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WINDINGS OF RANDOM WALKS[1]

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Let $X_1, X_2, X_3, \dots$ be a sequence of iid $\mathbb{R}^2$-valued bounded random variables with mean vector zero and covariance matrix identity. Let $S = (S_n; n \geq 0)$ be the random walk defined by $S_n = \sum_{i=1}^n X_i$. Let $\phi(n)$ be the winding of S at time n, that is, the total angle wound by S around the origin up to time n. Under a mild regularity condition on the distribution of X_1, we show that $2\phi(n)/\log n \to_d W$ where $\to_d$ denotes convergence in distribution and where W has density $(1/2)\text{sech}(\pi w/2)$.

1. Introduction and statement of the main result. Let $Z = (Z(t);\ t \geq 0)$ be a standard two-dimensional Brownian motion starting at a point z_0 other than the origin and let $\theta(t)$ be the winding of Z at time t, that is, the total continuous angle wound by Z around the origin up to time t. Spitzer's law says that

$$2\theta(t)/\log t \to_d C \quad \text{as } t \to \infty,$$

where C is a standard Cauchy random variable and where $\to_d$ denotes convergence in distribution [Spitzer (1958), Williams (1974) and Durrett (1982)]. The purpose of this paper is to present an analogue of Spitzer's law for windings of planar random walks. Let $X_1, X_2, \dots$ be a sequence of iid $\mathbb{R}^2$-valued random variables and assume that their distribution satisfies the following conditions:

CONDITION A (Normalization). X_1 has mean vector zero and covariance matrix identity.

CONDITION B (Boundedness). There exists a finite constant b such that $\mathbf{P}[\|X_1\| \leq b] = 1$.

CONDITION C (Regularity). One of the following holds:

(C.1) (The absolutely continuous case). The distribution of X_1 is absolutely continuous with respect to Lebesgue measure in the plane.

(C.2) (The lattice case). The additive subgroup of $\mathbb{R}^2$ generated by the support of the distribution of X_1 is the lattice $\mathscr{L}_d = \{dz;\ z \in \mathbb{Z}^2\}$ for some $d > 0$.

Now fix x_0 [with $x_0 \in \mathbb{R}^2$ if (C.1) holds, $x_0 \in \mathscr{L}_d$ if (C.2) holds] and consider the random walk $S = (S_n;\ n \geq 0)$ defined by $S_n = x_0 + \sum_{i=1}^n X_i$. Let $\phi(n)$ be the

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winding of S at time n, that is, the total angle wound be S around the origin up to time n. More precisely, $\phi(n) = \sum_{j=1}^{n}\lambda(j)$ where the $\lambda(j)$'s are defined as follows: If S_{j-1}, S_j and the origin are colinear then $\lambda(j) = 0$; otherwise $\lambda(j)$ is the unique number between $-\pi$ and π such that

$$\frac{S_{j-1}}{\|S_{j-1}\|}e^{i\lambda(j)} = \frac{S_j}{\|S_j\|}.$$

(In the last equation we have identified the real plane and the complex plane and we are using complex variable notation and operations for convenience.) The main result of this paper is

THEOREM 1.1. *If conditions* A, B *and* C *are satisfied, then*

$$2\phi(n)/\log n \to_d W \quad \text{as } n \to \infty,$$

where W has density $(1/2)\operatorname{sech}(\pi w/2)$.

[The distribution with density $(1/2)\operatorname{sech}(\pi w/2)$ will be referred to as the standard hyperbolic secant distribution. It has mean 0 and variance 1.] To understand the difference between Spitzer's law and Theorem 1.1, consider the following Brownian winding analysis, as in Messulam and Yor (1982) and Pitman and Yor (1984, 1986). Without any loss of generality assume that $z_0 = (1, 0)$. Define

$$\theta_{\text{small}}(t) = \int_0^t 1_{\{\|Z(s)\| \le 1\}}\, d\theta(s),$$

$$\theta_{\text{big}}(t) = \int_0^t 1_{\{\|Z(s)\| > 1\}}\, d\theta(s),$$

$$\zeta_r = \inf\{s \ge 0 \colon \|Z(s)\| = r\}$$

and write

$$\left(\frac{2\theta_{\text{small}}(t)}{\log t}, \frac{2\theta_{\text{big}}(t)}{\log t}\right) = \left(\frac{2\theta_{\text{small}}(\zeta_{\sqrt{t}})}{\log t}, \frac{2\theta_{\text{big}}(\zeta_{\sqrt{t}})}{\log t}\right)$$
$$+ \left(\frac{2\left(\theta_{\text{small}}(t) - \theta_{\text{small}}(\zeta_{\sqrt{t}})\right)}{\log t}, \frac{2\left(\theta_{\text{big}}(t) - \theta_{\text{big}}(\zeta_{\sqrt{t}})\right)}{\log t}\right).$$

