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THE LATTICE PROPERTY OF UNIFORM AMARTS

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In this note it is shown that the L^1-bounded uniform amarts in l^1 form a vector lattice. This extends a result of Austin, Edgar and Ionescu Tulcea on real-valued L^1-bounded amarts and parallels a result of Ghoussoub on L^1-bounded order amarts.

1. Introduction. As a part of their proof of the amart convergence theorem, Austin, Edgar and Ionescu Tulcea [1] proved that the L^1-bounded amarts are stable under lattice operations; see also [2]. This lattice property is also of independent interest since it fails for L^1-bounded martingales, and it has subsequently been studied for several other classes of adapted sequences which generalize L^1-bounded amarts and for which almost sure convergence still obtains: The lattice property holds for L^1-bounded pramarts [15], but it fails for L^1-bounded martingales in the limit [4] and for L^1-bounded mils [15].

In a Banach lattice having the Radon–Nikodym property, the most important generalizations of real-valued amarts are order amarts and uniform amarts: The L^1-bounded order amarts form a vector lattice [8] and the L^1-bounded uniform amarts converge (strongly) almost surely [3]. If the Banach lattice is also isomorphic (as a topological vector lattice) to an AL-space and hence, by the Radon–Nikodym property, to $l^1(\Gamma)$ for some index set Γ, then every order amart is a uniform amart [8] and the inclusion is strict whenever the Banach lattice has infinite dimension [9]. It is therefore interesting to know whether, at least in this case, the L^1-bounded uniform amarts form a vector lattice as well.

In Section 3 of this paper we shall show that every L^1-bounded uniform amart in a Banach lattice having the Radon–Nikodym property is the difference of two positive L^1-bounded uniform amarts and that the L^1-bounded uniform amarts in a Banach lattice isomorphic to $l^1(\Gamma)$ form a vector lattice. The proofs of these results will be based on properties of vector measures of bounded variation which we shall recall in Section 2.

Throughout this paper, let $\mathbb{E}$ be a Banach lattice. We recall that $\mathbb{E}$ has *property* (P) if it is (under evaluation) the range of a positive contractive projection in its bidual and that $\mathbb{E}$ is an AL-*space* if $\|x + y\| = \|x\| + \|y\|$ holds for all $x, y \in \mathbb{E}_+$. Since a Banach lattice having the Radon–Nikodym property cannot contain a Banach sublattice isomorphic to $c_0(\mathbb{N})$ and since such a Banach lattice is a band in its bidual, every Banach lattice having the Radon–Nikodym property has property (P). Furthermore, every AL-space has property (P) and every AL-space having the Radon–Nikodym property is isometrically isomorphic to $l^1(\Gamma)$ for some index set Γ. For further information on the

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Radon–Nikodym property and on Banach lattices, we refer to the monographs by Diestel and Uhl [7] and by Schaefer [10], respectively. Probabilistic characterizations of Banach lattices isomorphic to $l^1(\Gamma)$ are to be found in [5, 8, 9, 14].

2. Vector measures of bounded variation. Let Ω be a set, let $\mathscr{G}$ be an algebra of subsets of Ω and let $\text{bva}(\mathscr{G}, \mathbb{E})$ denote the normed ordered vector space of all vector measures of bounded variation $\mathscr{G} \to \mathbb{E}$ (under the canonical linear operations and order relation, and the variation norm $\|\cdot\|(\Omega)$). The following results are proven in [11, Theorem 4.1.3 and Corollary 4.1.4]; see also [12].

PROPOSITION 2.1. *If $\mathbb{E}$ has property (P), then $\text{bva}(\mathscr{G}, \mathbb{E})$ is an order complete Banach lattice.*

PROPOSITION 2.2. *If $\mathbb{E}$ is an AL-space, then $\|\mu\|(\Omega) = \|\mu(\Omega)\|$ holds for each positive $\mu \in \text{bva}(\mathscr{G}, \mathbb{E})$.*

Let λ: $\mathscr{G} \to \mathbb{R}$ be a bounded additive set function and let $\text{bva}^{\lambda c}(\mathscr{G}, \mathbb{E})$ and $\text{bva}^{\lambda s}(\mathscr{G}, \mathbb{E})$ denote the normed ordered vector spaces of all λ-continuous (resp. λ-singular) vector measures in $\text{bva}(\mathscr{G}, \mathbb{E})$. The following Banach lattice version of the Lebesgue decomposition of vector measures of bounded variation is proven in [12]; for the Banach space case, see [7, Theorem I.5.9].

