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NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONTINUITY OF LOCAL TIME OF LÉVY PROCESSES

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Let $u_1(x)$ be the 1-potential kernel density for a Lévy process, let $\phi^2(x) = 2u_1(0) - u_1(x) - u_1(-x)$, let $\bar{\phi}$ be the monotone rearrangement of ϕ and let $I(\bar{\phi}) = \int_{0+}\bar{\phi}(u)u^{-1}(\log(1/u))^{-1/2}\,du$. Barlow and Hawkes proved that if $I(\bar{\phi}) < \infty$, then the local time has a jointly continuous version. In this paper it is shown that if $I(\bar{\phi}) = \infty$, then the local time is not continuous.

0. Introduction. Let X_t, $t \geq 0$, be a one-dimensional Lévy process; that is, a process with stationary independent increments. We denote by P^x the law of X starting at $x \in \mathbb{R}$, we write P for P^0 and denote expectation with respect to P^x, P by E^x, E. We will say that X has characteristics (a, σ^2, ν) if

$$Ee^{i\theta X_t} = e^{-t\chi(\theta)},$$

where

$$\chi(\theta) = -ia\theta + \sigma^2\theta^2 - \int\left[e^{i\theta y} - 1 - i\theta y 1_{(|y|<1)}\right]\nu(dy). \tag{0.1}$$

Here ν is a measure on $\mathbb{R}$ satisfying $\int(1 \wedge y^2)\nu(dy) < \infty$, $\nu(\{0\}) = 0$.

We will be interested in Lévy processes satisfying

$$|\{X_s, 0 \leq s \leq t\}| > 0 \quad \text{a.s. for each } t > 0 \tag{0.2}$$

and

$$0 \text{ is regular for } \{0\}. \tag{0.3}$$

The first of these conditions states that the range of X has positive Lebesgue measure, and this implies that X has an occupation density, or local time. The second ensures that this local time is, at each fixed point, a.s. continuous as a function of t.

Analytic conditions, in terms of (a, σ^2, ν), for (0.2) and (0.3) to hold were given by Kesten [18] and Bretagnolle [8]. (0.2) holds if and only if

$$\int_{-\infty}^{\infty} \operatorname{Re}\frac{1}{1 + \chi(\theta)}\,d\theta < \infty, \tag{0.4}$$

and if (0.4) holds then (0.3) holds if and only if

$$\text{either} \quad \sigma^2 > 0 \quad \text{or} \quad \int(1 \wedge |y|)\nu(dy) = \infty. \tag{0.5}$$

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The following theorem summarises results from [5, 8, 12 and 18] on the existence and properties of the local time of X. We introduce the notation

$$U_\alpha f(x) = E^x \int_0^\infty f(X_s) e^{-\alpha s}\, ds,$$

$$T_A = \inf\{t \geq 0: X_t \in A\}, \qquad T_x = T_{\{x\}},$$

$$\psi_\alpha(x) = E^0 e^{-\alpha T_x}.$$

THEOREM A (see [12, Theorem 4]). *Suppose X satisfies* (0.4) *and* (0.5). *Then*:

(i) *U_α has a bounded continuous density $u_\alpha(x)$, so that*

$$E^x \int_0^\infty e^{-\alpha s} f(X_s)\, ds = U_\alpha f(x) = \int u_\alpha(y - x) f(y)\, dy \tag{0.6}$$

for bounded measurable f.

(ii) *There exists a jointly measurable mapping $(x, t, \omega) \to L_t^x(\omega)$ such that* (a) *for each x, L^x is a continuous additive functional,* (b) *for each $t \geq 0$, $(x, \omega) \to L^x(\omega)$ is $\mathscr{B}(\mathbb{R}) \times \mathscr{F}_t$-measurable and* (c) *$L$ satisfies the density of occupation formula*: *For bounded measurable f we have*

$$\int_0^t f(X_s)\, ds = \int f(a) L_t^a\, da. \tag{0.7}$$

(iii) *u_α and ψ_α satisfy*

$$u_\alpha(x) = E^0 \int_0^\infty e^{-\alpha s}\, dL_s^x, \tag{0.8}$$

$$u_\alpha(x) = \psi_\alpha(x) u_\alpha(0), \tag{0.9}$$

$$u_\alpha(x) + u_\alpha(-x) = \frac{1}{\pi} \int \cos \theta x \operatorname{Re} \frac{1}{\alpha + \chi(\theta)}\, d\theta. \tag{0.10}$$

We will be concerned with the following questions:

1. When is there a jointly continuous version of $(x, t) \to L_t^x$?
2. If there is a jointly continuous version, what is the modulus of continuity of $x \to L_t^x$?

For Brownian motion the answers are well known. Trotter [25] proved that Brownian local time has a jointly continuous version and McKean [20] and Ray [24] found its exact modulus of continuity, Ray's proof used the Ray–Knight theorems [19, 24], which give the law of the process $x \to L_T^x$ for certain stopping times T, and from which it is easy to read off many absolute sample path properties of $L_t^{\cdot}$.

There is no (known) analogue of the Ray–Knight theorems for any Lévy process with jumps. However in [7] Boylan proved that, for a class of Lévy processes which included the stable process of index greater than 1, there is a jointly continuous local time. Meyer [21] obtained an estimate on the tail of the distribution of $|L_t^a - L_t^b|$, and using this and Garsia's lemma, Getoor and Kesten [12] found fairly general sufficient conditions for continuity.

Getoor and Kesten also proved that if

$$\limsup_{\alpha \to \infty} (\log \alpha) u_\alpha(0) > 0, \tag{0.11}$$

then $x \to L_t^x$ has no continuous version. [The asymmetric Cauchy process has $u_\alpha(0) \sim c(\log \alpha)^{-1}$ and so satisfies (0.11).] Thus not all local times are continuous, and Getoor and Kesten posed problem 1 above. In [22] Millar and Tran improved this discontinuity result and showed that if (0.11) holds, then $L_t^{\cdot}$ is unbounded in every interval around X_0.

In [1] a new estimate on the tail of $|L^a - L^b|$ was found, and using this and Dudley's theorem, Barlow and Hawkes [2] obtained a sufficient condition for joint continuity.

Let

$$\begin{aligned} \phi(x)^2 &= 2u_1(0) - u_1(x) - u_1(-x) \\ &= \frac{1}{\pi} \int (1 - \cos\theta x) \mathrm{Re} \frac{1}{1 + \chi(\theta)} d\theta, \end{aligned} \tag{0.12}$$

and let $\bar{\phi}$ denote the monotone rearrangement of ϕ. Set

$$I(\bar{\phi}) = \int_0^{1/e} \frac{\bar{\phi}(u)\, du}{u(\log 1/u)^{1/2}}. \tag{0.13}$$

THEOREM B (Barlow and Hawkes [2]). *Suppose X satisfies* (0.4) *and* (0.5).

(a) *If $I(\bar{\phi}) < \infty$, then $(x, t) \to L_t^x$ has a jointly continuous version.*

(b) *Let $H(u)$ denote the ϕ-metric entropy of $[0,1]$. For each $t > 0$ there exists $\delta = \delta_t(\omega) > 0$ such that, for all a, b with $\phi(b-a) < \delta$,*

$$\sup_{s \le t} |L_s^a - L_s^b| \le 416 \Big(\sup_x L_t^x \Big)^{1/2} \int_0^\delta H(u)^{1/2}\, du. \tag{0.14}$$

The main result of this paper is that the condition $I(\bar{\phi}) < \infty$ is necessary, as well as sufficient, for the joint continuity of the local time. (This was conjectured by Hawkes [14].)

THEOREM 1. *Let X be a Lévy process satisfying* (0.4) *and* (0.5), *and suppose that*

$$I(\bar{\phi}) = \infty. \tag{0.15}$$

Then $(x, t) \to L_t^x$ has no continuous version. Further, for each $t > 0$, $\varepsilon > 0$, $\{L_t^a, a \in \mathbb{Q} \cap (-\varepsilon, \varepsilon)\}$ is P^0-a.s. dense in $[0, \infty)$.

The estimates which lead to Theorem 1 also yield lower bounds on the modulus of continuity of $x \to L_t^x$. By working a little harder than in [1], and so getting better constants for the upper bounds, we obtain an exact modulus for a fairly wide class of Lévy processes.

Theorem 2. *Suppose that* $\phi(u) = u^{\alpha} f(u)$, *where* $\alpha > 0$ *and* f *is slowly varying at* 0. *Then*

$$\lim_{\delta \downarrow 0} \sup_{\substack{|a-b|<\delta \\ a,b \in I \\ 0 \le s \le t}} \frac{|L_s^a - L_s^b|}{\phi(b-a)(\log(1/|b-a|))^{1/2}} = 2\Big(\sup_{x \in I} L_t^x\Big)^{1/2} \tag{0.16}$$

for all intervals $I \subseteq \mathbb{R}$, $t \ge 0$, *a.s.*

Specialising to the case of stable processes, we have

Theorem 3. *Let* X_t *be a stable process of index* $\alpha > 1$, *so that*

$$\chi(\theta) = |\theta|^{\alpha} + ih|\theta|^{\alpha}\operatorname{sgn}(\theta),$$

where $|h| \le \tan(\alpha\pi/2)$. *Then*

$$\phi^2(x) \sim \frac{c_\alpha |x|^{\alpha-1}}{1+h^2} \quad \textit{as } x \to 0,$$

where

$$c_\alpha = \frac{2}{\pi}\int_0^\infty (1-\cos y) y^{-\alpha}\,dy = \frac{1}{\pi}\frac{\Gamma(2-\alpha)}{\alpha-1}\sin\frac{(2-\alpha)\pi}{2}. \tag{0.17}$$

We have

$$\lim_{\delta \downarrow 0} \sup_{\substack{|a-b|<\delta \\ a,b \in I \\ 0 \le s \le t}} \frac{|L_s^a - L_s^b|}{|b-a|^{(\alpha-1)/2}(\log(1/|b-a|))^{1/2}} = \frac{2c_\alpha^{1/2}}{(1+h^2)^{1/2}}\Big(\sup_{x \in I} L_t^x\Big)^{1/2} \tag{0.18}$$

for all intervals $I \subseteq \mathbb{R}$ *and all* $t > 0$, *a.s.*

Remarks. 1. These results may be compared with the Dudley–Fernique theorem [9, 10] (see also [16]) for stationary Gaussian processes. Let Y be a stationary zero-mean Gaussian process and let $\psi(t-s) = \|Y_t - Y_s\|_2$. Then Y has a continuous version if and only if $I(\bar{\psi}) < \infty$.

In Hawkes [15] it is shown that the local time of a Lévy process can be made into a stationary process, and that ϕ is then the incremental variance; thus ϕ plays roughly the same role for $L_t^{\cdot}$ as ψ does for Y.

2. Belyaev [4] showed that a Gaussian process either has continuous paths a.s. or the sample paths are a.s. unbounded on every interval. The essential idea of

the proof is that Y has an L^2 expansion

$$Y_t = \sum_{j=1}^{\infty} \phi_j(t)\xi_j, \tag{0.19}$$

in which the ξ_j are independent random variables and $\phi_j(t)$ are continuous (see, for example, Jain and Marcus [16]). The continuity or discontinuity of the sample paths is a tail event for the ξ_j and hence has probability 0 or 1.

We have been unable to find a similar 0–1 law for the local time. The L^2 expansion (0.19) is still valid (see Hawkes [15]), but the ξ_j are no longer independent. Of course Theorems A and 1 do lead to a similar dichotomy, but the proof is "hard" (i.e., involving detailed calculations and estimates) rather than "soft" (using 0–1 laws and the structure of the process).

3. The term $(\sup_x L_t^x)^{1/2}$ on the right-hand side of (0.16) shows how the size of the oscillations of $x \to L_t^x$ near a point x_0 depends on $L_t^{x_0}$. For Brownian local time the Ray–Knight theorems give one reason for the presence, and form, of this term. It would, however, be interesting to have a more general explanation for its presence.

EXAMPLE. Let $\chi(\theta)$ be given by (0.1), with $a \in \mathbb{R}$, $\sigma^2 = 0$ and

$$\nu(dx) = x^{-2}g_{\alpha\beta}(1/|x|)\left(p1_{(x>0)} + q1_{(x<0)}\right)dx,$$

where $p, q > 0$, $p + q = 1$ and

$$g_{\alpha\beta}(y) = (\log y)^{\alpha}(\log\log y)^{\beta}.$$

(0.4) and (0.5) hold if $\alpha > -1$ or if $\alpha = -1$, $\beta \geq -1$. We then have

$$\chi(\theta) \sim \pi|\theta|g_{\alpha\beta}(|\theta|) + i(p-q)\theta(1+\alpha)^{-1}g_{\alpha\beta}(|\theta|)(\log|\theta|)\eta(\theta) \quad \text{as } |\theta| \to \infty,$$

where $\eta(\theta) = 1$ if $\alpha > -1$, $\eta(\theta) = \log\log\theta$ if $\alpha = -1$, $\beta > -1$ and $\eta(\theta) = (\log\log\theta)(\log\log\log\theta)$ if $\alpha = \beta = -1$.

If $\alpha > -1$ then

$$\phi^2(x) \sim c(\log(1/|x|))^{-(\alpha+1)}(\log\log(1/|x|))^{-\beta} \quad \text{as } x \to 0.$$

Thus if $\alpha < 0$ or $\alpha = 0$ and $\beta \leq 2$ the local time is discontinuous, while if $\alpha > 0$ or $\alpha = 0$ and $\beta > 2$ it is continuous.

These examples were discussed in [1], but there is an error there: The form of $\operatorname{Im}\chi(\theta)$ is incorrect in the case $\alpha = -1$. In the table [1, page 34] the case $\alpha = -1$, $-1 \leq \beta \leq 1$, belongs with $\alpha = -1$, $\beta > 1$.

The plan of this paper is as follows. In Section 1 we collect a number of preliminary results and in Section 2 we obtain estimates which relate the function $h(x, A) = E^x L_{T_A}^x$ to $\phi^2(x - y)$. The main estimates on the local time are given in Section 3.

In order to prove that $L_T^{\cdot}$ is discontinuous, it is necessary to find lower bounds on quantities like $P(\max_{x\in A}(L_T^x - L_T^{x_0}) > z)$, where A is a finite set and $x_0 \in A$. The basic idea is as follows. Let $V_t = \sum_{x\in A} L_t^x$; if X is time-changed by the inverse of V we obtain a continuous time Markov chain with state space A. As is well known, such a Markov chain can be built up from its jump chain and a set of independent negative exponential holding times. Using this decomposition we deduce that if $\lambda > 0$ and $T = \inf\{s \geq 0: L_s^{x_0} > \lambda\}$, then

$$L_T^x = \sum_{j=1}^{N(x)} V_{xj} \quad \text{for } x \in A, \tag{0.20}$$

where $N(x)$ is the number of visits by the jump chain to x, and the V_{xj} are independent negative exponential random variables.

From (0.20) we see that there are two possible reasons why we may have $L_T^x \gg L_T^{x_0}$; we may have $N(x) \gg N(x_0)$ or we may have $\sum_{j=1}^{N(x_0)} V_{xj} \gg \sum_{j=1}^{N(x_0)} V_{x_0,j}$. The first source of variation is hard to control, since in general the random variables $N(x)$, $x \in A$, are as difficult to deal with as the local times L_T^x, $x \in A$. On the other hand, the second source can be handled very easily: Conditioning on $\sigma(N(x_0), V_{x_0,j}, j \geq 1)$ the random variables $\sum_{j=1}^{N(x_0)} V_{xj}$, $x \in A - \{x_0\}$, are independent and have a gamma distribution with parameters for which good estimates can be found. Fortunately it turns out that, in some sense, most of the variation in L_T^x, $x \in A$, is due to the second source, and so the estimates which we obtain by throwing away the variation in $N(x)$ provide a good lower bound for $P(\max_{x \in A}(L_T^x - L_T^{x_0}) > z)$.

