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COVERING PROBLEMS FOR BROWNIAN MOTION ON SPHERES[1]

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Bounds are given on the mean time taken by a strong Markov process to visit all of a finite collection of subsets of its state space. These bounds are specialized to Brownian motion on the surface of the unit sphere Σ_p in R^p. This leads to bounds on the mean time taken by this Brownian motion to come within a distance ε of every point on the sphere and bounds on the mean time taken to come within ε of every point or its opposite. The second case is related to the Grand Tour, a technique of multivariate data analysis that involves a search of low-dimensional projections. In both cases the bounds are asymptotically tight as $\varepsilon \to 0$ on Σ_p for $p \geq 4$.

1. Introduction. The Grand Tour, as described in Asimov (1985), is a technique of data analysis that involves visual examination of a sequence of low-, typically two-, dimensional projections of a p-dimensional data set. Here a one-dimensional Grand Tour, a sequence of one-dimensional projections, is considered. One technique to construct a one-dimensional Grand Tour is to generate a random walk on the surface of the unit sphere Σ_p in R^p and to look at the projections of the data onto the sequence of lines spanned by the sequence of points visited by the random walk. If the random walk takes small steps, then adjacent projections in the Grand Tour will be close together, a desirable quality for visual inspection. A quantity of interest is the number of steps taken by the random walk until the sequence of projections has come within an angle ε of every projection. This is the number of steps taken until the points visited by the random walk and their reflections in the origin are within a geodesic distance ε of every point on the sphere, or the number of steps taken until caps of geodesic radius ε about these points cover Σ_p. Call this the two-cap problem for the random walk. There is an analogous one-cap problem, the number of steps taken until caps of radius ε about the points visited (and not their reflections) cover Σ_p.

Next consider Brownian motion on Σ_p. Analogous covering times for Brownian motion are of interest in their own right, and bounds on their expectations can give asymptotic bounds on expected covering times for random walks as the step sizes of the random walks shrink to zero. Let $C_1(\varepsilon, p)$ be the first time a Brownian path on Σ_p has come within a distance ε of all points of Σ_p and define $C_2(\varepsilon, p)$ analogously for the two-cap problem. This article gives upper and lower bounds on $EC_1(\varepsilon, p)$ and $EC_2(\varepsilon, p)$. For Brownian motion with scale parameter

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λ on Σ_p, the bounds are asymptotically tight as $\varepsilon \to 0$ for $p \geq 4$ and yield

$$(1.1)\quad EC_2(\varepsilon, p) = \frac{2\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1 + O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right)$$

and

$$(1.2)\quad EC_1(\varepsilon, p) = \frac{4\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1 + O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right).$$

For $p = 3$ the bounds are not asymptotically tight. As $\varepsilon \to 0$ the asymptotic results are

$$(1.3)\qquad 4 \leq \liminf \frac{\lambda EC_2(\varepsilon, 3)}{\log^2(\varepsilon^{-1})} \leq \limsup \frac{\lambda EC_2(\varepsilon, 3)}{\log^2(\varepsilon^{-1})} \leq 16$$

and

$$(1.4)\qquad 8 \leq \liminf \frac{\lambda EC_1(\varepsilon, 3)}{\log^2(\varepsilon^{-1})} \leq \limsup \frac{\lambda EC_1(\varepsilon, 3)}{\log^2(\varepsilon^{-1})} \leq 32.$$

Note that the numbers obtained for the one-cap problem are always twice those obtained for the two-cap problem.

The main result of this article is that the bounds referred to above can be given in terms of expected hitting times of caps on Σ_p and the number of caps needed to cover Σ_p. Thus to obtain these bounds it is necessary to be able to calculate the expected hitting times of small caps on Σ_p. For Brownian motion these calculations are straightforward. If expected hitting times could be calculated for random walks, then the methods of this paper would give bounds on the one- and two-cap mean covering times for random walks. Similarly, bounding expected covering times for a random Grand Tour of projections onto planes requires expected hitting times for a random walk or Brownian motion on a Grassmann manifold. In a practical sense the answers to the covering problems for two-dimensional Grand Tours are already known; the space of two-dimensional subspaces of R^p, for p reasonably large, is so large that even the most efficient Grand Tour would take an impractically long time. See Huber (1985) for a discussion.