PROPOSITION 2.3. *If $\mathbb{E}$ has property (P), then $\text{bva}^{\lambda c}(\mathscr{G}, \mathbb{E})$ and $\text{bva}^{\lambda s}(\mathscr{G}, \mathbb{E})$ are order complete Banach lattices and projection bands of $\text{bva}(\mathscr{G}, \mathbb{E})$, and $\text{bva}(\mathscr{G}, \mathbb{E})$ is the order direct sum of these projection bands.*

Assume now that $(\Omega, \mathscr{G}, \lambda)$ is a probability space, let $L^1(\mathscr{G}, \lambda, \mathbb{E})$ denote the Banach lattice of all $\mathscr{G}$-measurable Bochner integrable random variables $\Omega \to \mathbb{E}$ and let J: $L^1(\mathscr{G}, \lambda, \mathbb{E}) \to \text{bva}^{\lambda c}(\mathscr{G}, \mathbb{E})$ be the linear operator given by

$$(JX)(A) := \int_A X \, d\lambda,$$

for all $X \in L^1(\mathscr{G}, \lambda, \mathbb{E})$ and $A \in \mathscr{G}$.

PROPOSITION 2.4. *If $\mathbb{E}$ has the Radon–Nikodym property, then J is an isometric vector lattice isomorphism from $L^1(\mathscr{G}, \lambda, \mathbb{E})$ onto $\text{bva}^{\lambda c}(\mathscr{G}, \mathbb{E})$.*

For a proof of Proposition 2.4 and further information on the properties of J, see [6].

3. Uniform amarts. Let $(\Omega, \mathscr{F}, \lambda)$ be a probability space, let $\{\mathscr{F}_n | n \in \mathbb{N}\}$ be an increasing sequence of sub-σ-algebras of $\mathscr{F}$ and define

$$\mathscr{F}_\infty := \{A \in \mathscr{F} | A \in \mathscr{F}_n \text{ for some } n \in \mathbb{N}\}.$$

Then $\mathscr{F}_\infty$ is an algebra. A mapping $\tau\colon \Omega \to \mathbb{N}$ is

1. a *stopping time* if $\{\tau = n\} \in \mathscr{F}_n$ holds for all $n \in \mathbb{N}$ and it is
2. *bounded* if $\sup_\Omega \tau(\omega) < \infty$.

Let T denote the directed $(\leq)$ set of all bounded stopping times. For $\tau \in T$, define

$$T(\tau) := \{\sigma \in T | \tau \leq \sigma\}$$

and

$$\mathscr{F}_\tau := \{A \in \mathscr{F}_\infty | A \cap \{\tau = n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}\}.$$

Then $\mathscr{F}_\tau$ is a σ-algebra. Furthermore, for $\tau \in T$ and a vector measure μ defined on a subalgebra of $\mathscr{F}$ containing $\mathscr{F}_\tau$, let $R_\tau \mu$ denote the restriction of μ to $\mathscr{F}_\tau$.

