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A MARTINGALE APPROACH TO POINT PROCESSES IN THE PLANE[1]

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A rigorous definition of two-parameter point processes is given as a distribution of a denumerable number of random points in the plane. A characterization with stopping lines and relation with predictability are obtained. Using the one-parameter multivariate point-process representation, a general representation theorem for a wide class of martingales is presented, which extends the representation theorem with respect to a Poisson process.

1. Introduction. Roughly speaking, a plane point process (p.p.p.) is a distribution of a denumerable number of random points in the positive quadrant of the plane. The purpose of this paper is to study these processes using the two-parameter martingale point of view as it was done, for example, in the classical case in the book of Brémaud [4]. Clearly, some difficulties will arise due to the fact that we do not have a complete order in the parameter set $\mathbb{R}_+^2$, so the tools of the general two-parameter stochastic processes theory will be necessary. In this context, only very special cases were already studied: two-parameter jump processes with one jump (Al-Hussaini and Elliott [1–3] and Mazziotto and Szpirglas [8]) and two-parameter Poisson processes (Yor [11], Mazziotto and Szpirglas [7] and Merzbach and Nualart [9]).

In Section 3, we present a satisfactory definition of plane-point processes (and multivariate plane point processes), some characterizations of such processes by random measures or by the notion of stopping lines and the connection with the two-parameter martingales. Some properties concerning the left continuity of the process, the continuity of its dual predictable projection and the predictability of the associated stopping lines are given. Section 4 attacks the general problem of martingale representation with respect to a p.p.p. The classical proofs cannot work in our context since they use the distribution of the difference of two consecutive jumps. The idea here is to truncate the p.p.p. and to consider this truncated process as a multivariate one-parameter point process. In this case, we can apply twice the general representation theorem of Jacod [5], and working carefully with the conditions of predictability, we obtain a representation theorem for martingales. A more satisfactory form can be reached in some special cases; for example, when the dual 1-predictable projection and the dual 2-predictable projection of the p.p.p. coincide. This class contains the class of p.p.p. which can be time-changed into a Poisson process and lead to the same representation as the representation with respect to the Poisson process.

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2. Notation and preliminaries. The usual notation and the main tools are introduced as follows: The processes are indexed by points of $\mathbb{R}_+^2$ in which the partial order induced by the Cartesian coordinates is defined: Let $z = (s, t)$ and $z' = (s', t')$; then $z \leq z'$ if $s \leq s'$ and $t \leq t'$, and $z < z'$ if $s < s'$ and $t < t'$. We denote $z \wedge z'$ if $s \leq s'$ and $t \geq t'$. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given equipped with an increasing right-continuous filtration $\{\mathcal{F}_z,\ z \in \mathbb{R}_+^2\}$ of sub-σ-algebras of $\mathcal{F}$. For $z = (s, t)$, denote $\mathcal{F}_z^1 = \mathcal{F}_{(s,\infty)}$ and $\mathcal{F}_z^2 = \mathcal{F}_{(\infty,t)}$. The conditional independence property: For every z, $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$ are conditionally independent given $\mathcal{F}_z$, will be assumed throughout the paper. Set $R_z = [(0,0), z]$ for any $z \in \mathbb{R}_+^2$.

Denote by S the set of all the decreasing lines, i.e., $L \in S$ if and only if:

(i) $\forall\ z, z' \in L \Rightarrow$ either $z \wedge z'$ or $z' \wedge z$.

(ii) $\forall\ z \in \mathbb{R}_+^2$ and $z \notin L$, $\exists\ z' \in L$: $z < z'$ or $z' < z$.

For each $z = (s, t)$, denote $\bar{z} = \{(s, t')\colon t \leq t'\} \cup \{(s', t)\colon s \leq s'\}$, $\underline{z} = \{(s, t')\colon t' \leq t\} \cup \{(s', t)\colon s' \leq s\}$ and $\tilde{z} = \bar{z} \cup \underline{z}$. Clearly $\bar{z}, \underline{z} \in S$ (but not $\tilde{z}$). If $L, L' \in S$, we denote $L \leq L'$ if $\forall\ z \in L$, $\exists\ z' \in L'$ such that $z \leq z'$. This relation defines a partial order in S. $L < L'$ will mean $L \leq L'$ and $L \cap L' = \varnothing$. Also $z \leq L$ will mean $\underline{z} \leq L$.

$$L \wedge L' = \sup\{L''\colon L'' \leq L \text{ and } L'' \leq L'\},$$

$$L \vee L' = \inf\{L''\colon L \leq L'' \text{ and } L' \leq L''\}.$$

Let A be a subset of $\mathbb{R}_+^2$; the Debut of A, denoted D_A will be the greatest element of S such that: $z < D_A \Rightarrow z \notin A$. (For example, $D_{\{z\}} = \bar{z}$.)

