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A NONLINEAR RENEWAL THEORY

BY CUN-HUI ZHANG

State University of New York at Stony Brook

Let T be the first time that a perturbed random walk crosses a nonlinear boundary. This paper concerns the approximations of the distribution of the excess over the boundary, the expected stopping time ET and the variance of the stopping time $\mathrm{Var}(T)$. Expansions are obtained by using linear renewal theorems with varying drift.

1. Introduction. Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables with $EX = \mu > 0$ and $\mathrm{Var}(X) = \sigma^2 > 0$. Let $\xi = \{\xi_n, n \geq 1\}$ be a sequence of random variables such that ξ_n is independent of $X_{n+1}, X_{n+2}, \ldots$ for each n. And let $A = A(t; \lambda)$, $\lambda \in \Lambda$, be a family of boundary functions, where Λ is an index set. Define

$$(1.1)\quad T = T_\lambda = \inf\{n \geq 1: Z_n > A(n; \lambda)\}, \qquad \inf \varnothing = \infty, \text{for each } \lambda \in \Lambda,$$

where $Z_n = S_n + \xi_n$ and $S_n = X_1 + \cdots + X_n$ for each n. This paper concerns the approximations of the distribution of the overshoot, the expected stopping time ET, the variance of the stopping time $\mathrm{Var}(T)$ and other quantities related to the stopping time T as the boundary tends to infinity.

Nonlinear renewal theory concerning boundary crossing times has been studied by Chow (1966), Chow, Hsiung and Lai (1979), Gut (1974), Hagwood (1980), Hagwood and Woodroofe (1982), Lai and Siegmund (1977, 1979), Lalley (1980), Siegmund (1967, 1969) and Woodroofe (1976, 1977), among others.

Lai and Siegmund (1977) studied the limiting distribution of the overshoot for the constant boundary case. In Section 2 we give sufficient conditions for the weak convergence of the overshoot, which generalize the main results of Lai and Siegmund (1977). For the expansion of ET_λ up to $o(1)$, different sufficient conditions were given by Woodroofe (1976, 1977) for $\xi_n = 0$ and by Lai and Siegmund (1979) and Hagwood and Woodroofe (1982) for $A(t; \lambda) = \text{constant}$. Their conditions are unified, weakened and generalized to stopping times of form (1.1) in Section 3. Lai and Siegmund (1979) also pointed out that the validity of the expansion of $\mathrm{Var}(T)$ up to $o(1)$ was unknown for any special case that the boundary $A(t; \lambda)$ is nonlinear or $\xi_n \neq 0$. In Section 4 the expansion of $\mathrm{Var}(T)$ is obtained under reasonable conditions.

Define

$$(1.2)\qquad \tau(c, u) = \inf\{n \geq 1: S_n - un > c\}, \qquad c \geq 0, u \leq \mu.$$

Unlike previous authors, we first consider the difference between T_λ and $\tau_\lambda = \tau(c_\lambda, d_\lambda)$ for suitable c_λ and d_λ, establish the uniform integrability of $|T_\lambda - \tau_\lambda|^p$

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(in Section 2) and then derive nonlinear renewal theorems directly from parallel results in the linear case by making use of the uniform integrabilities and the weak convergence of the overshoot. As a result, our theorems are uniform with respect to the drift $\mu - d_\lambda$ of the random walk $S_n - d_\lambda n$, which defines τ.

We shall also study the approximations of the expectations of

$$U = U_\lambda = 1 + \sum_{n=1}^{\infty} I\{Z_n \le A(n;\lambda)\}, \tag{1.3}$$

$$N = N_\lambda = 1 + \sup\{n \ge 1: Z_n \le A(n;\lambda)\}, \qquad \sup\varnothing = 0, \tag{1.4}$$

and Blackwell-type renewal theorems relative to the boundary $A(t;\lambda)$ and the process Z_n. A number of examples are given in Section 5.

2. Uniform integrabilities and weak convergences. Let us define

$$b = b_\lambda = \sup\{t \ge 1: A(t;\lambda) \ge \mu t\}, \qquad \sup\varnothing = 1, \tag{2.1}$$

$$d = d_\lambda = (\partial A/\partial t)(b_\lambda;\lambda), \tag{2.2}$$

$$\bar{d} = \sup\{(\partial A/\partial t)(t;\lambda): t \ge b_\lambda, \lambda \in \Lambda\}, \tag{2.3}$$

$$R = R_\lambda = Z_T - A(T;\lambda), \tag{2.4}$$

$$R(c,u) = S_{\tau(c,u)} - u\tau(c,u) - c, \qquad u \le \mu,\ c \ge 0, \tag{2.5}$$

$$r(u) = ER^2(0,u)/(2ER(0,u)), \qquad u \le \mu, \tag{2.6}$$

$$G(x,u) = \int_x^\infty P\{R(0,u) > y\}\,dy/ER(0,u), \qquad u \le \mu,\ x \ge 0, \tag{2.7}$$

where $\tau(c,u)$ is defined by (1.2).

We shall always assume that $A(t;\lambda)$ is twice differentiable in t and b_λ is finite so that d and $\bar{d}$ are well defined. Our first theorem is a Blackwell-type nonlinear renewal theorem. A random variable Y has an arithmetic distribution if cY is integer-valued for some constant $c \ne 0$.

THEOREM 1. *Let $R = R_\lambda$ be defined by* (2.4). *Suppose there exist functions $\rho(\delta) > 0$, $b^{1/2} \le \gamma(b) \le b$, $\gamma(b)/b \to 0$ as $b \to \infty$, and constant $d^* < \mu = EX \in (0,\infty)$ such that*

$$(T_\lambda - b_\lambda)/\gamma(b_\lambda) = O_P(1), \quad \text{as } b_\lambda \to \infty, \tag{2.8}$$

$$\lim_n P\Big\{\max_{1\le j\le \rho(\delta)\gamma(n)} |\xi_{n+j} - \xi_n| \ge \delta\Big\} = 0, \quad \text{for any } \delta > 0, \tag{2.9}$$

$$\sup\Big\{|\gamma^2(b)(\partial^2 A/\partial t^2)(t;\lambda)|: |t-b| \le K\gamma(b), \lambda \in \Lambda\Big\} < \infty, \forall K, \tag{2.10}$$

and

$$\lim_{b\to\infty} d_\lambda = d^*.$$

If $X - d^*$ does not have an arithmetic distribution, then

$$P\{R_\lambda > x\} = G(x) + o(1), \quad \text{as } b_\lambda \to \infty \text{ for any } x \geq 0, \tag{2.11}$$

where $G(x) = G(x, d^*)$ and $G(x, u)$ is defined by (2.7). If, in addition, $(T - b)/\gamma(b)$ converges in distribution to a random variable W as $b \to \infty$, then

$$\lim_b P\{R_\lambda > x, T_\lambda \geq b_\lambda + t\gamma(b_\lambda)\} = G(x)P\{W \geq t\}, \tag{2.12}$$

for every real number t with $P\{W = t\} = 0$.

REMARKS. Since $b^{-\alpha}(T - b) = o_P(1)$ implies (2.8) with $\gamma(b)/b^\alpha \to 0$, Theorem 1 implies the main results of Lai and Siegmund (1977). For the special case $\gamma(b) = b^{1/2}$, sufficient conditions for the asymptotic normality of T are given by Proposition 1 at the end of the section.

PROOF. We shall first fix $\theta > 0$ and t real. Let $m(b)$ be an integer-valued function and $n_j = n_j(\lambda)$, $j = 0, \dots, k$, be integers such that

$$[\theta\gamma(b)] \leq n_j - n_{j-1} \leq 2[\theta\gamma(b)], \qquad 1 \leq j \leq k,\ n_0 = [b + t\gamma(b)], \tag{2.13}$$

where $[x]$ is the integer part of x,

$$P\{n_j \leq T_\lambda \leq n_j + 2m(b_\lambda)\} \to 0, \quad \text{as } b_\lambda \to \infty, \tag{2.14}$$

and $0 < m(b) \to \infty$. $[m(b)/\gamma(b) \to 0.]$ Define

$$\begin{aligned}
\bar{d}_j &= [A(n_j; \lambda) - A(n_{j-1}; \lambda)]/(n_j - n_{j-1}), \\
A'_j(n; \lambda, v) &= A(n; \lambda) + v, \qquad n < n_{j-1}, \\
&= A(n_{j-1}; \lambda) + (n - n_{j-1})\bar{d}_j + v, \qquad n \geq n_{j-1}, \\
\xi'_n &= \xi'_{n,j} = \xi_{\min(n, n_{j-1}-1)} + vI\{n < n_{j-1}\}, \\
Z'_n &= Z'_{n,j} = S_n + \xi'_{n,j}, \\
T'_j &= T'_j(v) = \inf\{n: Z'_{n,j} > A'_j(n; \lambda, v)\},
\end{aligned} \tag{2.15}$$

and

$$R'_j = R'_j(v) = Z'_{T'_j} - A'_j(T'_j; \lambda, v).$$

Then, by (2.10) and (2.13) for $n_{j-1} \leq n \leq n_j$,

$$\left|A(n; \lambda) - [A(n_{j-1}; \lambda) + (n - n_{j-1})\bar{d}_j]\right| \leq \theta^2 M,$$

for some constant M which does not depend on θ and λ, and we can choose $\theta > 0$ such that

$$\delta/2 + A'_j(n; \lambda, -\delta) < A(n; \lambda) < A'_j(n; \lambda, -\delta) + 2\delta, \tag{2.16}$$

$$A'_j(n; \lambda, \delta) - 2\delta < A(n; \lambda) < A'_j(n; \lambda, \delta) - \delta/2, \tag{2.17}$$

for $n_{j-1} \leq n \leq n_j$. Setting $\theta < \rho(\delta/2)/2$ and $B_j = \{\max_{n_{j-1} \leq n \leq n_j}|\xi_n - \xi_{n_{j-1}}| >$

$\delta/2\}$, we have by (2.9), (2.13) and (2.15)–(2.17) for $n \ge n_{j-1}$ and $x > 3\delta$,

$$P\{B_j\} \to 0, \quad \text{as } b \to \infty,\ j = 1, \dots, k, \tag{2.18}$$

$$T_j'(-\delta) \le T, \quad \text{on } B_j^c \cap \{n_{j-1} \le T \le n_j \text{ or } n_{j-1} \le T_j'(-\delta) \le n_j\}, \tag{2.19}$$

$$T \le T_j'(\delta), \quad \text{on } B_j^c \cap \{n_{j-1} \le T \le n_j \text{ or } n_{j-1} \le T'(\delta) \le n \ \}, \tag{2.20}$$

$$\begin{aligned} &P\{R_j'(-\delta) > x,\ n \le T_j'(-\delta) \le n_j\} \\ &\quad \le P\{R > x - 3\delta,\ n \le T \le n_j\} + o(1) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &P\{R > x,\ n \le T \le n_j\} \\ &\quad \le P\{R_j'(\delta) > x - 3\delta,\ n \le T_j'(\delta) \le n_j\} + o(1). \end{aligned} \tag{2.22}$$

For $n \ge n_{j-1}$, we have

$$\begin{aligned} &P\{R_j' > x,\ T_j' \ge n\} \\ &\quad = \int_0^\infty P\{R(y, \bar{d}_j) > x\}\, dP\{T_j' \ge n,\ A_j'(n-1; \lambda, v) - Z_{n-1}' \le y\}. \end{aligned} \tag{2.23}$$

Since $m(b) \to \infty$ and S_n is the sum of i.i.d. nondegenerate random variables,

$$\begin{aligned} &\lim_b P\{T_j' \ge n, 0 \le A_j'(n-1; \lambda, v) - Z_{n-1}' \le c\}, \qquad n \ge m(b) + n_{j-1}, \\ &\quad \le \lim_b \sup_y P\{y \le S_{m(b)} \le y + c\} = 0, \quad \text{for every } c. \end{aligned}$$

It follows from (2.23) and Theorem 7 stated at the end of Section 4 that

$$\begin{aligned} &P\{R_j' > x,\ T_j' \ge n\} = G(x) P\{T_j' \ge n\} + o(1), \\ &\qquad\qquad n \ge m(b) + n_{j-1}. \end{aligned} \tag{2.24}$$

By (2.14), (2.18)–(2.20) and (1.2)

$$\begin{aligned} &P\{n_{j-1} \le T_j'(-\delta) \le n_{j-1} + m(b)\} \\ &\quad \le P\{B_j\} + P\{n_{j-1} \le T \le n_{j-1} + 2m(b)\} \\ &\qquad + P\{T_j'(\delta) - T_j'(-\delta) \ge m(b)\} \\ &\quad = o(1) + P\{\tau(2\delta, \bar{d}_j) \ge m(b)\} = o(1) \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} P\{n_{j-1} \le T_j'(\delta) \le n_j\} &\le P\{n_{j-1} \le T \le n_j\} + o(1) \\ &\le P\{n_{j-1} \le T_j'(-\delta) \le n_j\} + o(1). \end{aligned} \tag{2.26}$$