The organization of the remainder of this article is as follows. In Section 2 general bounds on the expected time taken by a strong Markov process to visit a finite collection of subsets of its state space are given. Caps on spheres are discussed briefly in Section 3. Section 4 covers Brownian motion on spheres and its relevant expected hitting times. These are combined in Section 5 to give the results (1.1)–(1.4).

2. General bounds. This section gives upper and lower bounds on the expected time taken by a strong Markov process to visit all of a finite collection of subsets of its state space. Let A be a topological space and (Ω, F) a probability space. Let $\{X(t),\ t \geq 0\}$ be a time homogeneous strong Markov process defined on (Ω, F) with state space A. For each $a \in A$ let P_a be a probability measure on (Ω, F) such that $X(0) = a$, P_a a.s. and X has the same transition probabilities under P_a for all a.

Fix an initial position a_0 and a collection of N Borel subsets of A, $\{A_1, \dots, A_N\}$, to be visited. Let S_N be the set of all $N!$ permutations of N elements and G_N the field of subsets of S_N. Let ν denote the uniform distribution on (S_N, G_N), and let σ denote a random permutation, the identity map from S_N to S_N. Form the product space $(\Omega \times S_N, F \times G_N)$. The probability measure $P_{a_0} \times \nu$ on this space will be denoted P. When the probability measure P is used X and σ will be regarded as being defined on $(\Omega \times S_N, F \times G_N)$ by $X(\omega, \pi) = X(\omega)$ and $\sigma(\omega, \pi) = \sigma(\pi) = \pi$ for $(\omega, \pi) \in (\Omega \times S_N)$. Thus X and σ are independent and σ is uniformly distributed under P. Finally, let F_t be the sub σ-field of $F \times S_N$ generated by σ and $\{X(s),\ 0 \leq s \leq t\}$.

For any nonempty collection $\{A_1, \dots, A_i\}$ of Borel subsets of A define

$$T(A_j) = \inf\{t \geq 0\colon\ X(t) \in A_j\}, \quad \text{for } j = 1, \dots, i$$

and

$$T(A_1, \dots, A_i) = \max_{j=1,\dots,i} T(A_j).$$

Returning to the specific a_0 and $\{A_1, \dots, A_N\}$ of interest, for $i = 1, \dots, N$ define $\hat{A}_i$ to be the set

$$\{a_0\} \cup \bigcup_{j \neq i} A_j$$

and its regular points. Further define

$$\mu_- = \min_{i=1,\dots,N}\ \inf_{a \in \hat{A}_i} E_a T(A_i) \tag{2.1}$$

and

$$\mu_+ = \max_{i=1,\dots,N}\ \sup_{a \in \hat{A}_i} E_a T(A_i). \tag{2.2}$$

Next consider times taken to hit random subcollections of $\{A_1, \dots, A_N\}$. For any permutation $\pi = (\pi_1, \dots, \pi_N)$ let A_i^π denote the π_ith member of $\{A_1, \dots, A_N\}$. Then for the random permutation σ, $T(A_1^\sigma), \dots, T(A_1^\sigma, \dots, A_N^\sigma)$ are hitting times of random subcollections of $\{A_1, \dots, A_N\}$. Clearly, $T(A_1, \dots, A_N) = T(A_1^\sigma, \dots, A_N^\sigma)$. Define $R_1 = T(A_1^\sigma)$ and $R_i = T(A_1^\sigma, \dots, A_i^\sigma) - T(A_1^\sigma, \dots, A_{i-1}^\sigma)$ for $i = 2, \dots, N$. R_i can be thought of as the additional time taken to visit A_i^σ after $A_1^\sigma, \dots, A_{i-1}^\sigma$ have all been visited. The following propositions are easy consequences of the definitions.