A tightness argument yields

$$\frac{2\left(\theta_{\text{small}}(t) - \theta_{\text{small}}(\zeta_{\sqrt{t}})\right)}{\log t} \to_P 0 \quad \text{and} \quad \frac{2\left(\theta_{\text{big}}(t) - \theta_{\text{big}}(\zeta_{\sqrt{t}})\right)}{\log t} \to_P 0 \quad \text{as } t \to \infty.$$

Conformal invariance and scaling yield

$$\left(\frac{2\theta_{\text{small}}(\zeta_{\sqrt{t}})}{\log t}, \frac{2\theta_{\text{big}}(\zeta_{\sqrt{t}})}{\log t}\right) =_d \left(\int_0^\sigma 1_{\{\alpha_s \le 1\}}\, d\beta_s, \int_0^\sigma 1_{\{\alpha_s > 1\}}\, d\beta_s\right).$$

Here $=_d$ denotes equality in distribution, $((\alpha_s, \beta_s);\ s \ge 0)$ is a standard two-dimensional Brownian motion starting at the origin, and $\sigma = \inf\{s \ge 0 \colon \alpha_s = 1\}$.

Thus

$$\left(\frac{2\theta_{\text{small}}(t)}{\log t}, \frac{2\theta_{\text{big}}(t)}{\log t}\right) \to_d \left(\int_0^\sigma 1_{\{\alpha_s \le 1\}}\, d\beta_s, \int_0^\sigma 1_{\{\alpha_s > 1\}}\, d\beta_s\right) \quad \text{as } t \to \infty.$$

[Spitzer's law follows at once since $\theta(t) = \theta_{\text{small}}(t) + \theta_{\text{big}}(t)$ and since $\beta(\sigma)$ is standard Cauchy.] Theorem 4.2 of Pitman and Yor (1986) gives the joint Fourier transform of the limiting distribution. In particular $\int_0^\sigma 1_{\{\alpha_s > 1\}}\, d\beta_s$ is standard hyperbolic secant.

Our approach to Theorem 1.1 is an adaptation of the above Brownian winding analysis. It goes essentially along the following lines. Define

$$\phi_{m\text{-small}}(n) = \sum_{i=1}^{n} (\phi(i) - \phi(i-1)) 1_{\{\|S(i-1)\| \le m\}},$$

$$\phi_{m\text{-big}}(n) = \sum_{i=1}^{n} (\phi(i) - \phi(i-1)) 1_{\{\|S(i-1)\| > m\}},$$

$$T(r) = \min\{i \ge 0\colon \|S(i)\| \ge r\}$$

and write

$$\frac{2\phi(n)}{\log n} = \frac{2\phi_{m\text{-small}}(T(\sqrt{n}))}{\log n} + \frac{2\phi_{m\text{-big}}(T(\sqrt{n}))}{\log n} + \frac{2(\phi(n) - \phi(T(\sqrt{n})))}{\log n}.$$

A tightness argument will yield

$$\frac{2(\phi(n) - \phi(T(\sqrt{n})))}{\log n} \to_P 0 \quad \text{as } n \to \infty.$$

For a sequence m_n increasing to ∞, strong Brownian approximation will yield

$$\frac{2\phi_{m_n\text{-big}}(T(\sqrt{n}))}{\log n} \to_d \int_0^\sigma 1_{\{\alpha_s > 1\}}\, d\beta_s \quad \text{as } n \to \infty$$

and ergodic theory will yield

$$\frac{2\phi_{m_n\text{-small}}(T(\sqrt{n}))}{\log n} \to_P 0 \quad \text{as } n \to \infty.$$

Hence Theorem 1.1.