Therefore by (2.21)–(2.22) and (2.24)–(2.26) we have for $x > 3\delta$,

$$\begin{aligned} G(x+3\delta)&P\{n_{j-1} \le T \le n_j\} \\ &\le G(x+3\delta)P\{n_{j-1} + m(b) \le T_j'(-\delta) \le n_j\} + o(1) \\ &\le P\{R_j'(-\delta) > x + 3\delta,\ n_{j-1} \le T_j'(-\delta) \le n_j\} + o(1) \\ &\le P\{R > x,\ n_{j-1} \le T \le n_j\} + o(1) \\ &\le G(x-3\delta)P\{n_{j-1} \le T \le n_j\} + o(1), \end{aligned} \tag{2.27}$$

which implies by summing up over j that

$$\begin{aligned} G(x+3\delta)P\{n_0 \le T \le n_k\} &\le P\{R > x,\ n_0 \le T \le n_k\} + o(1) \\ &\le G(x-3\delta)P\{n_0 \le T \le n_k\} + o(1). \end{aligned} \tag{2.28}$$

Hence the proof is complete by taking limits in the order $b \to \infty$, $k \to \infty$ [$t \to -\infty$ for (2.11)] and then $\delta \to 0$. □

To study the uniform integrabilities of the powers of the differences of linear and nonlinear stopping times, we shall first give the regularity conditions on ξ. The process $\xi = \{\xi_n,\ n \ge 1\}$ is said to be regular with $p \ge 0$ and $1/2 < \alpha \le 1$ if there exists a random variable L, a function $f(x)$ and a sequence of random variables V_n, $n \ge 1$, such that

$$\xi_n = f(n) + V_n, \quad \text{for } n \ge L \text{ and } EL^p < \infty, \tag{2.29}$$

$$\max_{1 \le j \le \sqrt{n}} |f(n+j) - f(n)| \le K, \qquad K < \infty, \tag{2.30}$$

$$\Big\{ \max_{1 \le j \le n^\alpha} |V_{n+j}|^p,\ n \ge 1 \Big\} \text{ is uniformly integrable}, \tag{2.31}$$

$$n^p P\Big\{ \max_{0 \le j \le n} V_{n+j} \ge \theta n^\alpha \Big\} \to 0, \quad \text{as } n \to \infty, \forall\, \theta > 0, \tag{2.32}$$

and for some $w > 0$, $w < \mu - \bar{d}$ if $\alpha = 1$,

$$\sum_{n=1}^{\infty} n^{p-1} P\{-V_n \ge w n^\alpha\} < \infty. \tag{2.33}$$

The sequential procedures that motivate the decomposition (2.29) have been discussed by Pollak and Siegmund (1975) and Lai and Siegmund (1977). In Proposition 1 (2.31)–(2.33) are replaced by a single-moment condition in some special cases.

We shall set $f(n)$ to be the median of ξ_n when ξ is not regular and extend f to a function on $[1, \infty)$ by linear interpolation. Therefore we can define

$$\tau = \tau_\lambda = \tau(c_\lambda, d_\lambda), \tag{2.34}$$

where $\tau(c, u)$ is defined by (1.2) and $c_\lambda = b_\lambda(\mu - d_\lambda) - f(b_\lambda)$. The following theorem will be proved in Section 6.

Theorem 2. *Suppose that* ξ *is regular with* $p \geq 1$, $1/2 < \alpha \leq 1$, $E|X|^{(p+1)/\alpha} < \infty$ *and that there exist constants* δ *and* μ^* *with* $0 < \delta < 1$ *and* $\mu^* < \mu$ *such that*

$$b^p P\{T \leq \delta b\} \to 0, \quad \text{as } b \to \infty, \tag{2.35}$$

and

$$(\partial A/\partial t)(t;\lambda) \leq \mu^*, \qquad t \geq \delta b,\ \lambda \in \Lambda. \tag{2.36}$$

(i) *If* $E|X|^{2p} < \infty$ *and*

$$\sup\left\{\left|b_\lambda(\partial^2 A/\partial t^2)(t;\lambda)\right|: b_\lambda - Kb_\lambda^\alpha \leq t \leq b_\lambda + Kb_\lambda^\alpha,\ \lambda \in \Lambda\right\} < \infty, \tag{2.37}$$

for any $K > 0$, *then*

$$\left\{|T_\lambda - \tau_\lambda|^p : \lambda \in \Lambda\right\} \text{ is uniformly integrable.} \tag{2.38}$$

(ii) (2.38) *still holds without the condition that* $E|X|^{2p} < \infty$, *if* $\partial^2 A/\partial t^2 = 0$ *on* $t \geq \delta b$.

(iii) *If, in addition to the conditions in* (i) *or in* (ii), *the condition* (2.33) *is strengthened to*

$$\sum_{n=1}^{\infty} n^{p-1} P\left\{\sup_{j \geq n} -j^{-\alpha} V_j > w\right\} < \infty, \tag{2.39}$$

for some $w > 0$, $w < \mu - \bar{d}$ *if* $\alpha = 1$, *then*

$$\left\{|N_\lambda - T_\lambda|^p : \lambda \in \Lambda\right\} \text{ is uniformly integrable} \tag{2.40}$$

and

$$\left\{|U_\lambda - T_\lambda|^p : \lambda \in \Lambda\right\} \text{ is uniformly integrable.} \tag{2.41}$$

Corollary 1. (i) *Suppose that the conditions of Theorem* 2(i) *or* (ii) *hold. Then*

$$ET_\lambda = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1), \quad \text{if } p = 1, \tag{2.42}$$

$$\operatorname{Var}(T_\lambda) = (\mu - d_\lambda)^{-2}\sigma^2 b_\lambda + b_\lambda^{1/2} O(1), \qquad \text{if } p = 2. \tag{2.43}$$

(ii) *Suppose that the conditions of Theorem* 2(iii) *hold. Then*

$$EN_\lambda\ (\text{and } EU_\lambda) = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1), \quad \text{if } p = 1, \tag{2.44}$$

$$\operatorname{Var}(N_\lambda)[\text{and } \operatorname{Var}(U_\lambda)] = (\mu - d_\lambda)^{-2}\sigma^2 b_\lambda + b_\lambda^{1/2} O(1), \qquad \text{if } p = 2. \tag{2.45}$$

Proof. It follows from (2.30) that

$$f(b) = b^{1/2} O(1). \tag{2.46}$$

Therefore

$$E\tau_\lambda = (\mu - d_\lambda)^{-1} c_\lambda + O(1) = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1)$$

and

$$\operatorname{Var}(\tau_\lambda) = (\mu - d_\lambda)^{-3}\sigma^2 c_\lambda + c_\lambda^{1/2} O(1) = (\mu - d_\lambda)^{-2}\sigma^2 b_\lambda + b_\lambda^{1/2} O(1). \qquad \square$$

Remarks. The stopping time τ_λ can be written as

$$\tau_\lambda = \inf\{n \geq 1: S_n + f(b_\lambda) > \mu b_\lambda + d_\lambda(n - b_\lambda)\}, \tag{2.47}$$

where $\mu b_\lambda + d_\lambda(t - b_\lambda)$ is the two-term Taylor expansion of $A(t; \lambda)$ in the neighborhood $(b_\lambda - K b_\lambda^\alpha, b_\lambda + K b_\lambda^\alpha)$ in which the residual $A(t; \lambda) - \mu b_\lambda - d_\lambda(t - b_\lambda)$ is dominated by $O(1)(t - b_\lambda)^2/b_\lambda$. Therefore T_λ should be very close to τ_λ in view of the compactness of $(T_\lambda - b_\lambda)^2/b_\lambda$ and $\xi_T - f(b_\lambda)$. In Sections 3 and 4 we will see that more detailed expansions of $A(t; \lambda)$ lead to higher-order expansions of quantities relative to T. The stopping times $\tau(c, u)$ have been used by Woodroofe and Lai and Siegmund to obtain the expansions of ET up to $o(1)$. But they did not associate each T_λ with a $\tau_\lambda = \tau(c_\lambda, u_\lambda)$ and consider the difference between them. Also our theorems are uniform in the drift of the random walk in the sense that we can always set $\lambda' = (\lambda, u)$ and $A(t; \lambda') = A(t; \lambda) + ut$ so that

$$\begin{aligned} T_{\lambda'} &= \inf\{n \geq 1: Z_n - un > A(n; \lambda)\} \\ &= \inf\{n \geq 1: Z_n > A(n; \lambda')\}. \end{aligned} \tag{2.48}$$

Proposition 1. *Let $T = T_\lambda$ be defined by* (1.1) *and* $\operatorname{Var}(X) = \sigma^2 < \infty$.

(i) *Suppose that* $\lim A(n; \lambda) = \infty$ *as $b \to \infty$ for every fixed $n \geq 1$,*

$$\liminf_{b \to \infty} \inf\left\{A(n; \lambda)/(2n \log_2 n)^{1/2}: n \leq \delta b_\lambda\right\} > \sigma,$$

and

$$P\left\{\limsup_{n \to \infty} \xi_n/(n \log_2 n)^{1/2} = 0\right\} = 1.$$

Then $P\{T_\lambda \leq \delta b_\lambda\} \to 0$ as $b_\lambda \to \infty$.

(ii) *Suppose that $P\{n^{-1/2}\xi_n \to 0\} = 1$, $\lim P\{T_\lambda \leq \delta b_\lambda\} = 0$ for any $0 < \delta < 1$ and for some $d^* < \mu$,*

$$\lim_{\delta \to 0} \limsup_{b \to \infty} \sup\left\{|(\partial A/\partial t)(t; \lambda) - d^*|: |t - b_\lambda| \leq \delta b_\lambda\right\} = 0. \tag{2.49}$$

Then $b_\lambda^{-1/2}(T_\lambda - b_\lambda)$ converges in distribution to $N(0, \sigma^2/(\mu - d^*)^2)$.

(iii) *Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. random vectors with mean zero. Suppose that $|V_n| \leq \|Y_1 + \cdots + Y_n\|^2/n$ for each n and $E\|Y\|^{2p/\alpha} < \infty$ with $p \geq 1$ and $1/2 < \alpha \leq 1$. Then* (2.31)–(2.33) *and* (2.39) *hold.*

Proof. (i) follows from the law of the iterated logarithm. (iii) follows from Chow and Lai (1978), Theorem 1, and Doob's inequalities for submartingales. (ii) may be proved as follows:

$$\begin{aligned} P\{T > (1 + \delta)b\} &\leq P\left\{\max_{n \leq (1+\delta)b} Z_n \leq \mu b + (d^* + o(1))\,\delta b\right\} \\ &= o(1), \quad \text{by SLLN.} \end{aligned} \tag{2.50}$$

By the conditions given and (2.50)

$$T_\lambda/b_\lambda \to 1, \qquad \xi_T/b_\lambda^{1/2} \to 0, \qquad (A(T; \lambda) - \mu b_\lambda)/(T_\lambda - b_\lambda) \to d^*, \tag{2.51}$$

in probability as $b_\lambda \to \infty$. By the definitions (1.1) and (2.4)

$$\begin{aligned} S_T - \mu T &= R + A(T;\lambda) - \mu b + \mu(b-T) - \xi_T \\ &= R + (\mu - d^* + o_P(1))(b-T) + o_P(1)b^{1/2}. \end{aligned} \tag{2.52}$$

Since (2.51) still holds when T is replaced by $T-1$ and $EX_{T'}^2 = o(1)b$,

$$R = o_P(1)b^{1/2}, \quad \text{as } b \to \infty, \text{ where } T' = \min(T, 2b). \tag{2.53}$$

Therefore by (2.52) and (2.53)

$$(S_T - \mu T)/b^{1/2} = (\mu - d^* + o_P(1))(b-T)/b^{1/2} + o_P(1).$$

By Anscombe's theorem and Slutsky's theorem the proof is complete. □

3. Expansions of ET, EU and EN.

The following theorem is obviously a consequence of Theorems 1 and 2.

THEOREM 3. *Let T, U and N be defined by* (1.1), (1.3) *and* (1.4). *Suppose that* (2.9) *holds with $\gamma(n) = n^{1/2}$ and $\rho(\delta) = 1$,* (2.29) *holds with $p = 0$ and that there exist constants $d_1^* < \mu$ and d_2^* such that*

$$\lim_{n\to\infty} \max_{0 \le j \le \sqrt{n}} |f(n+j) - f(n)| = 0, \tag{3.1}$$

(3.2) *V_n converges in distribution to an integrable random variable V,*

$$\lim_{b\to\infty} d_\lambda = d_1^* \text{ and } X - d_1^* \text{ is nonarithmetic, and} \tag{3.3}$$

$$\lim_{b\to\infty} \sup\left\{ \left| b_\lambda (\partial^2 A/\partial t^2)(t;\lambda) - d_2^* \right| : (t - b_\lambda)^2 \le K b_\lambda \right\} = 0, \tag{3.4}$$

for any constant K.

