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AN ALGEBRAIC CONSTRUCTION OF A CLASS OF ONE-DEPENDENT PROCESSES

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A special class of stationary one-dependent two-valued stochastic processes is defined. We associate to each member of this class two parameter values, whereby different members receive different parameter values. For any given values of the parameters, we show how to determine whether:

1. a process exists having the given parameter values, and if so,
2. this process can be obtained as a two-block factor from an independent process.

This determines a two-parameter subfamily of the class of stationary one-dependent two-valued stochastic processes which are not two-block factors of independent processes.

Introduction. A discrete time stochastic process $X = (X_n)$ is *one-dependent* if at any given time n, its past $(X_k)_{k<n}$ is independent of its future $(X_k)_{k>n}$. In contrast to the Markovian concept, a weakening of independence which has been investigated thoroughly, no knowledge of the present value X_n is assumed. One-dependent processes arise naturally as limits of rescaling operations in renormalization theory (see, e.g., O'Brien [8]). In an analogous manner m-dependence ($m \geq 1$) can be defined, considering the present to be given by m successive observations. The works [2], [4]–[7] and [10] deal with various aspects of m-dependent processes.

Examples of m-dependent processes are given by so-called $(m+1)$-*block factors*: Let $Y = (Y_n)$ be an independent process and f a function of $m+1$ variables. If we define

$$X_n = f(Y_n, \ldots, Y_{n+m}),$$

then the $(m+1)$-block factor $X = (X_n)$ is an m-dependent process.

In this article we restrict our attention to one-dependent processes X which are *stationary* and *assume two values only*, denoted in the following by 0 and 1. It is not difficult to see that if X is a *two-block factor*, then it may be assumed that the underlying independent sequence Y is identically distributed with the

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uniform distribution on the unit interval as the common distribution, and that f can be identified with the subset A of the unit square on which it assumes one of the values, say 1. Hence the distribution of a two-block factor is completely described by a measurable subset A of the unit square, which we call an *indicator* of the two-block factor. Of course, different A's may give rise to two-block factors having the same distributions.

It is natural to ask ([6], [7], [9]) whether all one-dependent processes arise as two-block factors. Under certain extremal conditions, this is true ([3]). However, in the following we produce a two-parameter family of stationary 0–1-valued one-dependent processes which are *not* two-block factors. This extends a one-parameter family of such examples recently obtained by two of us [1] based on unpublished results of the other two of us.

The plan of the article is as follows. In Section 1 we show that every one-dependent process can be parametrized by the collection of probabilities it associates to runs of 1's. Here we define *cylinder functions* for arbitrary parameter values and note that a one-dependent process exists if and only if the corresponding one-dependent cylinder function assumes only nonnegative values.

In general, it seems to be difficult to decide whether a given set of parameter values yields a positive cylinder function and thus a process. However, if we restrict our attention to a class of cylinder functions which we call *special* (for lack of a better name), defined by requiring that three or more 1's in a row have probability 0, then an effective algorithm can be given to decide whether a one-dependent process, with prescribed values of the probabilities α of a single 1 and β of two successive 1's, exists. In Section 2 we present the basis for this algorithm.

Section 3 contains a classification of those pairs (α, β) corresponding to special two-block factors. This section is essentially independent of the other results.

In Section 4 we continue the development of our algorithm, which has the following form. Two mappings ϕ_0 and ϕ_1, depending on α and β, are defined on $\mathbb{R}^2$, and a special one-dependent process exists for (α, β) if and only if the orbit of $(1,1)$ under successive applications of ϕ_0 and ϕ_1 in any order always remains in the unit square. Section 4 is devoted to dynamical properties of the more complicated mapping ϕ_0.

Theorem 5 of Section 5 contains the final form of our algorithm, and the remainder of this section is devoted to the determination of those (α, β) giving rise to special one-dependent processes. Although we have an effective decision procedure for any given pair (α, β), the time needed for decision grows as (α, β) approaches $(\frac{1}{3}, \frac{1}{27})$ and no closed form expression for the admissible set of parameters in a neighborhood of this point has been found. Away from this point, things become easier, and several results are given. For example, if $0 \le \alpha \le \frac{1}{4}$, then a special one-dependent process exists for every $0 \le \beta \le \frac{1}{4}\alpha$ (and no other β), whereas a two-block factor requires (for $0 \le \alpha \le \frac{2}{9}$)

$$0 \le \beta \le \tfrac{1}{8}(1 + \sqrt{1 - 4\alpha})\alpha.$$

The sum of our investigations is recorded in Figure 2 of Section 5.

It is the opinion of the authors that this paper raises more questions than it resolves. We mention two such questions. First of all, our methods are algebraic in nature and seem to give no probabilistic mechanism to produce the processes which we have discovered. In particular, we have not been able to determine if they are m-block factors for some $m \geq 3$. Second, our methods for studying ϕ_0 and ϕ_1 are at best amateuristic, and a more canonical approach is desirable.