For an adapted sequence $\{X_n \in L^1(\mathscr{F}_n, R_n\lambda, \mathbb{E}) | n \in \mathbb{N}\}$, each $\tau \in T$ induces a random variable $X_\tau \in L^1(\mathscr{F}_\tau, R_\tau\lambda, \mathbb{E})$, given by

$$X_\tau(\omega) := \Sigma X_n(\omega) \chi_{\{\tau = n\}}(\omega)$$

for all $\omega \in \Omega$, and a vector measure $\mu_\tau \in \operatorname{bva}(\mathscr{F}_\tau, \mathbb{E})$, given by

$$\mu_\tau(A) := \int_A X_\tau \, d\lambda$$

for all $A \in \mathscr{F}_\tau$. An adapted sequence $\{X_n | n \in \mathbb{N}\}$ is

1. L^1-*bounded* if $\sup_{\mathbb{N}} \|\mu_n\|(\Omega) < \infty$;
2. of *class* (B) if $\sup_T \|\mu_\tau\|(\Omega) < \infty$;
3. a *martingale* if $\mu_\tau(\Omega) = \mu_1(\Omega)$ holds for all $\tau \in T$;
4. a *quasimartingale* if $\Sigma \|\mu_n - R_n \mu_{n+1}\|(\Omega) < \infty$;
5. a *uniform potential* if $\lim \|\mu_\tau\|(\Omega) = 0$;
6. a *uniform amart* if there exists a vector measure $\mu\colon \mathscr{F}_\infty \to \mathbb{E}$ satisfying $\lim \|\mu_\tau - R_\tau \mu\|(\Omega) = 0$.

The vector measure μ associated with a uniform amart $\{X_n | n \in \mathbb{N}\}$ is sometimes said to be the *limit measure* of $\{X_n | n \in \mathbb{N}\}$ (although it need not be countably additive). In particular, if $\{X_n | n \in \mathbb{N}\}$ is a martingale, then $\{X_n | n \in \mathbb{N}\}$ is a uniform amart and its limit measure μ is given by $\mu(A) := \lim \int_A X_n \, d\lambda$ for all $A \in \mathscr{F}_\infty$ and satisfies $R_\tau \mu = \mu_\tau$ for all $\tau \in T$. The following result indicates that the properties of uniform amarts are closely connected with those of their limit measure.

PROPOSITION 3.1. *For a uniform amart $\{X_n | n \in \mathbb{N}\}$ and its limit measure μ, the following are equivalent*:

(a) $\{X_n | n \in \mathbb{N}\}$ *is L^1-bounded.*
(b) $\{X_n | n \in \mathbb{N}\}$ *is of class (B).*
(c) μ *has bounded variation.*

For a proof of Proposition 3.1, see, e.g., [11, Corollary 3.5.4].

LEMMA 3.2. *The uniform potentials form a vector lattice.*

This is obvious from the fact that the L^1-norm is a lattice norm.

LEMMA 3.3. *If* $\mathbb{E}$ *has the Radon–Nikodym property, then every* L^1*-bounded martingale* $\{Y_n | n \in \mathbb{N}\}$ *with limit measure* φ *is the difference of two positive* L^1*-bounded martingales* $\{Y_n' | n \in \mathbb{N}\}$ *and* $\{Y_n'' | n \in \mathbb{N}\}$ *satisfying* $\int_A Y_n' \, d\lambda \leq \varphi^+(A)$ *and* $\int_A Y_n'' \, d\lambda \leq \varphi^-(A)$ *for all* $n \in \mathbb{N}$ *and* $A \in \mathscr{F}_n$.

PROOF. By Propositions 3.1 and 2.1, φ has bounded variation and satisfies $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+ := \sup\{\varphi, 0\}$ and $\varphi^- := \sup\{-\varphi, 0\}$.

By Propositions 2.3 and 2.4, there exist positive random variables $U_n', U_n'' \in L^1(\mathscr{F}_n, R_n\lambda, \mathbb{E})$ and positive vector measures $\eta_n', \eta_n'' \in \mathrm{bva}^{R_n\lambda s}(\mathscr{F}_n, \mathbb{E})$ satisfying

$$(R_n\varphi^+)(A) = \int_A U_n' \, d\lambda + \eta_n'(A)$$

and

$$(R_n\varphi^-)(A) = \int_A U_n'' \, d\lambda + \eta_n''(A)$$

and thus

$$\begin{aligned}\int_A Y_n \, d\lambda &= (R_n\varphi)(A) \\ &= (R_n\varphi^+)(A) - (R_n\varphi^-)(A) \\ &= \int_A U_n' \, d\lambda + \eta_n'(A) - \int_A U_n'' \, d\lambda - \eta_n''(A)\end{aligned}$$

for all $n \in \mathbb{N}$ and $A \in \mathscr{F}_n$. By Proposition 2.3, this yields $\eta_n' - \eta_n'' = 0$ and thus

$$Y_n = U_n' - U_n''$$

for all $n \in \mathbb{N}$.