(i) *If* $\operatorname{Var}(X) = \sigma^2 < \infty$ *and* (2.38) *holds for $p = 1$, then*

$$ET_\lambda = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + C_0 + o(1), \quad \text{as } b_\lambda \to \infty, \tag{3.5}$$

where

$$C_0 = (\mu - d_1^*)^{-1}\left(r(d_1^*) + (\mu - d_1^*)^{-2} d_2^* \sigma^2/2 - EV \right),$$

and $r(\cdot)$ is defined by (2.6).

(ii) *If* $\operatorname{Var}(X) = \sigma^2 < \infty$ *and* (2.40) *holds with $p = 1$, then*

$$\begin{aligned} EN_\lambda &= b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + C_0 \\ &\quad - \int_0^\infty EN(-x, d_1^*)\, dG(x) + o(1), \quad \text{as } b_\lambda \to \infty, \end{aligned} \tag{3.6}$$

where $G(x)$ is given in (2.11) *and*

$$N(x,u) = \sup\{n : S_n - un \le x\}, \qquad \sup \varnothing = 0,\ u < \mu,\ -\infty < x < \infty. \tag{3.7}$$

(iii) *If* $\operatorname{Var}(X) = \sigma^2 < \infty$ *and* (2.41) *holds with $p = 1$, then*

$$EU_\lambda = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + C_1 + o(1), \quad \text{as } b_\lambda \to \infty, \tag{3.8}$$

where

$$C_1 = (\mu - d_1^*)^{-2}\left[\sigma^2 + (\mu - d_1^*)^2 + (\mu - d_1^*)^{-1} d_2^* \sigma^2 - 2(\mu - d_1^*) EV\right]/2.$$

COROLLARY 2. *Under the conditions of Theorem* 3(ii), *for* $x > 0$,

$$\lim_b \sum_{n=1}^{\infty} P\{A(n;\lambda) - x \le Z_n \le A(n;\lambda)\} = (\mu - d_1^*)^{-1} x \tag{3.9}$$

and

$$\lim_b \sum_{n=1}^{\infty} P\{A(n;\lambda) \le Z_n \le A(n;\lambda) + x\} = (\mu - d_1^*)^{-1} x.$$

COROLLARY 3. *Under the conditions of Theorem* 3(ii)

$$\begin{aligned} &\lim_b \sum_{n=1}^{\infty} P\{T > n,\, A(n;\lambda) - Z_n < x\} \\ &\quad = (\mu - d_1^*)^{-1} \int_0^x P\Big\{\min_{n \ge 1}(S_n - d_1^* n) \ge -y\Big\}\, dy. \end{aligned} \tag{3.10}$$

REMARKS. Theorem 3 still holds when $f(n)$ is the sum of $cn^{1/2}$ and a term satisfying (3.1), since $cn^{1/2}$ can be absorbed by $A(n;\lambda)$. Theorems 2 and 3 imply Theorem 3 of Lai and Siegmund (1979) and Theorem 2 of Hagwood and Woodroofe (1982).

PROOF OF THEOREM 3. (i) By the definitions

$$S_T = R + A(T;\lambda) - \xi_T$$

and

$$S_\tau = R(c_\lambda, d) + \mu b + d(\tau - b) - f(b).$$

It follows that

$$\begin{aligned} &S_T - S_\tau - d(T - \tau) \\ &\quad = R - R(c_\lambda, d) + [A(T;\lambda) - \mu b - d(T - b)] - [\xi_T - f(b)]. \end{aligned} \tag{3.11}$$

Since

$$\begin{aligned} S_T - \mu T - S_\tau + \mu\tau = &\left[\left(S_{\max(T,\tau)} - \mu \max(T,\tau)\right) - (S_\tau - \mu\tau)\right] \\ &- \left[(S_\tau - \mu\tau) - \left(S_{\min(T,\tau)} - \mu \min(T,\tau)\right)\right], \end{aligned}$$

it follows from Chow, Robbins and Teicher (1965) that

$$\begin{aligned} E(S_T - S_\tau - \mu(T - \tau))^2 &= \sigma^2[E(\max(T,\tau) - \tau) + E(\tau - \min(T,\tau))] \\ &= \sigma^2 E|T - \tau|. \end{aligned}$$

Therefore by (2.38)

$$S_T - S_\tau - d(T - \tau) \text{ is uniformly integrable.} \tag{3.12}$$

By Anscombe's central limit theorem $(\tau - b)b^{-1/2}$ converges in distribution to $(\mu - d_1^*)^{-1}\sigma N(0,1)$. It follows that, as $b \to \infty$,

$$(3.13)\qquad (T - b)b^{1/2} \to \sigma^* N(0,1), \quad \text{in distribution, by (2.38)},$$

$$(3.14)\qquad \lim P\{R > x\} = \lim P\{R(c,d) > x\} = G(x), \quad \text{by Theorem 1},$$

$$(3.15)\qquad A(T;\lambda) - \mu b - d(T-b) \to d_2^*(\sigma^* N(0,1))^2/2, \quad \text{in distribution, by (3.4)},$$

$$(3.16)\qquad \xi_T - f(b) \to V, \quad \text{in distribution, by (2.9), (3.1) and (3.2)},$$

where $c = c_\lambda = (\mu - d_\lambda)b_\lambda - f(b_\lambda)$ and $\sigma^* = (\mu - d_1^*)^{-1}\sigma$.

It follows from a simple renewal argument that $R(c_\lambda, d_\lambda)$ is uniformly integrable. Hence by (3.11)–(3.16)

$$E(S_T - S_\tau - d(T-\tau)) = (\mu - d_1^*)^{-2}\sigma^2 d_2^*/2 - EV + o(1),$$

$$ET = E\tau + (\mu - d_1^*)^{-3}\sigma^2 d_2^*/2 - (\mu - d_1^*)^{-1}EV + o(1),$$

and

$$\begin{aligned} E\tau &= (\mu - d)^{-1}c + (\mu - d)^{-1}r(d^*) + o(1) \\ &= b - (\mu - d)^{-1}f(b) + (\mu - d_1^*)^{-1}r(d_1^*) + o(1). \end{aligned}$$

This completes the proof of (i).

(ii) Define for $v < \mu - d_1^*$,

$$(3.17)\qquad N'_\lambda(v) = \sup\{n: \xi_T + S_{T+n} \le A(T;\lambda) + (d_1^* + v)n\}, \quad \sup\varnothing = 0.$$

Then by (2.9), (3.3), (3.4), (3.13), (2.40) and (3.7)

$$(3.18)\qquad \lim_b P\{N'_\lambda(-v) \le N - T \le N'_\lambda(v)\} = 1, \quad \text{for every } v > 0,$$

$$(3.19)\qquad P\{N'_\lambda(v) \le m\} = \int_0^\infty P\{N(-x, d_1^* + v) \le m\}\,dP\{R \le x\}.$$

By (3.7) $N(x,u)$ is monotone in x and in u and continuous in u at d_1^* a.s. when $\sum_{n=1}^\infty P\{S_n - d_1^* n = x\} = 0$. Since $G(x)$ has a density,

$$(3.20)\qquad \lim_{v\to 0}\lim_b P\{N'_\lambda(v) \le m\} = \int_0^\infty P\{N(-x, d_1^*) \le m\}\,d(1 - G(x)).$$

Therefore by (3.18)–(3.20) the limiting distribution of $N - T$ is given by the right-hand side of (3.20) and (3.6) follows from the uniform integrability of $N - T$. The proof of (iii) is similar and omitted. □

Proof of Corollaries 2 and 3. Let $\lambda' = (\lambda, x)$ and $A(t;\lambda') = A(t;\lambda) - x$. Then $b_{\lambda'} = b_\lambda - (\mu - d_1^*)^{-1}x + o(1)$, $d_{\lambda'} = d_\lambda + b_\lambda^{-1}O(1)$ and $f(b_{\lambda'}) = f(b_\lambda) + o(1)$. Therefore (3.9) follows from (3.8). For (3.10) we shall write (3.10) in two parts,

$$\sum_{n=1}^{\infty} P\{T > n,\ A(n; \lambda) - Z_n < x\}$$

$$= \sum_{n=1}^{\infty} P\{A(n; \lambda) - x < Z_n \le A(n; \lambda)\}$$

$$- \sum_{n=1}^{\infty} P\{T \le n,\ A(n; \lambda) - x < Z_n \le A(n; \lambda)\}.$$

The limit of the first term is given by (3.9) and the techniques in the proof of Theorem 3(ii) bring the second term to the linear case. The final formulation is taken from Woodroofe (1976), Theorem 3.1. □

4. Expansion of Var(T). We shall study the case

$$T = T_\lambda = \inf\{n \ge 1: S_n > A(n; \lambda)\}, \qquad \inf \varnothing = \infty, \tag{4.1}$$

where $S_n = X_1 + \cdots + X_n$ and $X, X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with $EX = \mu > 0$ and $0 < \mathrm{Var}(X) = \sigma^2 < \infty$.

For $A(t; \lambda) = \lambda t^\alpha$, $0 \le \alpha < 1$, the first term in the expansion

$$\mathrm{Var}(T) = (1-\alpha)^{-2}(\sigma/\mu)^2(\lambda/\mu)^{1/(1-\alpha)}(1 + o(1)) \quad \text{as } \lambda \to \infty,$$

was obtained by Woodroofe (1976) and Chow, Hsiung and Lai (1979) under certain moment conditions on X. For the linear case that $A(t; \lambda) = \lambda =$ constant, the second-order expansion of Var(T),

$$\mathrm{Var}(T) = \mu^{-3}\sigma^2\lambda + \mu^{-2}C_1^* + o(1) \quad \text{as } \lambda \to \infty, \tag{4.2}$$

was obtained by Smith (1959) for $P\{X < 0\} = 0$ and by Lai and Siegmund (1979) for $P\{X < 0\} > 0$. The variances of boundary crossing times have also been studied by Chow (1966), Siegmund (1969), Gut (1974) and Lai (1975). The following theorem gives the expansion of Var(T) up to $o(1)$ for the general case.

THEOREM 4. *Let T be defined by* (4.1). *Suppose that for some constants* $0 < \delta < 1$, $\mu^* < \mu$, d_1^*, d_2^* *and* d_3^*,

$$\lim_{b \to \infty} b_\lambda^2 P\{T_\lambda \le \delta b_\lambda\} = 0, \qquad 0 < \delta < 1, \tag{4.3}$$

$$(\partial A/\partial t)(t; \lambda) \le \mu^* < \mu, \quad \textit{for any } t \ge \delta b_\lambda \textit{ and } \lambda \in \Lambda, \tag{4.4}$$

$$\lim d_\lambda = d_1^* \quad \textit{and} \quad \lim b_\lambda(\partial^2 A/\partial t^2)(b_\lambda; \lambda) = d_2^*, \quad \textit{as } b_\lambda \to \infty, \tag{4.5}$$

$$\sup\left\{b_\lambda^2 |(\partial^3 A/\partial t^3)(t; \lambda)|: |t - b_\lambda| \le \theta b,\ \lambda \in \Lambda\right\} < \infty, \quad \textit{some } \theta > 0, \tag{4.6}$$

$$\limsup_b \left\{|b_\lambda^2(\partial^3 A/\partial t^3)(t; \lambda) - d_3^*|: (t-b)^2 < Kb\right\} = 0, \quad \textit{any } K, \tag{4.7}$$

where $b = b_\lambda$ and $d = d_\lambda$ are defined by (2.1) *and* (2.2).

If X has a density $p(x)$ with respect to Lebesgue measure dx such that

$$\sum_{n=-\infty}^{\infty} \sup_{n \le x < n+1} p(x) < \infty \quad \text{and} \quad EX^4 < \infty, \tag{4.8}$$

then

$$\operatorname{Var}(T_\lambda) = (\mu - d_\lambda)^{-2}\sigma^2 b_\lambda + \mu_0^{-2}C^* + o(1), \quad \text{as } b_\lambda \to \infty, \tag{4.9}$$

where $\mu_0 = \mu - d_1^*$ and C^* can be written as

$$\begin{aligned} C^* = {} & C_1^* - 2\mu_0^{-2} d_2^* \sigma^2 C_2^* - \mu_0^{-2} d_2^* E(X-\mu)^3 + \mu_0^{-3} d_3^* \sigma^4 \\ & + \mu_0^{-4} d_2^* \sigma^4 (7 d_2^* + 11\mu_0)/2, \end{aligned} \tag{4.10}$$

with

$$\begin{aligned} C_1^* = {} & \sigma^2 r(d_1^*)/\mu_0 + 3\big(r(d_1^*)\big)^2 + 2\int_0^\infty x^2\, dG(x, d_1^*) \\ & + 2r(d_1^*) E \max_{n \ge 0}(n d_1^* - S_n)^+ \\ & - 2\int_0^\infty ER(x, d_1^*) P\Big\{\max_{n\ge 0}(n d_1^* - S_n) \ge x\Big\}\, dx, \end{aligned}$$

$$C_2^* = \mu_0 \sum_{n=1}^{\infty} P\{S_n - n d_1^* \le 0\} + \sum_{n=1}^{\infty} n^{-1} E(S_n - n d_1^*)^-,$$

and $R(\cdot,\cdot)$, $r(\cdot)$ and $G(\cdot,\cdot)$ given by (2.5)–(2.7).