1. Cylinder functions. Let W be the set of all finite sequences of 0's and 1's. An element of W is called a *word*. The empty word will be denoted by e and the word consisting of n 1's by 1^n. If $w_1, \ldots, w_n \in W$, then $w = w_1 \cdots w_n \in W$ is the concatenation of the words $w_1, \ldots, w_n$, and the w_i are *subwords* of w.

DEFINITION. A (normalized) *cylinder function* is a mapping

$$\mu\colon W \to \mathbb{R}$$

such that

(i) $$\mu(e) = 1,$$

(ii) $$\mu(w) = \mu(0w) + \mu(1w), \qquad w \in W,$$

(iii) $$\mu(w) = \mu(w0) + \mu(w1), \qquad w \in W.$$

The cylinder function μ is *positive* if

$$\mu(w) \geq 0, \qquad w \in W,$$

and *one-dependent* if

$$\mu(v)\mu(w) = \mu(v0w) + \mu(v1w), \qquad v, w \in W.$$

By elementary measure theory, the set of positive cylinder functions is in one-to-one correspondence with the set of distributions of stationary 0–1-valued discrete time stochastic processes, $\mu(w)$ being the probability of "seeing" the word w. Moreover, such a process is one-dependent if and only if its corresponding cylinder function is one-dependent.

THEOREM 1. *Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be any sequence of real numbers. Then there exists a unique one-dependent cylinder function μ_γ such that*

$$\mu_\gamma(1^n) = \gamma_n, \qquad n \geq 1.$$

PROOF. In the proof of this theorem and the next theorem, we denote the number of zeroes in a word w by $n_0(w)$. Set $\gamma_0 = 1$. The requirement, together with (i) of the definition of a cylinder function, defines $\mu_\gamma(w)$ for all $w \in W$ with $n_0(w) = 0$. We now proceed by induction on $n_0(w)$, as follows. If $w \in W$ with $n_0(w) > 0$, then clearly

$$w = 1^n 0 v$$

for some $n \geq 0$ and $v \in W$, and

$$n_0(v) = n_0(w) - 1.$$

One-dependence now dictates that

$$\mu_\gamma(1^n)\mu_\gamma(v) = \mu_\gamma(w) + \mu_\gamma(1^{n+1}v),$$

and since $n_0(1^{n+1}v) = n_0(v) < n_0(w)$, the formula

$$\mu_\gamma(w) = \gamma_n\mu_\gamma(v) - \mu_\gamma(1^{n+1}v)$$

defines μ_γ inductively on all of W. Straightforward induction arguments now show that μ_γ is a one-dependent cylinder function, whose uniqueness is obvious from the inductive definition. □

THEOREM 2. *If for some $m \geq 1$ we have*

$$\gamma_m = \gamma_{m+1} = \cdots = 0,$$

and if 1^m is a subword of $w \in W$, then

$$\mu_\gamma(w) = 0.$$

PROOF. The hypothesis states that $\mu_\gamma(w) = 0$ if $n_0(w) = 0$ and if 1^m is a subword of w. Now proceed by induction: If $n_0(w) > 0$, write as above

$$w = 1^n 0 v,$$

with

$$\mu_\gamma(w) = \gamma_n\mu_\gamma(v) - \mu_\gamma(1^{n+1}v).$$

If 1^m is a subword of w, then either $n \geq m$ and $\gamma_n = 0$ or $n < m$ and 1^m is a subword of v. In both cases, 1^m is a subword of $1^{n+1}v$, and hence $\mu_\gamma(w) = 0$ by induction. □

In the sequel we restrict our attention exclusively to one-dependent cylinder functions $\mu = \mu_\gamma$ for which $\gamma_3 = \gamma_4 = \cdots = 0$. For the sake of brevity (and in want of a more suitable name), such μ are called *special*. By Theorem 2, if μ is special and if 111 is a subword of w, then $\mu(w) = 0$. Hence positive special cylinder functions correspond bijectively to stationary 0–1-valued one-dependent processes for which the probability of three 1's in a row is 0; we refer to these as *special processes*.

REMARK 1. Suppose that μ is a one-dependent cylinder function such that $\mu(w) = 0$ whenever 101 is a subword of w. Set $\alpha = \mu(1)$ and $\beta = \mu(11)$. Then

$$\mu(11111) = \mu(11) \cdot \mu(11) - \mu(11011) = \beta^2,$$

but also

$$\begin{aligned}\mu(11111) &= \mu(111) \cdot \mu(1) - \mu(11101)\\ &= \mu(1)(\mu(1) \cdot \mu(1) - \mu(101))\\ &= \mu(1) \cdot \mu(1) \cdot \mu(1) = \alpha^3.\end{aligned}$$

Hence $\beta^2 = \alpha^3$. This remark is intended to persuade the reader to examine the induction arguments of the above proofs carefully.