A random decreasing line $L\colon \Omega \to S$ is called a stopping line if for every $z \in \mathbb{R}_+^2$, $\{\omega\colon z \leq L(\omega)\} \in \mathcal{F}_z$. A stopping point Z is a random point such that $\bar{Z}$ is a stopping line. L is called a stepped stopping line if for every $\omega \in \Omega$, the set of the minimal points of $L(\omega)$ is denumerable and is finite in every bounded domain.

In the product space $\Omega \times \mathbb{R}_+^2$, the predictable (resp. 1-predictable, 2-predictable) σ-algebra is defined to be the σ-algebra generated by the sets $F \times (z, z']$, where $F \in \mathcal{F}_z$ (resp. $F \in \mathcal{F}_z^1$, $F \in \mathcal{F}_z^2$) and $(z, z']$ is the rectangle $\{\xi\colon z < \xi \leq z'\}$; it is denoted $\mathcal{P}$ (resp. $\mathcal{P}^1$, $\mathcal{P}^2$). In $\Omega \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$, another predictable σ-algebra is needed: $\hat{\mathcal{P}}$ is defined to be the σ-algebra generated by the sets $F \times (z_1, z_1'] \times (z_2, z_2']$, where $F \in \mathcal{F}_{\sup(z_1, z_2)}$, and every couple taken from $(z_1, z_1'] \times (z_2, z_2']$ satisfies the relation $\wedge$. A stopping line L is called predictable if its graph $[\![L]\!] = \{(\omega, z)\colon z \in L(\omega)\}$ is a predictable set.

A process $A = \{A_z,\ z \in \mathbb{R}_+^2\}$ is called increasing if its increment on every rectangle $(z, z']$ is nonnegative: $A(z, z'] = A_{z'} - A_{(s,t')} - A_{(s',t)} + A_z \geq 0$. The difference of two increasing processes is called a process of bounded variation. Let us introduce the different kinds of martingales used below. Let $M = \{M_z,\ z \in \mathbb{R}_+^2\}$ be an adapted and integrable process. M is a weak martingale if $E[M(z, z']|\mathcal{F}_z] = 0$, M is an i-martingale if $\mathcal{F}_z$ is replaced by $\mathcal{F}_z^i$, $i = 1, 2$, M is a martingale if it is a 1-martingale and a 2-martingale and a one-parameter martingale on the axes (which gives the usual definition of martingale), and M is a strong martingale if it is a martingale and $E[M(z, z']|\mathcal{F}_z^*] = 0$, where

$\mathcal{F}_z^* = \mathcal{F}_z^1 \vee \mathcal{F}_z^2$, for every $z < z'$ in $\mathbb{R}_+^2$. To every increasing integrable and adapted process A, we can associate its dual predictable (resp. i-predictable, $i = 1, 2$) projection denoted A^π (resp. $A^{(i)}$, $i = 1, 2$) [10]. It is characterized to be the unique predictable (resp. i-predictable, $i = 1, 2$) increasing process such that $A - A^\pi$ (resp. $A - A^{(i)}$, $i = 1, 2$) is a weak martingale (resp. i-martingale, $i = 1, 2$). Let $X = \{X_z,\ z \in \mathbb{R}_+^2\}$ be a right-continuous process ($\lim_{z < z',\ z' \to z} X_{z'} = X_z$) possessing limits in the other quadrants, and denote its jump at $z = (s, t)$ by the following: $\Delta X_z = X_z - X_{(s^-, t)} - X_{(s, t^-)} + X_{z^-}$, $\Delta^1 X_z = X_z - X_{(s^-, t)}$ and $\Delta^2 X_z = X_z - X_{(s, t^-)}$. Therefore $\Delta X_z = \Delta^1 X_z - \Delta^1 X_{(s, t^-)} = \Delta^2 X_z - \Delta^2 X_{(s^-, t)}$. Moreover, if X is increasing, then the set of its points of discontinuity consists of a countable number of semi-lines parallel to the axes, and if X is also adapted, then this set is a countable union of stepped stopping lines [6].