REMARKS. When $A(t;\lambda) = c_\lambda + t d_1^*$, $d_2^* = d_3^* = 0$ and C^* agrees with C_1^*, which has the same expression as the constant in Theorem 5 of Lai and Siegmund (1979). Keener (1987) pointed out that C_1^* can be written in terms of moments of ladder variables. Usually, (4.3) can be verified by Lemma 5 of Chow and Lai (1978). But a special argument is often required for each family of boundary curves so that the lemma can be used.

We shall split the proof of Theorem 4 into several lemmas. Under the conditions of Theorem 4, our lemmas also imply the following statements:

$$\begin{aligned} E(T_\lambda - b_\lambda)^3/b_\lambda = {} & \mu_0^{-3}\big[3\sigma^2 r(d_1^*) - E(X-\mu)^3\big] \\ & + 3\mu_0^{-4}\sigma^4 + 9\mu_0^{-5} d_2^* \sigma^4/2 + o(1), \quad \text{as } b_\lambda \to \infty. \end{aligned} \tag{4.11}$$

Under the further assumption that

$$\inf\{A(t;\lambda)/t^2 \colon t > b_\lambda,\ \lambda \in \Lambda\} > -\infty, \tag{4.12}$$

$$\begin{aligned} \operatorname{Cov}(T_\lambda, R_\lambda) = {} & \mu_0^{-1}\int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, P\Big\{\max_n (S_n - n d_1^*)^- \le x\Big\}\, dx \\ & + (\sigma/\mu_0)^2 d_2^* \left[\big(1 - (\sigma/\mu_0)^2\big)/2 + \sum_{n=1}^{\infty} P\{S_n - n d_1^* \le 0\}\right] \\ & + o(1), \quad \text{as } b_\lambda \to \infty. \end{aligned} \tag{4.13}$$

The proof of Theorem 4 is contained in Section 7 where the linear renewal theorems in Zhang (1986) are used. We shall state these results here for reference.

THEOREM 5. *Let $R(c, u)$ and $r(u)$ be defined by* (2.5) *and* (2.6). *Suppose that X is strongly nonlattice in the sense of Stone* (1965) *with $E|X|^3 < \infty$ and that for some $k < \infty$ and $-\infty < a \le b < \mu$,*

$$\sup\left\{\int_{-\infty}^{\infty} |E\exp[itR(0,u)]|^k\,dt:\ a \le u \le b\right\} < \infty, \qquad i^2 = -1. \tag{4.14}$$

Then

$$\int_0^{\infty} \sup\{|ER(y,u) - r(u)|:\ y \ge x,\ a \le u \le b\}\,dx < \infty. \tag{4.15}$$

THEOREM 6. *Let $R(c, u)$ and $r(u)$ be defined by* (2.5) *and* (2.6). *Suppose that X has a density $p(x)$ such that* (4.8) *holds. Then* (4.14) *holds and for any $-\infty < a \le b < \mu$,*

$$\sum_{n=0}^{\infty} \sup\{|(\partial/\partial u)(ER(x,u) - r(u))|:\ n \le x < n+1,\ a \le u \le b\} < \infty. \tag{4.16}$$

THEOREM 7. *Let $R(c, u)$ and $G(x, u)$ be defined by* (2.5) *and* (2.7), *respectively. Assume that c_n and u_n are two sequences of constants such that $c_n \to \infty$ and $u_n \to u^* < \mu$ as $n \to \infty$. If $X - u^*$ does not have an arithmetic distribution, then*

$$P\{R(c_n, u_n) > x\} \to G(x, u^*), \quad \text{as } n \to \infty. \tag{4.17}$$

REMARK. The existence of EX is assumed in Section 1.

Basically, Theorems 5 and 6 can be proved by the Fourier method.

5. Discussion and examples. We shall first compare the results in this paper with previous studies. To fix the ideas, let us consider the expectations of stopping times which can be written as

$$T_\lambda = \inf\{n:\ S_n > A(n;\lambda)\} \tag{5.1}$$

$$= \inf\{n:\ ng(S_n/n) + f(n) > c\} \tag{5.2}$$

$$= \inf\{n:\ S_n - nd^* + \xi_n > c\}, \qquad c = c_\lambda, \tag{5.3}$$

for some smooth functions $g(t)$ and $f(t)$, where S_n is a random walk. Woodroofe (1976, 1977) considered the case that $A(t;\lambda) = \lambda a(t)$ for a regular varying function $a(t)$, took the Taylor expansion of $a(t)$ and used a local limit theorem to obtain the expansion of ET_λ. Lai and Siegmund (1979) took the Taylor expansion of $g(t)$, rewrote T_λ into the form (5.3) and used the classical (linear) renewal theory to obtain the expansion of ET_λ. In this paper, linear renewal theorems with varying drift are used to derive the expansion of ET_λ. Since better linear renewal theorems are used, we can take Taylor expansions of either $A(t; \lambda)$ or $g(t)$ and we do not need the help of local limit theorems. As a result, our theorems give a unified treatment for stopping times of a more general form (1.1), require less and weaker regularity conditions and allow the shape of the boundary curve and the slope of the random walk to change. For example, (2.9) used in our Theorem 3 is much weaker than corresponding (3) in Hagwood and Woodroofe (1982) with $\alpha = 1$ and (17) in Lai and Siegmund (1979) in view of the proof of Proposition 1 in Lai and Siegmund (1979). However, taking Taylor expansions of $A(t; \lambda)$ has two technical advantages when T_λ can be written as both (5.1) and (5.3). The first is that the regularity conditions with respect to $A(t; \lambda)$ are often easy to check. The second is that we can include more terms in the Taylor expansion of $A(t; \lambda)$ to obtain further expansions of quantities relative to the stopping times T by the techniques in Section 7, whereas the second-order expansion of ξ_n is almost impossible to describe.

We shall demonstrate the methods of checking the conditions in our theorems by the following examples, which also give comparisons of our results with previous studies. We shall always put $S_n = X_1 + \cdots + X_n$, where $X, X_1, X_2, \dots$ is a sequence of i.i.d. random variables.

EXAMPLE 1 [Robbins (1970)]. Let $T = T_\lambda$ be defined by (4.1) with $EX = \mu > 0$, $0 < \mathrm{Var}(X) = \sigma^2 < \infty$ and

$$A(t; \lambda) = (t(\log t + 2\lambda))^{1/2}, \qquad \lambda > 0. \tag{5.4}$$

Then as $\lambda \to \infty$,

$$b_\lambda = \mu^{-2}(\log b_\lambda + 2\lambda) = \mu^{-2}(2\lambda + \log(2\lambda) - 2\log\mu) + o(1), \tag{5.5}$$

$$d_\lambda = \mu/2 + (2\mu b_\lambda)^{-1} \to d_1^* = \mu/2. \tag{5.6}$$

It follows from Theorem 2 that

$$ET_\lambda = \mu^{-2}(2\lambda + \log(2\lambda)) + O(1), \tag{5.7}$$

$$\mathrm{Var}(T_\lambda) = 8\mu^{-4}\sigma^2\lambda + O(\lambda^{1/2}), \quad \text{if } EX^4 < \infty. \tag{5.8}$$

Suppose that $X - \mu/2$ does not have an arithmetic distribution. Then, by Theorems 1–3 we have, with $s_n = S_n - \mu n/2$ and $\tau = \inf\{n\colon s_n > 0\}$,

$$ET_\lambda = \mu^{-2}\left[2\lambda + \log(2\lambda) - 2\log\mu - \mu\sigma^2/2 + \mu E s_\tau^2/E s_\tau\right] + o(1). \tag{5.9}$$

Furthermore, if X has a density $p(x)$ satisfying (4.8), then

$$\begin{aligned}\mu^2\mathrm{Var}(T_\lambda) &= 4\mu^{-2}\sigma^2(2\lambda + \log(2\lambda)) - 8\mu^{-2}\sigma^2\log\mu + 8\mu^{-2}\sigma^2 + 4C_1^* \\ &\quad + 8\mu^{-1}\sigma^2 C_2^* + 4\mu^{-1}E(X-\mu)^3 - 18\mu^{-2}\sigma^4 + o(1),\end{aligned} \tag{5.10}$$

where

$$C_1^* = \mu^{-1}\sigma^2 Es_\tau^2/Es_\tau + (3/4)\left(Es_\tau^2/Es_\tau\right)^2 - (2/3)Es_\tau^3/Es_\tau$$
$$-\left(Es_\tau^2/Es_\tau\right)E\min_{n\geq 0} s_n - 2\int_0^\infty r^*(x)P\{\min s_n \leq -x\}\,dx,$$

$$C_2^* = 2^{-1}\mu\sum_{n=1}^{\infty} P\{s_n \leq 0\} + \sum_{n=1}^{\infty} n^{-1}Es_n^-,$$

and

$$r^*(x) = E(s_{\tau(x)} - x), \quad \text{with } \tau(x) = \inf\{n: s_n > x\}.$$

PROOF.

$$(5.11) \qquad (\partial A/\partial t)(t;\lambda) = A(t;\lambda)/(2t) + 1/(2A(t;\lambda)).$$

Also, we have

$$\partial^2 A/\partial t^2 = -\left(A/t^2 + A^{-3}\right)/4,$$
$$b(\partial^2 A/\partial t^2)(b;\lambda) = -\mu/4 - \left(\mu^3 4b^2\right)^{-1} \to d_2^* = -\mu/4,$$

and

$$\partial^3 A/\partial t^3 = (3/8)\left(A/t^3 + (tA^3)^{-1} + A^{-5}\right) + (8t^2A)^{-1}.$$

Since $A(tb_\lambda;\lambda)/b_\lambda$ is uniformly continuous at $t = 1$, (4.4)–(4.7) are satisfied with $d_3^* = 3\mu/8$. Rewriting T_λ into the form of Lai and Siegmund (1979), page 61, and applying Lemma 5 of Chow and Lai (1978), we have (2.35) with $p = 1, 2$ under the condition that $E|X|^{2p} < \infty$. □

EXAMPLE 2 [Woodroofe (1977) and Chow, Hsiung and Yu (1983)]. Let $(X, Y), (X_1, Y_1), \ldots$ be i.i.d. random vectors with $EX = \mu > 0$ and $EY = 0$. Let ξ_n be a sequence of uniformly bounded random variables such that ξ_n is independent of $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), \ldots$. Assume that $a(t) > 0$, $f(t)$ and $h(t)$ are three functions such that $t^{-\beta}a(t)$ is slowly varying for some $\beta > 0$ and $f(t)/t = o(1)$ as $t \to \infty$ and that $h(t)$ is bounded and continuous at $t = 0$. Define for each $0 < \lambda < \infty$,

$$(5.12) \qquad T = \inf\{n \geq n_\lambda: U_n < \lambda a(n)\},$$

where $U_n = \overline{X}_n + h(\overline{Y}_n\xi_n)\overline{Y}_n^2 + f(n)/n$ and $n_\lambda = K\lambda^{-\gamma}$ for some $0 < \gamma < 1/\beta$.

Suppose that $E(X^+)^p < \infty$, $E|Y|^p < \infty$ and $\lim t^{-\beta}a(t) = 1$ as $t \to \infty$. Chow and Yu (1981) and Chow, Hsiung and Yu (1983) showed that

$$(5.13) \qquad E\left|\lambda^{1/\beta}T_\lambda - \mu^{1/\beta}\right|^p = o(1), \quad \text{as } \lambda \to 0.$$

Let us apply our theorems. We shall first rewrite the stopping time as

$$(5.14) \qquad T_\lambda = \inf\{n: Z_n > A(n;\lambda)\},$$

where $Z_n = 2n\mu - nU_n$ and $A(t;\lambda) = 2t\mu - \lambda t a(t)$. Assume there exist two functions $u(t)$ and $v(t)$ such that

(5.15) $t^{1/2}(d/dt)(f(t) - u(t)) = o(1)$ and $u(t) = o(1)$, as $t \to \infty$,

(5.16) $a(t) = t^{\beta}e^{v(t)}$, $tv'(t) = o(1)$ and $t^2v''(t) = o(1)$, as $t \to \infty$.