REMARK 2. Theorem 1 can be viewed as a parametrization result for one-dependent cylinder functions with parameter γ: Each cylinder function yields a parameter, different cylinder functions possess different parameters and γ is the parameter of a process if and only if μ_γ is positive. In the sequel, we set

$$\gamma_1 = \alpha, \qquad \gamma_2 = \beta, \qquad \gamma_3 = \gamma_4 = \cdots = 0$$

and discuss the *admissible pairs* (α, β) yielding special processes.

2. Positivity of special cylinder functions. In this section we derive a necessary and sufficient condition for the positivity of the special cylinder function defined by

$$\mu(1) = \alpha, \qquad \mu(11) = \beta, \qquad \mu(1^n) = 0, \qquad n \geq 3.$$

By Theorem 2, we need only examine words not having 111 as a subword. Let V be the set of all such words and denote by V_n those words of V having exactly n 0's. Then

$$V_0 = \{e, 1, 11\},$$

and if we define the set of words

$$U = \{0, 10, 110\},$$

then for each $n \geq 0$ the set of words V_n can be identified with

$$U^n \times V_0.$$

That is, each $v \in V_n$ has a unique representation

$$v = u_n u_{n-1} \cdots u_1 v_0,$$

with $v_0 \in V_0$ and $u_k \in U$, $1 \leq k \leq n$.

We now describe an algorithm for calculating the values of $\mu(v)$, $v \in V$. For each $v \in V$, define the column vector $\mathbf{v} \in \mathbb{R}^3$ by

$$\mathbf{v} = \begin{pmatrix} x(v) \\ y(v) \\ z(v) \end{pmatrix}$$

with

$$x(v) = \mu(0v), \qquad y(v) = \mu(10v), \qquad z(v) = \mu(110v).$$

Also set

$$\mathbf{f} = \begin{pmatrix} \mu(e) \\ \mu(1) \\ \mu(11) \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}.$$

Finally, define the 3×3 matrices

$$M_0 = \begin{pmatrix} 1 & -1 & 0 \\ \alpha & 0 & -1 \\ \beta & 0 & 0 \end{pmatrix}, \qquad M_{10} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{pmatrix}, \qquad M_{110} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \alpha \\ 0 & 0 & \beta \end{pmatrix}$$

indexed by elements of U.

Theorem 3. *If $v \in V_n$, then*

$$\mathbf{v} = M_{u_n} M_{u_{n-1}} \cdots M_{u_1} M_{v_0 0} \mathbf{f}.$$

Proof. The case $n = 0$ is easily checked from the definitions. Now use induction on n, together with the following formulas:

$$x(v) = \mu(0v) = \mu(v) - \mu(1v) = \begin{cases} \mu(v) - \mu(10v'), & \text{if } v = 0v', \\ \mu(v) - \mu(110v'), & \text{if } v = 10v', \\ \mu(v), & \text{if } v = 110v', \end{cases}$$

$$y(v) = \mu(10v) = \alpha\mu(v) - \mu(11v) = \begin{cases} \alpha\mu(v) - \mu(110v'), & \text{if } v = 0v', \\ \alpha\mu(v), & \text{if } v = (1)10v', \end{cases}$$

$$z(v) = \mu(110v) = \beta\mu(v) - \mu(111v) = \beta\mu(v).$$

The formula for $x(v)$ shows that the first rows of the matrices M are correct, and those for $y(v)$ and $z(v)$ verify the second and third row, respectively. □

Corollary. *For $(x, y) \in \mathbb{R}^2 \setminus \{(x, y) \colon xy = 0\}$ set*

$$\phi_0(x, y) = \left(1 - \frac{\alpha y}{x}, 1 - \frac{\beta}{\alpha x}\right),$$

$$\phi_1(x, y) = \left(1 - \frac{\beta}{\alpha y}, 1\right).$$

Then the pair (α, β) is admissible if and only if either $\alpha = \beta = 0$ or $0 < \alpha \leq 1$, $0 \leq \beta \leq \alpha$, and all iterates of the point $(1, 1)$ under successive applications of ϕ_0 and ϕ_1 in any order remain in the unit square $S = \{(x, y) \colon 0 < x \leq 1, 0 < y \leq 1\}$.