By [13, Theorem 3.7], the positive adapted sequences $\{U_n' | n \in \mathbb{N}\}$ and $\{U_n'' | n \in \mathbb{N}\}$ are L^1-bounded quasimartingales and hence L^1-bounded uniform amarts [3], and it now follows from the Riesz decomposition of uniform amarts [3] that there exist martingales $\{Y_n' | n \in \mathbb{N}\}$ and $\{Y_n'' | n \in \mathbb{N}\}$ and uniform potentials $\{V_n' | n \in \mathbb{N}\}$ and $\{V_n'' | n \in \mathbb{N}\}$ satisfying

$$U_n' = Y_n' + V_n'$$

and

$$U_n'' = Y_n'' + V_n''$$

and thus

$$Y_n = Y_n' + V_n' - Y_n'' - V_n''$$

for all $n \in \mathbb{N}$. Since $\{Y_n - Y_n' + Y_n'' | n \in \mathbb{N}\}$ is a martingale, whereas $\{V_n' - V_n'' | n \in \mathbb{N}\}$ is a uniform potential, we have $V_n' - V_n'' = 0$ and thus

$$Y_n = Y_n' - Y_n''$$

for all $n \in \mathbb{N}$.

Therefore, $\{Y_n | n \in \mathbb{N}\}$ is the difference of the martingales $\{Y_n' | n \in \mathbb{N}\}$ and $\{Y_n'' | n \in \mathbb{N}\}$. Furthermore, since the limit measure of $\{Y_n' | n \in \mathbb{N}\}$ agrees with that of $\{U_n' | n \in \mathbb{N}\}$, which is positive and dominated by φ^+, we see that $\{Y_n' | n \in \mathbb{N}\}$ is positive and L^1-bounded and satisfies

$$\int_A Y_n' \, d\lambda \leq \varphi^+(A)$$

for all $n \in \mathbb{N}$ and $A \in \mathscr{F}_n$. By a similar argument, $\{Y_n'' | n \in \mathbb{N}\}$ is positive and L^1-bounded and satisfies

$$\int_A Y_n'' \, d\lambda \leq \varphi^-(A)$$

for all $n \in \mathbb{N}$ and $A \in \mathscr{F}_n$. □

Combining the Riesz decomposition of L^1-bounded uniform amarts with Lemmas 3.2 and 3.3 and using the fact that martingales and uniform potentials are uniform amarts, we obtain the following decomposition of uniform amarts.

COROLLARY 3.4. *If $\mathbb{E}$ has the Radon–Nikodym property, then every L^1-bounded uniform amart is the difference of two positive L^1-bounded uniform amarts.*

We now turn to the main result of this note.

THEOREM 3.5. *If $\mathbb{E}$ is a Banach lattice isomorphic (as a topological vector lattice) to $l^1(\Gamma)$ for some index set Γ, then the L^1-bounded uniform amarts form a vector lattice.*

PROOF. We may assume that $\mathbb{E}$ is an AL-space having the Radon–Nikodym property.

Consider an L^1-bounded uniform amart $\{X_n | n \in \mathbb{N}\}$. By the Riesz decomposition of uniform amarts, there exists an L^1-bounded martingale $\{Y_n | n \in \mathbb{N}\}$ and a uniform potential $\{Z_n | n \in \mathbb{N}\}$ satisfying

$$X_n = Y_n + Z_n$$

for all $n \in \mathbb{N}$.