In Sections 4 and 5 these estimates are used to prove Theorems 1–3.

Throughout this paper X will be a Lévy process satisfying (0.4) and (0.5) with local times L_t^a, $a \in \mathbb{R}$, $t \geq 0$. We will write $\tau_t(a)$ for the right continuous inverse of L^a:

$$\tau_t(a) = \inf\{s \geq 0: L_s^a > t\}.$$

Occasionally, when a, t are complicated expressions, we will write $L(a, t)$, $\tau(a, t)$ for L_t^a, $\tau_t(a)$. For $x \in \mathbb{R}$, $A \in \mathscr{B}(\mathbb{R})$, $\alpha \geq 0$, we set

$$h_\alpha(x, A) = E^x \int_0^{T_A} e^{-\alpha s}\, dL_s^x,$$

and we write $h(x, A) = h_0(x, A)$, $h_\alpha(x, y) = h_\alpha(x, \{y\})$.

The notation c_0, c_1, etc., will be used to denote a (fixed) universal constant, with $0 < c_i < \infty$. $c_i(\varepsilon)$ will denote a (fixed) function of ε, with $0 < c_i(\varepsilon) < \infty$ for each $\varepsilon > 0$.

1. Preliminaries. Let $Z(\beta, h)$, $Z'(\beta, h)$ be independent random variables with a $\Gamma(\beta, h)$ distribution. Thus $EZ(\beta, h) = \beta h$, $\operatorname{var} Z(\beta, h) = \beta h^2$ and the distribution of $h^{-1}(Z(\beta, h) - \beta h)$ does not depend on h. Set

$$\gamma^+(\beta, u) = P\big(Z(\beta, h) \geq \beta h + h\beta^{1/2}u\big),$$

$$\gamma^-(\beta, u) = P\big(Z(\beta, h) \leq \beta h + h\beta^{1/2}u\big).$$

Elementary and tedious calculations with the Γ density function and Stirling's formula yield

LEMMA 1.1. *There exist constants* $c_0, c_1(\varepsilon) > 0$ *such that, for all* $\beta \geq 1$, $\frac{1}{2} > \varepsilon > 0$:

(i) $\gamma^+(\beta, 1) \geq c_0$.
(ii) $\gamma^-(\beta, 0) \geq c_0$.
(iii) $\gamma^+(\beta, u) \geq c_1(\varepsilon)\exp(-((1+\varepsilon)/2)u^2)$ *for* $0 \leq u \leq \varepsilon\beta^{1/2}$.
(iv) $\gamma^-(\beta, u) \geq c_1(\varepsilon)\exp(-((1+\varepsilon)/2)u^2)$ *for* $0 \leq u \leq \varepsilon\beta^{1/2}$.

LEMMA 1.2. *Let* $\beta_0, \beta_1 \geq 1$. *Then*

$$P(|Z(\beta_0, h_0) - Z'(\beta_1, h_1)| > x)$$
$$\geq c_1(\varepsilon)^2 \exp\left(-\frac{(1+\varepsilon)x^2}{2(\beta_0 h_0^2 + \beta_1 h_1^2)}\right) \quad \text{for } 0 \leq x \leq \varepsilon(\beta_0 h_0 \wedge \beta_1 h_1). \tag{1.1}$$

PROOF. We may suppose that $\beta_0 h_0 \geq \beta_1 h_1$. Let $p = \beta_0 h_0^2/(\beta_0 h_0^2 + \beta_1 h_1^2)$ and $q = 1 - p$. Then

$$\begin{aligned} P(|Z(\beta_0, h_0) - Z'(\beta_1, h_1)| > x) & \\ &\geq P(Z(\beta_0, h_0) - \beta_0 h_0 + (\beta_0 h_0 - \beta_1 h_1) + \beta_1 h_1 - Z'(\beta_1, h_1) > x) \\ &\geq P(Z(\beta_0, h_0) - \beta_0 h_0 > px) P(Z'(\beta_1, h_1) - \beta_1 h_1 < -qx) \\ &= \gamma^+(\beta_0, u)\gamma^-(\beta_1, \nu), \end{aligned}$$

where $u\beta_0^{1/2}h_0 = px$, $\nu\beta_1^{1/2}h_1 = qx$. It is easily checked that $u \leq \varepsilon\beta_0^{1/2}$, $\nu \leq \varepsilon\beta_1^{1/2}$, and (1.1) now follows from Lemma 1.1. □

LEMMA 1.3. *Let* X_t *be a Lévy process satisfying* (0.4) *and* (0.5).

(i) $P(\sup_{s \leq \tau_t(a)}(L_s^a - L_s^b) \geq x) \leq \exp(-x^2/4th(a, b))$.
(ii) $P(\sup_{s \geq 0}|\lambda \wedge L_s^a - \lambda \wedge L_s^b| \geq x) \leq 2\exp(-x^2/4\lambda h(a, b))$,
(iii) $P(\sup_{s \leq \tau_t(a)}(L_s^b - L_s^a) \geq x) \leq \exp(-x^2/(4(x+t)h(a, b)))$.

PROOF. (i) and (ii) are proved in [1, Proposition 2.7 and Lemma 2.8]. (iii) is an easy consequence of (i), since

$$\left\{\sup_{s \leq \tau_t(a)}(L_s^b - L_s^a) \geq x\right\} \subseteq \left\{\sup_{s \leq \tau_{t+x}(b)}(L_s^b - L_s^a) \geq x\right\}. \qquad \square$$

REMARK. The estimate of [21] (see also [6, Proposition V.3.28]) states that

$$P^x\left(\sup_{0 \leq s \leq t}|L_s^a - L_s^b| \geq x\right) \leq 2e^t \exp\left(\frac{-x}{\gamma(b-a)}\right), \tag{1.2}$$

where $\gamma(y)^2 = 1 - \psi_1(y)\psi_1(-y)$. We will see in Section 2 that $\gamma^2(y)/h(y) \to$

$u_1(0)$ as $y \to 0$. The estimates in Lemma 1.3 give better control over the tail of $|L^a - L^b|$, but (1.2) is simpler to use.

The following two 0–1 laws are both consequences of the Blumenthal 0–1 law.

LEMMA 1.4 ([12, Theorem 4], [2, Proposition 3]). (a) *Either* (i) $L_t^{\cdot}$ *has a continuous version a.s. or else* (ii) $L_t^{\cdot}$ *does not have a continuous version a.s.*

(b) *Either* (i) $\sup_{a \in \mathbb{Q}} L_t^a < \infty$ *for all* $t \geq 0$ *a.s. or else* (ii) $\sup_{a \in \mathbb{Q}} L_t^a = \infty$ *for all* $t > 0$ *a.s.*

There is nothing special about the set $\mathbb{Q}$.

LEMMA 1.5. *Let* D_1, D_2 *be countable subsets of* $\mathbb{R}$, *with* $D_1 \supseteq \operatorname{cl} D_2$. *Then*

$$\{L_s^x, x \in D_1\} \subseteq \operatorname{cl}\{L_s^x, x \in D_2\} \quad \textit{for all } s \geq 0 \ a.s.$$

PROOF. This is a simple consequence of the continuity in probability of $L_t^{\cdot}$. Let $t > 0$: It is enough to prove the lemma for $0 \leq s \leq t$. Using the estimate (1.2) we deduce that there exists a sequence $\delta_n \downarrow 0$ such that if $x \in \mathbb{R}$ and $|y_n - x| < \delta_n$, then

$$P\Big(\lim_{n\to\infty} \sup_{0 \leq s \leq t} |L_s^x - L_s^{y_n}| = 0\Big) = 1.$$

For each $x \in D$, let $y_n(x)$ be a sequence in D_2 with $|x - y_n(x)| < \delta_n$. Since D_1 is countable, we have

$$\sup_{0 \leq s \leq t} |L_s^x - L_s^{y_n}| \to 0 \quad \text{as } n \to \infty \text{ for all } x \in D_1 \text{ a.s.,}$$

and the desired result follows immediately. □

We now define the monotone rearrangement of ϕ. From (0.9) we have $u_1(x) \to 0$ as $|x| \to \infty$ and $u_1(x) < u_1(0)$ for $x \neq 0$. Thus $\phi(x) > 0$ for $x \neq 0$ and $\phi(x) \to (2u_1(0))^{1/2}$ as $|x| \to \infty$.

Let

$$\rho(y) = |\{x: \phi(x) < y\}|. \tag{1.3}$$

Then $\rho(0) = 0$, ρ is right continuous and strictly increasing and ρ is finite on $[0, (2u_1(0))^{1/2})$.

DEFINITION. Let $\bar{\phi}$ be the right continuous inverse of ρ; that is,

$$\bar{\phi}(x) = \inf\{y: \rho(y) > x\}. \tag{1.4}$$

$\bar{\phi}$ is the *monotone rearrangement* of ϕ on $[0, \infty)$.

Since ρ is strictly increasing, $\bar{\phi}$ is continuous; also $\bar{\phi}(0) = 0$, $\bar{\phi}$ is weakly increasing and $0 \leq \bar{\phi}(x) \leq (2u_1(0))^{1/2}$ for all x. It is clear that $\overline{\lambda\phi} = \lambda\bar{\phi}$ for $\lambda > 0$, and that if $\phi_1 \leq \phi_2$, then $\bar{\phi}_1 \leq \bar{\phi}_2$.

Let $d_\phi(a, b) = \phi(b - a)$ for $a, b \in \mathbb{R}$. It is shown in [2] that d_ϕ is a metric on $\mathbb{R}$. For, if $\gamma(a, b) = \frac{1}{2}[u_1(a - b) + u_1(b - a)]$ then, by (0.10) and Bochner's theorem, γ is a covariance kernel. If Y_x, $x \in \mathbb{R}$, is the stationary zero mean Gaussian process with covariance γ, then $\|Y_a - Y_b\|_2^2 = 2\gamma(0,0) - 2\gamma(a,b) = \phi^2(b-a)$, from which it follows that d_ϕ is a metric.

The following combinatorial lemma is based on Lemma 6.1 of Jain and Marcus [16].

LEMMA 1.6. *Let ϕ and ρ be functions defined by* (0.12) *and* (1.2). *Let* $0 < a < b$, $\varepsilon > 0$, $x_0 \in \mathbb{R}$. *There exists a set* $A = A(x_0, a, b, \varepsilon)$ *containing* x_0 *such that if* $n = \#(A)$ *and* $A = \{x_0, x_1, \dots, x_{n-1}\}$, *then*

$$\frac{\rho(b)}{\rho(a)} \le n < \frac{\rho(a)}{\rho(b)} + 1, \tag{1.5}$$

$$\phi(x_i - x_j) \ge a \quad \text{for all } i \ne j, \tag{1.6a}$$

$$\phi(x_0 - x_j) \le b \quad \text{for all } 1 \le j \le n-1, \tag{1.6b}$$

$$\text{for each } i \text{ there exists } j_i \ne i \text{ with } \quad \phi(x_{j_i} - x_i) \le (1+\varepsilon)a. \tag{1.6c}$$

PROOF. Let

$$B_\phi(x, \delta) = \{y \colon \phi(x-y) < \delta\}$$

be the open ball in the ϕ-metric with centre x and radius δ. As ϕ is continuous $B_\phi(x,\delta)$ is also open in the usual metric. By (1.3),

$$|B_\phi(x, \delta)| = 2\rho(\delta).$$

Starting with x_0 we shall define inductively a sequence of points $\{x_0, x_1, \dots, x_k\}$ satisfying (1.6). Suppose that $x_0, \dots, x_k$ have been chosen, and let $C_k = B_\phi(x_0, b) - \bigcup_{i=1}^k B_\phi(x_i, a)$. Then, by the continuity of ϕ, if C_k is nonempty it is possible to choose $x_{k+1} \in C_k$ so that $\phi(x_{k+1}, x_j) \le a(1+\varepsilon)$ for some $1 \le j \le k$. Continue in this way until $C_{m-1} = \varnothing$. Then

$$B_\phi(x_0, b) \subseteq \bigcup_{i=0}^{m-1} B_\phi(x_i, a),$$

and so by (1.3), $2\rho(b) \le m 2\rho(a)$. Let n be the integer satisfying (1.5); we have $n \le m$ and therefore the set $A = \{x_0, \dots, x_{n-1}\}$ satisfies (1.5) and (1.6). □

REMARKS. 1. Using the fact that the balls $B_\phi(x_i, \frac{1}{2}a)$ must be disjoint, a similar argument shows that if A satisfies (1.6), then $\#A \le \rho(b + \frac{1}{2}a)/\rho(a)$.

2. Let b be fixed and let $N(a)$ denote the smallest number of d_ϕ-balls of radius a which cover $B = B_\phi(x_0, b)$ and let $H(a) = \log N(a)$, the d_ϕ-metric entropy of B. We have

$$\log\frac{1}{\rho(a)} - \log\frac{1}{\rho(b)} \le H(a) \le \log\frac{1}{\rho(\frac{1}{2}a)} - \log\frac{1}{\rho(b + \frac{1}{2}a)}.$$

Thus there is a close link between the function $\log 1/\rho(u)$ and metric entropy.

However, we will not need to use this as Lemma 1.6 enables us to work directly with the function $\log 1/\rho$.

We will be able to avoid some complications by reducing to the case of a recurrent process X. This is possible because while (interval) recurrence is a global property, and so is affected by the large jumps of X, these large jumps have no effect on the continuity of the local time.

Let X have exponent $\chi((\theta)$ and characteristics (a, σ^2, ν), and let $K > 1$. Define the truncated process $X_t(K)$ by

$$X_t(K) = X_t - \sum_{0 \le s \le t} \Delta X_s 1_{(|\Delta X_s| \ge K)};$$

then $X_t(K)$ is a Lévy process with characteristics $(a, \sigma^2, \nu^{(K)})$, where $\nu^{(K)}(dx) = 1_{(-K,K)}(x)\nu(dx)$. Expanding $\chi(\theta)$ in powers of θ, we deduce that

$$EX_t(K) = at + t\int_{1 \le |y| < K} y\nu(dy), \tag{1.7a}$$

$$\operatorname{var} X_t(K) = \sigma^2 t + t\int_{|y| < K} y^2\nu(dy). \tag{1.7b}$$

Now set

$$\nu_1(dx) = \nu^{\{K\}}(dx) + b_1\delta_{\{K\}}(dx) + b_2\delta_{\{-K\}}(dx), \tag{1.8}$$

where b_1, $b_2 \ge 0$ are chosen so that at most one of b_1, b_2 is nonzero and

$$b_1K - b_2K + at + \int_{1 \le |y| < K} y\nu(dy) = 0. \tag{1.9}$$

Let Z^1, Z^2 be Poisson processes with rates b_1, b_2, respectively, independent of X. (We can suppose that the probability space carrying X is large enough to support these extra processes.) Set

$$Y_t = X_t(K) + KZ_t^1 - KZ_t^2; \tag{1.10}$$

Y is a Lévy process with characteristics (a, σ^2, ν_1). Let $\chi_1(\theta)$ be the exponent of Y and let $\phi_1(x)$ be given by

$$\phi_1(x) = \frac{1}{\pi}\int (1 - \cos\theta x)\operatorname{Re}\frac{1}{1 + \chi_1(\theta)}\,d\theta.$$

PROPOSITION 1.7. *Let X and Y be as above.*

(a) *There exists a stopping time $S > 0$ such that $X = Y$ on $[0, S)$.*

(b) *There exist constants $c(K)$ (depending only on a, ν, K), with $c(K) \to 1$ as $K \to \infty$, such that*

$$\begin{aligned} c(K)^{-1}\operatorname{Re}\frac{1}{1 + \chi(\theta)} &\le \operatorname{Re}\frac{1}{1 + \chi_1(\theta)} \\ &\le c(K)\operatorname{Re}\frac{1}{1 + \chi(\theta)} \quad \textit{for all } \theta \in \mathbb{R}. \end{aligned} \tag{1.11}$$

(c) *Y is point recurrent.*
(d) $c(K)^{-1/2}\phi(x) \le \phi_1(x) \le c(K)^{1/2}\phi(x)$ *for all* $x \in \mathbb{R}$.
(e) $I(\bar{\phi}_1) < \infty$ *if and only if* $I(\bar{\phi}) < \infty$.