PROPOSITION 2.3. *For $i = 1, \dots, N$, $P(R_i \neq 0) \leq 1/i$, and assuming $T(A_1), \dots, T(A_N)$ are distinct and nonzero P a.s., $P(R_i \neq 0) = 1/i$.*

PROOF. For the second assertion, for $\pi = (\pi_1, \dots, \pi_N) \in S_N$, let Π denote the event $\{T(A_1^\pi) < T(A_2^\pi) < \cdots < T(A_N^\pi)\}$. By assumption

$$P\left(\bigcup_{\pi \in S_N} \Pi\right) = 1.$$

Conditional on any particular Π, the event $\{R_i \neq 0\}$ is the event that σ_i occurs further to the right in $(\pi_1, \dots, \pi_N)$ than all of $\sigma_1, \dots, \sigma_{i-1}$. This event has conditional probability $1/i$ by the uniformity of σ and the independence of X and σ. Thus $P(R_i \neq 0) = 1/i$ unconditionally as well. A similar but slightly messier argument proves the first assertion. □

PROPOSITION 2.4. *For $i = 1, \dots, N$, $T(A_1^\sigma, \dots, A_i^\sigma)$ is a stopping time with respect to the family of σ-fields $\{F_t,\ t \geq 0\}$.*

PROOF. Write

$$\{T(A_1^\sigma, \dots, A_i^\sigma) \leq t\} = \bigcup_{\pi \in S_N} \{\sigma = \pi\} \cap \{T(A_1^\pi, \dots, A_i^\pi) \leq t\}.$$

Each of these events is in F_t. □

Now define F^0 to be the σ-field generated by σ and $X(0)$, and for $i = 1, \dots, N-1$ let F^i be the σ-field generated by σ and $\{X(t),\ 0 \leq t \leq T(A_1^\sigma, \dots, A_i^\sigma)\}$.

PROPOSITION 2.5. $\{R_i \neq 0\} \in F^{i-1}$ *for $i = 1, \dots, N$.*

PROOF.

$$\{R_i = 0\} = \bigcup_{j=1}^{N} \{\sigma_i = j\} \cap \{T(A_j) \leq T(A_1^\sigma, \dots, A_{i-1}^\sigma)\}.$$

Each of these events is in F^{i-1}. □

Now the main result of this section can be given.

THEOREM 2.6.

$$ET(A_1, \dots, A_N) \leq \mu_+ \sum_{i=1}^{N} 1/i, \tag{2.7}$$

and assuming $T(A_1), \dots, T(A_N)$ are distinct and nonzero P a.s.,

$$ET(A_1, \dots, A_N) \geq \mu_- \sum_{i=1}^{N} 1/i. \tag{2.8}$$

PROOF. If either μ_- or μ_+ is infinite, it is easy to see that the corresponding bound holds. Therefore assume both are finite and write

$$ET(A_1, \dots, A_N) = \sum_{i=1}^{N} E(R_i) = \sum_{i=1}^{N} E\big(E(R_i | F^{i-1})\big).$$

By the strong Markov property, time homogeneity and Proposition 2.5,

$$E(R_i|F^{i-1}) = I_{\{R_i \neq 0\}} E_{X(T(A_1^\sigma,\dots,A_{i-1}^\sigma))} T(A_i^\sigma), \quad P \text{ a.s.}$$

On the set $\{R_i \neq 0\}$, $X(T(A_1^\sigma,\dots,A_{i-1}^\sigma)\} \in \hat{A}_i^\sigma$. Thus by definitions (2.1) and (2.2),

$$I_{\{R_i \neq 0\}}\mu_- \leq E(R_i|F^{i-1}) \leq I_{\{R_i \neq 0\}}\mu_+, \quad P \text{ a.s.}$$

Taking expectations, Proposition 2.3 yields the result. □

Similar arguments can be used to get bounds on the moment generating function of $T(A_1,\dots,A_N)$. See Matthews (1985) for examples of this in the context of random walks on finite groups.