Let φ denote the limit measure of $\{Y_n | n \in \mathbb{N}\}$ and let $\{Y_n' | n \in \mathbb{N}\}$ and $\{Y_n'' | n \in \mathbb{N}\}$ be the positive L^1-bounded martingales given by Lemma 3.3. Then we have

$$Y_n = Y_n' - Y_n''$$

for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\le \int_\Omega Y_\tau' \wedge Y_\tau'' \, d\lambda \\ &\le \inf_{\mathscr{F}_\tau} \left(\int_A Y_\tau' \, d\lambda + \int_{\Omega \setminus A} Y_\tau'' \, d\lambda \right) \\ &\le \inf_{\mathscr{F}_\tau} (\varphi^+(A) + \varphi^-(\Omega \setminus A)) \end{aligned}$$

for all $\tau \in T$,

$$\inf_{\mathscr{F}_\sigma} (\varphi^+(A) + \varphi^-(\Omega \setminus A)) \le \inf_{\mathscr{F}_\tau} (\varphi^+(A) + \varphi^-(\Omega \setminus A))$$

for all $\tau \in T$ and $\sigma \in T(\tau)$ and

$$\inf_T \inf_{\mathscr{F}_\tau} (\varphi^+(A) + \varphi^-(\Omega \setminus A)) \le \inf_{\mathscr{F}_\infty} (\varphi^+(A) + \varphi^-(\Omega \setminus A)) = 0.$$

This implies that the net $\{\int_\Omega Y_\tau' \wedge Y_\tau'' \, d\lambda | \tau \in T\}$ decreases to 0. By Proposition 2.2, this yields

$$\lim \int_\Omega \|Y_\tau' \wedge Y_\tau''\| \, d\lambda = \lim \left\| \int_\Omega Y_\tau' \wedge Y_\tau'' \, d\lambda \right\| = 0,$$

which means that $\{Y_n' \wedge Y_n'' | n \in \mathbb{N}\}$ is a uniform potential. By Lemma 3.2, $\{Z_n^+ | n \in \mathbb{N}\}$ and $\{Z_n^- | n \in \mathbb{N}\}$ are uniform potentials satisfying

$$Z_n = Z_n^+ - Z_n^-$$

and thus

$$X_n = (Y_n' + Z_n^+) - (Y_n'' + Z_n^-)$$

for all $n \in \mathbb{N}$. Furthermore, letting $W_n := (Y_n' + Z_n^+) \wedge (Y_n'' + Z_n^-)$, we have

$$0 \le W_n \le Y_n' \wedge Y_n'' + |Z_n|$$

for all $n \in \mathbb{N}$, which implies that $\{W_n | n \in \mathbb{N}\}$ is a uniform potential.

Therefore, $\{Y_n' + Z_n^+ - W_n | n \in \mathbb{N}\}$ and $\{Y_n'' + Z_n^- - W_n | n \in \mathbb{N}\}$ are positive L^1-bounded uniform amarts satisfying

$$X_n = (Y_n' + Z_n^+ - W_n) - (Y_n'' + Z_n^- - W_n)$$

and

$$0 = (Y_n' + Z_n^+ - W_n) \wedge (Y_n'' + Z_n^- - W_n)$$

and thus

$$X_n^+ = Y_n' + Z_n^+ - W_n$$

and

$$X_n^- = Y_n'' + Z_n^- - W_n$$

for all $n \in \mathbb{N}$, which implies that $\{|X_n| \, | n \in \mathbb{N}\}$ is an L^1-bounded uniform amart, as was to be shown. □

It remains an open question whether the assertion of Theorem 3.5 is valid in arbitrary Banach lattices having the Radon–Nikodym property.

We finally remark that the L^1-bounded uniform amarts in a Banach lattice isomorphic to $l^1(\Gamma)$ form a normed vector lattice under the norm $\|\cdot\|_T$, given by

$$\|\{X_n | n \in \mathbb{N}\}\|_T := \sup_T \int_\Omega \|X_\tau\| \, d\lambda$$

for each L^1-bounded uniform amart $\{X_n | n \in \mathbb{N}\}$. This follows from Proposition 3.1 and Theorem 3.5, and it can be shown that this normed vector lattice is even a Banach lattice. The same is also true for the uniform potentials in an arbitrary Banach lattice. We omit the details.

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