Proof. Theorem 3 yields all values $\mu(v), v \in V$, as iterates of $\mathbf{f}$ under the three M-matrices. In testing positivity we can disregard M_{110} since it brings us back to a multiple of $\mathbf{f}$. Next, reduce the dimension by normalizing such that the third coordinate is always equal to β, i.e., set

$$\Phi_0(x, y, \beta) = \frac{1}{x} M_0 \begin{pmatrix} x \\ y \\ \beta \end{pmatrix}, \qquad \Phi_1(x, y, \beta) = \frac{1}{y} M_{10} \begin{pmatrix} x \\ y \\ \beta \end{pmatrix}$$

and then drop β to obtain

$$\Phi_0(x, y) = \left(1 - \frac{y}{x}, \alpha - \frac{\beta}{x}\right),$$

$$\Phi_1(x, y) = \left(1 - \frac{\beta}{y}, \alpha\right),$$

with initial value $(1, \alpha)$. Clearly $\alpha = \mu(1)$ must lie in the unit interval, and $0 \leq \beta = \mu(11) \leq \alpha$ is also necessary. The case $\alpha = \beta = 0$ yields the special process which is given by all 0's, and if $\alpha > 0$, then we can replace y by αy, which results in the given ϕ_0 and ϕ_1, with initial value $(1,1)$. Noting now that if (x, y), $\phi_0(x, y)$, $\phi_1(x, y)$ have positive coordinates, then $\phi_0(x, y)$ and $\phi_1(x, y)$ cannot have a coordinate greater than 1 and that $x = 0$ or $y = 0$ leads to a negative coordinate, we see that the proof is finished. □

3. Determination of the parameter set corresponding to two-block factors. Let μ_A be the cylinder function corresponding to a two-block factor with indicator A, such that $\mu_A(111) = 0$. In this section we determine the range of possible values for $\alpha = \mu_A(1)$ and $\beta = \mu_A(11)$. By the definition, we have for any $n \geq 1$,

$$\mu_A(1^n) = \int_0^1 \cdots \int_0^1 1_A(x_0, x_1) \cdots 1_A(x_{n-1}, x_n)\, dx_0 \cdots dx_n.$$

Moreover, if T: $[0,1] \to [0,1]$ preserves Lebesgue measure, then A and $(T \times T)^{-1}(A)$ give rise to the same process.

Examples of sets A for which $\mu_A(111) = 0$ can be obtained in the following manner. Let $a, b \in [0,1]$ with $a \leq b$ and define

$$F(a, b) = ([a, b) \times [0, a)) \cup ([b, 1] \times [0, b)).$$

If $A \subseteq F(a, b)$ and if $(x_0, x_1) \in A$, $(x_1, x_2) \in A$, then clearly $x_1 < b$ and hence $x_2 < a$, so that no choice of x_3 permits $(x_2, x_3) \in A$. That is,

$$A \subseteq F(a, b) \Rightarrow \mu_A(111) = 0.$$

The following lemma shows that, up to a measure preserving transformation T, the reverse implication is valid.

LEMMA. *If $\mu_A(111) = 0$, then there exists a transformation T: $[0,1] \to [0,1]$ preserving Lebesgue measure and $a, b \in [0,1]$ with $a \leq b$ such that*

$$(T \times T)^{-1} A \subseteq F(a, b)$$

modulo Lebesgue measure on the unit square.

PROOF. Define

$$A_2 = \left\{x_2 \in [0,1]: \int_0^1 1_A(x_2, x_3)\, dx_3 > 0\right\},$$

$$A_1 = \left\{x_1 \in [0,1]: \int_{A_2} 1_A(x_1, x_2)\, dx_2 > 0\right\},$$

$$A_0 = \left\{x_0 \in [0,1]: \int_{A_1} 1_A(x_0, x_1)\, dx_1 > 0\right\}.$$

Then $A_2 \supseteq A_1 \supseteq A_0$, and the formula

$$0 = \mu_A(111) = \int_{A_0}\left(\int_{A_1} 1_A(x_0, x_1)\left(\int_{A_2} 1_A(x_1, x_2)\left(\int_0^1 1_A(x_2, x_3)\, dx_3\right) dx_2\right) dx_1\right) dx_0$$

allows us to conclude that the Lebesgue measure of A_0 is 0. Choosing

$$a = 1 - \text{Lebesgue measure}(A_2),$$
$$b = 1 - \text{Lebesgue measure}(A_1)$$

and T measure preserving with

$$T((a,1]) = A_2,$$
$$T((b,1]) = A_1$$

yields the desired result. □

In accordance with our previous usage, a set A such that $\mu_A(111) = 0$ will be called *special.* In order to calculate α and β, note that the first formula of this section for $n = 1$ and $n = 2$ reduces to

$$\alpha = \mu_A(1) = \text{Lebesgue measure}(A)$$

and

$$\beta = \mu_A(11) = \int_0^1 H_A(x) V_A(x)\, dx,$$

where $H_A(x)$ and $V_A(x)$ denote the Lebesgue measures of the horizontal and vertical sections of A at x, respectively. In particular, if $A \subseteq F(a, b)$, the part of A lying in the lower right rectangle $[b,1] \times [0, a)$ does not contribute to $\beta = \mu_A(11)$. A simple but tedious calculation (which we omit) now shows that for fixed α, the minimal value of β occurs when $A = F(a, b)$ for suitable a and b, and the maximal value of β (for $0 \leq \alpha \leq 2/9$) occurs when

$$A = G(a, b) := F(a, b) \setminus ([b,1] \times [0, a)),$$

again for suitable a and b. Further reduction eventually produces

THEOREM 4. *Let μ be a cylinder function with $\alpha = \mu(1)$, $\beta = \mu(11)$ and $0 = \mu(1^n)$ for $n \geq 3$. Then μ is the cylinder function of a two-block factor if and only if*