3. Plane point processes.

DEFINITION 3.1. A right-continuous process $M = \{M_z,\ z \in \mathbb{R}_+^2\}$ is called a plane point process (p.p.p.) if:

(i) M vanishes on the axes and takes its values in $N \cup \{\infty\}$.
(ii) M is increasing.
(iii) $\forall\ z \in \mathbb{R}_+^2$, $\Delta M_z, \Delta^1 M_z, \Delta^2 M_z \in \{0, 1\}$.
(iv) M is adapted (with respect to a given filtration $\{\mathcal{F}_z\}$).

REMARKS. (1) If $\Delta M_z = 1$, then $\Delta^1 M_z = \Delta^2 M_z = 1$, but the converse does not hold.

(2) For all z, we have $M_z = \sum_{z' \le z} \Delta M_{z'}$; therefore M can be characterized as an adapted discrete measure which is the sum of Dirac measures $\sum_n \delta_{Z_n}$ on the jump points and is finite for every bounded set belonging to $\{M < \infty\}$.

(3) Contrary to the one-parameter case, the jump points are not, in general, stopping points so we cannot expect to characterize a p.p.p. by its jump points; however, the jump points $\{Z_n\}_n$ of a p.p.p. M are characterized by the following properties:

(i) $Z_0 = (0, 0)$ and if $Z_n = \infty$, then $Z_m = \infty$, $\forall\ m > n$.
(ii) $\forall\ n$ such that $Z_n < \infty$, then $\forall\ m > n$, $Z_m \not\leqslant Z_n$ and $Z_n \notin \bar{\underline{Z}}_m$.
(iii) $\forall\ n \ge 1$, $\Delta M_{Z_n} = 1$ a.s. and $M_z = \sum_n I_{\{Z_n \le z\}}$.
(iv) For every random point Z such that $[\![Z]\!] \cap (\cup_n [\![Z_n]\!])$ is evanescent, we have $\Delta M_Z = 0$ a.s., and if moreover $[\![Z]\!] \cap (\cup_n [\![\bar{Z}_n]\!])$ is evanescent, $\Delta^1 M_Z = \Delta^2 M_Z = 0$ a.s. (This condition means that if $\Delta^i M_Z = 1$, then there exists an integer n such that $Z \in \bar{Z}_n$ and if moreover $\Delta^{3-i} M_Z = 1$, then there exist integers m and n such that $Z = \bar{Z}_n \cap \bar{Z}_m$.)

(4) Conversely, let $\{Z_n\}_n$ be a sequence of stopping points satisfying (i) and (ii); then the process M defined by $M_z = k - 1$, where k is the number of sets $[Z_n, \infty)$ which contain the point z, is the p.p.p. associated with $\{Z_n\}_n$.

(5) In the same spirit, we can define the concept of a multivariate plane-point process using the notion of discrete measure, since the couples of random

variables $(Z_n, X_n)_{n\geq 1}$ cannot characterize a multivariate p.p.p. We consider a Lusin space E and an extra point Δ. A multivariate p.p.p. is the following discrete random measure on $\mathbb{R}_+^2 \times E$:

$$\mu(\omega; dz, dx) = \sum_{n\geq 1} I_{\{Z_n(\omega)<\infty\}} \varepsilon_{(Z_n(\omega), X_n(\omega))}(dz, dx),$$

- where ε_a denotes the Dirac measure located at point a,
- the random points $\{Z_n\}_n$ satisfy properties (i) and (ii) of Remark (3),
- $\{X_n\}_n$ are random variables in $E \cup \{\Delta\}$,
- $X_n(\omega) = \Delta$ if and only if $Z_n(\omega) = \infty$, and
- for each Borel subset C of E, the process

$$M_z(C) = M(R_z \times C) = \sum_{n\geq 1} I_{\{Z_n \leq z\}} I_C(X_n) \text{ is adapted.}$$

Note that if E reduces to one point, then $M_z(E)$ reduces to an ordinary p.p.p.

As in the one-parameter case, we can prove and characterize the existence of the dual predictable projection of a multivariate p.p.p. [5].