Then

$$\mu/\lambda = a(b_\lambda) = b_\lambda^\beta \exp[v(b_\lambda)], \tag{5.17}$$

$$d_\lambda = \mu(1 - \beta - b_\lambda v'(b_\lambda)) = \mu(1 - \beta + o(1)), \tag{5.18}$$

the conditions (2.36), (2.37), (3.3) and (3.4) are satisfied, and $nh(\overline{Y}_n \xi_n)\overline{Y}_n^2$ is regular with $\alpha = p = 1$ if $EY^2 < \infty$ by Proposition 1. Assume, in addition, that $E|X|^{p^*} + E|Y|^{p^*} < \infty$ for some $p^* > 1 + 1/(\gamma\beta)$. Then (2.35) holds with $p = 1$. Therefore if $X - \mu(1 + \beta)$ does not have an arithmetic distribution, then all conditions in Theorems 1–3 hold with $p = 1$. We have

$$\begin{aligned} ET_\lambda = {} & b_\lambda + [\mu(\beta + b_\lambda v'(b_\lambda))]^{-1} f(b_\lambda) - (\mu\beta)^{-2}(\beta + 1)\mathrm{Var}(X)/2 \\ & + (\mu\beta)^{-1}EY^2h(0) + (\mu\beta)^{-1}Es_\tau^2/(2Es_\tau) + o(1), \end{aligned} \tag{5.19}$$

where $s_n = \mu(1 + \beta)n - (X_1 + \cdots + X_n)$ and $\tau = \inf\{n: s_n > 0\}$.

Proof. We only need to check (2.35) since other statements in the example are obvious. It follows from Lemma 5 of Chow and Lai (1978) that for some $\theta > 0$,

$$\begin{aligned} P\{T_\lambda \le \delta b_\lambda\} \le {} & \sum_{k=0}^{\infty} P\{\max(S_j': j \le 2^{k+1}n_\lambda) > \theta 2^k n_\lambda\} \\ & + \sum_{k=0}^{\infty} P\{\max(|S_j''|: j \le 2^{k+1}n_\lambda) > \theta 2^k n_\lambda\} \\ \le {} & \sum_{k=0}^{\infty} 2^{k+1}n_\lambda P\{\mu - X > \theta 2^k n_\lambda/(2j)\} + O(1)\sum_{k=0}^{\infty}[2^k n_\lambda]^{-j} \\ & + \sum_{k=0}^{\infty} 2^{k+1}n_\lambda P\{|Y| > \theta 2^k n_\lambda/(2j)\} \\ \le {} & 4(2j/\theta)E|X|I\{X > \theta n_\lambda/(2j)\} + O(1)n_\lambda^{-j} \\ & + 4(2j/\theta)E|Y|I\{|Y| > \theta n_\lambda/(2j)\} \\ = {} & o(1)n_\lambda^{1-p^*} = o(1)b_\lambda^{-1}, \end{aligned}$$

where $S_n' = n(\mu - \overline{X}_n)$, $S_n'' = n\overline{Y}_n$, j is an integer large enough and $\delta < 1$. Since

$$(\partial A/\partial t)(t; \lambda) = 2\mu - \lambda t a(t)[(1 + \beta)t^{-1} + v'(t)]$$

and

$$\begin{aligned} (\partial^2 A/\partial t^2)(t; \lambda) = {} & -\lambda t a(t)[(1 + \beta)t^{-1} + v'(t)]^2 \\ & - \lambda t a(t)[-(1 + \beta)t^{-2} + v''(t)], \end{aligned}$$

the constants in (3.5) are given by (5.18) and $d_1^* = \mu(1-\beta)$ and $d_2^* = -\mu\beta(1+\beta)$. Note that $Z_n = S_n^* + \xi_n^*$ with $S_n^* = 2\mu n - S_n$ and $\xi_n^* = -n[h(\bar{Y}_n \xi_n)\bar{Y}_n^2 + f(n)/n]$, we have (5.19). □

EXAMPLE 3.

$$T_\lambda = \inf\{n: S_n > \lambda n^\beta\}, \qquad 0 \le \beta < 1. \tag{5.20}$$

Then the conditions (2.10), (2.36), (2.37), (3.4) and (4.4)–(4.7) [all conditions in Theorems 1–4 on $A(t; \lambda)$] are satisfied and the constants are given by

$$b = b_\lambda = (\lambda/\mu)^{1/(1-\beta)}, \tag{5.21}$$

$$d = d_\lambda = \bar{d}_\lambda = d^* = d_1^* = \beta\mu, \tag{5.22}$$

$$d_2^* = \beta(\beta-1)\mu \quad \text{and} \quad d_3^* = \beta(\beta-1)(\beta-2)\mu. \tag{5.23}$$

PROOF. Clearly, (2.1) implies that $\mu b_\lambda = \lambda b_\lambda^\beta$, which implies (5.21). By (5.20)

$$(\partial A/\partial t)(t; \lambda) - \beta\mu = \beta\mu\left((t/b_\lambda)^{\beta-1} - 1\right), \tag{5.24}$$

$$b_\lambda(\partial^2 A/\partial t^2)(t; \lambda) - \beta(\beta-1)\mu = \beta(\beta-1)\mu\left[(t/b_\lambda)^{\beta-2} - 1\right], \tag{5.25}$$

$$\begin{aligned} &b_\lambda^2(\partial^3 A/\partial t^3)(t; \lambda) - \beta(\beta-1)(\beta-2)\mu \\ &\qquad = \beta(\beta-1)(\beta-2)\mu\left[(t/b_\lambda)^{\beta-3} - 1\right]. \end{aligned} \tag{5.26}$$

The proof is complete. □

6. Proof of Theorem 2. We assume that $EX = \mu = 1$ in this section without loss of generality. The following lemma states some useful results on random walk and renewal theory, which may be found in Chow (1973), Chow and Lai (1978) and Chow, Hsiung and Lai (1979).

LEMMA 1. *Let* $S_n = X_1 + \cdots + X_n$, $n \ge 1$, *and* $\tau(c, u)$ *be defined by* (1.2), *and let* p *and* α *be two constants with* $p \ge 1$ *and* $1/2 < \alpha \le 1$.

(i) *If* $E|X|^{(p+1)/\alpha} < \infty$, *then*

$$\sum_{n=1}^\infty n^{p-1} P\left\{\max_{j \le n} |S_j - j| \ge \theta n^\alpha\right\} < \infty, \quad \text{any } \theta > 0,$$

and

$$\sum_{n=1}^\infty n^{p-1} P\left\{\sup_{j \ge n} j^{-\alpha} |S_j - j| \ge \theta\right\} < \infty, \quad \text{any } \theta > 0.$$

(ii) *If* $c^{(p+1)/\alpha} P\{|X| \ge c\} \to 0$ *as* $c \to \infty$ *and* $EX^2 < \infty$, *then*

$$n^p P\left\{\max_{j \le n} |S_j - j| \ge \theta n^\alpha\right\} \to 0, \quad \text{as } n \to \infty \text{ for any } \theta > 0,$$

and for any $\theta > 0$ *and* $K > 0$,

$$\lim_{c \to \infty} \sup_{K^{-1} \le 1-u \le K} c^p P\left\{\tau(c, u) \le (1-u)^{-1} c - \theta c^\alpha\right\} = 0.$$

(iii) *If* $E|X|^{2p} < \infty$, *then for any* $K > 0$,

$$\Big\{\Big(\big(\tau(c,u) - (1-u)^{-1}c\big)^2/c\Big)^p;\ c \ge 1,\ K^{-1} \le 1-u \le K\Big\} \text{ is u.i.}$$

LEMMA 2. *Suppose that* ξ *is regular with* $p \ge 1$ *and* $1/2 < \alpha \le 1$ *and that conditions* (2.35) *and* (2.36) *hold. If* $E|X|^{(p+1)/\alpha} < \infty$, *then*

$$\lim_{b\to\infty} b^p P\{T \le b - \theta b^\alpha\} = 0, \quad \text{for any } \theta > 0.$$

PROOF. On the event $\{L < \delta b < n \le b - \theta b^\alpha\}$, by (2.36)

$$A(n;\lambda) \ge b + \mu^*(n-b) \ge n + (1-\mu^*)\theta b^\alpha.$$

By (1.1) and (2.9)

$$\begin{aligned} &b^p P\{\delta b < T \le b - \theta b^\alpha,\ L < \delta b\} \\ &\quad \le b^p P\{S_n + V_n + f(n) > n + (1-\mu^*)\theta b^\alpha, \text{ some } \delta b < n \le b\}. \end{aligned}$$

And by (2.30) $f(n) = O(n^{1/2})$ and $\{\max_{\delta b \le n \le b}|f(n)| \ge \theta' b^\alpha\} = \varnothing$ for large b and any given $\theta' > 0$. It follows from (2.35), (2.36), Lemma 1(ii) and (2.32) that

$$\begin{aligned} &b^p P\{T \le b - \theta b^\alpha\} \\ &\quad \le b^p P\{T \le \delta b\} + b^p P\{L \ge \delta b\} + b^p P\Big\{\max_{n \le b}|S_n - n| \ge \theta' b^\alpha\Big\} \\ &\qquad + b^p P\Big\{\max_{\delta b \le n \le b} V_n \ge \theta' b^\alpha\Big\} + b^p P\Big\{\max_{\delta b \le n \le b} f(n) \ge \theta' b^\alpha\Big\} \\ &\quad = o(1), \end{aligned}$$

where $\theta' = (1-\mu^*)\theta/3 > 0$. □

LEMMA 3. *Suppose that* ξ *is regular with* $p \ge 1$ *and* $1/2 < \alpha \le 1$ *and that condition* (2.36) *holds. If* $E|X|^{(p+1)/\alpha} < \infty$, *then there exists a constant* $K > 0$ *such that*

$$\lim_{b\to\infty} \sum_{n=n^*}^{\infty} n^{p-1} P\{T > n\} = 0,$$

where $n^* = [b + Kb^\alpha]$.

PROOF. By (2.33) there exists a $w > 0$ such that

$$\sum_{n=1}^{\infty} n^{p-1} P\{V_n^- \ge w n^\alpha\} < \infty \quad \text{and} \quad w < 1 - \bar{d} \text{ if } \alpha = 1.$$

Let K be large enough and $\theta > 0$ be small enough such that for large b,

$$(1-\bar{d})(n-b) > (\theta + w)n^\alpha, \quad \text{for any } n \ge b + Kb^\alpha,$$

$$\begin{aligned} A(n;\lambda) &\le b + \bar{d}(n-b) = n - (1-\bar{d})(n-b) \\ &< n - (\theta+w)n^\alpha, \quad \text{any } n \ge n^*. \end{aligned} \tag{6.1}$$

By (2.29), Lemma 1(i), (2.33) and (2.30)

$$P\{T > n,\ L < n\} \le P\{S_n + V_n + f(n) < n - (\theta + w)n^{\alpha}\}, \quad \text{any } n \ge n^*,$$

$$\sum_{n=n^*}^{\infty} n^{p-1}P\{T > n\}$$

$$\le \sum_{n^*}^{\infty} n^{p-1}P\{L > n\} + \sum_{n^*}^{\infty} n^{p-1}P\{n - S_n > \theta n^{\alpha}/2\} \tag{6.2}$$

$$+ \sum_{n^*}^{\infty} n^{p-1}P\{V_n^- > wn^{\alpha}\} + \sum_{n^*}^{\infty} n^{p-1}P\{-f(n) > \theta n^{\alpha}/2\}$$

$$= o(1).$$

□

LEMMA 4. *Under the conditions of Theorem* 2(i),

$$\left\{\left((T - \tau)^+\right)^p;\ \lambda \in \Lambda\right\} \text{ is } u.i.$$

PROOF. Let $n_1 = [b - \theta b^{\alpha}]$, $n^* = [b + Kb^{\alpha}]$, $T' = \max(n_1, \min(T, n^*))$ and $\tau' = \max(n_1, \min(\tau, n^*))$. For $n > b + \theta b^{\alpha}$ we have

$$P\{\tau > n\} \le P\{S_n - dn \le b(1 - d) - f(b)\}$$

$$\le P\{n - S_n \ge (n - b)(1 - d) - f(b)\}.$$

It follows from Lemma 1(i) and (2.30) that for $n_2 = [b + \theta b^{\alpha}]$,

$$\lim_{b \to \infty} \sum_{n=n_2}^{\infty} n^{p-1}P\{\tau > n\} = 0, \quad \text{for any } \theta > 0. \tag{6.3}$$