(i) $0 \leq \alpha \leq \frac{1}{3}$ *and*
(ii) $m(\alpha) \leq \beta \leq M(\alpha)$, *where*

$$m(\alpha) = \begin{cases} 0, & 0 \leq \alpha \leq \frac{1}{4}, \\ \frac{1}{3}\alpha - \frac{2}{27}\left\{1 + (1 - 3\alpha)^{3/2}\right\}, & \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

and

$$M(\alpha) = \begin{cases} \frac{1}{8}(1 + \sqrt{1 - 4\alpha})\alpha, & 0 \leq \alpha \leq \frac{2}{9}, \\ \frac{1}{27}, & \frac{2}{9} \leq \alpha \leq \frac{1}{3}. \end{cases}$$

For related results and similar calculation we refer to de Valk [2]. In the next sections we shall need the following observation.

LEMMA. *If $A = F(a, b)$ and $\alpha = \mu_A(1)$, $\beta = \mu_A(11)$, then the equation*

$$x^3 - x^2 + \alpha x - \beta = 0$$

has the three real roots $r_1 = a$, $r_2 = b - a$ and $r_3 = 1 - b$.

PROOF. One easily calculates

$$\alpha = a(b - a) + (1 - b)b = r_1 r_2 + r_2 r_3 + r_1 r_3$$

and

$$\beta = a(1 - b)(b - a) = r_1 r_2 r_3. \qquad \square$$

4. A study of ϕ_0. Before using the corollary of Section 2 to determine admissible pairs (α, β), we investigate the mapping ϕ_0. Recall that for fixed $0 < \alpha \leq 1$ and $0 \leq \beta \leq \alpha$,

$$\phi_0(x, y) = \left(1 - \frac{\alpha y}{x}, 1 - \frac{\beta}{\alpha x}\right).$$

4.1. *Fixed points.* These are given by solutions to the equations

$$x = 1 - \frac{\alpha y}{x}$$

and

$$y = 1 - \frac{\beta}{\alpha x};$$

eliminating y results in

$$\rho(x) := x^3 - x^2 + \alpha x - \beta = 0.$$

This equation can have either one real root and two complex roots, or three real roots. As $\rho(0) = -\beta \leq 0$ and $\rho(1) = \alpha - \beta \geq 0$, one root must lie in the unit interval. The sum of the roots is 1, so that if the other two are also real, they also lie in the unit interval, because they have the same sign. If we denote these roots by r_1, r_2, r_3 and set $a = r_1$, $b = r_1 + r_2$, then it follows from the lemma at the end of Section 3 that the cylinder function μ corresponding to the pair (α, β) is equal to μ_A, with $A = F(a, b)$. Hence we have proved the

PROPOSITION. *If $x^3 - x^2 + \alpha x - \beta = 0$ has three real roots and if $0 < \alpha \leq 1$, $0 \leq \beta \leq \alpha$, then the pair (α, β) is admissible and corresponds to a two-block factor with indicator $A = F(a, b)$ for suitable a and b.*

Having discovered the situation for three real roots, we now restrict our attention to those (α, β) for which

$$x^3 - x^2 + \alpha x - \beta = 0$$

has only one real root $x_0 \in [0, 1]$. If now $x_0 = 0$, then we have $\beta = 0$ and $\alpha > \frac{1}{4}$, and a simple application of the corollary of Section 2 shows that (α, β) cannot be admissible. Hence we may also assume that $\beta > 0$ and $x_0 > 0$. Now set

$$y_0 = 1 - \frac{\beta}{\alpha x_0}.$$

By the foregoing, $\phi_0(x_0, y_0) = (x_0, y_0)$.