3. Caps on spheres. Here the number of caps of radius ε needed to cover a sphere and the number of disjoint caps of radius ε that can be packed onto a sphere are considered. The second problem is related to coding theory, and there is a substantial literature on the subject. See Sloane (1982) for results and references. The first problem has not been investigated as much; Rogers (1963) is one reference. Analogous covering problems for pairs of caps will also be considered. Here only crude answers to these problems will be offered. As is apparent from the last section, the bounds on the mean time taken by Brownian motion to visit a set of caps will depend on the number of caps only through its logarithm. Due to this fact, the following result suffices to give asymptotically tight bounds on the mean time taken by Brownian motion to become nearly dense on a sphere.

PROPOSITION 3.1. *For fixed dimension p, there exist positive constants U and L depending on p such that*:

(i) *There are $N(\rho)$ caps of radius ρ that cover Σ_p with*

$$N(\rho) \leq U\rho^{1-p}.$$

(ii) *There are $M(\rho,\theta)$ disjoint caps of radius ρ on Σ_p with points in different caps a distance at least 2θ apart such that*

$$L(\rho+\theta)^{1-p} \leq M(\rho,\theta).$$

(iii) *In both* (i) *and* (ii) *the caps can be chosen so the set of cap centers is symmetric in the origin.*

PROOF. With the surface area of Σ_p normalized to one, the volume of a cap of radius ρ is

$$\int_0^\rho \sin^{p-2}(x)\,dx \Big/ \int_0^\pi \sin^{p-2}(x)\,dx.$$

This is $O(\rho^{1-p})$ as $\rho \to 0$. Thus at most $O((\rho/2)^{1-p})$ disjoint caps of radius $\rho/2$ can be placed on Σ_p. Place disjoint pairs of caps of radius $\rho/2$ with centers symmetric in the origin on Σ_p until there is no room for any more. Now concentric caps of radius ρ must cover Σ_p, implying Σ_p can be covered by $O(\rho^{1-p})$ caps of radius ρ as $\rho \to 0$. This implies (i).

For (ii), place symmetric pairs of caps of radius $\rho + \theta$ on Σ_p until no more fit. Since concentric caps of twice the radius will cover Σ_p, there must be at least $O((\rho + \theta)^{1-p})$ of these caps. Taking concentric caps of radius ρ gives the same number of caps with points in different caps a distance at least 2θ apart. This yields (ii).

Part (iii) is implicit in the discussion of parts (i) and (ii) and gives the needed covering results for pairs of caps. □

4. Brownian motion on spheres. Brownian motion X_λ on Σ_p as a limit of random walks was studied by Roberts and Ursell (1960). See Watson (1983) for a more modern description. Here the full diffusion need not be considered; it is sufficient to focus attention on the cosine of the distance between X_λ and a particular point of Σ_p, without loss of generality $(1, 0, \dots, 0)$. This itself is a diffusion with state space $[-1, 1]$, drift

$$\mu(x) = -\lambda x/2$$

and infinitesimal variance

$$\sigma^2(x) = \frac{\lambda(1 - x^2)}{p - 1}.$$

Suppressing the dependence on p, call this diffusion W_λ. Karlin and Taylor (1981), page 338, discuss this diffusion briefly.

The arbitrary parameter λ is like the infinitesimal variance of Brownian motion on the line. A random walk on Σ_p is symmetric if the directions of its steps are uniformly distributed and independent of the lengths of its steps. For a symmetric random walk on Σ_p whose step lengths have distribution function $F(x)$, if time and space are rescaled as usual in the convergence of random walks to Brownian motion, then the limiting diffusion will be Brownian motion on Σ_p with parameter $\lambda = \int x^2\, dF(x)$.