And by Lemmas 1(ii), 2 and 3

$$\lim_b E|T - T'|^p = \lim_b E|\tau - \tau'|^p = 0 \tag{6.4}$$

and

$$P\{T' > \tau' + n\} \le P\{L \ge n_1\} + P\{\tau \le n_1\}$$

$$+ P\{L < n_1 < \tau \le \tau + n < T' \le n^*\}.$$

Making use of (2.29), (2.30) and Lemma 1(ii), we find that

$$\sum_{n_0}^{\infty} n^{p-1}P\{T' > \tau' + n\}$$

$$= \sum_{n_0}^{n^*} n^{p-1}P\{T' > \tau' + n\} \tag{6.5}$$

$$= o(1) + \sum_{n_0}^{n^*} n^{p-1}P\{S_{\tau+n} + V_{\tau+n} + f(\tau + n)$$

$$\le A(\tau + n; \lambda),\ \tau \in (n_1, n^* - n)\}.$$

On the event $\{n_1 < \tau \le \tau + n < n^*\}$, we have by (2.36), (2.34) and (2.30)

$$A(\tau + n; \lambda) \le \mu^* n + (A(\tau; \lambda) - b - d\tau) + S_\tau + f(b),$$

and by (2.37) there exists a constant $K^* < \infty$ such that

$$\begin{aligned} |A(m; \lambda) - b - dm| &\le K^*(m - b)^2/b, \qquad n_1 \le m \le n^*, \\ |f(\tau + n) - f(b)| &\le K^*|\tau + n - b|n_1^{-1/2} \\ &\le n(1 - \mu^*)/5 + K^*|\tau - b|n_1^{-1/2}, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} &S_{\tau+n} + V_{\tau+n} + f(\tau + n), \quad \text{on } \{T' > \tau' + n\}, \\ &\quad \le \mu^* n + (A(\tau; \lambda) - b - d\tau) + S_\tau + f(b) \\ &\quad \le \mu^* n + K^*(\tau - b)^2/b + S_\tau + K^*|\tau - b|n_1^{-1/2} + f(\tau + n) + n(1 - \mu^*)/5. \end{aligned}$$

Therefore, letting $\theta' = (1 - \mu^*)/5$, we have by (6.5)

$$\begin{aligned} \sum_{n_0}^{\infty} n^{p-1} P\{T' > \tau' + n\} \le &\sum_{n_0}^{\infty} n^{p-1} P\{S_\tau + n - S_{\tau+n} \ge \theta' n\} \\ &+ \sum_{n_0}^{\infty} n^{p-1} P\Big\{\max_{n_1 \le j \le n^*} V_j^- > \theta' n\Big\} \\ &+ \sum_{n_0}^{\infty} n^{p-1} P\big\{K^*(\tau - b)^2/b > \theta' n\big\} \\ &+ \sum_{n_0}^{\infty} n^{p-1} P\big\{K^*|\tau - b|n_1^{-1/2} > \theta' n\big\} + o(1). \end{aligned}$$

It follows from Lemma 1(i), (2.31), Lemma 1(iii) and the condition $E|X|^{2p} < \infty$ that

$$\sum_{n_0}^{\infty} n^{p-1} P\{T > \tau' + n\} \to 0, \quad \text{as} \min(n_0, b) \to \infty.$$

This proves the uniform integrability of $\{((T - \tau)^{+p}\}$ since the uniform integrability of $\{T^p;\ b \le b^*,\ \lambda \in \Lambda\}$ for any given b^* is implied by (6.2) in the proof of Lemma 3. □

LEMMA 5. *Let Y be an integer-valued random variable with finite pth moment, $p \ge 1$. Suppose that there exist constants c, θ and δ^* such that $0 \le \delta^* < 1$, $\theta(4\delta^*)^p \le 2$ and for some $m > 1/(1 - \delta^*)$,*

$$\sum_{n=m}^{\infty} n^{p-1} P\{Y \le n\} \le c + \theta \sum_{n=m}^{\infty} n^{p-1} P\{\delta^* Y \ge n\} < \infty.$$

Then

$$\sum_{n=m}^{\infty} n^{p-1} P\{Y \ge n\} \le 2c.$$

PROOF. Clearly,

$$\sum_m^\infty n^{p-1}P\{\delta^* Y \ge n\} \le 2^{p-1}\int_{m-1}^\infty y^{p-1}P\{\delta^* Y \ge y\}\,dy$$
$$\le 2^{p-1}(\delta^*)^p\int_{(m-1)/\delta^*}^\infty y^{p-1}P\{Y \ge y\}\,dy$$
$$\le 4^{p-1}(\delta^*)^p\sum_m^\infty n^{p-1}P\{Y \ge n\} \le (4\theta)^{-1}2\sum_m^\infty n^{p-1}P\{Y \ge n\}.$$

□

PROOF OF THEOREM 2. (i) Let $n_1 = [b - \theta b^\alpha]$, $n_2 = [b + \theta b^\alpha]$, $T' = \max(n_1, \min(T, n_2))$ and $\tau' = \max(n_1, \min(\tau, n_2))$. Though n_2 is different from n^* in Lemma 3, $(T - T')^+ \le (T - \tau)^+ + (\tau - n_2)^+$, and by Lemma 4, (6.3) and (6.4) we have

$$\lim_{b\to\infty} E|T - T'|^p = \lim_{b\to\infty} E|\tau - \tau'|^p = 0. \tag{6.7}$$

Clearly,

$$P\{\tau' > T' + n\} \le P\{L \ge n_1\} + P\{T \le n_1\} + P\{\tau \ge n_2\}$$
$$+ P\{L < n_1 < T \le T + n < \tau < n_2\}.$$

By (2.29), Lemma 2 and (6.3)

$$\begin{aligned} &\sum_{n=n_0}^\infty n^{p-1}P\{\tau' > T' + n\} \\ &= \sum_{n=n_0}^{n_2-n_1} n^{p-1}P\{\tau' > T' + n\} \\ &\le \sum_{n_0}^{n_2-n_1} n^{p-1}P\{L < n_1 < T \le T + n < \tau < n_2\} + o(1). \end{aligned} \tag{6.8}$$

On the event $\{L < n_1 < T \le T + n < \tau < n_2\}$,

$$\begin{aligned} S_{T+n} + f(b) &\le b + d(T + n - b) \\ &\le \mu^* n + (b + d(T - b) - A(T; \lambda)) + S_T + V_T + f(T), \end{aligned} \tag{6.9}$$

and there exists some finite constant K^* that does not depend on θ such that

$$\begin{aligned} &f(T) - f(\tau) \le |\tau - T|K^* n_1^{-1/2}, \quad \text{by (2.30)}, \\ &f(\tau) - f(b) \le K^*|\tau - b|b^{-1/2}, \quad \text{by (2.30)}, \\ &(b + d(T - b) - A(T; \lambda)) - (b + d(\tau - b) - A(\tau; \lambda)) \\ &\quad \le |\tau - T|(\theta K^* b^{\alpha-1}), \quad \text{for small } \theta \text{ by (2.37)}, \\ &(b + d(\tau - b) - A(\tau; \lambda)) \le K^*(\tau - b)^2/b, \quad \text{by (6.6)}, \\ &(b + d(T - b) - A(T; \lambda)) + f(T) - f(b) \\ &\quad \le \delta^*|\tau' - T'| + K^*(\tau - b)^2/b + K^*|\tau - b|b^{-1/2}, \end{aligned} \tag{6.10}$$

where $\delta^* = \theta K^* b^{\alpha-1} + K^* b^{-1/2}$, and by (6.9)

$$S_{T+n} - S_T \le \mu^* n + V_T + \delta^* |\tau' - T'| + K^*(\tau - b)^2/b + K^*|\tau - b| b^{-1/2}. \tag{6.11}$$

Therefore it follows from (6.8) and (6.11) that for $\theta' = (1 - \mu^*)/5$,

$$\begin{aligned}
&\sum_{n_0}^{\infty} n^{p-1} P\{\tau' > T' + n\} \\
&\quad \le \sum_{n_0}^{\infty} n^{p-1} P\{S_T + n - S_{T+n} \ge \theta' n\} \\
&\qquad + \sum_{n_0}^{\infty} n^{p-1} P\Big\{\max_{n_1 \le j \le n_2} V_j \ge \theta' n\Big\} + \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(T - \tau) \ge \theta' n\} \\
&\qquad + \sum_{n_0}^{\infty} n^{p-1} P\big\{K^*(\tau - b)^2/b + K^*|\tau - b| n^{-1/2} \ge \theta' n\big\} \\
&\qquad + \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(\tau' - T') \ge \theta' n\} + o(1), \quad \text{as } \min(n_0, b) \to \infty.
\end{aligned} \tag{6.12}$$

Since θ is arbitrary, we can choose θ small enough such that $(4\delta^*/\theta')^p \le 2$ for the δ^* specified in (6.10). Hence it follows from (6.12), Lemma 1(i), (2.31), Lemmas 4 and 1(iii) that

$$\sum_{n_0}^{\infty} n^{p-1} P\{\tau' - T' > n\} \le \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(\tau' - T') \ge \theta' n\} + o(1), \quad \text{as } \min(n_0, b) \to \infty.$$

And by (6.7) and Lemmas 4 and 5

$$\sum_{n_0}^{\infty} n^{p-1} P\{\tau' - T' > n\} = o(1)$$

and

$$\{|T - \tau|^p;\ \lambda \in \Lambda\} \text{ is u.i.}$$

(ii) For the case where $\partial^2 A/\partial t^2 = 0$, the term $(\tau - b)^2/b$ disappears throughout the proof of (i).

(iii) Let $n^* = [b + Kb^\alpha]$. By (6.1)

$$P\{N > n, L < n\} \le P\Big\{\sup_{j \ge n} j - S_j - (\theta + w) j^\alpha + V_j^- - f(j) \ge 0\Big\}, \quad \text{any } n \ge n^*.$$

The term $\sup_{j \ge n} V_{\tau+j}^-/j$ is controlled on $n_1 \le \tau \le n^*$ by

$$\begin{aligned}
\sup_{j \ge n} V_{\tau+j}^-/j &\le \max_{n \le j \le K^* b^\alpha} V_{\tau+j}^-/j + \sup_{j \ge \max(n, K^* b^\alpha)} V_{\tau+j}^-/j \\
&\le \max_{n \le j \le n^* + K^* b^\alpha} V_j^-/n + \sup_{j \ge n} V_j^-/j^\alpha \sup_{j \ge K^* b^\alpha} (\tau + j)^\alpha/j.
\end{aligned}$$

Since K^* is arbitrary, we can take K^* large enough such that

$$\sup_{j \geq K^* b^\alpha} (n^* + j)^\alpha / j \leq 1 + \delta^*, \qquad \alpha = 1,$$
$$\leq \delta^*, \qquad \alpha < 1,$$

for any given $\delta^* > 0$.

The rest of the proof is similar to the proof of (i) and is omitted. □

7. Proof of Theorem 4. We shall keep all the notation and definitions introduced in Sections 1–4 and assume $Z_n = S_n$ so that the stopping time T is defined by (4.1).

Let $n_1 = [b_\lambda - \theta b_\lambda]$ and $n_2 = [b_\lambda + \theta b_\lambda]$ for some small $\theta > 0$, where $[y]$ is the largest integer in $(-\infty, y]$. We shall use the notation

$$T' = T'_\lambda = \inf\{n \geq n_1 \colon S_n > A(n; \lambda) \text{ or } n = n_2\}, \tag{7.1}$$

$$R' = R'_\lambda = S_{T'} - A(T'; \lambda), \tag{7.2}$$

$$g(t; \lambda) = A(t; \lambda) - (\mu b_\lambda + d_\lambda(t - b_\lambda)). \tag{7.3}$$

LEMMA 6. *Suppose that* (4.3) *and* (4.4) *hold. If* $EX^4 < \infty$, *then*

$$\operatorname{Var}(T_\lambda) - \operatorname{Var}(T'_\lambda) = o(1), \quad \text{as } b_\lambda \to \infty, \tag{7.4}$$

$$\operatorname{Cov}(T_\lambda, R_\lambda) - \operatorname{Cov}(T'_\lambda, R'_\lambda) = o(1), \quad \text{as } b_\lambda \to \infty \text{ if (4.12) holds}, \tag{7.5}$$

and

$$E(T_\lambda - b_\lambda)^3 / b_\lambda - E(T'_\lambda - b_\lambda)^3 / b_\lambda = o(1), \quad \text{as } b_\lambda \to \infty.$$

Since by (4.12) $RI\{T > n_2\} \leq |X_T| + KT^2$ for some $K < \infty$ and

$$ET|X_T|I\{T > n_2\} \leq E|X|E(T^2 - n_2^2)^+,$$

the proof of Lemma 6 is similar to the proofs of Lemmas 2 and 3 and is therefore omitted.

LEMMA 7. *Suppose that the conditions* (4.3)–(4.6) *hold and that* X *has a strongly nonlattice distribution with* $EX^4 < \infty$. *Then*

$$\begin{aligned} \operatorname{Var}(R'_\lambda + g(T'_\lambda)) = &- \int_0^\infty x^2\, dG(x, d_1^*) - r^2(d_1^*) \\ &+ (\sigma/\mu_0)^4 (d_2^*)^2/2 + o(1), \quad \text{as } b \to \infty, \end{aligned} \tag{7.6}$$

where $\mu_0 = \mu - d_1^*$.