4.2. *Regions of increase and decrease.* Recall that

$$S = \{(x, y): 0 < x \leq 1, 0 < y \leq 1\}$$

and define

$$X_+ = \left\{(x, y) \in S: 1 - \frac{\alpha y}{x} \geq x\right\},$$

$$X_- = \left\{(x, y) \in S: 1 - \frac{\alpha y}{x} \leq x\right\},$$

$$Y_+ = \left\{(x, y) \in S: 1 - \frac{\beta}{\alpha x} \geq y\right\},$$

$$Y_- = \left\{(x, y) \in S: 1 - \frac{\beta}{\alpha x} \leq y\right\},$$

$$\mathrm{I} = X_- \cap Y_-,$$

$$\mathrm{II} = X_+ \cap Y_-,$$

$$\mathrm{III} = X_+ \cap Y_+,$$

$$\mathrm{IV} = X_- \cap Y_+,$$

thus dividing S into four regions whose boundaries are segments of the parabola

$$P: y = \frac{1}{\alpha} x(1 - x)$$

and/or the hyperbola

$$H: y = 1 - \frac{\beta}{\alpha x}.$$

Figure 1 has two parts, according to whether $\alpha \leq \frac{1}{4}$ or $\alpha > \frac{1}{4}$.
By definition:

(i) If $(x, y) \in \mathrm{I}$, then $\phi_0(x, y)$ is to the left and below (x, y).
(ii) If $(x, y) \in \mathrm{II}$, then $\phi_0(x, y)$ is to the right and below (x, y).
(iii) If $(x, y) \in \mathrm{III}$, then $\phi_0(x, y)$ is to the right and above (x, y).
(iv) If $(x, y) \in \mathrm{IV}$, then $\phi_0(x, y)$ is to the left and above (x, y).

4.3. *Line segments.* It is trivial to check that if L is a straight line segment in S, then $\phi_0(L)$ is a straight line segment.

4.4. *Image of P.* It is trivial to check that $\phi_0(P) \subseteq H$.

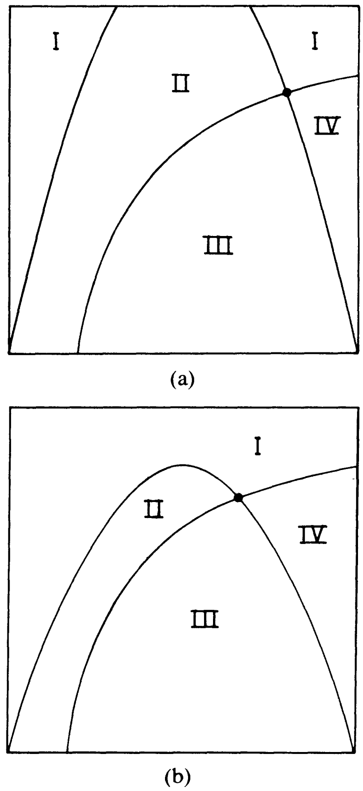

FIG. 1. (a) $\alpha \leq \frac{1}{4}$; (b) $\alpha > \frac{1}{4}$ [$\bullet = (x_0, y_0)$].

4.5. *Images of regions.* It follows from Sections 4.3, 4.4 and the definitions that

$$\phi_0(\mathrm{I}) \cap S \subseteq \mathrm{I} \cup \mathrm{II},$$
$$\phi_0(\mathrm{II}) \cap S \subseteq \mathrm{III},$$
$$\phi_0(\mathrm{III}) \cap S \subseteq \mathrm{III} \cup \mathrm{IV},$$
$$\phi_0(\mathrm{IV}) \cap S \subseteq \mathrm{I}.$$

4.6. *Entering region* II. We now show that our hypothesis of one real root (= one point of intersection of P and H) implies that for each $(x, y) \in \mathrm{I}$, there exists n such that

$$\phi_0^{(n)}(x, y) \notin \mathrm{I},$$

i.e., either

$$\phi_0^{(n)}(x, y) \in \mathrm{II}$$

or it leaves S. Assume the contrary. Then Sections 4.2 and 4.3 imply that some line through (x_0, y_0) must be taken into itself by ϕ_0. If L is such a line then either

(i) L is vertical or
(ii) L is tangent to P at (x_0, y_0) or
(iii) L intersects P (not necessarily in S) at some point $(x_1, y_1) \neq (x_0, y_0)$.

Now (i) is impossible because $1 - (\alpha y)/x_0$ cannot be equal to x_0 for more than one value of y and (ii) implies (by Section 4.4) that P and H are tangent at (x_0, y_0), which says that x_0 is a root of multiplicity three of $\rho(x) = 0$ and is excluded by hypothesis. But (iii) also is impossible, since ϕ_0 maps $(x_1, y_1) \in P$ to a point $(x_1, y_1') \in H$ (by Section 4.4) with $y_1' \neq y_1$.