Given $\mu(x)$ and $\sigma^2(x)$ it is an elementary exercise to calculate the expected time taken by Brownian motion to hit a pair of caps with centers symmetric in the origin. Without loss of generality suppose the caps have radius $\cos^{-1}(r)$ and are centered at $(1, 0, \dots 0)$ and $(-1, 0, \dots, 0)$, the initial position of the Brownian motion is $(x, (1 - x^2)^{1/2}, 0, \dots, 0)$, and $1 > r > x \geq 0$. Then the time taken to hit a member of the pair has the same distribution as the time taken by W_λ to leave the interval $(-r, r)$ starting from x. A similar argument equates single cap hitting times for Brownian motion and hitting times for W_λ.

For W_λ, let $v(x, r) = E_x T((-1, -r) \cup (r, 1))$. Following Karlin and Taylor (1981), $v(x, r)$ satisfies

$$-1 = \frac{-\lambda x}{2} \frac{d}{dx} v(x, r) + \frac{\lambda(1 - x^2)}{2(p - 1)} \frac{d^2}{dx^2} v(x, r) \tag{4.1}$$

subject to $v(r, r) = v(-r, r) = 0$. Maximal and minimal expected hitting times for use in Theorem 2.6 will be $v(0, r)$ and $v(x(r), r)$ where $x(r)$ is slightly smaller than r. Standard calculations as in Karlin and Taylor (1981), page 194, give exact solutions to (4.1). The answers are quite complicated but have the following simpler representations, valid as $r \to 1$:

$$v(0, r) = \frac{2\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \left(\frac{1}{1-r^2}\right)^{(p-3)/2} \left(1 + O\left((1-r^2)^{1/2}\right)\right), \quad p \geq 4, \tag{4.2}$$

$$v(0, r) = \frac{4}{\lambda} \log\left(\frac{1}{1-r^2}\right) + O(1), \quad p = 3. \tag{4.3}$$

For $p \geq 4$ and

$$x(r) = \left(1 - \frac{1}{4}(1-r^2)\log^2(1-r^2)\right)^{1/2},$$

$$v(x(r), r) = \frac{2\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \left(\frac{1}{1-r^2}\right)^{(p-3)/2} \left(1 + O\left(\frac{1}{\log(1-r^2)}\right)\right). \tag{4.4}$$

For $p = 3$ and $x(r) = (1 - (1-r^2)a^2(r))^{1/2}$ for any $a(r)$ satisfying $1 < a(r) < (1-r^2)^{-1/2}$,

$$v(x(r), r) = \frac{4}{\lambda} \log\left(a^2(r)\right) + O(1). \tag{4.5}$$

Next consider the expected time taken by Brownian motion on Σ_p to hit a single cap of radius $\cos^{-1}(r)$ from a point a distance $\cos^{-1}(x)$ from the center of the cap, denoted $u(x, r)$. Similar calculations yield exact values or $u(x, r)$. As $r \to 1$, to the order of accuracy given in (4.2) and (4.3), $u(-1, r) = 2v(0, r)$. Similarly, to the order of accuracy given in (4.4) and (4.5), $u(x(r), r) = 2v(x(r), r)$.

5. Expected covering times. In this section upper and lower bounds are calculated for the mean time taken by X_λ to come within a geodesic distance ε of all points of Σ_p or their opposites, the two-cap problem. Results will also be stated for the one-cap problem and can be obtained in the same manner. The actual bounds are quite involved, and therefore only asymptotic results, valid as $\varepsilon \to 0$, will be stated. Since the bounds are asymptotically tight for $p \geq 4$ and not tight for $p = 3$, the two cases will be considered separately. Recall that $EC_1(\varepsilon, p)$ and $EC_2(\varepsilon, p)$ are the mean covering times in the one- and two-cap problems, respectively.