Let us introduce now the following sequences of random lines, associated with a given p.p.p. M.

Define $L_1 = L_1' = D_{\{M_z \geq 1\}} = \wedge_n \bar{Z}_n$, and for $n > 1$, define

$$L_n = D_{\{z:\, \Delta M_z = 1,\ L_{n-1} < z\}}$$

(which is equal to $\wedge_k \bar{Z}_k$ for all the integers k such that $L_{n-1} < \bar{Z}_k$) and $L_n' = D_{\{M_z \geq n\}}$.

PROPOSITION 3.2. *Any of the sequences $\{L_n\}_{n=1}^{\infty}$ or $\{L_n'\}_{n=1}^{\infty}$ which satisfies the respective following properties characterizes the p.p.p. M:*

(i) $\forall\ n$, *L_n and L_n' are stepped stopping lines.*
(ii) *The sequences $\{L_n\}_n$ and $\{L_n'\}_n$ are increasing.*
(iii) *$\{L_n\}_n$ is disjoint: $[\![L_n]\!] \cap [\![L_m]\!] = \varnothing$ for $m \neq n$.*
(iv) $\forall\ m \neq n, \forall\ \omega \in \Omega$, *the set $[\![L_n'(\omega)]\!] \cap [\![L_m'(\omega)]\!]$ is countable.*

Moreover these lines satisfy

$$\bigcup_n [\![Z_n]\!] \subset \bigcup_n [\![L_n]\!] \subset \bigcup_n [\![\bar{Z}_n]\!] \subset \bigcup_n [\![L_n']\!].$$

PROOF. Let the sequence $\{L_n'\}_n$ satisfying (i), (ii) and (iv) be given, and construct the bounded variation process

$$B_z = \begin{cases} 0, & \text{if } z < L_1', \\ n, & \text{if } L_n' \leq z < L_{n+1}', \\ \infty, & \text{if } \forall\ n,\ L_n' < z. \end{cases}$$

This process is adapted and can be decomposed by $B = M - N$, where $M_z = \Sigma_{z' \leq z} I_{\{\Delta B_{z'} = +1\}}$, $N_z = -\Sigma_{z' \leq z} I_{\{\Delta B_{z'} = -1\}}$ are adapted and increasing processes. M is the p.p.p. associated to the sequence $\{L_n'\}_n$. The same holds for the sequence $\{L_n\}_n$. □

REMARK. Let Γ be an optional increasing path, i.e., a random increasing path formed by stopping points, and consider a p.p.p. M along this path: M^{Γ}. In general, the one-parameter process obtained is not a point process since jumps of magnitude 2 can occur (however, it is a one-parameter multivariate point process). In fact, in most of the examples M^{Γ} is a point process since the intersection of a stopping line and an optional increasing path must be a stopping point. Conversely, if M is an increasing process and if it is a one-parameter point process along every optional increasing path, then M is a p.p.p.

Turning now to the properties of predictability for a p.p.p. M, let us call the dual predictable projection of M, the predictable measure associated with M, and if this measure is absolutely continuous with respect to the Lebesgue measure, denote the density by λ_z and call it the intensity of the p.p.p. M. This process can be chosen to be predictable. Notice that $M - M^{(1)} - M^{(2)} + M^{\pi}$ is a martingale of bounded variation.

The following propositions are essentially proved in [6].

PROPOSITION 3.3. *Let M be a* p.p.p. *The following properties are equivalent*:

(i) *M is a predictable process.*
(ii) *The stopping lines $\{L_n\}_n$ are predictable.*
(iii) *The stopping lines $\{L'_n\}_n$ are predictable.*

For any nonnegative process $X = \{X_z,\ z \in \mathbb{R}^2_+\}$, the predictable projection of X is defined to be the unique predictable process $Y = \{Y_z,\ z \in \mathbb{R}^2_+\}$ such that $E[\int X_z\, dA_z] = E[\int Y_z\, dA_z]$ for every increasing and predictable process $A = \{A_z,\ z \in \mathbb{R}^2_+\}$.