4.7. *Invariant polygons.* Let $0 < t < 1$ and set $x_1 = t$, $y_1 = 1$. Suppose that we successively apply ϕ_0 to (x_1, y_1), obtaining a sequence (x_n, y_n) which remains in S. Then by Sections 4.6 and 4.2 there is an $n \geq 1$ such that

$$(x_k, y_k) \in \mathrm{I} \text{ for } 1 \leq k < n,$$

$$(x_n, y_n) \in \mathrm{II}$$

and

$$(x_{n+1}, y_{n+1}) \in \mathrm{III}.$$

We now claim that the points

$$(1,1), (x_1, y_1), \ldots, (x_{n+1}, y_{n+1}), (1, y_{n+1})$$

are the vertices of a convex polygon $C(t)$, and that $\phi_0(C(t)) \subseteq C(t)$. By the properties in Section 4.2, connecting the given points in the given order forms a nonself-intersecting polygon, and the inclusion is obvious if one notes that $\phi_0(1,1)$ lies on the line segment joining $(1,1)$ and (x_2, y_2) and that $\phi_0(1, y_{n+1})$ lies on the line segment joining $\phi_0(1,1)$ and $(1, 1 - \beta/\alpha) = \phi_0(1,0)$. The convexity of $C(t)$ is also easy to show, but we omit the calculation as it is not needed in the sequel.

5. Determination of admissibility. Now we can use the results of the previous section, together with the corollary of Section 2, to determine the admissibility of a given pair (α, β). Suppose first that (α, β) is admissible; if C denotes the convex hull of the orbit closure of $(1,1)$ under ϕ_0 and ϕ_1, then $\phi_0(C) \subseteq C$, $\phi_1(C) \subseteq C$ and $C \subseteq S$. Now set

$$y^* = \min\{y: (x, y) \in C\},$$

$$t^* = 1 - \frac{\beta}{\alpha y^*},$$

$$L^* = \{(x,1): t^* \leq x \leq 1\}.$$

Then $\phi_1(x, y^*) = (t^*, 1)$ implies that $L^* \subseteq C$. If we set

$$t = \min\{x: (x,1) \in C\},$$

then $t \leq t^*$ and the ϕ_0 invariant polygon $C(t)$ of the previous section is also contained in C and hence ϕ_1-invariant. Conversely, if for some $0 < t < 1$ the polygon $C(t)$ is also ϕ_1-invariant, then clearly (α, β) is admissible, since the orbit of $(1,1)$ is contained in $C(t)$. We have shown

THEOREM 5. *The pair (α, β) is admissible if and only if*

(i) $0 \leq \alpha \leq 1$, $0 \leq \beta \leq \alpha$ *and either*
(ii) *the equation*

$$x^3 - x^2 + \alpha x - \beta = 0$$

has three (not necessarily distinct) real roots or
(ii′) *the equation*

$$x^3 - x^2 + \alpha x - \beta = 0$$

has exactly one real root, and there exists $t \in (0,1)$ such that $C(t)$ is well defined [i.e., the ϕ_0-orbit of $(1,1)$ enters region III without previously leaving S] and such that

$$1 - \frac{\beta}{\alpha y^*} \geq t,$$

where

$$y^* = \min\{y \colon (x, y) \in C(t)\}.$$

A computer program has been written which decides, within the limits of machine accuracy, whether for given (α, β) the conditions of the above theorem are verified or not, and a copy is available on request. Moreover, we have the following rigorous results concerning admissibility.

1. If (α, β) is admissible, then $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta \leq \alpha/4$.
2. If $0 \leq \alpha \leq \frac{1}{4}$ and $0 \leq \beta \leq \alpha/4$, then (α, β) is admissible.
3. If $\frac{1}{4} < \alpha < \frac{1}{2}$ and $2\alpha^{3/2} - \alpha \leq \beta \leq \frac{1}{2}(\alpha - \alpha^{3/2})$, then (α, β) is admissible.
4. In the following ranges, (α, β) is not admissible:
   (i) $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$ and $27\beta < 9\alpha - 2(1 - 3\alpha)^{3/2} - 2$,
   (ii) $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$ and $27\beta < 9\alpha - 2$ and
   (iii) $\frac{1}{4} < \alpha \leq \frac{1}{2}$ and $\beta > \frac{1}{2}(\alpha - \alpha^{3/2})$.

These results, together with the two-block factor region, are summarized in Figure 2.

Finally, we sketch our proofs of results 1–4.

1. If $\alpha > \frac{1}{2}$, then either the x-coordinate of $\phi_0\phi_1(1,1)$,

$$\frac{1 - \alpha - \beta/\alpha}{1 - \beta/\alpha},$$

is negative, or if this is nonnegative, the x-coordinate of $\phi_0^2\phi_1(1,1)$,