THEOREM 5.1. *For* $p \geq 4$,

$$EC_1(\varepsilon, p) = \frac{4\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1+O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right)$$

and

$$EC_2(\varepsilon, p) = \frac{2\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1+O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right). \tag{5.2}$$

PROOF. First consider a lower bound on $EC_2(\varepsilon, p)$. By Proposition 3.1 there is a set of at least $\frac{1}{2}L(\varepsilon + \varepsilon\log(\epsilon^{-1}))^{1-p}$ pairs of caps of radius ε such that points in different caps are a distance at least $2\varepsilon\log(\varepsilon^{-1})$ apart. Choose the caps so $X_\lambda(0)$ is in one of the caps and consider the time taken by X_λ to hit all the remaining pairs. This is a lower bound on $EC_2(\varepsilon, p)$ since if X_λ has not visited one of the remaining pairs, then it has not been within a distance ε of either of the two centers, so it has not been within ε of all points or their opposites. Now μ_-, the minimum expected time to hit a pair of caps from $X(0)$ or inside another pair of caps, is given by

$$\mu_- = v\big(\cos(\varepsilon(1+2\log(\varepsilon^{-1}))), \cos(\varepsilon)\big).$$

Let $x(r) = (1 - \frac{1}{4}(1-r^2)\log^2(1-r^2))^{1/2}$. For ε reasonably small $\cos(\varepsilon(1 + 2\log(\varepsilon^{-1}))) < x(\cos(\varepsilon))$, so

$$v\big(\cos(\varepsilon(1+2\log(\varepsilon^{-1}))), \cos(\varepsilon)\big) \geq v\big(x(\cos(\varepsilon)), \cos(\varepsilon)\big). \tag{5.3}$$

The right-hand side of (5.3) is given by (4.4), which in terms of ε is

$$\frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{1}{\varepsilon^{p-3}}\left(1+O\left(\frac{1}{\log(\varepsilon^{-1})}\right)\right). \tag{5.4}$$

Theorem 2.6 can now be applied. The conditions for (2.8) are satisfied. Using (5.4) and the fact that $\sum_{i=1}^{N} 1/i = \log N + O(1)$, (2.8) says

$$EC_2(\varepsilon, p) \geq \frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{1}{\varepsilon^{p-3}}\left(1-O\left(\frac{1}{\log(\varepsilon^{-1})}\right)\right)$$

$$\times\left(\log\left(\frac{L}{2}(\varepsilon+\varepsilon\log(\varepsilon^{-1}))^{1-p}\right)+O(1)\right).$$

This simplifies to

$$(5.5)\quad EC_2(\varepsilon, p) \geq \frac{2\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1+O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right).$$

Next consider an upper bound on $EC_2(\varepsilon, p)$. By Proposition 3.1 there is a set of at most

$$\frac{U}{2}\left(\frac{\varepsilon}{\log(\varepsilon^{-1})}\right)^{1-p}$$

pairs of caps of radius $\varepsilon/\log(\varepsilon^{-1})$ that cover Σ_p. Place concentric caps of radius

$$\delta = \varepsilon\left(1 - \frac{1}{\log(\varepsilon^{-1})}\right)$$

about the center of each of these caps. If X_λ visits one of the enlarged caps, then it simultaneously comes within ε of every point in the smaller concentric cap and within ε of the reflection in the origin of every point in the other small cap of the pair. Since the small caps cover Σ_p, $C_2(\varepsilon, p)$ is larger than the time taken to visit all the pairs of enlarged caps. The maximum expected hitting time needed in (2.7) is

$$\mu_+ = v(0, \cos(\delta)),$$

which is given in (4.2).