PROPOSITION 3.4. *Let M be a p.p.p. and consider the following properties*:

(i) *For all n, the predictable projection of the process $I_{[\![L'_n]\!]}$ vanishes.*
(ii) *M is quasicontinuous to the left: For every predictable stopping line L, $\int I_{[\![L]\!]}\, dM = 0$.*
(iii) *For every predictable stepped stopping line L, $\int I_{[\![L]\!]}\, dM = 0$.*
(iv) *The dual predictable projection M^{π} is a continuous process.*
(v) *The predictable projection of the process $I_{\bigcup_n [\![Z_n]\!]}$ vanishes.*

Then

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow \text{(v)}.$$

Intuitively, property (i) means that the stopping lines $\{L'_n\}_n$ are "inaccessible": They do not intersect any predictable stopping line.

EXAMPLES. (1) *One-jump process.* This kind of process was extensively studied by Al-Hussaini and Elliott [1–3] and also by Mazziotto and Szpirglas [8]. Let $Z = (S, T)$ be a stopping point and consider the p.p.p. $M = I_{[Z, \infty)}$. Denote by F the distribution function of Z and by G its survivor function: $G(z) = \mathbb{P}\{z < Z\}$. The predictable measure of M clearly depends on the chosen

filtration. In the minimal filtration such that Z is a stopping point (which does not satisfy the conditional independence property, except in some degenerate situations), one obtains [8]

$$M_z^\pi = \int_{(0,0)}^{z} I_{\{u \le S \text{ or } v \le T\}}(dF(u,v))/(1 - F(u^-, v^-)).$$

In the product filtration (which satisfies the conditional independence property if and only if S and T are independent), one obtains [2]

$$M_z^\pi = \int_{(0,0)}^{z \wedge Z} (dG(\xi))/(G(\xi^-)).$$

In both cases, $M - M^\pi$ is a weak martingale. Moreover, if F is continuous then M^π is continuous, M is quasicontinuous to the left and the stopping line $\bar{Z}$ is inaccessible.

(2) *The Poisson process.* The two parameter Poisson process was defined and studied by several authors; see, for example, the work of Yor [11] and the work of Mazziotto and Szpirglas [7] for a martingale approach. The doubly stochastic Poisson process was defined in [9]. The Poisson process is characterized by the fact that the intensity is deterministic, $\mathcal{F}^*_{(0,0)}$-measurable in the doubly stochastic Poisson process case, and $M - M^\pi$ is a martingale (or a strong martingale). M is quasicontinuous to the left and both the stopping lines $\{L_n\}_n$ and the stopping lines $\{L'_n\}_n$ are inaccessible [6]. Moreover, the trace of a (doubly stochastic) Poisson process along an optional increasing path is still a point process which is a (nonstationary) (doubly stochastic) Poisson process. Therefore, the jump points $\{Z_n\}_n$ are not stopping points.

4. The representation theorem. The aim of this section is to prove that if the filtration is generated by a p.p.p., then each martingale can be written as a sum of integrals with respect to the p.p.p. or to its associated predictable measure. The method will be to consider the truncated p.p.p. as a multivariate one-parameter point process, where the mark is the value of the process at the second coordinate, and then to use the one-parameter multivariate point process representation theorem due to Jacod [5].

In the Poisson case, such a theorem was already proved by Yor [11]: If M is a standard two-parameter Poisson process with parameter λ and if X is a square-integrable martingale, then there exists a constant X_0, a square-integrable predictable process $\varphi = \{\varphi_z,\ z \in \mathbb{R}^2_+\}$ and a square-integrable $\hat{\mathcal{P}}$-predictable process $\psi = \{\psi_{z_1, z_2},\ z_1, z_2 \in \mathbb{R}^2_+\}$ such that for every z

$$X_z = X_0 + \int_{(0,0)}^{z} \varphi_\xi (dM_\xi - \lambda\, d\xi) + \int_{(0,0)}^{z}\int_{(0,0)}^{z} \psi_{z_1, z_2}(dM_{z_1} - \lambda\, dz_1)(dM_{z_2} - \lambda\, dz_2);$$

and this decomposition is unique.

Clearly, the converse also holds: These Stieltjes integrals are stochastic integrals and, therefore, the process X is a martingale.