PROOF. For (a) it is enough to take $S = \inf\{t \ge 0: \max(|\Delta X_t|, |\Delta Y_t|) \ge K\}$. Since ν and ν_1 agree on $(-K, K)$, we have

$$\begin{aligned}|\chi(\theta) - \chi(\theta)| &\le \int_{|y|\ge K} |1 - e^{i\theta y}||\nu(dy) - \nu_1(dy)|\\ &\le 2\nu\big((-K, K)^c\big) + 2b_1 + 2b_2\\ &= \eta(K), \quad \text{say.}\end{aligned}$$

Using (1.9) and dominated convergence, we have $\eta(K) \to 0$ as $K \to \infty$. Writing $\eta = \eta(K)$, we have

$$\begin{aligned}\big||1+\chi|^2 - |1+\chi_1|^2\big| &\le |(1+\chi)^2 - (1+\chi_1)^2|\\ &= |\chi - \chi_1||2 + 2\chi_1 + \chi - \chi_1|\\ &\le 2\eta|1+\chi_1| + \eta^2\\ &\le (2\eta + \eta^2)|1+\chi_1|,\end{aligned}$$

since $\operatorname{Re}\chi_1 \ge 0$ and $|1+\chi_1| \ge 1 + \operatorname{Re}\chi_1 \ge 1$.

Hence

$$\begin{aligned}&\left|\operatorname{Re}\frac{1}{1+\chi} - \operatorname{Re}\frac{1}{1+\chi_1}\right| |1+\chi|^2\,|1+\chi_1|^2\\ &\quad = \big||1+\chi_1|^2(\operatorname{Re}\chi - \operatorname{Re}\chi_1) + (1+\operatorname{Re}\chi_1)\big(|1+\chi_1|^2 - |1+\chi|^2\big)\big|\\ &\quad \le |1+\chi_1|^2(3\eta + \eta^2).\end{aligned}$$

Writing $c(K) = 1 + 3\eta(K) + \eta(K)^2$, we have

$$\begin{aligned}\operatorname{Re}\frac{1}{1+\chi_1} &\le c(K)\frac{1}{|1+\chi|^2}\\ &\le c(K)\frac{(1+\operatorname{Re}\chi)}{|1+\chi|^2} = c(K)\operatorname{Re}\frac{1}{1+\chi}.\end{aligned}$$

Interchanging χ and χ_1 in these calculations gives the other side of (1.10).

From (1.7)–(1.10) we have $EY_t^2 < \infty$, $EY_t = 0$, for all $t \ge 0$. Hence Y_n, $n \ge 0$, is interval recurrent (see [11, Theorem 3.1]). The estimate (1.11) shows that Y satisfies (0.2) and (0.3), and therefore Y is point recurrent by [11, Corollary 5.4].

(d) and (e) now follow easily from (1.11), the definition of ϕ_1 and the properties of monotone rearrangements. □

REMARK. From the construction of S, we have that S has a negative exponential distribution and $ES = (b_1 + b_2 + \nu((-K, K)^c))^{-1} = 2/\eta(K)$. So $P(S > t) = \exp(-\frac{1}{2}t\eta(K)) \to 1$ as $K \to \infty$.

The final result in this section is a rather long analytic lemma, which will be used in Section 4 with $H(u) = \log 1/\rho(u)$. If $H(u)$ is slowly varying at 0 the result is quite easy, but the general case requires some work.

LEMMA 1.8. *Let $H(u)$ be a decreasing positive function satisfying*

$$\int_0^1 H(u)^{1/2}\,du = +\infty. \tag{1.12}$$

Let $\delta, K, \lambda, \theta > 0$. Then there exist sequences $(a_n), (b_n), (\mu_n)$ decreasing to 0 such that $\mu_0 = \lambda$, and

$$b_{n+1} < a_n < 2a_n < b_n \quad \textit{for all } n, \tag{1.13}$$

$$\sum_{n=1}^{\infty} \exp \alpha_n < \delta, \tag{1.14}$$

$$\sum_{n=1}^{\infty} a_n (H(a_n) - H(b_n))^{1/2} = \infty, \tag{1.15}$$

$$H(a_n) - H(b_n) \geq K, \tag{1.16}$$

where

$$\alpha_n = \sum_{j=1}^{n-1} H(a_j) - \frac{\theta^2}{b_n^2}\left[\mu_n \wedge a_n^2(H(a_n) - H(b_n))\right]. \tag{1.17}$$

PROOF. We begin by noting that it is enough to find $(a_n), (b_n), (\mu_n)$ such that $\sum \exp \alpha_n < \infty$. For if $m \geq 1$ and $a'_n = a_{n+m}$, $b'_n = b_{n+m}$ and $\mu'_n = \mu_n$, then $\alpha'_n \leq \alpha_{n+m}$ and (1.14) follows by taking m sufficiently large.

We consider two cases:

Case A. $\limsup_{u \to 0} u^2 H(u) > 0$.
Case B. $\limsup_{u \to 0} u^2 H(u) = 0$.

Case A is fairly straightforward. There exists an $\eta > 0$ and a sequence $x_n \downarrow 0$ such that $x_n^2 H(x_n) > \eta$ for all n. Set $\mu_n = \eta/n$, and choose b_1 such that $H(b_1) \geq 2K$. Now define $(a_n), (b_n)$ inductively as follows. Suppose $b_1, \dots, b_n, a_1, \dots, a_{n-1}$ have been chosen. Let a_n be the largest x_i such that

$$x_i < \tfrac{1}{2} b_n, \qquad H(x_i) > 2H(b_n),$$

and now let b_{n+1} be the largest 2^{-i}, $i \geq 1$, such that

$$\frac{\mu_{n+1}\theta^2}{(2^{-i})^2} > (n+1) + \sum_{r=1}^{n} H(a_r). \tag{1.18}$$

Such a choice of $(a_n), (b_n)$ is clearly possible. Then $H(a_n) - H(b_n) \geq \frac{1}{2}H(a_n) \geq \frac{1}{2}H(b_1) \geq K$, so that (1.16) holds. Also as $H(a_n) \geq \eta a_n^{-2}$, $a_n(H(a_n) - H(b_n))^{1/2} \geq (\frac{1}{2}\eta)^{1/2}$ and (1.15) holds.

This inequality also implies that

$$\alpha_n \le \sum_{j=1}^{n-1} H(a_j) - \frac{\theta^2}{b_n^2}\mu_n \le -n, \tag{1.19}$$

by the choice of b_n and so (1.14) is satisfied.

Case B is harder. We begin with some minor simplifications. It is enough to prove the result with $\mu_n = \lambda$ for all n, for then setting $\mu_n = a_n^2 H(a_n)$ we do not affect the value of α_n and have $\mu_n \to 0$. Further, there exists $u_0 > 0$ such that $u^2 H(u) < \lambda$ for $0 \le u \le u_0$, and so, taking $b_1 < u_0$, we have

$$\alpha_n = \sum_{j=1}^{n-1} H(a_j) - \theta^2 \frac{a_n^2}{b_n^2}(H(a_n) - H(b_n)). \tag{1.20}$$

Thus it is sufficient to take α_n as given by (1.20) and find sequences $(a_n), (b_n)$ satisfying (1.13), (1.14), (1.15) and (1.16).

Set $g(u) = e^{-u} H(e^{-u})^{1/2}$, $0 \le u < \infty$. Then $H(v) = v^{-2} g(\log 1/v)^2$ and (1.12) implies that

$$\int_0^\infty g(u)\, du = \infty. \tag{1.21}$$

As H is decreasing we have, for any $x, u > 0$,

$$g(x+u) \ge e^{-x-u} H(e^{-x-u})^{1/2} \ge e^{-u} e^{-x} H(e^{-x})^{1/2} = e^{-u} g(x). \tag{1.22}$$

Let $I_n = \int_n^{n+1} g(u)\, du$. We have, using (1.22),

$$I_n = \int_0^1 g(n+u)\, du \ge g(n) \int_0^1 e^{-u}\, du = (1 - e^{-1}) g(n)$$

and

$$I_n = \int_0^1 g((n+1) + u - 1)\, du \le \int_0^1 g(n+1) e^{1-u}\, du \le e g(n+1).$$

So

$$(1 - e^{-1}) g(n) \le I_n \le e g(n+1), \tag{1.23}$$

$$\sum_{n=1}^{\infty} g(n) = \infty. \tag{1.24}$$

Now set $\eta = \frac{1}{2}$ and let

$$J = \{n : g(n+1) \ge e^{-\eta} g(n)\}. \tag{1.25}$$

If J were finite then, taking $r \ge \max\{n : n \in J\}$,

$$\sum_{n=r}^{\infty} g(n) \le \sum_{n=r}^{\infty} e^{-\eta(n-r)} g(r) < \infty,$$

which contradicts (1.24). Thus J is infinite. We will now show that

$$\sum_{n \in J} g(n+1) = \infty. \tag{1.26}$$

As $g(n+1) \geq e^{-1}g(n)$ this is immediate if $\sum_{n \in J} g(n) = \infty$. So suppose that $\sum_{n \in J} g(n) < \infty$. Then J^c must be infinite and $\sum_{n \in J^c} g(n) = \infty$. Let

$$J^c = \bigcup_{k=1}^{\infty} \{r_k, r_k+1, \dots, r_k+s_k\},$$

where $r_k < r_k + s_k < r_{k+1} - 1$ for each k. If $n \in J^c$ then $g(n+1) \leq e^{-\eta}g(n)$, and so

$$\begin{aligned}\sum_{n \in J^c} g(n) &= \sum_{k=1}^{\infty} \sum_{m=0}^{s_k} g(r_k+m)\\ &\leq \sum_{k=1}^{\infty} g(r_k) \sum_{m=0}^{\infty} e^{-\eta m} \leq c \sum_{k=1}^{\infty} g(r_k).\end{aligned}$$

Thus $\sum_k g(r_k) = \infty$. As $r_k - 1 \in J$ for $k \geq 2$ this proves (1.26).

Let $J = \{n_1, n_2, \dots\}$, with $n_i < n_{i+1}$ for all i. Choose $k \geq 1$ and, for $0 \leq s \leq k-1$, let $J_s = \{n_{rk+s}, r \geq 1\}$. By (1.26) there is at least one s, s_0, say, such that $\sum_{n \in J_{s_0}} g(n+1) = \infty$. Let y be large enough so that $H(u) \geq 2K$ for $0 \leq u \leq y$, let $J' = J_{s_0} \cap [y, \infty)$ and write $J' = \{m_1, m_2, \dots\}$, where $m_i < m_{i+1}$. We have

$$\sum_{i=1}^{\infty} g(m_i+1) = \infty, \qquad m_{i+1} - m_i \geq k. \tag{1.27}$$

Now define

$$a_i = e^{-(m_i+1)}, \qquad b_i = e^{-m_i}.$$

Thus $b_{i+1} < a_i < 2a_i < ea_i = b_i$ so that $(a_n), (b_n)$ satisfy (1.13). Since $m_i \in J$, $g(m_i+1) \geq e^{-\eta}g(m_i)$. We have

$$H(b_i) = e^{2m_i}g(m_i)^2, \qquad H(a_i) = e^{2+2m_i}g(m_i+1)^2,$$

and thus

$$\begin{aligned}H(a_i) - H(b_i) &= e^{2m_i}\left[e^2 g(m_{i+1})^2 - g(m_i)^2\right]\\ &\geq e^{2m_i}g(m_i+1)^2\left[e^2 - e^{2\eta}\right]\\ &= H(a_i)(1 - e^{2\eta-2}) = H(a_i)(1 - e^{-1}).\end{aligned} \tag{1.28}$$

So $a_i(H(a_i) - H(b_i))^{1/2} \geq ca_i H(a_i)^{1/2} = g(m_i+1)$, and hence (1.15) follows from (1.27). As $H(a_i) \geq 2K$ (1.28) implies (1.16).

It remains to prove that (1.14) holds. By the construction of J' the set $\{m_i+1, \dots, m_{i+1}\}$ contains k elements of J and $s_i = m_{i+1} - m_i - k$ elements of J^c. Therefore,

$$g(m_{i+1}+1) \geq e^{-s_i}e^{-\eta k}g(m_i+1).$$

We have

$$\begin{aligned}H(a_{i+1}) &= e^{2+2m_{i+1}}g(m_{i+1}+1)^2\\ &\geq e^{2+2m_i}e^{2(m_{i+1}-m_i)}e^{-2s_i-2\eta k}g(m_i+1)^2\\ &= H(a_i)e^{2k(1-\eta)} = H(a_i)e^k.\end{aligned}$$

Thus $\sum_{j=1}^{n-1} H(a_j) \le H(a_n)\sum_{j=1}^{n-1} e^{-k(n-j)} \le H(a_n)(e^k - 1)^{-1}$. By (1.20) and (1.28),

$$\alpha_n \le H(a_n)\left[\frac{1}{e^k - 1} - \frac{\theta^2}{e^2}(1 - e^{-1})\right].$$

Now choose k large enough so that the term in square brackets is negative; then $\alpha_n \le -cH(a_n) \le -ce^{nk}H(a_1)$, and we have proved (1.14). □

2. Potential theory estimates. Recall from the introduction the notation

$$h_\alpha(x, A) = E^x \int_0^{T_A} e^{-\alpha s}\, dL_s^x \tag{2.1}$$

and let $v_\alpha(x) = u_\alpha(0) - u_\alpha(x)$. In this section we obtain estimates for h in terms of $v_1(x)$ and $\phi(x)$.