REMARK 1. Theorem 1.1 gives, in particular, the limit distribution for the winding of a simple symmetric random walk on the integer lattice. This special case has been investigated by Fisher, Privman and Redner (1984) and by Rudnick and Hu (1987) using completely different methods.

REMARK 2. The method developed in this paper, and used to prove Theorem 1.1, is more important than the result itself. It can be used to obtain limit theorems for various functionals of two-dimensional random walks, analogous to the Brownian motion log-scaling laws of Pitman and Yor (1986).

2. Proof of the main result. The above decomposition of the winding $\phi(n)$ into m-small and m-big windings will now be replaced by a decomposition that is easier to analyze. Fix m, a positive integer large enough so that $\|x_0\| \le m$. Let $\sigma(m,0) = \min\{j \ge 0: \|S(j)\| \ge m\}$, $\eta(m,0) = 0$, $\xi_0(m,0) = \phi(\sigma(m,0))$, and for every positive integer i, define (recursively)

$$\sigma(m,i) = \begin{cases} \min\{j \ge \sigma(m,i-1): \|S(j)\| \ge me\} \\ \quad \text{if } \eta(m,i-1) = 0, \\ \min\{j \ge \sigma(m,i-1): \|S(j)\| \le me^{\eta(m,i-1)-1} \text{ or } \ge me^{\eta(m,i-1)+1}\} \\ \quad \text{if } \eta(m,i-1) > 0, \end{cases}$$

$$\sigma'(m,i) = \min\{j \ge \sigma(m,i-1): \|S(j)\| \le me^{\eta(m,i-1)-1} \text{ or } \ge me^{\eta(m,i-1)+1}\},$$

$$\eta(m,i) = \begin{cases} \eta(m,i-1) - 1 & \text{if } \|S(\sigma(m,i))\| \le me^{\eta(m,i-1)-1}, \\ \eta(m,i-1) + 1 & \text{if } \|S(\sigma(m,i))\| \ge me^{\eta(m,i-1)+1}, \end{cases}$$

$$\xi_0(m,i) = \phi(\sigma(m,i)) - \phi(\sigma'(m,i)),$$

$$\xi(m,i) = \phi(\sigma'(m,i)) - \phi(\sigma(m,i-1)).$$

The random variables $\xi_0(m,i)$ and $\xi(m,i)$ represent increments of m-small winding and m-big winding, respectively. Fix k, a positive integer. Let

$$N(m,k) = \min\{i \ge 0: \eta(m,i) = k\} \quad \text{and} \quad I(m,k) = \sum_{i=1}^{N(m,k)} 1_{\{\eta(m,i-1)=0\}}.$$

Observe that

$$\frac{1}{k}\phi(T(me^k)) = \frac{1}{k}\sum_{i=1}^{N(m,k)} \xi(m,i) + \frac{1}{k}\sum_{i=0}^{N(m,k)} \xi_0(m,i), \tag{2.1}$$

where $T(r) = \min\{j \ge 0: \|S(j)\| \ge r\}$, as in Section 1. Now consider the m-big winding at time $T(me^k)$, that is, the first term on the right-hand side of (2.1). Let $Z = (Z(t);\ t \ge 0)$ be a standard two-dimensional Brownian motion starting at the origin. Let $\sigma(0) = \inf\{t \ge 0: \|Z(t)\| = 1\}$, $\eta(0) = 0$, and for every positive integer i, define (recursively)

$$\sigma(i) = \begin{cases} \inf\{t \ge \sigma(i-1): \|Z(t)\| = e\} & \text{if } \eta(i-1) = 0, \\ \inf\{t \ge \sigma(i-1): \|Z(t)\| = e^{\eta(i-1)-1} \text{ or } e^{\eta(i-1)+1}\} & \text{if } \eta(i-1) > 0, \end{cases}$$

$$\sigma'(i) = \inf\{t \ge \sigma(i-1): \|Z(t)\| = e^{\eta(i-1)-1} \text{ or } e^{\eta(i-1)+1}\},$$

$$\eta(i) = \begin{cases} \eta(i-1) - 1 & \text{if } \|Z(\sigma(i))\| = e^{\eta(i-1)-1}, \\ \eta(i-1) + 1 & \text{if } \|Z(\sigma(i))\| = e^{\eta(i-1)+1}, \end{cases}$$

$$\xi(i) = \text{total windings of } Z \text{ from time } \sigma(i-1) \text{ to time } \sigma'(i).$$