From now on, let M be a p.p.p. and let $\{\mathscr{F}_z\}$ be the filtration generated by M, eventually completed by a given σ-algebra $\mathscr{F}_0$. We fix $T > 0$ and denote by $0 = S_0 < S_1 < S_2 < \cdots$ the sequence of the first coordinates of the set of points $\{Z_n,\ n \geq 0\} \cap ([0, \infty) \times [0, T])$, ordered in increasing order. Set $Z_n = (S_n, T_n)$. Then, we have:

(1) For any $n \geq 0$, S_n is a stopping time with respect to the one-parameter filtration $(\mathscr{F}_{s,T},\ s \geq 0\}$. Indeed, $\{S_n \leq s\} = \{M_{s,T} \geq n\} \in \mathscr{F}_{s,T}$.
(2) For any $n \geq 0$, T_n is $\mathscr{F}_{S_n,T}$-measurable. Indeed, $\{T_n \leq t\} \cap \{S_n \leq s\} = \{M_{s,T} \geq n\} \cap \{M_{S_n,t} > M_{S_n,t^-}\} \in \mathscr{F}_{S_n,T}$, for any $t \leq T$, since M is adapted.

Therefore, $\{(S_n, T_n),\ n \geq 1\}$ is a one-parameter multivariate point process with respect to the filtration $\{\mathscr{F}_{s,T},\ s \geq 0\}$.

We assume that M satisfies the following hypotheses:

(H1) M is nonexplosive ($M_z < \infty$, $\forall\, z \in \mathbb{R}_+^2$ a.s.).

This implies that the multivariate point process (S_n, T_n) is also nonexplosive for any $T > 0$.

(H2) The filtration $\{\mathscr{F}_z,\ z \in \mathbb{R}_+^2\}$ is the natural filtration generated by the p.p.p. M and verifies the property F4 of the conditional independence.

In particular, (H2) implies that $\{\mathscr{F}_{s,T},\ s \geq 0\}$ is the one-parameter filtration generated by the multivariate point process $\{(S_n, T_n),\ n \geq 0\}$.

Then following Theorem 5.4 of Jacod [5], for any right-continuous local martingale $\{N_s^T, \mathscr{F}_{s,T},\ s \geq 0\}$ there exists a finite $\tilde{\mathscr{P}}^T$-measurable process $\{X_{s,t}^T,\ s \geq 0,\ 0 \leq t \leq T\}$ satisfying

$$\text{(i)} \qquad \int_0^s \int_0^T |X_{\sigma,\tau}^T| M^{\pi^T}(d\sigma, d\tau) < \infty,$$

$$\text{(ii)} \qquad N_s^T = N_0^T - \int_0^s \int_0^T X_{\sigma,\tau}^T \left(M^{\pi^T}(d\sigma, d\tau) - M(d\sigma, d\tau)\right) \quad \text{a.s.},$$

where $\tilde{\mathscr{P}}^T = \mathscr{P}^{1,T} \otimes \mathscr{B}[0, T]$, $\mathscr{P}^{1,T} = \sigma\{(s, s'] \times F,\ F \in \mathscr{F}_{s,T}\}$, and M^{π^T} is the dual $\tilde{\mathscr{P}}^T$-predictable projection of M, with respect to $\mathscr{F}_{s,T} = \mathscr{F}_{s,T}^M$, introduced by Jacod [5]. Notice that the dual predictable projection M^{π^T} does not depend on T and is equal to the dual 1-predictable projection of the two-parameter increasing process M, $M^{(1)}$. Moreover, if $T \leq T'$, then $\tilde{\mathscr{P}}^T \subset \tilde{\mathscr{P}}^{T'} \subset \mathscr{P}^1$.

Using the result of Jacod we can establish the following representation for the martingales adapted to the σ-field generated by the point process M.

THEOREM 4.1. *Let $M = \{M_z,\ z \in \mathbb{R}_+^2\}$ be a plane-point process satisfying* (H1) *and* (H2). *Assume that $M^{(1)}$ is continuous in the first coordinate, and $M^{(2)}$ is continuous in the second coordinate. Suppose that $N = \{N_z,\ z \in \mathbb{R}_+^2\}$ is a martingale with respect to the filtration $\mathscr{F}_z$ $(= \mathscr{F}_z^M)$ which is bounded in any rectangle R_z. Then, there exist processes X_z and $Y(z, z')$ verifying the following*

properties:

(i) X_z *is* 1*-predictable and adapted, and* $Y(z, z')$ *is* $\hat{\mathcal{P}}$*-measurable. Also,*

$$\int_0^s \int_0^t |X_z| M^{(1)}(dz) < \infty,$$

and

$$\int_{R_z} \int_{R_z} |Y(\xi, \xi')| M^{(1)}(d\xi) M^{(2)}(d\xi') < \infty, \quad \text{for all } z, z' \in \mathbb{R}_+^2.$$