PROOF. Let $\tau = \tau_\lambda = \tau(c_\lambda, d_\lambda)$, $\tau(c, u)$ be defined by (1.2) and $c_\lambda = b_\lambda(\mu - d_\lambda)$. By the definitions

$$\begin{aligned} &S_{T'} - S_\tau - d(T' - \tau) + R(c_\lambda, d_\lambda) = R'_\lambda + g(T'_\lambda), \\ &P\{R(c, u) > x\} \leq \int_x^\infty E\tau(y - x, u)\, dP\{X - u \leq y\}. \end{aligned} \tag{7.7}$$

Since

$$\sup\{E\tau(c,u)/c\colon u\le\mu^*,\, c\ge 1\}<\infty, \tag{7.8}$$
$$\left\{\left(R(c_\lambda, d_\lambda)\right)^3,\ \lambda\in\Lambda\right\}\text{ is u.i.}$$

By Theorem 2 and Lemma 6

$$\left\{(T'_\lambda-\tau_\lambda)^2,\ \lambda\in\Lambda\right\}\text{ is u.i.}$$

It follows that

$$\left\{(S_{T'}-S_\tau)^4,\ \lambda\in\Lambda\right\}\text{ is u.i.}, \tag{7.9}$$
$$\left\{\left(R'_\lambda+g(T'_\lambda)\right)^2,\ \lambda\in\Lambda\right\}\text{ is u.i.},\quad\text{by (7.7)}.$$

Taking a Taylor expansion, we have

$$g(T'_\lambda)=d_2^*(T'_\lambda-b)^2b^{-1}(1+o(1))/2.$$

It follows from Theorem 1 that

$$P\left\{R'_\lambda>x,\ g(T'_\lambda)>t(d_2^*/2)(\sigma/\mu_0)^2\right\}=G(x,d_1^*)P\left\{(N(0,1))^2>t\right\}+o(1).$$

This and (7.9) finish the proof. □

LEMMA 8. *Under the conditions of Theorem* 4, *the following relation holds*:

$$\operatorname{Cov}(T'_\lambda,R'_\lambda)=C_3^*/\mu_0-(\sigma/\mu_0)^2d_2^*\left[\left(1-(\sigma/\mu_0)^2\right)/2+\sum_{n=1}^{\infty}P\{S_n-nd_1^*\le 0\}\right]+o(1), \tag{7.10}$$

as $b\to\infty$, *where* $\mu_0=\mu-d_1^*$ *and*

$$C_3^*=\int_0^\infty\left[ER(x,d_1^*)-r(d_1^*)\right]P\left\{\max_n(S_n-nd_1^*)^-\le x\right\}dx.$$

PROOF. For $j=0,\dots,4$, $i=1,2$, there is a finite K such that

$$b^{j-2}|S_{n_i}|^{4-j}\le Kb^2+\left(S_{n_i}-\mu n_i\right)^4Kb^{-2}I\{|S_{n_i}-\mu n_i|\ge b\}.$$

It follows from (4.8) and Chow and Lai (1978) that

$$b^{j-2}E|S_{n_i}|^{4-j}I\{T'=n_i\}\to 0,\quad\text{for } i=1,2\text{ and }0\le j\le 4. \tag{7.11}$$

Let $d(t;\lambda)=A(t;\lambda)-A(t-1;\lambda)$ and

$$F(x;n)=F(x;n,\lambda)=P\{T'_\lambda\ge n,\ A(n-1;\lambda)-S_{n-1}\le x\}. \tag{7.12}$$

Then we have

$$P\{R'_\lambda>x,\ T'_\lambda=n\}=\int_0^\infty P\{R(y,d(n;\lambda))>x\}d\left[F(y;n,\lambda)-F(y;n+1,\lambda)\right], \tag{7.13}$$

for $n_1<n<n_2$.

By (7.11) and (7.13)

$$ER'_\lambda T'_\lambda \tag{7.14}$$
$$= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty nER(y, d(n; \lambda)) d[F(y; n) - F(y; n+1)] + o(1),$$

and similarly,

$$Er(d(T'_\lambda; \lambda))T'_\lambda \tag{7.15}$$
$$= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty nr(d(n; \lambda))\, d[F(y; n) - F(y; n+1)] + o(1).$$

It follows from Theorems 5 and 6 and (4.8) that there exists a decreasing function $h(x) > 0$ such that for any u with $-\infty < u^* \le u \le \mu^* < \mu$,

$$|Er(x, u) - r(u)| \le h(x) \quad \text{and} \quad \int_0^\infty h(x)\, dx < \infty. \tag{7.16}$$

By (4.4)–(4.6) we can choose u^* such that

$$u^* \le d(n; \lambda) \le \mu^*, \quad \text{for all } |n - b_\lambda| \le \theta b \quad \text{and } b \text{ large}. \tag{7.17}$$

Let n be an integer with $\theta b/2 \le b - n \le \theta b$. By (4.4)

$$A(n; \lambda) - \mu n \ge (\mu - \mu^*)(b - n) \ge (\mu - \mu^*)\theta b/2. \tag{7.18}$$

For $u^* \le u \le \mu^*$,

$$\begin{aligned}
&b\left|\int_0^\infty (ER(x, u) - r(u))\, dF(x; n+1)\right| \\
&\le b\int_0^\infty h(x)\, dP\{A(n; \lambda) - S_n \le x\} \quad [\text{by } (7.16)] \\
&\le bh(b\theta') + bh(0)P\{A(n; \lambda) - S_n \le \theta' b\} \\
&\le bh(\theta' b) + bh(0)P\{S_n - \mu n \ge (\mu - \mu^*)\theta b/2 - \theta' b\} \quad [\text{by } (7.18)] \\
&\le o(1) + bh(0)P\{S_n - \mu n \ge \theta' b\} \\
&= o(1), \quad \text{by taking } \theta' \le (\mu - \mu^*)\theta/4.
\end{aligned}$$

Also, for $\theta b/2 \le n - b \le \theta b$,

$$A(n; \lambda) \le \mu b + (n - b)\mu^*, \quad \text{by (4.4)}, \tag{7.19}$$

and

$$P\{T' > n\} \le P\{n\mu - S_n \le (n - b)(\mu - \mu^*)\} = o(b^{-1}).$$

Therefore, for $\theta b/2 \le |n - b| \le \theta b$, $u^* \le u \le \mu^*$, we have

$$b\left|\int_0^\infty (ER(x, u) - r(u))\, dF(x; n)\right| = o(1), \quad \text{by (7.16) and (7.17)}. \tag{7.20}$$

It follows from (7.14), (7.15) and (7.20) that

$$
\begin{aligned}
&\operatorname{Cov}(R'_\lambda, T'_\lambda) - \operatorname{Cov}(r(d(T'_\lambda;\lambda)), T'_\lambda) \\
&\quad = \operatorname{Cov}([R'_\lambda - r(d(T'_\lambda;\lambda))], T'_\lambda) \\
&\quad = \sum_{n=n_1+1}^{n_2-1} \int_0^\infty (n - ET'_\lambda)[ER(x, d(n;\lambda)) - r(d(n;\lambda)] \\
&\qquad\qquad \times d[F(x;n) - F(x;n+1)] + o(1) \\
&\quad = \sum_{n=n\ +1}^{n_2-1} \int_0^\infty ER(x, d(n;\lambda)) - r(d(n;\lambda))\, dF(x;n) \\
&\qquad + \sum_{n=n_1+1}^{n_2-1} \int_0^\infty (n - ET' - 1)\delta^*(y;n,\lambda)\, dF(y;n) + o(1) \\
&\quad = I_1 + I_2 + o(1), \quad \text{say,}
\end{aligned} \tag{7.21}
$$

where

$$\delta^*(x;n,\lambda) = \int_{d(n-1;\lambda)}^{d(n;\lambda)} (\partial/\partial u)[ER(x,u) - r(u)]\, du.$$

By Theorem 6 there exists a function $h^*(x)$ such that

$$|(\partial/\partial u)[ER(x,u) - r(u)]| \le h^*(x) \quad \text{for } u^* \le u \le \mu^*, \tag{7.22}$$

and

$$\sum_{n=0}^{\infty} \sup_{n \le x < n+1} h^*(x) \le K < \infty. \tag{7.23}$$

By (7.17) and the definitions

$$|\delta^*(x;n,\lambda)| \le h^*(x)|d(n;\lambda) - d(n-1;\lambda)|, \quad \text{for large } b.$$

And by (4.5) and (4.6), we can bound the second term in (7.21) as

$$
\begin{aligned}
|I_2| &\le \sum_{n=n_1+1}^{n_2-1} \int_0^\infty |n - ET'_\lambda - 1| h^*(x) K b^{-1}\, dF(x;n,\lambda) \\
&= \sum_{n=n_1+1}^{n_2-1} \int_{\theta''b}^\infty + \sum_{|n-b|\le\theta'b} \int_0^{\theta''b} + \sum_{\theta'b<|n-b|\le\theta b} \int_0^{\theta''b}.
\end{aligned} \tag{7.24}
$$

We shall prove that the three terms given previously tend to zero as b_λ tends to infinity by choosing suitable θ' and θ''. Clearly,

$$P\{A(n;\lambda) - S_n \ge \theta''b\} \le P\{\mu n - S_n \ge \mu n - A(n;\lambda) + \theta''b\}.$$

For $|n - b| \le K\theta b$ we have by (4.5) and (4.6)

$$
\begin{aligned}
|A(n;\lambda) - (\mu b + d(n-b))| &= |g(n;\lambda)| \\
&\le K^*(n-b)^2/b, \quad \text{for some } K^* < \infty,
\end{aligned} \tag{7.25}
$$

and

$$n\mu - A(n;\lambda) \ge -(\mu - d)|n - b| - K^*(n-b)^2/b \ge -\theta''b/2,$$

provided that $|n - b| \le \theta b$ and that

(7.26) $$0 < \theta(\mu - d + K^*\theta) \le \theta''/2.$$

Hence

$$P\{A(n;\lambda) - S_n \ge \theta'' b\} \le P\{\mu n - S_n \ge \theta'' b/2\} = o(b^{-2}), \qquad n_1 \le n \le n_2.$$

And the first term on the right-hand side of (7.24) is $o(1)$. Let $\lambda' = (\lambda, x)$ and $A'(t;\lambda') = A(t;\lambda) - x$. Then by (4.4)

$$0 \le b - b' \le x/(\mu - \mu^*), \quad \text{for } 0 \le x \le (\mu - \mu^*)(1-\theta)b,$$

where $b' = \sup\{t\colon A'(t;\lambda') \le \mu t\}$.

Therefore, if θ'' is small, the family $\{A'(t;\lambda'),\ 0 \le x \le \theta'' b,\ \lambda \in \Lambda\}$ still satisfies the conditions (4.3)–(4.7). It follows from Corollary 2 that

$$\sum_{n=n_1+1}^{n_2-1} P\{A'(n;\lambda') < S_n \le A'(n;\lambda') + 1\}$$
$$= \sum_{n=n_1+1}^{n_2-1} P\{x - 1 \le A(n;\lambda) - S_n < x\}$$
$$\le K < \infty, \quad \text{for any } x \le \theta'' b \text{ and } \lambda \in \Lambda.$$

The second term on the right-hand side of (7.24) is bounded by

$$\big(\theta' b + |ET' - 1 - b|\big) K b^{-1} K \sum_{j=0}^{\infty} \sup_{j \le x < j+1} h^*(x)$$
$$\le \theta' K^3 + o(1), \quad \text{for some } K \text{ not depending on } \theta'.$$

It follows from (7.25) that

$$|A(n;\lambda) - \mu n| \ge |(\mu - \mu^* - K^*\theta/b)(n - b)|,$$
$$P\{T > n,\ A(n;\lambda) - S_n \le x\} = F(x; n+1)$$
$$\le P\{|S_n - \mu n| \ge (\mu - \mu^* - K^*\theta/b)\theta' b\}$$
$$= o(b^{-1}),$$

for $\theta' b \le |n - b| \le \theta b$.