LEMMA 2.1. *For $x, y \in \mathbb{R}$, we have*

$$\begin{aligned} h_1(x, y) &= u_1(0)(1 - \psi_1(x - y)\psi_1(y - x)) \\ &= \phi^2(x - y) - u_1(0)^{-1}(v_1(y - x)v_1(x - y)). \end{aligned} \tag{2.2}$$

PROOF. By the strong Markov property of X and (0.9),

$$u_1(0) = E^x \int_0^{T_y} e^{-s}\, dL_s^x + \psi_1(y - x)\psi_1(x - y)u_1(0).$$

So

$$h_1(x, y) = u_1(0)(1 - \psi_1(x - y)\psi_1(y - x)),$$

which gives the first half of (2.2). The second half is obtained by the substitutions $\psi_1(z) = 1 - u_1(0)^{-1}v_1(z)$, $\phi^2(z) = v_1(z) + v_1(-z)$. □

LEMMA 2.2. *For $x \in \mathbb{R}$, $A \in \mathscr{B}(\mathbb{R})$,*

$$h_1(x, A) \le h(x, A) \le \frac{h_1(x, A)}{e^x e^{-T_A}}. \tag{2.3}$$

PROOF. The left-hand inequality is evident. Let R be a negative exponential random variable, with mean 1, independent of X. Then

$$h_1(x, A) = E^x L_{R \wedge T_A}^x, \qquad P^x(R > T_A) = E^x e^{-T_A}.$$

Therefore, since $h(x, A) = E^x L_{T_A}^x$, we have

$$\begin{aligned} h(x, A) - h_1(x, A) &= E^x 1_{(R \le T_A)}\left(L_{T_A}^x - L_R^x\right) \\ &= E^x 1_{(R \le T_A)} E^{X_R} L_{T_A}^x \\ &\le E^x 1_{(R \le T_A)} E^x L_{T_A}^x \\ &= (1 - E^x e^{-T_A})h(x, A), \end{aligned}$$

giving the right-hand inequality. □

COROLLARY 2.3.

(a)
$$\phi^2(x-y)\left(1-\frac{\phi^2(x-y)}{u_1(0)}\right)$$
$$\le h_1(x,y)$$
$$\le h(x,y) \le \phi^2(x-y)\left(1-\frac{\phi^2(x-y)}{u^1(0)}\right)^{-1}.$$

(b)
$$h_1(x,y) \le \phi^2(x-y).$$

(c)
$$\frac{h(x,y)}{\phi^2(x-y)} \to 1 \quad as\ |x-y| \to 0.$$

PROOF. Let $z = y - x$. Since $v(z)v(-z) \le \phi^4(z)$, the left-hand inequality in (a) follows from (2.2), while the middle inequality is clear. Now $u_1(z) \ge u_1(0) - \phi^2(z)$, and therefore $\psi_1(z)^{-1} \le (1 - \phi^2(z)/u_1(0))^{-1}$. So, by (2.3) and (2.2),

$$\begin{aligned} h(x,y) &\le \psi_1^{-1}(z)h_1(x,y) \\ &\le \psi_1^{-1}(z)\phi^2(z) \\ &\le \phi^2(z)\left(1-\phi^2(z)/u^1(0)\right)^{-1}, \end{aligned}$$

which proves the right-hand inequality.

(b) and (c) are immediate from Lemma 2.1 and (a). □

Let us now suppose that X is recurrent. Let $A \subseteq B$ be Borel sets in $\mathbb{R}$ with $0 \notin A$. Let $p = P^0(T_A < T_0 \circ \theta_{T_B})$; p is the probability that an excursion of X from 0 which hits B also hits A. If n is the characteristic measure of the Poisson point process of excursions from 0 and, for $A \in \mathscr{B}(\mathbb{R})$, $m(A)$ is the n-measure of the set of excursions which hit A, then $p = m(A)/m(B)$. We also have $h(0, A) = m(A)^{-1}$.

Let $\hat{X} = -X$ be the dual process of X (relative to dx). If $\cdot$ is some quantity associated with X [like the measures P^x or the function $u_1(x)$], then $\hat{\cdot}$ will denote the same quantity for $\hat{X}$. The semigroups (P_t) and $(\hat{P}_t)$ are in duality relative to dx.

Because X is recurrent, $\hat{X}$ is also recurrent. Also, a.s., all the excursions from 0 begin and end at 0. (This is an easy consequence of the spatial homogeneity of X and the fact that X has only countably many jumps.) Therefore, by a theorem of Getoor and Sharpe [13] (see also [17] and [23] for the case when X does not have a transition density), the excursions of $\hat{X}$ have the same law as the time reversal of the excursions of X. An excursion hits A if and only if its time reversal does, and so $\hat{m}(A) = m(A)$ for all $A \subseteq \mathbb{R}$. Thus

$$\hat{h}(0, A) = h(0, A) \tag{2.4}$$

and

$$\hat{P}^0(T_A < T_0 \circ \theta_{T_B}) = P^0(T_A < T_0 \circ \theta_{T_B}). \tag{2.5}$$

REMARK. By considering marked excursions we can also show that $\hat{h}_\alpha(x, A) = h_\alpha(x, A)$, but we will not need this.

PROPOSITION 2.4. *Let A be a closed set and $x \notin A$. Set $\mu = \inf\{\phi(x - y), y \in A\}$. Then*

$$\tfrac{1}{4}\mu^2 \le h(x, A). \tag{2.6}$$

PROOF. It is sufficient to deal with the case $x = 0$. Suppose first that X is recurrent. Set

$$B = \{y \in A: v_1(y) > v_1(-y)\},$$
$$C = \{y \in A: v_1(y) = v_1(-y)\},$$
$$D = \{y \in A: v_1(y) < v_1(-y)\}.$$

By (0.8),

$$\begin{aligned} u_1(0) &= h_1(0, A) + E^0 e^{-T_A} E^{X_{T_A}} \int_0^\infty e^{-s}\, dL_s^0 \\ &\le h_1(0, A) + E^0 E^{X_{T_A}}\Big[\big(1_{(T_{D\cup C} < T_0)} + 1_{(T_{D\cup C} > T_0)}\big) u_1(-X_{T_A})\Big] \\ &\le h_1(0, A) + E^0 E^{X_{T_A}}\Big[1_{(T_{D\cup C} < T_0)} u_1(-X_{T_A}) + 1_{(T_{D\cup C} < T_0)} u_1(0)\Big]. \end{aligned}$$

Therefore writing $S = T_0 \circ \theta_{T_A}$,

$$\begin{aligned} h_1(0, A) &\ge E^0\big(1_{(T_{D\cup C} < S)} v_1(-X_{T_{D\cup C}})\big) \\ &\ge \tfrac{1}{2}\mu^2 P^0(T_{D\cup C} < S), \end{aligned} \tag{2.7}$$

as $v_1(y) \ge \frac{1}{2}[v_1(y) + v_1(-y)] \ge \frac{1}{2}\mu^2$ for $y \in D \cup C$.

We have $\hat{v}_1(x) = v_1(-x)$, $\hat{h}(x, A) = h(x, A)$ and so, writing $\hat{B} = \{y \in A: \hat{v}_1(y) > \hat{v}_1(-y)\} = D$, $\hat{C} = C$, $\hat{D} = B$, and repeating the proof of (2.7) for $\hat{X}$, we obtain

$$\hat{h}_1(0, A) \ge \tfrac{1}{2}\mu^2 \hat{P}^0(T_{\hat{D}\cup\hat{C}} < S).$$

Now

$$\begin{aligned} \hat{P}^0(T_{\hat{D}\cup\hat{C}} < S) &= \hat{P}^0(T_{B\cup C} < S) \quad (\text{as } \hat{D} = B, \hat{C} = C) \\ &= P^0(T_{B\cup C} < S) \quad [\text{by (2.5)}]. \end{aligned}$$

As $B \cup C \cup D = A$, $P^0(T_{B\cup C} < S) + P^0(T_{D\cup C} < S) \ge 1$ and therefore, since $h(0, A) = \hat{h}(0, A) \ge \max(h_1(0, A), \hat{h}_1(0, A))$, we have $h(0, A) \ge \frac{1}{4}\mu^2$.

Now suppose that X is not recurrent. Let $K \geq 1$, let $Y^{(K)}$ be the recurrent process constructed in Section 1 and let $u_1^{(K)}$, $\phi^{(K)}$ and $h^{(K)}$ be the functions u_1, ϕ and h for $Y^{(K)}$. Then $h_1^{(K)}(0, A) \geq \frac{1}{4}\inf\{\phi^{(K)}(y), y \in A\}$ for each $K \geq 1$. Now if $S = S^{(K)}$ is the stopping time given in Proposition 1.7(a) and $c(K)$, $\eta(K)$ are the constants given there, then

$$\begin{aligned}
|h_1^{(K)}(0, A) - h_1(0, A)| &= \left| E^0 1_{(T_A > S)} \left(\int_S^{T_A} e^{-s}\, dL_s^0(X) - \int_S^{T_A(Y)} e^{-s}\, dL_s^0(Y) \right) \right| \\
&\leq E^0 e^{-S} E^{X_S} \int_0^\infty e^{-s} \left(dL_s^0(Y) + dL_s^0(X) \right) \\
&\leq E^0 e^{-S} \left(u_1^{(K)}(0) + u_1(0) \right) \\
&\leq c(K) u_1(0) \left(\eta(K)/(2 + \eta(K)) \right).
\end{aligned}$$

Letting $K \to \infty$, we deduce (2.6). □

PROPOSITION 2.5. *Let A be a closed subset of $\mathbb{R}$, and let $x, y \notin A$. Suppose that for some $K > 6$,*

$$\phi^2(x - z) \geq K\phi^2(x - y), \qquad \phi^2(y - z) \geq K\phi^2(x - y) \quad \textit{for all } z \in A.$$

Then if $\phi^2(x - y) \leq \frac{1}{5}u_1(0)$,

$$\left(1 - \frac{5}{K}\right) h_1(x, y) \leq h(x, A \cup \{y\}) \leq h(x, y).$$

PROOF. Let $F = \{T_y < T_A\}$. Then

$$\begin{aligned}
h(x, A) - h(x, y) &= E^x 1_F \int_{T_y}^{T_A} dL_s^x - E^x 1_{F^c} \int_{T_A}^{T_y} dL_s^x \\
&\leq E^x \left(1_F E^y \int_0^{T_A} dL_s^x \right) \leq P^x(F) h(x, A).
\end{aligned}$$

Thus writing $\varepsilon = \phi^2(x - y)/u_1(0)$, we have

$$\begin{aligned}
P^x(F) &\geq \frac{h(x, A) - h(x, y)}{h(x, A)} \\
&\geq 1 - \frac{\phi^2(x - y)(1 - \varepsilon)^{-1}}{h(x, A)} \geq 1 - \frac{4}{K(1 - \varepsilon)} \geq 1 - \frac{5}{K},
\end{aligned}$$

because by Proposition 2.4, $h_1(x, A) \geq \frac{1}{4}K\phi^2(x - y)$.

Let $B = A \cup \{y\}$; we have

$$\begin{aligned}
h_1(x, y) - h_1(x, B) &= E^x \int_{T_B}^{T_y} e^{-s}\, dL_s^x \\
&= E^x 1_{F^c} e^{-T_B} E^{X_{T_B}} \int_0^{T_y} e^{-s}\, dL_s^x \leq P^x(F^c) h_1(x, y).
\end{aligned}$$

Hence

$$h_1(x, B) \geq P^x(F)h_1(x, y) \geq \left(1 - \frac{5}{K}\right)h_1(x, y),$$

proving the left-hand inequality; the other one is evident since $T_y \geq T_B$. □

3. Basic estimates on the local time. Let A be a finite set in $\mathbb{R}$, let $n = \#(A)$ and let $A = \{x_0, x_1, \dots, x_{n-1}\}$. Set

$$(3.1) \qquad h_i = h(x_i, A - \{x_i\}), \qquad 1 \leq i \leq n-1,$$
$$d = \min_i h_i, \qquad D = \max_{1 \leq i \leq n-1} h(x_0, x_i).$$

We will assume A satisfies

$$(3.2) \qquad \#(A) = n \geq 20.$$

Let λ, ε and η be strictly positive constants satisfying

$$(3.3) \qquad \varepsilon < \tfrac{1}{10},$$

$$(3.4) \qquad \lambda > 10 \max_i h_i,$$

$$(3.5) \qquad \eta \leq \varepsilon \lambda^{1/2}.$$

We begin by time-changing X to reduce it to a Markov chain on A. Let $V_t = \sum_{x \in A} L_t^x$, and let $\alpha_t = \inf\{s: V_s > t\}$ be the right continuous inverse of V. Let ζ be a cemetary state, and set $\tilde{X}_t = X_{\alpha_t}$ if $\alpha_t < \infty$, $\tilde{X}_t = \zeta$ if $\alpha_t = \infty$. Then $\tilde{X}$ is a Markov chain on $A \cup \{\zeta\}$, and if $Q = (q_{x_i x_j})$ is the Q-matrix of $\tilde{X}$ we have $q_{x_i x_i} = -h_i$ for $1 \leq i \leq n-1$.

Let Y_r, $r \geq 0$, be the jump chain of $\tilde{X}$ and let $M_j = \sum_{r=0}^{\infty} 1_{(Y_r = x_j)}$ be the total number of visits by Y to x_j. For $0 \leq i \leq n-1$, $1 \leq j \leq M_j$, let V_{ij} be the duration of the jth visit by $\tilde{X}$ to x_i; for $0 \leq i \leq n-1$, $j > M_j$, let (V_{ij}) be independent negative exponential random variables with $EV_{ij} = h_i$.

Set

$$\mathscr{H} = \sigma(Y_r, r \geq 0),$$
$$\mathscr{G}_i = \sigma(V_{ij}, j \geq 1),$$
$$T = \inf\left\{t \geq 0: \int_0^t 1_{(\tilde{X}_s = x_0)}\, ds > \lambda\right\},$$

and let $N(x_i)$, $0 \leq i \leq n-1$, be the number of visits made by $\tilde{X}$ to x_i before time T. Let $\tau_\lambda = \tau_\lambda(x_0)$ be the inverse of $L_t^{x_0}$.

LEMMA 3.1. (i) $\mathscr{H}, \mathscr{G}_i$, $i = 0, \dots, n-1$, *are independent.*

(ii) V_{ij} *are independent negative exponential r.v. and* $EV_{ij} = h_i$.

(iii) $L_{\tau_\lambda}^{x_i} = \sum_{j=1}^{N(x_i)} V_{ij}$ *for* $0 \leq i \leq n-1$.

(iv) $N(x_i) \in \mathscr{H} \vee \mathscr{G}_0$ *for* $0 \leq i \leq n-1$ *and* $\sigma(V_{ij}, 1 \leq i \leq n-1, j \geq 1)$ *is independent of* $\mathscr{H} \vee \mathscr{G}_0$.

PROOF. This is essentially immediate from the construction of $\tilde{X}$, Y and the V_{ij}. □

Now set

$$\lambda_0 = \lambda - \eta\lambda^{1/2}, \qquad \lambda_1 = \lambda + \eta\lambda^{1/2}, \qquad r_0(x_i) = \frac{\lambda_0}{h_i}, \qquad r_1(x_i) = \frac{\lambda_1}{h_i}.$$

Note that by (3.2)–(3.5) we have, for $0 \le i \le n-1$,

$$r_0(x_i) \ge \frac{\lambda(1-\varepsilon)}{\max_i h_i} \ge 10(1-\varepsilon) \ge 9, \tag{3.6}$$

and similarly $r_1(x_i) \ge 10$. Note also that $1 - \varepsilon < \lambda_0/\lambda < 1 < \lambda_1/\lambda < 1 + \varepsilon$.

Let $\mathcal{J}$ be the random subset of A defined by

$$\mathcal{J} = \{x_i, 1 \le i \le n-1 : r_0(x_i) < N(x_i) < r_1(x_i)\} \tag{3.7}$$

and let

$$B = \{\#(\mathcal{J}) < \tfrac{4}{5}n - 1\}. \tag{3.8}$$

$\mathcal{J}$ is the set of "good" x_i, on which we have reasonable control over $N(x_i)$; B is the "bad" event that $\mathcal{J}$ is too small. We begin by obtaining a bound on the size of B.