(ii) *For any* z *such that* $N_z < \infty$, *we have*

$$N_z = N_{0,0} + \int_{R_z} X_\xi \big(M^{(1)}(d\xi) - M(d\xi)\big)$$

$$+ \int_{R_z} \int_{R_z} Y(\xi, \xi') \big(M^{(1)}(d\xi) - M(d\xi)\big)\big(M^{(2)}(d\xi) - M(d\xi)\big).$$

PROOF. We may restrict our parameter set to some bounded rectangle R_{z_0}, $z_0 = (s_0, t_0)$. Let $N_{z_0}^+$ (resp. $N_{z_0}^-$) be the positive (resp. negative) part of N_{z_0}. In view of the decomposition

$$N_z = N_z^+ - N_z^-,$$

where

$$N_z^+ = \frac{E\big(N_{z_0}^+ | \mathcal{F}_z\big)}{E\big(N_{z_0}^+\big)}, \qquad N_z^- = \frac{E\big(N_{z_0}^- | \mathcal{F}_z\big)}{E\big(N_{z_0}^-\big)},$$

we assume that N_z is a nonnegative bounded martingale such that $E(N_{z_0}) = 1$. Now we fix the coordinate $T \le t_0$. Without loss of generality we may assume that N is equal to 1 on the axes. Then, $\{N_{s,T},\ s \ge 0\}$ is a one-parameter martingale. So, we can choose a right-continuous version of this martingale and apply the representation result of Jacod,

$$N_{s,T} = 1 + \int_0^s \int_0^T X_{\sigma,\tau}^T \big(M^{(1)}(d\sigma, d\tau) - M(d\sigma, d\tau)\big). \tag{1}$$

Consider the random measure $N_{\sigma T} M(d\sigma, d\tau)$ on $[0, \infty) \times [0, T]$. Following Jacod (see the remark after Theorem 4.1 of [5]), the 1-predictable projection of this measure is absolutely continuous with respect to $M^{(1)}(d\sigma, d\tau)$. We will denote by $Z_{\sigma,\tau,T}$ the corresponding density. That means for T fixed,

$$Z_{\sigma\tau T} = \frac{d\big[N_{\sigma T} M\big]^{(1)}}{dM^{(1)}}\bigg|_{\mathbf{R}_+ \times [0,T]} = \frac{d\big[N_{\sigma\tau_0} M\big]^{(1)}}{dM^{(1)}}\bigg|_{\mathbf{R}_+ \times [0,T]}. \tag{2}$$

The second equality of (2) follows from the martingale property of $N_{\sigma T}$ on the coordinate T.

From expression (15) of Jacod [5], we obtain

$$X_{\sigma\tau}^T = N_{\sigma^- T} - I_{\{N_{\sigma^- T} > 0\}} Z_{\sigma\tau T}.$$

Observe that we can choose a version of the bounded martingale N which is right-continuous and has left limits in the remaining three quadrants. Note that $N_{\sigma^- T}$ is 1-predictable and adapted. Moreover, we can select a version of $Z_{\sigma\tau T}$ which is a measurable function of all its variables and as a function of (σ, T, ω), $\tau \le T \le t_0$ is 1-predictable and 2-optional. Indeed, we claim that $\{I_{\{N_{\sigma^- T}>0\}} Z_{\sigma\tau T},\ \tau \le T \le t_0\}$ is a (generalized) martingale with respect to the filtration $\{\mathscr{F}_{\sigma^- T},\ \tau \le T \le t_0\}$. To see this fact, we first observe that $Z_{\sigma\tau T}$ is $\mathscr{F}_{\sigma^- T}$-measurable because $\{Z_{\sigma\tau T},\ (\sigma, \tau) \in [0, S] \times [0, T]\}$ is $\tilde{\mathscr{P}}^T$-measurable. Furthermore, for any $\tilde{\mathscr{P}}^T$-measurable and bounded process $\xi(\sigma, \tau)$, we have