The third term on the right-hand side of (7.24) is bounded by

$$K^2 \sum_{j=m}^{\infty} \sup_{j \le x < j+1} h^*(x) + K^2(n_2 - n_1) o(b^{-1}) = o(1).$$

Note that the constant θ is arbitrary when we define the stopping time T' in (7.1) and the choice of θ'' only depends on the parameters in the conditions (4.3)–(4.6), we can always find a θ'' such that (7.26) is satisfied. By letting $b_\lambda \to \infty$ first and then $\theta' \to 0$, we have $I_2 = o(1)$ and it follows from (7.21) that

$$\operatorname{Cov}\big(\big[R'_\lambda - r(d(T'_\lambda;\lambda))\big], T'_\lambda\big) = I_1 + o(1).$$

Since by (7.22) and (7.17)

$$\big|\big[ER(x, d(n;\lambda)) - r(d(n;\lambda))\big] - \big[ER(x, d_1^*) - r(d_1^*)\big]\big|$$
$$\le h^*(x)|d(n;\lambda) - d_1^*|,$$

we can repeat the argument again and have

$$\operatorname{Cov}([R' - r(d(T';\lambda))], T') = \int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, d\left[\sum_{n=n_1}^{n_2} F(x; n, \lambda)\right] + o(1).$$

It follows from Corollary 3 that

$$\operatorname{Cov}([R' - r(d(T';\lambda))], T')\mu_0 = \int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, P\Big\{\max_n (S_n - nd_1^*)^- \le x\Big\}\, dx + o(1). \tag{7.27}$$

Now let us consider $\operatorname{Cov}(r(d(T_\lambda'; \lambda)), T_\lambda')$. By Wiener–Hopf factorization [Feller (1966), page 605], the derivative of $r(u)$ is continuous and

$$(\partial r/\partial u)(d_1^*) = -\big(1 - (\sigma/\mu_0)^2\big)/2 - \sum_{n=1}^\infty P\{S_n - nd_1^* \le 0\}.$$

It follows from (4.5) and (4.6) that

$$r(d(T_\lambda'; \lambda)) - r(d_1^*) = [(\partial r/\partial u)(d_1^*)]\, d_2^*(T_\lambda' - b)(1 + o(1))/b.$$

Hence

$$\operatorname{Cov}(r(d(T';\lambda)), T') = -d_2^*(\sigma/\mu_0)^2\left[\big(1 - (\sigma/\mu_0)^2\big)/2 + \sum_{n=1}^\infty P\{S_n - nd_1^* \le 0\}\right]. \qquad \square$$

LEMMA 9. *Suppose that the conditions* (4.4)–(4.7) *hold and that X has a strongly nonlattice distribution with finite fourth moment. Let T_λ' be defined by* (7.1) *and $\mu_0 = \mu - d_1^*$. Then*

$$\begin{aligned}\operatorname{Cov}(T_\lambda', g(T_\lambda'; \lambda)) &= \mu_0^{-3}\big[d_2^*\sigma^2 r(d_1^*) - d_2^* E(X-\mu)^3/2\big] \\ &\quad + \mu_0^{-4}(d_3^* + 3d_2^*)\sigma^4/2 + \mu_0^{-5} 2(d_2^*)^2\sigma^4 + o(1)\end{aligned} \tag{7.28}$$

and

$$\begin{aligned}E(T_\lambda' - b)^3/b &= \mu_0^{-3}\big[3\sigma^2 r(d_1^*) - E(X-\mu)^3\big] \\ &\quad + \mu_0^{-4}(3\sigma^4) + \mu_0^{-5}\big(9 d_2^* \sigma^4/2\big) + o(1).\end{aligned}$$

PROOF. By (4.5) and (4.6)

$$g(T_\lambda'; \lambda) = d_2^* b^{-1}(T_\lambda' - b)^2/2 + d_3^* b^{-2}(T_\lambda' - b)^3(1/6 + o(1)), \tag{7.29}$$

where $o(1)$ is uniformly bounded by a constant K. By (7.1)–(7.3)

$$S_{T'} - \mu T_\lambda' = (\mu - d)(b - T_\lambda') + g(T_\lambda'; \lambda) + R_\lambda', \tag{7.30}$$

$$|g(T_\lambda'; \lambda)/b| \le K\theta^2 < \infty \quad \text{and} \quad T'/b \le 2, \quad \text{a.s.} \tag{7.31}$$

It follows from Theorem 3 that

$$ET'_\lambda = b + \mu_0^{-1}\big(r(d_1^*) + \big(d_2^*(\sigma/\mu_0)^2/2\big) + o(1). \tag{7.32}$$

Since

$$\begin{aligned} &P\{R'_\lambda > x,\, T'_\lambda \neq n_1,\, T'_\lambda \neq n_2\} \\ &\quad \leq P\{X_{T'} > x\} \leq ET'P\{X > x\} \leq 2bP\{X > x\}, \\ &\quad \big\{(R')^3/b,\, \lambda \in \Lambda\big\} \text{ is u.i.,} \quad \text{by (7.11).} \end{aligned} \tag{7.33}$$

By Lemma 5 of Chow and Yu (1981)

$$\big\{(S_{T'} - \mu T'_\lambda)^4/b^2, (T'_\lambda - b)^4/b^2\big\} \text{ is u.i.} \tag{7.34}$$

And by Theorem 1 with $(\sigma^*)^2 = (\sigma/\mu_0)^2$,

$$P\{R'_\lambda > x,\, T'_\lambda - b > t\sigma^*\sqrt{b}\} = G(x, d_1^*)P\{N(0,1) > t\} + o(1). \tag{7.35}$$

By (7.29), (7.34), (7.35) and (7.32)

$$\begin{aligned} &\operatorname{Cov}(T', g(T'; \lambda)) \\ &\quad = d_2^* E(T' - b)^3/(2b) + d_3^*(\sigma^*)^4 E(N(0,1))^4/6 \\ &\qquad - d_2^* E(T' - b)^2(2b)^{-1}(ET' - b) + o(1) \\ &\quad = d_2^* E(T' - b)^3/(2b) + d_3^*(\sigma^*)^4/2 \\ &\qquad - d_2^*(\sigma^*)^2\big[r(d_1^*) + d_2^*(\sigma^*)^2/2\big]/(2\mu_0) + o(1). \end{aligned} \tag{7.36}$$

By (7.29), (7.31) and (7.34)

$$\big\{(g(T'; \lambda))^2, |g(T'; \lambda)|^3/b\big\} \text{ is u.i.}$$

By (7.33) and (7.35)

$$E(g(T'; \lambda) + R')^3/b = o(1).$$

Since $(T' - b)/b$ is bounded, by (7.9) and (7.35)

$$E(T' - b)(g(T'; \lambda) + R')^2/b = o(1).$$

Therefore by (7.30)

$$\begin{aligned} E(T' - b)^3/b = \big[&3\mu_0^2 E(T' - b)^2(g(T'; \lambda) + R')/b \\ &- E(S_{T'} - \mu T')^3/b\big]/\mu_0^3 + o(1). \end{aligned} \tag{7.37}$$

Since $S_n - \mu n$, $n \geq 1$, are partial sums of i.i.d. mean zero random variables and T' is a bounded stopping time, it follows from Chow, Robbins and Teicher (1965) that

$$E(S_{T'} - \mu T')^3 = ET'E(X - \mu)^3 + 3\sigma^2 ET'(S_{T'} - \mu T'). \tag{7.38}$$

Again, by (7.34), (7.30) and (7.35)

$$\begin{aligned} ET'(S_{T'} - \mu T')/b &= -\mu_0 E(T' - b)^2/b + o(1). \\ &= -\mu_0(\sigma^*)^2 + o(1). \end{aligned} \tag{7.39}$$

By (7.29), (7.34), (7.9) and (7.35)

$$(7.40)\quad E(T'-b)^2(g(T';\lambda)+R')/b = 3d_2^*(\sigma^*)^4/2 + (\sigma^*)^2 r(d_1^*) + o(1).$$

It follows from (7.37)–(7.40) that

$$E(T'-b)^3/b = 3\big[3d_2^*(\sigma^*)^4/2 + (\sigma^*)^2 r(d_1^*)\big]/\mu_0 \\ - E(X-\mu)^3/\mu_0^3 + 3\sigma^2(\sigma^*)^2/\mu_0^2 + o(1).$$

Since $\sigma^* = \sigma/(\mu - d_1^*) = \sigma/\mu_0$, by algebra

$$E(T'-b)^3/b = \mu_0^{-3}\big[3\sigma^2 r(d_1^*) - E(X-\mu)^3\big] + \mu_0^{-4}(3\sigma^4) \\ + \mu_0^{-5}\big(9d_2^*\sigma^4/2\big) + o(1).$$

Hence by (7.36)

$$\begin{aligned}\operatorname{Cov}(T', g(T';\lambda)) &= (d_2^*/2)E(T'-b)^3/b + \mu_0^{-4}\big(d_3^*\sigma^4/2\big) \\ &\quad - \mu_0^{-3}\big(d_2^*\sigma^2 r(d_1^*)/2\big) - \mu_0^{-5}\big(d_2^*\sigma^2/2\big)^2 + o(1) \\ &= \mu_0^{-3}\big[d_2^*\sigma^2 r(d_1^*) - d_2^* E(X-\mu)^3/2\big] \\ &\quad + \mu_0^{-4}\big[(d_3^* + 3d_2^*)\sigma^4/2\big] + \mu_0^{-5}\big(2(d_2^*)^2\sigma^4\big) + o(1). \quad \square\end{aligned}$$

PROOF OF THEOREM 4. By (7.30) and Wald's lemma of Chow, Robbins and Teicher (1965)

$$\operatorname{Var}(S_{T'} - \mu T') = (\mu - d)^2 \operatorname{Var}(T') - 2(\mu - d)\operatorname{Cov}(T', g(T';\lambda) + R') \\ + \operatorname{Var}(R' + g(T';\lambda)),$$

$$(\mu - d)^2\operatorname{Var}(T') = \sigma^2 ET' + 2(\mu - d)\operatorname{Cov}(T', R') \\ + 2(\mu - d)\operatorname{Cov}(T', g(T',\lambda)) - \operatorname{Var}(R' + g(T';\lambda)).$$

It follows from (7.32) and Lemmas 7–9 that

$$\begin{aligned}(\mu - d)^2\operatorname{Var}(T') &= \sigma^2 b + \sigma^2\big[r(d_1^*) + d_2^*(\sigma/\mu_0)^2/2\big]/\mu_0 \\ &\quad + \left[\int_0^\infty x^2\, dG(x, d_1^*) + (r(d_1^*))^2 - \mu_0^{-4}(d_2^*)^2\sigma^4/2\right] \\ &\quad + 2C_3^* - \mu_0^{-1} d_2^*\sigma^2\left[(1 - \mu_0^{-2}\sigma^2) + 2\sum_{n=1}^{\infty} P\{S_n - nd_1^* \le 0\}\right] \\ &\quad + \mu_0^{-2} d_2^*\big[2\sigma^2 r(d_1^*) - E(X-\mu)^3\big] + \mu_0^{-3}(d_3^* + 3d_2^*)\sigma^4 \\ &\quad + \mu_0^{-4}\big(4(d_2^*)^2\sigma^4\big) + o(1),\end{aligned}$$

where

$$C_3^* = \int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, P\Big\{\max_{n\ge 0}(S_n - nd_1^*)^- \le x\Big\}\, dx.$$

Since

$$r(d_1^*) = (\mu_0^2 + \sigma^2)/(2\mu_0) - \sum_{n=1}^{\infty} n^{-1}E(S_n - nd_1^*)^-$$

and

$$\int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, dx = (r(d_1^*))^2 + \int_0^\infty x^2\, dG(x, d_1^*)/2,$$

by algebra

$$\begin{aligned}
(\mu - d)^2 &\mathrm{Var}(T') - \sigma^2 b \\
&= \left[\sigma^2 r(d_1^*)/\mu_0 + \int_0^\infty x^2\, dG(x, d_1^*) + (r(d_1^*))^2 + 2C_3^*\right] \\
&\quad + \mu_0^{-3} d_2^* \sigma^4/2 - \mu_0^{-4}(d_2^*)^2\sigma^4/2 - \left[\mu_0^{-1} d_2^*\sigma^2 - \mu_0^{-3} d_2^*\sigma^4\right] \\
&\quad - \mu_0^{-1} d_2^*\sigma^2 2 \sum_{n=1}^{\infty} P\{S_n - nd_1^* \le 0\} \\
&\quad + \left[\mu_0^{-1} d_2^*\sigma^2 + \mu_0^{-3} d_2^*\sigma^4\right] - \mu_0^{-2} d_2^*\sigma^2 2 \sum_{n=1}^{\infty} n^{-1}E(S_n - nd_1^*)^- \\
&\quad - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3}(d_3^* + 3d_2^*)\sigma^4 + \mu_0^{-4}\left(4(d_2^*)^2\sigma^4\right) + o(1) \\
&= \left[\sigma^2 r(d_1^*)/\mu_0 + 2\int_0^\infty x^2\, dG(x, d_1^*) + 3(r(d_1^*))^2\right] \\
&\quad - 2\int_0^\infty [ER(x, d_1^*) - r(d_1^*)]\, P\left\{\max_{n \ge 0}(S_n - nd_1^*)^- > x\right\} dx \\
&\quad - 2\mu_0^{-2} d_2^*\sigma^2 \left[\mu_0 \sum_{n=1}^{\infty} P\{S_n - nd_1^* \le 0\} + \sum_{n=1}^{\infty} n^{-1}E(S_n - nd_1^*)^-\right] \\
&\quad - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3}\left[11 d_2^*\sigma^4/2 + d_3^*\sigma^4\right] \\
&\quad + 7\mu_0^{-4}(d_2^*)^2\sigma^4/2 + o(1) \\
&= C_1^* - 2\mu_0^{-2} d_2^*\sigma^2 C_2^* - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3} d_3^*\sigma^4 \\
&\quad + \mu_0^{-4} d_2^*\sigma^4(7d_2^* + 11\mu_0)/2 + o(1).
\end{aligned}$$

This finishes the proof. □

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DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903