Set $H = \{\tau_\lambda < \infty\}$.

LEMMA 3.2. $P(B \cap H) < c_2 \exp(-\eta^2/5D)$.

PROOF. Let $\xi = n - 1 - \#(\mathcal{J})$, so that $B = \{\xi \ge n/5\}$. Thus

$$\begin{aligned} P(B \cap H) &\le \left(\frac{n}{5}\right)^{-1} E(\xi 1_H) \\ &\le \frac{5}{n} \sum_{i=1}^{n-1} P(N(x_i) \le r_0(x_i) \text{ or } N(x_i) \ge r_1(x_i), H) \\ &\le 5 \max_{1 \le i \le n-1} P(N(x_i) \le r_0(x_i), H) \\ &\quad + 5 \max_{1 \le i \le n-1} P(N(x_i) \ge r_1(x_i), H). \end{aligned} \tag{3.9}$$

Let $1 \le i \le n-1$ and write $x = x_i$, $h = h_i$. By Lemma 1.3(i) we have

$$\begin{aligned} P(L^x_{\tau_\lambda} < \lambda_0, H) &= P(\lambda - L^x_{\tau_\lambda} > \eta\lambda^{1/2}, H) \\ &\le P(L^{x_0}_{\tau_\lambda} - L^x_{\tau_\lambda} > \eta\lambda^{1/2}) \\ &\le \exp\left(\frac{-\eta^2}{4h(x_0, x)}\right) \\ &\le \exp\left(-\frac{\eta^2}{4D}\right). \end{aligned} \tag{3.10}$$

Similarly, by Lemma 1.3(ii),

$$P\big(L^x_{\tau_\lambda} > \lambda_1\big) = P\big(L^x_{\tau_\lambda} - \lambda > \eta\lambda^{1/2}, H\big) \le P\big(L^x_{\tau_\lambda} - L^{x_0}_{\tau_\lambda} > \eta\lambda^{1/2}\big)$$

$$\le \exp\left(\frac{-\eta^2\lambda}{4\lambda_1 h(x_0, x)}\right) \le \exp\left(\frac{-\eta^2}{4(1+\varepsilon)D}\right) \le \exp\left(\frac{-\eta^2}{5D}\right). \tag{3.11}$$

We also have

$$P\big(L^x_{\tau_\lambda} < \lambda_0, H\big) \ge P\big(L^x_{\tau_\lambda} < \lambda_0 | N(x) \le r_0(x), H\big) \times P\big(N(x) \le r_0(x), H\big). \tag{3.12}$$

Using Lemma 3.1, the second term in (3.12) is

$$P\left(\sum_{j=1}^{N(x)} V_{ij} < \lambda_0 | N(x) \le r_0(x)\right) \ge P\left(\sum_{1 \le j \le r_0(x)} V_{ij} < \lambda_0\right) \ge P(Z_0 < \lambda_0), \tag{3.13}$$

where Z_0 has a $\Gamma(r_0(x), h)$ distribution. By Lemma 1.1, as $r_0(x) > 1$,

$$P(Z_0 < \lambda_0) = P(Z_0 \le r_0(x)h) = \gamma^-(r_0(x), 0) \ge c_0. \tag{3.14}$$

Combining the estimates (3.10) and (3.12)–(3.14) we have, for $1 \le i \le n-1$,

$$P\big(N(x_i) \le r_0(x_i), H\big) \le c_0^{-1}\exp(-\eta^2/4D). \tag{3.15}$$

A similar argument works for the other estimate. We have

$$P\big(L^x_{\tau_\lambda} > \lambda_1, H\big) \ge P\left(\sum_{j=1}^{N(x)} V_{ij} > \lambda_1 | N(x) \ge r_1(x), H\right) P\big(N(x) \ge r_1(x), H\big)$$

and

$$P\left(\sum_{j=1}^{N(x)} V_{ij} > \lambda_1 | N(x) \ge r_1(x), H\right) \ge P(Z_1 > \lambda_1),$$

where Z_1 has a $\Gamma(\tilde r_1(x), h)$ distribution and $\tilde r_1(x) = [r_1(x)]$.

Now

$$\begin{aligned} P(Z_1 > \lambda_1) &= P\big(Z_1 > \tilde r_1(x)h + (r_1(x) - \tilde r_1(x))h\big) \\ &\ge P\big(Z_1 > \tilde r_1(x)h + h\big) \\ &= \gamma^+(\tilde r(x), 1) \ge c_0 \end{aligned}$$

by Lemma 1.1. From these estimates we obtain, for $1 \le i \le n-1$,

$$P\big(N(x_i) \ge r_1(x_i), H\big) \le c_0^{-1}\exp(-\eta^2/5D). \tag{3.16}$$

The lemma now follows from (3.9, (3.14) and (3.15). □

Let Λ_i, $1 \le i \le n-1$, be intervals in $\mathbb{R}$, let $z > 0$ and let

$$p(x_i, \beta, \Lambda_i) = P(Z(\beta, h_i) \in \Lambda_i),$$

$$q(\beta_1, \beta_2, h_1, h_2, x) = P(|Z(\beta_1, h_1) - Z'(\beta_2, h_2)| > x),$$

$$F = \{\tilde{\omega}: L^{x_i}_{\tau_\lambda} \in \Lambda_i \text{ for some } 1 \le i \le n-1\},$$

$$Y_i = \left| \sum_{j=1}^{N(x_{i-1})} V_{i-1,j} - \sum_{j=1}^{N(x_i)} V_{ij} \right|,$$

$$G = \{\omega: |Y_{2i}| > z \text{ for some } 1 \le i \le \tfrac{1}{2}(n-1)\}.$$

PROPOSITION 3.3. (a) *Suppose that*

$$0 < \alpha_0 < p(x_i, \beta, \Lambda_i) \quad \text{for } 1 \le i \le n-1,\ r_0(x_i) \le \beta \le r_1(x_i).$$

Then

$$P(F^c \cap \{\tau_\lambda < \infty\}) \le \exp\left(-\frac{1}{2} n\alpha_0\right) + c_2 \exp\left(\frac{-\eta^2}{5D}\right). \tag{3.17}$$

(b) *Suppose that*

$$0 < \alpha_1 < q(\beta_{2i-1}, \beta_{2i}, h_{2i-1}, h_{2i}, z)$$

$$\text{for } 1 \le i \le \frac{n-1}{2},\ r_0(x_j) < \beta_j < r_1(x_j),\ 1 \le j \le n.$$

Then

$$P(G^c \cap \{\tau_\lambda < \infty\}) \le \exp\left(-\frac{1}{5} n\alpha_1\right) + c_2 \exp\left(\frac{-\eta^2}{5D}\right). \tag{3.18}$$

PROOF. Set $\xi_i = 1_{\Lambda_i}(L^{x_i}_{\tau_\lambda})$, so that

$$1_{H \cap F^c} = 1_H \prod_{i=1}^{n-1} (1 - \xi_i).$$

As H, B, J are $\mathscr{H} \vee \mathscr{G}_0$-measurable, we have

$$\begin{aligned} P(H \cap B^c \cap F^c) &= E\, 1_{B^c} 1_H \prod_{i=1}^{n-1} (1 - \xi_i) \\ &\le E\, 1_{B^c} 1_H \prod_{i \in J} (1 - \xi_i) \\ &= E\left[1_{B^c} 1_H E\left(\prod_{1 \in J} (1 - \xi_i) | \mathscr{H} \vee \mathscr{G}_0\right)\right]. \end{aligned}$$

Since $L^{x_i}_{\tau_\lambda} = \sum_{j=1}^{N(x_i)} V_{ij}$, the r.v. ξ_i are independent, conditional on $\mathscr{H} \vee \mathscr{G}_0$, and by

Lemma 3.1(ii),

$$1_H E(1-\xi_i|\mathscr{H}\vee\mathscr{G}_0) = \big(1-p(x_i, N(x_i), \Lambda_i)\big)1_H$$
$$\le \big(1_B + 1_{B^c}(1-\alpha_0)\big)1_H.$$

Hence

$$1_{H\cap B^c} E\Big(\prod_{i\in J}(1-\xi_i)|\mathscr{H}\vee\mathscr{G}_0\Big) \le 1_{H\cap B^c}(1-\alpha_0)^{\#J}$$

and so

$$\begin{aligned} P(H\cap F^c) &\le P(H\cap B) + P(H\cap B^c\cap F)\\ &\le P(H\cap B) + E\,1_{H\cap B^c}(1-\alpha_0)^{\#J}\\ &\le P(H\cap B) + (1-\alpha_0)^{n/2} \quad \text{as } \#J > \tfrac{1}{2}n \text{ on } B^c\\ &\le P(H\cap B) + \exp\big(-\tfrac{1}{2}n\alpha_0\big). \end{aligned}$$

(a) now follows immediately.

We now turn to (b). Set $\xi_i = 1_{(Y_i > z)}$ so that

$$1_{G_c} = \prod_{1\le i\le (n-1)/2}(1-\xi_{2i}).$$

By the same argument as in (a),

$$P(H\cap B^c\cap G^c) \le E\left[1_{B^c\cap H} E\left[\prod_{2i\in K}(1-\xi_{2i})|\mathscr{H}\vee\mathscr{G}_0\right]\right],$$

where $K = \{2i: \{2i-1, 2i\}\subseteq J, 1\le i\le (n-1)/2\}$. On B^c we have $\#K\ge n/5$ and

$$1_{H\cap B^c}E(1-\xi_{2i}|\mathscr{H}\vee\mathscr{G}_0) \le (1-\alpha_1)1_{H\cap B^c}.$$

So

$$\begin{aligned} P(H\cap G^c) &\le P(H\cap B) + E\big[1_{H\cap B^c}(1-\alpha_1)^{n/5}\big]\\ &\le P(H\cap B) + \exp\big(-\tfrac{1}{5}n\alpha_1\big), \end{aligned}$$

proving (3.18). □

LEMMA 3.4. *Let* $0 < z < \varepsilon\lambda^{1/2}$. *If* Λ_i *is either* $[0, \lambda - z\lambda^{1/2}]$ *or* $[\lambda + z\lambda^{1/2}, \infty)$, *then for each* $1\le i\le n-1$, $r_0(x_i)\le\beta\le r_1(x_i)$,

$$p(x_i, \beta, \Lambda_i) \ge c_3(\varepsilon)\exp\left(-\frac{1+5\varepsilon}{2}\frac{(z+\eta)^2}{d}\right). \tag{3.19}$$

PROOF. Let $1\le i\le n-1$ be fixed and write $r_0 = r_0(x_i)$, $r_1 = r_1(x_i)$, $h = h_i$, $x = x_i$. First consider the case $\Lambda_i = [0, \lambda - z\lambda^{1/2}]$. Then, for $\beta\in[r_0, r_1]$,

$$\begin{aligned} p(x,\beta,\Lambda) \ge p(x, r_1, \Lambda) &= P\big(Z(r_1, h) < \lambda - z\lambda^{1/2}\big)\\ &= P\big(Z(r_1, h) < r_1 h - (z+\eta)\lambda^{1/2}\big) = \gamma^-(r, u), \end{aligned}$$

where u is defined by $uhr_1^{1/2} = (z+\eta)\lambda^{1/2}$. Now $u = (z+\eta)h^{-1/2}(\lambda/\lambda_1)^{1/2} \le (z+\eta)h^{-1/2}$, and so

$$p(x,\beta,\Lambda_i) \ge \gamma^-\big(r_1,(z+\eta)h^{-1/2}\big).$$

Since $(z+\eta)h^{1/2} \le 2\varepsilon\lambda^{1/2}h^{-1/2} \le 2\varepsilon r_1^{1/2}$, we have by Lemma 1.1(iv),

$$p(x,\beta,\Lambda) \ge c_1(2\varepsilon)\exp\left(-\frac{(1+2\varepsilon)}{2}\frac{(z+\eta)^2}{h}\right), \tag{3.20}$$

and as $h \ge d$, this establishes (3.19) for this case.

Now let $\Lambda = [\lambda + z\lambda^{1/2}, \infty)$. For $\beta \in [r_0, r_1]$ we have

$$p(x,\beta,\Lambda) \ge p(x,r_0,\Lambda) = \gamma^+(r_0,u),$$

where $u = (\eta+z)\lambda^{1/2}(hr_0^{1/2})^{-1}$. Then

$$u = \frac{(\eta+z)}{h^{1/2}}\left(\frac{\lambda}{\lambda_0}\right)^{1/2} \le \frac{\eta+z}{h^{1/2}}(1-\varepsilon)^{-1/2} = u_0, \quad \text{say}.$$

Now $u_0 \le 3\varepsilon r_0^{1/2}$, and so, by Lemma 1.1(iii),

$$\begin{aligned} p(x,\beta,\Lambda) &\ge \gamma^+(r_0,u_0) \\ &\ge c_1(3\varepsilon)\exp\left(-\frac{1+3\varepsilon}{2}u_0^2\right) \ge c_1(3\varepsilon)\exp\left(-\frac{1+5\varepsilon}{2}\frac{(\eta+z)^2}{h}\right), \end{aligned}$$

proving (3.19). □

Lemma 3.5. *Let $1 \le i \le \frac{1}{2}(n-1)$, and let $r_0(x_j) \le \beta_j \le r_1(x_j)$ for $j = 2i-1, 2i$. Then if $z \le \frac{1}{2}\varepsilon\lambda$,*

$$q(\beta_{2i-1},\beta_{2i},h_{2i-1},h_{2i},z) \ge c_1(\varepsilon)^2\exp\left(-\frac{(1+3\varepsilon)}{4\lambda d}z^2\right). \tag{3.21}$$

Proof. We have $\beta_j h_j \ge r_0(x_j)h_j = \lambda_0 \ge \lambda(1-\varepsilon)$, and so $z < \varepsilon\beta_j h_j$. Thus by Lemma 1.2,

$$q(\beta_{2i-1},\beta_{2i},h_{2i-1},h_{2i},z) \ge c_1(\varepsilon)^2\exp\left(-\frac{(1+\varepsilon)z^2}{2k}\right),$$

where $k = \beta_{2i-1}h_{2i-1}^2 + \beta_{2i}h_{2i}^2$. However $k \ge \lambda_0(h_{2i-1}+h_{2i}) \ge 2\lambda_0 d$, and (3.21) now follows. □

Theorem 3.6. *Let A, ε and λ satisfy* (3.1)–(3.4). *Let*

$$z_0 = \varepsilon\lambda^{1/2} \wedge \big(2d(1-8\varepsilon)\log n\big)^{1/2}, \tag{3.22}$$

$$z_1 = \left(\tfrac{1}{2}\varepsilon\lambda^{1/2}\right) \wedge \big(4d(1-8\varepsilon)\log n\big)^{1/2} \tag{3.23}$$

and let

$$F^{(-)} = \left\{ \min_{1 \le i \le n-1} L_{\tau_\lambda}^{x_i} < \lambda - \lambda^{1/2} z_0 \right\},$$

$$F^{(+)} = \left\{ \max_{1 \le i \le n-1} L_{\tau_\lambda}^{x_i} > \lambda + \lambda^{1/2} z_0 \right\},$$

$$G = \left\{ \max_{1 \le i \le (n-1)/2} |L_{\tau_\lambda}^{x_{2i}} - L_{\tau_\lambda}^{x_{2i-1}}| > \lambda^{1/2} z_1 \right\}.$$