$$
\begin{aligned}
&E\int_0^s \int_0^T \xi(\sigma,\tau) I_{\{N_{\sigma^- T}>0\}} Z_{\sigma\tau T} M^{(1)}(d\sigma, d\tau) \\
&\quad = E\int_0^s \int_0^T \xi(\sigma,\tau) I_{\{N_{\sigma^- T}>0\}} N_{\sigma t_0} M(d\sigma, d\tau) \\
&\quad = E\int_0^s \int_0^T \xi(\sigma,\tau) I_{\{N_{\sigma^- t_0}>0\}} N_{\sigma t_0} M(d\sigma, d\tau) \\
&\quad = E\int_0^s \int_0^T \xi(\sigma,\tau) I_{\{N_{\sigma^- t_0}>0\}} Z_{\sigma\tau t_0} M^{(1)}(d\sigma, d\tau) \\
&\quad = E\int_0^s \int_0^T \xi(\sigma,\tau) E\left[I_{\{N_{\sigma^- t_0}>0\}} Z_{\sigma\tau t_0} / \mathscr{F}_{\sigma^- T} \right] M^{(1)}(d\sigma, d\tau).
\end{aligned}
$$

Here we have used the fact that $N_{\sigma^- t_0} > 0$ implies $N_{\sigma^- T} > 0$, because $\{N_{\sigma^- t},\ t \ge 0\}$ is a nonnegative martingale.

In conclusion, $\{X^T_{\sigma\tau},\ \tau \le T \le t_0\}$ is a bounded martingale with respect to the filtration $\{\mathscr{F}_{\sigma^- T},\ \tau \le T \le t_0\}$. We can take a right-continuous version of this martingale.

Now we fix the interval $[0, \sigma)$ and order the points of the set $\{Z_n,\ n \ge 0\} \cap ([0, \sigma) \times [\tau, \infty))$ in such a way that the second coordinates of the jump points Z_n are increasing. As before we obtain a one-parameter multivariate point process with respect to the filtration $\{\mathscr{F}_{\sigma^- t},\ \tau \le t\}$. Thus, using again the representation theorem of Jacod, we can write

$$
X^T_{\sigma\tau} = X^\tau_{\sigma\tau} + \int_0^{\sigma^-} \int_\tau^T Y(\xi\nu, \sigma\tau)\left(M^{(2)}(d\xi, d\nu) - M(d\xi, d\nu)\right), \tag{3}
$$

where

$$
Y(\xi\nu, \sigma\tau) = X^{\nu^-}_{\sigma\tau} - I_{\{X^{\nu^-}_{\sigma\tau}>0\}} Z'_{\xi\nu\sigma\tau}. \tag{4}
$$

Here

$$
Z'_{\xi\nu\sigma\tau} = \left. \frac{d\left[X^\nu_{\sigma\tau} M(d\xi, d\nu)\right]^{(2)}}{dM^{(2)}(d\xi, d\nu)} \right|_{[0,\sigma)\times[\tau,\infty)}.
$$

We may take a version of $Y(\xi, \nu, \sigma, \tau)$ which is a measurable function of all its coordinates. By convention, we take $Y(\xi, \nu, \sigma, \tau) = 0$ unless $\sigma > \xi$ and $\nu > \tau$. It is clear that the process Y is $\hat{\mathscr{P}}$-measurable, because as a function of (σ, ν) this process is predictable.

Finally, we obtain the desired representation from expressions (1) and (3), setting $X(\sigma, \tau) = X^\tau_{\sigma\tau}$ and $Y(z, z')$ being the process defined by (4). It remains to

check the integrability conditions of part (i). But this is immediate because the martingale N is bounded on R_{z_0} and, therefore, the processes $X(\xi)$ and $Y(\xi, \xi')$ are also bounded by construction. □

REMARKS. (1) Using the continuity properties of the processes $M^{(1)}$ and $M^{(2)}$ (on the first and second coordinates, respectively), it can be proved that the processes $X(\xi)$ and $Y(\xi, \xi')$ appearing in the above representation are essentially unique. From this fact it follows that the representation result holds for locally bounded martingales.

Actually, the boundedness property has only been used to assume N has a right-continuous version with left limits and also to check the integrability conditions (i) and (ii).

(2) Clearly, a symmetric version of the representation theorem could also be stated. If we assume that $M^{(1)} = M^{(2)}$, then both representations coincide and the process $X(\xi)$ is predictable. In this case we obtain a generalization of the integral representation for Poisson martingales obtained by Yor [11].

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