Theorem 2.6 now implies

$$EC_2(\varepsilon, p) \leq \frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\left(\frac{1}{1-\cos^2(\delta)}\right)^{(p-3)/2}$$

$$\times(1+O(\sin\delta))\left(\log\left(\frac{\varepsilon}{\log(\varepsilon^{-1})}\right)^{1-p} + O(1)\right).$$

In terms of ε, this is

$$(5.6)\quad EC_2(\varepsilon, p) \leq \frac{2\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\log(\varepsilon^{-1})}{\varepsilon^{p-3}}\left(1+O\left(\frac{\log\log(\varepsilon^{-1})}{\log(\varepsilon^{-1})}\right)\right).$$

Results (5.5) and (5.6) yield (5.2). □

Now consider the case $p = 3$.

THEOREM 5.7. *On* Σ_3

$$8 \le \liminf \frac{\lambda EC_1(\varepsilon,3)}{\log^2(\varepsilon^{-1})} \le \limsup \frac{\lambda EC_1(\varepsilon,3)}{\log^2(\varepsilon^{-1})} \le 32$$

and

$$4 \le \liminf \frac{\lambda EC_2(\varepsilon,3)}{\log^2(\varepsilon^{-1})} \le \limsup \frac{\lambda EC_2(\varepsilon,3)}{\log^2(\varepsilon^{-1})} \le 16. \tag{5.8}$$

PROOF. For an upper bound on $EC_2(\varepsilon, 3)$, the same argument as the case $p \ge 4$ is used. Concentric caps of radii $\varepsilon/\log(\varepsilon^{-1})$ and $\varepsilon - \varepsilon/\log(\varepsilon^{-1})$ yield the right inequality of (5.8).

The best lower bound is obtained in the same manner as the lower bound for $p \ge 4$. Choose disjoint caps of radius $\varepsilon + \sqrt{\varepsilon}/2$ and concentric caps of radius ε. In (4.5) let $r = \cos(\varepsilon)$ and $x(r) = \cos(\varepsilon + \sqrt{\varepsilon})$, so $a(r) = \sin(\varepsilon + \sqrt{\varepsilon})/\sin(\varepsilon)$. The lower bound obtained is

$$EC_2(\varepsilon,3) \ge \frac{4}{\lambda}\log(\varepsilon^{-1})\log\left(\frac{L}{2}\left(\varepsilon + \frac{1}{2}\sqrt{\varepsilon}\right)^{-2}\right).$$

This simplifies to the left inequality of (5.8). □

6. Discussion. A natural question is: When will the bounds obtained by this method be tight? Intuitively, they will be tight when μ_- and μ_+ are close together; when the expected time taken by the process to hit a small cap is about the same whether the process starts quite near the cap or far from it. A process is rapidly mixing in the sense of Aldous (1983a) if its distribution after a short time is close to its stationary distribution. For Markov chains, a short time is a number of transitions small compared to the size of the state space. Aldous (1983b) gives bounds on mean covering times in this setting. An analogous notion of rapid mixing for a diffusion is that the distribution should be close to stationarity before the process has come close to a nonnegligible portion of its state space. Intuitively, processes in higher-dimensional spaces should be more rapidly mixing, and the bounds given in Section 2 should be tighter for these processes. This is exactly what happened here for Brownian motion on spheres.

There is some reason to suspect that for Brownian motion the upper bounds of Theorem 5.7 are tight and that the lower bounds could be improved. In the notation of Section 2, most pairs $X(T(A_1^\sigma, \dots, A_{i-1}^\sigma))$ and A_i^σ should be fairly far apart, leading to the possibility of a better lower bound than that using μ_-. Making precise statements along these lines appears difficult, though a crude technique was successful in Matthews (1985) in getting the asymptotic distributions of covering times for certain random walks on the discrete cube. If it can be shown that with sufficiently high probability $X(T(A_1^\sigma, \dots, A_{i-1}^\sigma))$ and A_i^σ are far apart, then the lower bounds of Theorem 5.7 can be improved, possibly to the extent of making them agree with the upper bounds.

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