Then there exist $c_4(\varepsilon)$, $c_5(\varepsilon)$ *such that, if* F *is either of* $F^{(+)}$ *or* $F^{(-)}$,

$$(3.24) \quad P(F^c; \tau_\lambda < \infty) \le c_2 \exp\left(\frac{-\varepsilon^4}{5D} (\lambda \wedge d \log n) \right) + \exp(-c_4(\varepsilon) n^\varepsilon),$$

$$(3.25) \quad P(G^c; \tau_\lambda < \infty) \le c_2 \exp\left(\frac{-\varepsilon^2 \lambda}{5D} \right) + \exp(-c_5(\varepsilon) n^\varepsilon).$$

PROOF. Let $\eta = \varepsilon z_0$, so that η satisfies (3.5), and set

$$\alpha = c_3(\varepsilon) \exp\left(-\frac{1 + 5\varepsilon}{2} \frac{(\eta + z_0)^2}{d} \right).$$

Take $F = F^{(+)}$ and let $\Lambda_i = [\lambda + \lambda^{1/2} z_0, \infty)$ for $1 \le i \le n - 1$. By Lemma 3.4 $p(x_i; \beta, \Lambda_i) \ge \alpha$ for $r_0(x_i) \le \beta \le r_1(x_i)$, and so, by Proposition 3.3,

$$(3.26) \qquad P(F^c; \tau_\lambda < \infty) \le \exp\left(-\frac{1}{2} n\alpha \right) + c_2 \exp\left(\frac{-\eta^2}{5D} \right).$$

Now

$$\begin{aligned} -\frac{1}{2} n\alpha &= -\frac{1}{2} c_3(\varepsilon) \exp\left[\log n - \frac{(1 + 5\varepsilon)}{2d} (1 + \varepsilon)^2 z_0^2 \right] \\ &\le -\frac{1}{2} c_3(\varepsilon) \exp\left[(\log n)\left(1 - (1 + 5\varepsilon)(1 + \varepsilon)^2 (1 - 8\varepsilon)\right) \right] \\ &\le -\frac{1}{2} c_3(\varepsilon) n^\varepsilon, \end{aligned}$$

since $1 - (1 + 5\varepsilon)(1 + \varepsilon)^2(1 - 8\varepsilon) \ge \varepsilon$. Also

$$\frac{\eta^2}{5D} = \frac{\varepsilon^2 z^2}{5D} \ge \frac{\varepsilon^4}{5D} (\lambda \wedge d \log n),$$

and substituting in (3.26) we obtain (3.24). The case $F = F^{(-)}$ is exactly the same.

We now turn to (3.25). Set $\eta = \varepsilon \lambda^{1/2}$ and let

$$\alpha_1 = c_1(\varepsilon)^2 \exp\left(-\frac{(1 + 3\varepsilon)}{4\lambda d} \lambda z_1^2 \right).$$

Then as $-z_1^2 \ge -4d(1-8\varepsilon)\log n$,

$$n\alpha_1 \ge c_1(\varepsilon)^2 \exp[(\log n)(1-(1+3\varepsilon)(1-8\varepsilon))]$$
$$\ge c_1(\varepsilon)^2 \exp(\varepsilon \log n).$$

By Lemma 3.5 and Proposition 3.3, since $\lambda^{1/2} z_1 \le \varepsilon\lambda^{1/2}$, we have

$$P(G^c; \tau_\lambda < \infty) \le c_2 \exp\left(-\frac{\varepsilon^2\lambda}{5D}\right) + \exp\left(-\frac{1}{5}n\alpha_1\right)$$
$$\le c_2 \exp\left(-\frac{\varepsilon^2\lambda}{5D}\right) + \exp\left(-\frac{1}{5}c_1(\varepsilon)^2 n^{\varepsilon}\right). \qquad \square$$

4. Sufficient conditions for discontinuity. Throughout this section X will be a Lévy process satisfying (0.4), (0.5) and (0.15). Until the end of Lemma 4.4 we will also assume that X is recurrent. Let ϕ and ρ be the functions defined by (0.12) and (1.3). Set

$$H(u) = \log 1/\rho(u), \qquad u > 0. \tag{4.1}$$

Let $u_k = \rho(2^{-k})$, so that $\bar\phi(u_k) = 2^{-k}$. Writing $\approx$ for "converges or diverges with" we have

$$\int_{0+} \frac{\bar\phi(u)\,du}{u(\log 1/u)^{1/2}} \approx \sum_k^\infty \int_{u_{k+1}}^{u_k} \frac{\bar\phi(u)\,du}{u(\log 1/u)^{1/2}}$$
$$\approx \sum_k^\infty 2^{-k} \int_{u_k}^{u_{k+1}} u^{-1}\left(\log\frac{1}{u}\right)^{-1/2} du$$
$$\approx \sum_k 2^{-k} 2\left(H(2^{-(k+1)})^{1/2} - H(2^{-k})^{1/2}\right)$$
$$\approx \sum_k H(2^{-k})^{1/2} 2^{-k} \approx \int_{0+} H(u)^{1/2}\,du$$

and thus, as $I(\bar\phi) = \infty$,

$$\int_{0+} H(u)^{1/2}\,du = +\infty. \tag{4.2}$$

LEMMA 4.1. *Let $\varepsilon = 1/20$ and let a, b, λ, z be chosen so that*

$$0 < a < b < \tfrac{1}{2}u_1(0)^{1/2}, \qquad \rho(b)/\rho(a) > 20, \qquad \lambda > 20a^2,$$
$$z = \varepsilon\lambda^{1/2} \wedge \tfrac{1}{2}a(H(a) - H(b))^{1/2}. \tag{4.3}$$

Let $A = \{x_0, x_1, \dots, x_{n-1}\}$ be a set satisfying (1.4) *and* (1.5) *and let F be either of the events*

$$\left\{\min_{1\le i\le n-1} L^{x_i}_{\tau_\lambda(x_0)} < \lambda - \lambda^{1/2}z\right\}, \qquad \left\{\max_{1\le i\le n-1} L^{x_i}_{\tau_\lambda(x_0)} > \lambda + \lambda^{1/2}z\right\}.$$

Then if $\theta = \varepsilon c_4(\varepsilon) \wedge (\varepsilon^4/40)$,

$$P(F^c) \le c_6 \exp\left(-\frac{\theta a^2}{b^2}\left[\frac{\lambda}{a^2} \wedge (H(a) - H(b))\right]\right). \tag{4.4}$$

Proof. Let h_i, d and D be given by (3.1). Then, by Corollary 2.3,

$$h(x_0, x_i) \le b^2\left(1 - \frac{b^2}{u_1(0)}\right)^{-1} \le \frac{4}{3}b^2 \le 2b^2,$$

and if x_{j_i} is such that $\phi(x_i - x_{j_i}) \le a(1+\varepsilon)$, we have

$$h_i \le h(x_i, x_{j_i}) \le a^2(1+\varepsilon)^2\big(1 - a^2(1+\varepsilon)^2/u_1(0)\big) \le 2a^2.$$

By Proposition 2.4, $h_i \ge h_1(x_i, A - \{x_i\}) \ge \frac{1}{4}a^2$. So we deduce that

$$\tfrac{1}{4}a^2 \le d, \qquad \max_i h_i \le 2a^2, \qquad D \le 2b^2.$$

We also have $\lambda \ge 10 \max_i h_i$. Since

$$\frac{1}{4}a^2(H(a) - H(b)) \le d^2 2(1 - 7\varepsilon)\log\frac{\rho(b)}{\rho(a)},$$

we have $z \le z_0$, where z_0 is defined by (3.22). Therefore, by Theorem 3.6, because $n \ge \rho(b)/\rho(a)$ and $P(\tau_\lambda(x_0) < \infty) = 1$,

$$P(F^c) \le c_2 \exp\left(\frac{-\varepsilon^4}{10b^2}\left(\lambda \wedge \frac{a^2}{4}(H(a) - H(b))\right)\right)$$
$$+ \exp\big(-c_4(\varepsilon)\exp\big(\varepsilon\big(H(a_n) - H(b_n)\big)\big)\big).$$

Writing $H = H(a) - H(b)$, we have $c_4(\varepsilon)e^{\varepsilon H} \ge c_4(\varepsilon)\varepsilon H \ge \theta a^2 b^{-2}(H \wedge \lambda a^{-2})$, and (4.4) follows. □

Remark. Though it will be obscured by the proof of Proposition 4.2, the proof of discontinuity really breaks into two cases, according to whether

$$\limsup_{a \downarrow 0} a^2 H(a) > 0 \tag{4.5}$$

or

$$\limsup_{a \downarrow 0} a^2 H(a) = 0. \tag{4.6}$$

The first case follows almost immediately from Lemma 4.1. If $a_m \downarrow 0$, with $a_m^2 H(a_m) > \delta$, and $z_m = z(a_m, b)$ is given by (4.3), then $z_m \ge \varepsilon\lambda^{1/2} \wedge \delta/2 > 0$ for all m. By Lemma 4.1 there exists a countable set D such that, if $B = \{y\colon \phi(y) < b\}$,

$$P\left(\sup_{x \in D \cap B}\left(L^x_{\tau_\lambda} - L^0_{\tau_\lambda}\right) \le \varepsilon\lambda^{1/2} \wedge \frac{1}{2}\delta\right) \le c_6 \exp\left(-\frac{\theta}{b^2}(\lambda \wedge \delta^2)\right).$$

Letting $b \downarrow 0$, it follows that $L^{\cdot}_{\tau_\lambda}$ has an oscillation of at least $\varepsilon\lambda^{1/2} \wedge (\delta/2)$ around 0, and from this it is possible, as in [22], to deduce that $L^{\cdot}_t$ is unbounded.

This argument is along the same lines as the proofs of discontinuity in [12] and [22]. It would be interesting to know whether (4.5) is equivalent to their condition

$$\limsup_{\alpha \to \infty} (\log \alpha) u_\alpha(0) > 0.$$

If (4.6) holds, then $z(a, b) \to 0$ as $a \to 0$ and the preceding simple argument does not work.

PROPOSITION 4.2. *Let X_t be a recurrent process satisfying* (0.4) *and* (0.5). *Suppose that $I(\bar{\phi}) = \infty$. Then there exists a countable set $D \subseteq \mathbb{R}$ such that*

$$\sup\{L_t^a, a \in D \cap [-\eta, \eta]\} = \infty \quad \textit{for all } t > 0,\ \eta > 0,\ a.s., \tag{4.7}$$

$$\inf\{L_t^a, a \in D \cap [-\eta, \eta]\} = 0 \quad \textit{for all } t > 0,\ \eta > 0,\ a.s. \tag{4.8}$$

PROOF. It is sufficient to prove (4.7) and (4.8) for a fixed $\eta > 0$. So let η be fixed and set $J = [-\eta, \eta]$. Let $\varepsilon = 1/20$, $K = \log 20$, $\theta = \varepsilon c_4(\varepsilon) \wedge (\varepsilon^4/40)$, $\lambda > 0$ and $\delta > 0$. By Lemma 1.8 there exist sequences $(a_n), (b_n), (\mu_n)$ decreasing to 0 and satisfying (1.13)–(1.18). Clearly b_1 can be chosen as small as we like: We choose it so that

$$20 b_1^2 < \lambda \qquad \rho(b_1) < \tfrac{1}{2}, \qquad b_1 \le \tfrac{1}{2} u_1(0)^{1/2}$$

and

$$\{y: \phi(y) < 2b_1\} \subseteq J.$$

Now let $(\lambda_n), (z_n)$ be chosen by

$$\lambda_1 = \lambda, \qquad z_n = \varepsilon\lambda_n^{1/2} \wedge \left(\tfrac{1}{2} a_n (H(a_n) - H(b_n))^{1/2}\right)$$

and

$$\text{either} \quad \lambda_{n+1} = \lambda_n + \lambda_n^{1/2} z_n, \quad \text{all } n \tag{4.9a}$$

$$\text{or} \quad \lambda_{n+1} = \max\left(\lambda_n - \lambda_n^{1/2} z_n, \mu_n, 20 a_n^2\right), \quad \text{all } n. \tag{4.9b}$$

[The first choice of the (λ_n) is for proving (4.7); the second for (4.8).] For each n, a_n, b_n, λ_n, z_n satisfy the hypotheses of Lemma 4.1. Since $\sum_n a_n (H(a_n) - H(b_n))^{1/2} = \infty$, if (λ_n) are chosen by (4.9a) then $\lambda_n \to \infty$, while if (λ_n) are chosen by (4.9b) then $\lambda_n \to 0$.

We now construct a sequence of finite subsets of $\mathbb{R}$ as follows. Given $x \in \mathbb{R}$, let $\Gamma(x, a, b)$ be a set constructed by Lemma 1.6, so that $\Gamma(x, a, b)$ satisfies (1.4) and (1.5). Set

$$A_0 = \{0\},$$

$$A_n = \bigcup_{x \in A_{n-1}} \Gamma(x, a_n, b_n) \quad \text{for } n \ge 1,$$

$$D = \bigcup_{n=1}^{\infty} A_n.$$

Recall from Section 1 that ϕ defines a metric on $\mathbb{R}$. Since $\phi(y-x) \le b_n$ for $y \in \Gamma(x, a_n, b_n)$, we have $\phi(y) \le \sum_1^n b_i \le 2b_1$ for $y \in A_n$. Thus $D \subseteq \{y: \phi(y) \le 2b_1\} \subseteq J$.

Let $\gamma_n = \#\Gamma(x, a_n, b_n)$; by the construction of Γ,

$$\gamma_n \le \frac{\rho(b_n)}{\rho(a_n)} = \exp\big(H(a_n) - H(b_n)\big) \le \exp H(a_n).$$

Hence $\#A_n = \prod_1^n \gamma_k \le \exp(\sum_1^n H(a_k))$. For $n \ge 1$, $x \in A_{n-1}$, let

$$F_n^+(x) = \left\{L\big(y, \tau_{\lambda_n}(x)\big) \ge \lambda_n + \lambda_n^{1/2} z_n \text{ for some } y \in \Gamma(x, a_n, b_n)\right\},$$

$$F_n^-(x) = \left\{L\big(y, \tau_{\lambda_n}(x)\big) \le \lambda_n - \lambda_n^{1/2} z_n \text{ for some } y \in \Gamma(x, a_n, b_n)\right\}$$

and let $F_n(x)$ be either $F_n^+(x)$ or $F_n^-(x)$. Set

$$G_n = \bigcap_{x \in A_{n-1}} F_n(x).$$

By Lemma 4.1,

$$P(G_n^c) \le \sum_{x \in A_{n-1}} P\big(F_n(x)^c\big)$$

$$\le c_2 \exp\left(\sum_1^{n-1} H(a_k) - \frac{\theta}{b_n^2}\big(\mu_n \wedge a_n^2(H(a_n) - H(b_n))\big)\right).$$

By the choice of (a_n) and (b_n), $\sum_n P(G_n^c) < \delta < \infty$, and so by Borel–Cantelli, $P(\liminf G_n) = 1$. Set $N = \min\{n: \sum_n^\infty 1_{G_n^c} = 0\}$ and let $R = \tau_{\lambda_N}(0)$. Then $N < \infty$, $R < \infty$ a.s. and $P(N = 1) > 1 - \delta$.

