|X| > \tau\sigma(t_n)))/\sigma(n)\}$ is relatively compact and by the convergence of types theorem, it easily follows that

$$\lim_n nE(XI(|X| > \tau\sigma(t_n)))/\sigma(n) = 0. \tag{4.12}$$

Hence, by combining (4.12) and (4.1), (1.17) is established. Thus (1.16) implies (1.17) as required.

For the proof that (1.17) implies (1.16) observe that as in the proof of Theorem 2 it follows that $\{S_n/\sigma(n)\}$ is tight and the related triangular array satisfies property(*). Hence, an immediate application of Theorem 4.1 in [13] (applied to subsequences if necessary) yields (1.16) for all $\tau > 0$. Thus, Corollary 3 is proved. □

5. Proof of Corollary 4. If $\{S_n/\sigma(n)\}$ is not stochastically compact, there are degenerate limits δ_a. By an application of Corollary 3, δ_a is centered Gaussian and, hence, $a = 0$. Thus, (1.20) holds.

Otherwise, $\{S_n/\sigma(n)\}$ is stochastically compact and, hence, (2.38) holds with $h(x) = x$ and $\{U_n\}$ derived from $\{X_n\}$. Further,

$$\limsup E\left[\frac{U_n - EU_n}{\sqrt{r_n}\, d(n/r_n)}\right]^2 < \infty. \tag{5.1}$$

By choosing $\{r_n\}$ and $\{\xi_n\}$ as in the proof of Corollary 3, that argument indicates that

$$\lim_n \rho\left(\mathscr{L}\left[\frac{U_n - EU_n}{\sqrt{r_n}\, d(n/r_n)}\right], \mathscr{L}\left(\frac{S_n}{\sigma_n}\right)\right) = 0, \tag{5.2}$$

where ρ denotes the Prohorov distance. Now, define

$$\tilde{\sigma}(n) = \sigma(n)c(n), \tag{5.3}$$

where

$$c^2(n) = E\left[\frac{U_n - EU_n}{\sqrt{r_n}\, d(n/r_n)}\right]^2.$$

Since $\{c(n)\}$ is bounded above and below by a positive constant,

$$\left\{\frac{U_n - EU_n}{\sqrt{r_n}\, d(n/r_n)c(n)}\right\} \tag{5.4}$$

is still tight with only centered Gaussian limits. Proposition 3.5 of [13] (since Lemma 2.4 holds) implies the uniform integrability of the sequence in (5.4) squared, which in turn necessitates that all Gaussian limits have variance 1. Consequently, the limit is independent of the subsequence, so the subsequence principle implies

$$\mathscr{L}\left(\frac{U_n - EU_n}{\sqrt{r_n}\, d(n/r_n)c(n)}\right) \to_w N(0,1). \tag{5.5}$$

Combining (5.2), (5.3) and (5.5),

$$\mathscr{L}(S_n/\tilde{\sigma}(n)) \to_w N(0,1).$$

This completes the proof of Corollary 4. □

6. Proof of Theorems 1* and 2* and Corollary 3*. To verify Remark 9 and to obtain the proof of Theorem 1* from that of Theorem 1, the following modifications are relevant.

(a) By the remark following Lemma 2.3 of [14], Proposition 3.5 of [13] and the "if" part of Theorem 4.1 of [13] are both valid without the assumption $\Phi(1) < 1$.

(b) Theorem 4.2 of [13] can be replaced by Proposition 2.4 of [14], which does not require $\Phi(1) < 1$. In Step 4, when this is used, we note that

$$\|u_j(n) - Eu_j(n)\| / \left(\sqrt{r_n}\, d(t_n)\right) \le 2\tau/\sqrt{r_n} \to 0.$$

(c) Note that $\Phi(1) < 1$ is not required for the validity of Remark 5 because of Lemma 2.3 in [14].

(d) Fix $\tau \ge \tau_0$ initially and note that when $\rho(n) = d(n)$ (H) implies

$$\text{(H*)} \qquad \{n\mathscr{L}(X/d(n)|B_\tau^c)\} \quad \text{is relatively compact.}$$

Hence (H*) can be used in Lemma 2.1 to establish $C(\tau) < \infty$.

(e) In the proof of Lemma 2.5, use (H) with $\rho(n) = d(n)$ to establish that

$$\sup_n nP(\|X\| > \tau d(n)) < \varepsilon/2$$

rather than Theorem 3.4 of [13].

The proof of Theorem 2* follows from Theorem 1* exactly as Theorem 2 follows from Theorem 1.

To verify Remark 10, notice that the only modification required in the proof of Corollary 3 is the use of Theorem 2* along with the fact that if $\tau > 0$ satisfies (1.16)*, then $\tau_0 = \tau$ fulfills (H), which is equivalent to the two conditions

$$\sup_n nP(|X| > \tau_0 \sigma(n)) \le \infty$$

and

$$\forall\, \varepsilon > 0, \qquad \exists\, c > 0 \text{ such that } \sup_n nP(|X| > c\sigma(n)) < \varepsilon.$$

This yields Corollary 3*.

Corollary 4 now holds without the condition $\Phi(1) < 1$ by using Corollary 3*. □

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M. G. HAHN
DEPARTMENT OF MATHEMATICS
TUFTS UNIVERSITY
MEDFORD, MASSACHUSETTS 02155

J. KUELBS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706

J. D. SAMUR
DEPARTAMENTO DE MATHEMÁTICA
FACULTAD DE CIENCIAS EXACTAS
UNIVERSIDAD NACIONAL DE LA PLATA
CASILLA DE CORREO 172
1900 LA PLATA
ARGENTINA