$$\frac{(1 - 2\alpha)(1 - \beta/\alpha)}{1 - \alpha - \beta/\alpha},$$

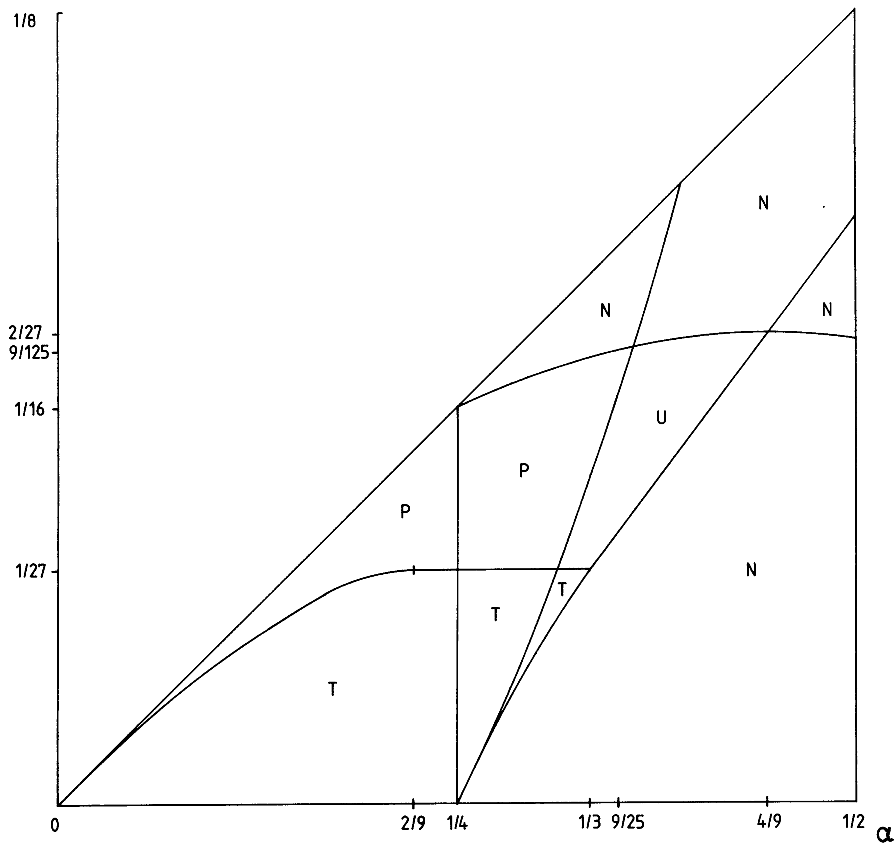

FIG. 2. *T, one-dependent processes which are two-block factors, P, one-dependent processes which are not two-block factors, N, no one-dependent processes, U, unexplored.*

is negative. If $\beta > \alpha/4$, then $(\phi_1\phi_0)^n(1,1)$ becomes negative in its x-coordinate for some n, since

$$(\phi_1\phi_0)^n(1,1) = \left(g^{2n}(1),1\right)$$

with

$$g(t) = 1 - \frac{\beta}{\alpha t},$$

and $g^{2n}(1)$ is eventually negative iff $t = g(t)$ has no real root, leading to $\beta > \alpha/4$.

2. This is the simplest polygon case, corresponding to $\alpha \leq \frac{1}{4}$ in Figure 1. Here $(\frac{1}{2},1)$ belongs to region II, so

$$\phi_0\left(\frac{1}{2},1\right) = \left(1 - 2\alpha, 1 - \frac{2\beta}{\alpha}\right) \in \mathrm{III}.$$

The quadrilateral with vertices

$$(1,1), \qquad \left(\frac{1}{2},1\right), \qquad \phi_0\left(\frac{1}{2},1\right), \qquad \left(1,1-\frac{2\beta}{\alpha}\right)$$

has lowest y-coordinate

$$y^* = 1 - \frac{2\beta}{\alpha}$$

with

$$1 - \frac{\beta}{\alpha y^*} \geq \frac{1}{2}$$

and is thus invariant under ϕ_0 and ϕ_1.

3. This is the next polygon case. For $t \in (0,1)$, $C(t)$ is a pentagon [i.e., $\phi_0(t,1) \in$ II] if t satisfies

$$\alpha t^2 - (\alpha + \beta)t + \alpha^2 \leq 0,$$

and ϕ_1-invariance holds if

$$(\alpha - \beta)t^2 - (\alpha^2 + \alpha - 2\beta)t + \alpha(\alpha - \beta) \leq 0.$$

Discriminant calculation and elementary considerations lead to the bounds given in result 3.

4(i) and 4(ii). Here one can show directly that $\mu(0^n) = z_n$ is negative for some n. By one-dependence one easily derives the recurrence

$$z_n = z_{n-1} - \alpha z_{n-2} + \beta z_{n-3},$$

whose characteristic equation is

$$\rho(x) = x^3 - x^2 + \alpha x - \beta = 0.$$

In the ranges indicated, there is one real root and two complex roots whose real part is larger than the real root, and it follows that z_n becomes negative.

4(iii). Here we have (similar result to 1)

$$(\phi_1\phi_0^2)^n(1,1) = (g^n(1),1)$$

with

$$g(t) = \frac{(\alpha - 2\beta)t - \alpha(\alpha - \beta)}{(\alpha - \beta)t - \alpha^2}.$$

Hence if $g(t) = t$ has no real root, then $g^n(1)$ becomes negative for some n. A discriminant calculation leads to the given bound.

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