Now let (λ_n) be given by (4.9a), and let $F_n(x) = F_n^+(x)$ for all x, n. Let $\omega \in \liminf G_n$, so that $\omega \in F_m(x)$ for all $m \ge N(\omega)$, $x \in A_{m-1}$. We define a (random) sequence of points in D by setting $x_N = 0 \in A_{N-1}$ and, given $x_N, \dots, x_m$ with $x_k \in A_{k-1}$, $N \le k \le m$, we choose $x_{m+1} \in \Gamma(x_m, a_m, b_m)$ so that

$$L\big(x_{m+1}, \tau_{\lambda_m}(x_m)\big) > \lambda_{m+1};$$

such a point exists since $\omega \in F_m(x_m)$. It follows that $\tau_{\lambda_{m+1}}(x_{m+1}) \le \tau_{\lambda_m}(x_m) \le \cdots \le R$, and thus we have

$$L_R^{x_m} > \lambda_m \quad \text{for all } m \ge N.$$

As $\lambda_m \to \infty$, we have $\sup_{x \in D} L_R^x = \infty$. Thus $\sup_{x \in \mathbb{Q} \cap J} L_R^x = \infty$ by Lemma 1.5, and so by Lemma 1.4, $\sup_{x \in \mathbb{Q}} L_t^x = \infty$ for all $t > 0$ a.s. Choosing t small enough so that $\sup_{s \le t} |X_s| < \eta$, we obtain (4.7).

Now let (λ_n) be given by (4.9b) and let $F_n(x) = F_n^-(x)$. We choose a random sequence in D in the same way as before: Let $x_N = 0$ and, given $x_N, \dots, x_m$ with $x_j \in A_{j-1}$, let $x_{m+1} \in \Gamma(x_m, a_m, b_m)$ be chosen so that $L(x_{m+1}, \tau_{\lambda_m}(x_m)) < \lambda_m - \lambda_m^{1/2} z_m \le \lambda_{m+1}$. Thus $\tau_{\lambda_{m+1}}(x_{m+1}) \ge \tau_{\lambda_m}(x_m) \ge \cdots \ge R$, and therefore,

$$L_R^{x_m} < \lambda_m \quad \text{for all } m \ge N.$$

Hence $\inf_{x \in D} L_R^x = 0$ and so $\inf_{x \in \mathbb{Q} \cap J} L_R^x = 0$ by Lemma 1.5. Since $P(R = \tau_\lambda(0)) = P(N = 1) > 1 - \delta$, we have

$$P\Big(\inf_{x \in \mathbb{Q} \cap J} L_{\tau_\lambda}^x = 0\Big) > 1 - \delta.$$

As δ, λ are arbitrary, this implies that $\inf_{x \in \mathbb{Q} \cap J} L_{\tau_\lambda}^x = 0$ a.s. for each $\lambda > 0$, and (4.8) now follows easily. □

COROLLARY 4.3. *Let $\varepsilon > 0$. Then*

$$\{a \in \mathbb{Q}: L_t^a < \varepsilon\} \text{ is dense in } \mathbb{R} \text{ for all } t \geq 0 \quad a.s. \tag{4.10}$$

PROOF. For $x \in \mathbb{R}$ let

$$K_t^x = \liminf_{\substack{a \to x \\ a \in \mathbb{Q}}} L_t^a.$$

It is sufficient to show that $K_t^x = 0$ for all $t \geq 0$, P^0-a.s., for each $x \in \mathbb{Q}$. Let $x \in \mathbb{Q}$ be fixed. By Proposition 4.2, $K_t^0 \equiv 0$, P^0-a.s., and so, using the spatial homogeneity of X, $K_t^x \equiv 0$, P^x-a.s. Therefore $K_t^x \circ \theta_{T_0} \leq K_{t+T_0}^x = 0$ for all $t \geq 0$, P^x-a.s., and thus, using the strong Markov property of X at T_0, we deduce that $K_t^x = 0$ for all $t \geq 0$, P^0-a.s. □

LEMMA 4.4. *$\{L_t^a, a \in \mathbb{Q} \cap [-\eta, \eta]\}$ is dense in $\mathbb{R}^+$ for each $\eta > 0$, $t > 0$, P^0-a.s.*

PROOF. This follows from (4.7) and (4.10) by the methods of [3, Proposition 4.2]. The hypotheses in [3] are stronger: It is assumed that X has a nowhere dense range, which implies that $\{x \in \mathbb{Q}: L_t^x = 0\}$ is dense in $\mathbb{R}$ for all $t \geq 0$. However [apart from (4.7)], all that is needed is that given $u > 0$ and given a stopping time S there exists a sequence (A_n) of $\mathscr{F}_s$-measurable random variables with $A_n \in \mathbb{Q}$, $A_n \to X_S$ and $L_S^{A_n} < u$, and this is guaranteed by (4.10). □

PROOF OF THEOREM 1. All that remains is to remove the hypothesis that X is recurrent. Let X be any Lévy process satisfying (0.4), (0.5) and (0.15). Suppose that the conclusion of Lemma 4.4 fails for X. Then there exists $\eta > 0$ and an interval $(u, v) \subseteq [0, \infty)$ such that

$$P\big(\text{there exists } t > 0 \text{ with } L_t^x \in (u, v)^c \text{ for all } x \in (-\eta, \eta) \cap \mathbb{Q}\big) > 0.$$

Hence, as $t \to L_t^x$ is continuous for all $x \in \mathbb{Q}$, there exist $t > 0$, $\delta > 0$, such that

$$P\big(L_t^x \in (u, v)^c \text{ for all } x \in (-\eta, \eta) \cap \mathbb{Q}\big) > \delta > 0.$$

Now let $K > 0$ and let Y be the recurrent Lévy process constructed in Section 1. By Proposition 1.7, $Y = X$ on $[0, S)$, where $S = \inf\{t \geq 0: |\Delta X_t| \vee |\Delta Y_t| \geq K\}$ and Y satisfies (0.4), (0.5) and (0.15). Therefore the conclusion of Lemma 4.4 holds for Y. Let K be chosen large enough so that $P(S \leq t) \leq \delta/2$: This is possible by the remark following Proposition 1.7. Then, since $L_t^x(X) = L_t^x(Y)$ if $t < S$,

$$P(L_t^x(Y) \notin (u, v) \text{ for any } x \in (-\eta, \eta) \cap \mathbb{Q}) > \tfrac{1}{2}\delta,$$

giving a contradiction. □

5. Modulus of continuity of $x \to L_t^x$. In this section we consider Lévy processes for which $I(\bar{\phi}) < \infty$, and which therefore have a jointly continuous local time, and will look at the modulus of continuity of $x \to L_t^x$. Upper bounds for this were given in [1] and [3] (see Theorem B); the estimates of Section 3 will enable us to obtain lower bounds. In many cases these differ from the upper bounds of [1] only by a constant factor. If in addition $\phi^2(x)$ is regularly varying at 0, but not slowly varying at 0, then a little extra work on the constants for the upper bound gives an exact modulus of continuity.

Set

$$\phi_0(x) = \inf_{y \geq |x|} \phi(y), \qquad \phi_1(x) = \sup_{y \leq |x|} \phi(y). \tag{5.1}$$

Thus ϕ_0 and ϕ_1 are monotone and $\phi_0 \leq \phi \leq \phi_1$, $\phi_0 \leq \bar{\phi} \leq \phi_1$. Throughout this section we assume

$$I(\bar{\phi}) < \infty. \tag{5.2}$$

From time to time we will make the following assumption on ϕ, which will enable us to obtain sharp results:

$$\phi^2(x) = |x|^\alpha f(x), \quad \text{where } \alpha > 0 \text{ and } f \text{ is slowly varying at } 0. \tag{5.3}$$

LEMMA 5.1. (a) *Suppose* (5.2) *holds. Then, for any* $\delta > 0$ *there exists* $u_0 > 0$ *such that*

$$\bar{\phi}(u) \leq \delta \left(\log \frac{1}{u} \right)^{-1/2} \quad \textit{for } 0 \leq u \leq u_0.$$

(b) *Suppose* (5.3) *holds. Then*

$$\frac{\phi_0(x)}{\phi_1(x)} \to 1 \quad \textit{as } x \to 0.$$

PROOF. (a) is elementary, while (b) follows by standard properties of regularly varying functions. □

Now let $\lambda > 0$, $0 < \varepsilon < \frac{1}{10}$, and let $\theta > 0$ be chosen so that

$$\psi_1(z) \geq 1 - \varepsilon \quad \text{for } |z| < \theta. \tag{5.4}$$

[This is certainly possible, since $\psi_1(0) = 1$ and ψ_1 is continuous.]

Since $\phi(x) \to 0$ as $x \to 0$, $\phi_0(x) > 0$ for $x > 0$ and ϕ and ϕ_0 are continuous there exists a sequence $\delta_n \downarrow 0$ with $\phi(\delta_n) = \phi_0(\delta_n)$ for all $n \geq 1$. We take $\delta_1 < \theta$. Let $K \geq 1$ be an integer and let

$$\Lambda_n = \left\{ rK\delta_n, (rK+1)\delta_n, 0 \leq r \leq \frac{\theta - \delta_n}{K\delta_n} \right\}, \tag{5.5}$$

so that $\Lambda_n \subseteq [0, \theta]$ and $\#(\Lambda_n) = 2[(\theta - \delta_n)/(K\delta_n)] + 2$. Set

$$d(n, K) = \min_{x \in \Lambda_n} h(x, \Lambda_n - \{x\}), \qquad D(n, K) = \max_{x \in \Lambda_n} h(0, x).$$

Lemma 5.2. (i) $\frac{1}{4}\phi_0^2(\delta_n) \leq h(x, \Lambda_n - \{x\}) \leq (1 - \varepsilon)^{-1}\phi^2(\delta_n)$ *for each* $x \in \Lambda_n$.
(ii) $\frac{1}{4}\phi_0^2(\delta_n) \leq d(n, K) \leq (1 - \varepsilon)^{-1}\phi^2(\delta_n)$.
(iii) $D(n, K) \leq (1 - \varepsilon)^{-1}/\phi_1^2(\theta)$.
(iv) *Suppose that* (5.3) *holds. Then, for all sufficiently large* n,

$$d(n, K) \geq (1 - 8K^{-\alpha})\phi^2(\delta_n).$$

Proof. By Proposition 2.4, for $x \in \Lambda_n$,

$$h(x, \Lambda_n - \{x\}) \geq \tfrac{1}{4}\min\{\phi^2(y - x), y \in \Lambda_n - \{x\}\}$$
$$\geq \tfrac{1}{4}\phi_0^2(\delta_n).$$

Also, if y is the unique element of Λ_n with $|x - y| = \delta_n$, then

$$h(x, \Lambda_n - \{x\}) \leq h(x, y)$$
$$\leq \psi_1^{-1}(y - x)h_1(x, y)$$
$$\leq (1 - \varepsilon)^{-1}\phi^2(\delta_n),$$

by (5.4) and Corollary 2.3(a), which proves (i). (ii) follows immediately from (i). Similarly,

$$h(0, x) \leq \psi^{-1}(x)\phi^2(x)$$
$$\leq (1 - \varepsilon)^{-1} \sup_{0 \leq z \leq \theta} \phi^2(z),$$

proving (iii).

Now suppose that (5.3) holds. Let η be small enough so that $f(Kx)/f(x) > (1 - \varepsilon)$, $\phi_1^2(\eta) \leq (K^\alpha - 5)K$ and $\phi_0^2(Kx)/\phi^2(Kx) \geq 1 - \varepsilon$ for $0 \leq x \leq \eta$. Then if $x \in \Lambda_n$, $y \in \Lambda_n$, with $|x - y| = \delta_n$, $z \in \Lambda_n - \{x, y\}$ and $\delta_n < \eta$,

$$\phi^2(x - z) \geq \phi_0^2(x - z) \geq \phi_0^2(K\delta_n)$$
$$\geq (1 - \varepsilon)\phi^2(K\delta_n)$$
$$= (1 - \varepsilon)K^\alpha \frac{f(K\delta_n)}{f(\delta_n)}\phi^2(\delta_n)$$
$$\geq (1 - \varepsilon)^2 K^\alpha \phi^2(\delta_n).$$

By Proposition 2.5 and Corollary 2.3(a), we have

$$\begin{aligned} h(x, \Lambda - \{x\}) &\geq h_1(x, \Lambda - \{x\}) \\ &\geq \left(1 - 6(1-\varepsilon)^{-2}K^{-\alpha}\right)h_1(x, \{y\}) \\ &\geq \left(1 - 6(1-\varepsilon)^{-2}K^{-\alpha}\right)\left(1 - \frac{\phi^2(\delta_n)}{K}\right)\phi^2(\delta_n) \\ &\geq (1 - 8K^{-\alpha})\phi^2(\delta_n). \end{aligned}$$ □

Now let

$$\gamma_n = d(n, K)^{1/2}/\phi(\delta_n), \qquad \gamma = \liminf_{n\to\infty} \gamma_n. \tag{5.6}$$

The last lemma has shown that $\gamma \geq \frac{1}{2}$, and that if (5.3) holds, then $\gamma \geq (1 - 8K^{-\alpha})^{1/2}$. It has also shown that for sufficiently large n, Λ_n satisfies the conditions of Section 3. Define $z = z(n, K)$ by

$$z(n, K) = \left(4d(n, K)(1 - 7\varepsilon)\log(\#\Lambda_n)\right)^{1/2}.$$

Now $\log(\#\Lambda_n)/\log(1/\delta_n) \to 1$ as $n \to \infty$, and therefore $z(n, K) \leq c\phi^2(\delta_n)(\log 1/\delta_n)^{1/2}$ for sufficiently large n. Thus, by Lemma 5.1(a) $z(n, K) \leq \frac{1}{2}\varepsilon\lambda^{1/2}$ for all large n. Let

$$\rho(x) = \phi(x)\left(\log\frac{1}{x}\right)^{1/2}$$

and let ρ_0, ρ_1 be defined similarly in terms of ϕ_0, ϕ_1, respectively. By Lemma 5.2(i),

$$\frac{z(n,K)}{\rho(\delta_n)} \geq \gamma_n(1 - 7\varepsilon)^{1/2}\left(\frac{\log \#\Lambda_n}{\log(1/\delta_n)}\right)^{1/2},$$

and therefore

$$\liminf_{n\to\infty} \frac{z(n, K)}{\rho(\delta_n)} \geq 2\gamma(1 - 7\varepsilon)^{1/2}.$$

Set

$$G(\theta, n) = \left\{ \sup_{0 \leq a \leq \theta - \delta_n} |L_{\tau_\lambda}^{a+\delta_n} - L_{\tau_\lambda}^{a}| > \lambda^{1/2} z(n, K)\right\}.$$

Let $y \in \mathbb{R}$. By Theorem 3.6,

$$P^y\left(G(\theta, n)^c;\ \tau_\lambda < \infty\right) \leq \exp\left(-c_5(\varepsilon)(2n)^{\varepsilon}\right) + c_2\exp\left(\frac{-\lambda\varepsilon^2}{5D(n, K)}\right)$$

and thus, using Lemma 5.2(iii),

$$P^y\left(\limsup_{n\to\infty} G(\theta, n)^c;\ \tau_\lambda < \infty\right) \leq c_2\exp\left(\frac{-\lambda\varepsilon^2}{6\phi_1^2(\theta)}\right).$$

Write $H(\theta) = \limsup_{n\to\infty} G(\theta, n)^c$; clearly if $\theta_1 < \theta$, then $H(\theta_1) \supseteq H(\theta)$. Therefore

$$P^y(H(\theta); \tau_\lambda < \infty) \le \limsup_{\theta_1 \downarrow 0} P(H(\theta_1); \tau_\lambda < \infty)$$
$$= 0.$$

Consequently $\liminf_n G(\theta, n)$ occurs P^y-a.s. on $\{\tau_\lambda < \infty\}$.

Thus there exist $\mathscr{F}_{\tau_\lambda}$-measurable random variables A_n, B_n, with $0 \le A_n \le B_n \le \theta$ such that $|A_n - B_n| = \delta_n$ and

$$|L(A_n, \tau_\lambda) - L(B_n, \tau_\lambda)| \ge 2\gamma\lambda^{1/2}(1 - 7\varepsilon)\rho(A_n - B_n) \tag{5.7}$$
$$\text{for all large } n,\ P^y\text{-a.s.}$$

We now show that (5.7) continues to hold for a short time after τ_λ. Let $k_n = \log\log 1/\delta_n$ and let $\mu > 0$. Given a, b with $|b - a| = \delta_n$, we have, by Lemma 1.3(i) and (iii),

$$P^y\left(\sup_{0 \le s \le \tau_\mu(a)} |L_s^a - L_s^b| > k_n\phi(b - a)\right) \le 2\exp\left(-\frac{k_n^2\phi^2(b-a)}{4(\mu + k_n\phi(b-a))h(a,b)}\right)$$
$$\le 2e^{-k_n},$$

provided n is sufficiently large and δ_n is sufficiently small. Set $R_n = \tau(A_n, \mu + L(A_n, \tau_\lambda))$. By the strong Markov property of X we have, for all sufficiently large n,

$$P^y\left(\sup_{\tau_\lambda \le t \le R_n} |(L(A_n, t) - L(A_n, \tau_\lambda)) - (L(B_n, t) - L(B_n, \tau_\lambda))| \ge k_n\phi(\delta_n)\right)$$
$$\le 2e^{-k_n}.$$

Passing to a (random) subsequence n_i, we have

$$\left|\left(L(A_{n_i}, t) - L(A_{n_i}, \tau_\lambda)\right) - \left(L(B_{n_i}, t) - L(B_{n_i}, \tau_\lambda)\right)\right| \ge k_{n_i}\phi(\delta_{n_i})$$
$$\text{for all } \tau_\lambda \le t \le R_{n_i}, \text{ for all } i,\ P^y\text{-a.s.}$$

Therefore,

$$\sup_{\tau_\lambda \le t \le R_{n_i}} |L(A_{n_i}, t) - L(B_{n_i}, t)|$$
$$\ge |L(A_{n_i}, \tau_\lambda) - L(B_{n_i}, \tau_\lambda)| - k_{n_i}\phi(\delta_{n_i})$$
$$= \phi(\delta_{n_i})\left(2\gamma\lambda^{1/2}(1 - 7\varepsilon)\log(1/\delta_{n_i}) - \log\log(1/\delta_{n_i})\right)$$
$$\ge (1 - 8\varepsilon)2\gamma\lambda^{1/2}\rho(\delta_{n_i}) \quad \text{for all large } i,\ P^y\text{-a.s.}$$

For any interval $I \subseteq \mathbb{R}$, $t \geq 0$, $\mu > 0$, let

$$S_t(I, \rho) = \lim_{\delta \downarrow 0} \sup_{\substack{a, b \in I \\ |a-b| \leq \delta}} \frac{|L_t^a - L_t^b|}{\rho(b-a)},$$

$$\sigma(t, I, \mu) = \inf\Big\{s \geq t: \sup_{x \in I} (L_s^x - L_t^x) \geq \mu\Big\}.$$

Since $\sigma(\tau_\lambda, I, \mu) \geq R_{n_i}$, we have proved that

(5.8) $S_t([0, \theta], \rho) \geq 2\gamma\lambda^{1/2}(1 - 8\varepsilon)$ for $\tau_\lambda \leq t \leq \sigma(\tau_\lambda, [0, \theta], \mu)$, P^y-a.s.

Now $\varepsilon > 0$ is arbitrary and X is spatially homogeneous, and therefore it follows from (5.8) that

$$(5.9)\qquad \begin{aligned} &S_t([a, b], \rho) \geq 2\gamma\lambda^{1/2} \\ &\qquad \text{for } \tau_\lambda(a) \leq t \leq \sigma(\tau_\lambda(a), [a, b], \mu) \text{ for all rational } a, b, \lambda, \mu, \text{a.s.} \end{aligned}$$

Given $t > 0$, we can find $\lambda \in \mathbb{Q}_+$ with $(1 - \varepsilon)^2 L_t^a < \lambda < L_t^a$ and $\mu \in \mathbb{Q}_+$ with $t < \sigma(\tau_\lambda(a), [a, b], \mu)$, and therefore $S_t([a, b], \rho) \geq 2\gamma(1 - \varepsilon)(L_t^a)^{1/2}$. So (5.9) implies that

$$(5.10)\qquad S_t([a, b], \rho) \geq 2\gamma(L_t^a)^{1/2} \quad \text{for all } t \geq 0,\ a, b \in \mathbb{Q}^+ \text{ a.s.}$$

If $a = a_0 < a_1 < \cdots < a_m = b$ we also have, from (5.10),

$$\begin{aligned} S_t([a, b], \rho) &\geq \max_{1 \leq i \leq m} S_t([a_{i-1}, a_i], \rho) \\ &\geq 2\gamma \max_{1 \leq i \leq m} (L_t^{a_i})^{1/2} \quad \text{for all } t > 0. \end{aligned}$$

As $L_t^{\cdot}$ is continuous, we obtain

$$(5.11)\qquad S_t(I, \rho) \geq 2\gamma\Big(\sup_{x \in I} L_t^x\Big)^{1/2} \quad \text{for all } t > 0, \text{ intervals } I \subseteq \mathbb{R}, \text{a.s.}$$

We have proved (5.11) under the hypothesis $I(\bar{\phi}) < \infty$. If however $I(\bar{\phi}) = \infty$, then by Theorem 1, $S_t(I, \rho) = +\infty$ for all $t > 0$.

We have proved

THEOREM 5.3. *Let X be a Lévy process satisfying* (0.4) *and* (0.5), *and let ϕ, ϕ_0 be given by* (0.12) *and* (5.1). *There exists a constant $\gamma \geq \frac{1}{2}$ such that*

$$(5.12)\qquad \lim_{\delta \downarrow 0} \sup_{\substack{a, b \in I \\ |a-b| < \delta}} \frac{|L_s^a - L_s^b|}{\phi(b-a)(\log|b-a|^{-1})^{1/2}} \geq 2\gamma\Big(\sup_{x \in I} L_s^x\Big)^{1/2}$$

for all $s \geq 0$, intervals $I \subseteq \mathbb{R}$, a.s.

If, further, for some $\alpha > 0$,

$$\phi(x) = x^\alpha f(x), \quad \textit{where } f \textit{ is slowly varying at } 0,$$

then (5.12) *holds with $\gamma = 1$.*

Remark. Suppose that ϕ is monotone in a neighbourhood of 0 and let

$$g(x) = \int_0^x (\log u^{-1})^{1/2}\, d\phi(u).$$

By [1, Theorem 1.1] for any interval $I \subseteq \mathbb{R}$, we have

$$\lim_{\delta \downarrow 0} \sup_{\substack{a,\, b \in I \\ |a-b|<\delta \\ 0 \le s \le t}} \frac{|L_s^a - L_s^b|}{g(|b-a|)} \le c_7 \Big(\sup_x L_t^x\Big)^{1/2} \text{ for all } t \ge 0 \text{ a.s.} \tag{5.13}$$

Integrating by parts and using Lemma 5.1(i), we have

$$g(x) = \phi(x)(\log x^{-1})^{1/2} + \frac{1}{2}\int_0^x \frac{\phi(u)\, du}{u(\log u^{-1})^{1/2}}.$$

If ϕ satisfies (5.3) or, for example, $\phi(x) = c(\log(1/x))^{-\beta}$ for some $\beta > \frac{1}{2}$, then $g(x) \le c\rho(x)$, and the upper and lower bounds (5.12) and (5.13) differ only by a constant.

If, on the other hand, $\phi(x) = (\log 1/x)^{-1/2}(\log\log 1/x)^{-\beta}$, where $\beta > 1$, then $g(x) \sim (1-\beta)^{-1}(\log\log 1/x)^{1-\beta}$ as $x \to 0$, so that $g(x)/\rho(x) \to \infty$ as $x \to 0$. In this case the upper bound (5.13) is undoubtedly of the right form, and the lower bound (5.12) needs to be improved by some kind of "ladder" argument, as in the proof of Theorem 1.

We now show that, if (5.3) holds, then the lower bound given in (5.12) is exact.

Proof of Theorem 2. By Lemma 5.2(b), $\phi_1(x)/\phi_0(x) \to 1$ as $x \to 0$, and so $S_t(I, \rho_0) = S_t(I, \rho) = S_t(I, \rho_1)$. Thus by Theorem 5.3 it is only necessary to show that

$$S_t(I, \rho_1) \le 2\Big(\sup_{x \in I} L_t^x\Big)^{1/2} \quad \text{for all } t \ge 0 \text{ a.s.} \tag{5.14}$$

To simplify notation let $I = [0,1]$ and let $\varepsilon > 0$, $\lambda > 0$.

Set

$$\begin{aligned}
\delta_k &= (1+\varepsilon)^{-k}, \qquad k > 1,\\
x_r^{k,i} &= r\delta_k + i\varepsilon\delta_k, \qquad 0 \le r \le \delta_k^{-1},\ 0 \le i \le \varepsilon^{-1},\\
K_t(x) &= \lambda \wedge L_t^x,\\
V_r^{k,i} &= \sup_r \big|K_s\big(x_{r+1}^{k,i}\big) - K_s\big(x_r^{k,i}\big)\big|, \qquad 0 \le r \le \delta_k^{-1},\ 0 \le i \le \varepsilon^{-1},\\
z_k &= (1+\varepsilon)\lambda^{1/2} 2\phi_1(\delta_k)(\log 1/\delta_k)^{1/2},\\
F_{k,i} &= \big\{V_r^{k,i} \le z_k \text{ for } 0 \le r \le \delta_k^{-1} - 1\big\},\\
G_k &= \bigcap_{0 \le i \le \varepsilon^{-1}} F_{k,i}.
\end{aligned}$$

By Lemma 1.3(ii),

$$P\left(V_r^{k,i} > z_k\right) \le 2\exp\left(\frac{-z_k^2}{4\lambda h(0,\delta_k)}\right).$$

By Corollary 2.3, $h(0,\delta_k) \le (1+\varepsilon)\phi^2(\delta_k) \le (1+\varepsilon)\phi_1^2(\delta_k)$ for all sufficiently large k.

Therefore, for large k,

$$\begin{aligned} P\left(V_r^{k,i} \ge z_k\right) &\le 2\exp\left(-(1+\varepsilon)^{-1}\frac{z_k^2}{4\gamma\phi_1^2(\delta_k)}\right) \\ &= 2\exp(-(1+\varepsilon)(\log 1/\delta_k)) \\ &= 2\delta_k^{1+\varepsilon} = 2(1+\varepsilon)^{-k(1+\varepsilon)}. \end{aligned}$$

So

$$P\left(F_{k,i}^c\right) \le \delta_k^{-1} 2\delta_k^{1+\varepsilon} = 2(1+\varepsilon)^{-k\varepsilon}$$

and hence

$$P(G_k^c) \le 2\varepsilon^{-1}(1+\varepsilon)^{-k\varepsilon}.$$

Summing over k, by Borel–Cantelli we have $P(\liminf G_k) = 1$, and therefore there exists a $k_0 = k_0(\omega)$ such that

$$(5.15) \qquad |V_r^{k,i}| \le z_k \quad \text{for all } k \ge k_0, 0 \le r \le \delta_k^{-1} - 1, 0 \le i \le \varepsilon^{-1}.$$

We also have, by formula (3.7) of [1], that

$$(5.16) \qquad \sup_{s\ge 0}|K_s(b) - K_s(a)| \le c_7\lambda^{1/2}\int_0^{|b-a|}\left(\log(1+u^{-2})\right)^{1/2} d\phi_1(u) \quad \text{for } a, b \in I \text{ with } |b-a| < \delta(\omega).$$

Since ϕ satisfies (5.3), $\int_0^y(\log(1+u^{-2}))^{1/2}\, d\phi_1(u) \sim 2^{1/2}(\log 1/y)^{1/2}\phi_1(y)$ as $y \to 0$, and so (5.16) implies that there exists $k_1 = k_1(\omega)$ such that

$$(5.17) \qquad \sup_{s\ge 0}|K_s(b) - K_s(a)| \le 2c_7\lambda^{1/2}\rho_1(b-a) \quad \text{for } a, b \in I, |b-a| < \delta_{k_1}.$$

It remains to verify that (5.15) and (5.17) together imply (5.14). Let $a, b \in I$, with $0 < b - a < \delta_{k_0} \wedge \delta_{k_1}$. Then there exists $n \ge \max(k_0, k_1)$ such that $\delta_n \le b - a < \delta_{n-1}$ and i, r such that $x_r^{n,i-1} \le a \le x_r^{n,i} < a + \varepsilon\delta_n$. Then $|b - x_{r+1}^{n,i}| \le |a - x_r^{n,i}| + |b - a - \delta_n| \le 2\varepsilon\delta_n$. We have, from (5.15) and (5.17),

$$\begin{aligned} \sup_{s\ge 0}|K_s(b) - K_s(a)| &\le \sup_{s\ge 0}|K_s(b) - K_s(x_{r+1}^{n,i})| + V_r^{n,i} \\ &\quad + \sup_{s\ge 0}|K_s(x_r^{n,i}) - K_s(a)| \\ &\le z_n + 4c_7\lambda^{1/2}\rho_1(2\varepsilon\delta_n) \\ &= 2\lambda^{1/2}\left[(1+\varepsilon)\rho(\delta_n) + 2c_7\rho_1(2\varepsilon\delta_n)\right]. \end{aligned}$$

Now $\rho_1(\theta x)/\rho_1(x) \to \theta^\alpha$ as $x \to 0$ and, therefore,

$$(5.18) \qquad \sup_{s \geq 0} |K_s(b) - K_s(a)| \leq 2(1 + \varepsilon + 4c_7 2^\alpha \varepsilon^\alpha)\lambda^{1/2}\rho_1(b-a)$$

for all sufficiently small $|b - a|$. As ε is arbitrary we deduce that

$$(5.19) \qquad \lim_{\delta \downarrow 0} \sup_{\substack{|a-b|<\delta \\ a, b \in I}} \sup_{s \geq 0} \frac{|\lambda \wedge L_s^b - \lambda \wedge L_s^a|}{\rho_1(b-a)} \leq 2\lambda^{1/2}.$$

Hence, for each fixed $t \geq 0$, we have

$$(5.20) \qquad \lim_{\delta \downarrow 0} \sup_{\substack{|a-b|<\delta \\ a, b \in I}} \sup_{0 \leq s \leq t} \frac{|L_s^b - L_s^a|}{\rho_1(a-b)} \leq 2\Big(\sup_{x \in I} L_t^x\Big)^{1/2} \quad \text{a.s.}$$

It remains to show that (5.20) holds simultaneously for all $t \geq 0$. However, $t \to \sup_{x \in I} L_t^x$ is continuous, while the left-hand side of (5.20) is increasing in t. (5.20) holds for all $t \in \mathbb{Q}_+$, and this extends to arbitrary t by a simple approximation argument. □

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