Finally, let

$$N(k) = \min\{i \geq 0 \colon \eta(i) = k\} \quad \text{and} \quad I(k) = \sum_{i=1}^{N(k)} 1_{\{\eta(i-1)=0\}}.$$

Donsker's invariance principle implies that

$$((\eta(m,i) \colon i \geq 0), (\xi(m,i) \colon i \geq 1)) \to_d ((\eta(i) \colon i \geq 0), (\xi(i) \colon i \geq 1)) \quad \text{as } m \to \infty.$$

In particular, for each k we have $I(m,k) \to_d I(k)$, $N(m,k) \to_d N(k)$ and $\sum_{i=1}^{N(m,k)} \xi(m,i) \to_d \sum_{i=1}^{N(k)} \xi(i)$ as $m \to \infty$. In Section 6 we will prove the following stronger result.

THEOREM 2.1. *If conditions* A *and* B *hold, then*

(a) $$\frac{1}{k} I(m,k) \to_d \frac{1}{k} I(k) \quad \textit{as } m \to \infty, \textit{uniformly in } k;$$

(b) $$\frac{1}{k^2} N(m,k) \to_d \frac{1}{k^2} N(k) \quad \textit{as } m \to \infty, \textit{uniformly in } k;$$

(c) $$\frac{1}{k} \sum_{i=1}^{N(m,k)} \xi(m,i) \to_d \frac{1}{k} \sum_{i=1}^{N(k)} \xi(i) \quad \textit{as } m \to \infty, \textit{uniformly in } k.$$

Now the strong Markov property and the conformal invariance of Z imply that:

1. $(\xi(i);\ i \geq 1)$ is a sequence of independent standard hyperbolic secant random variables.
2. $(\eta(i);\ i \geq 0)$ is a simple symmetric random walk on the nonnegative integers, starting at 0 and positively reflected at 0.
3. $(\xi(i);\ i \geq 1)$ and $(\eta(i);\ i \geq 0)$ are independent.

Thus, straightforward computations yield the following result.

PROPOSITION 2.2.

(a) $$\frac{1}{k} I(k) \to_d E \quad \textit{as } k \to \infty;$$

(b) $$\frac{1}{k^2} N(k) \to_d T \quad \textit{as } k \to \infty;$$

(c) $$\frac{1}{k} \sum_{i=1}^{N(k)} \xi(i) \to_d W \quad \textit{as } k \to \infty,$$

where E is standard exponential, where T is the hitting time of 1 *by a standard one-dimensional reflected Brownian motion starting at* 0 *and where W is standard hyperbolic secant.*

As a corollary to Theorem 2.1 and Proposition 2.2 we have

PROPOSITION 2.3. *Assume that conditions* A *and* B *are satisfied. If* m_k *increases to* ∞, *then*

$$\frac{1}{k}\sum_{i=1}^{N(m_k,k)} \xi(m_k,i) \to_d W \quad \text{as } k \to \infty.$$

Now consider the m-small winding at time $T(me^k)$, that is, the second term on the right-hand side of (2.1). Define

$$\begin{aligned}
U(m,0) &= \min\{j \ge 0: \|S(j)\| \ge m\},\\
V(m,1) &= \min\{j \ge U(m,0): \|S(j)\| \le me^{-1}\},\\
U(m,1) &= \min\{j \ge V(m,1): \|S(j)\| \ge me\},\\
V(m,i) &= \min\{j \ge U(m,i-1): \|S(j)\| \le me^{-1}\},\\
U(m,i) &= \min\{j \ge V(m,i): \|S(j)\| \ge me\}
\end{aligned}$$

and

$$\beta(m,i) = \phi(U(m,i)) - \phi(V(m,i)).$$

In the next section we will prove the following ergodic theorem for $\beta(m,i)$.

THEOREM 2.4. *Assume that conditions* A, B *and* C *hold. Then for each* m *there exists a constant* c_m *such that*

$$\frac{1}{n}\sum_{i=1}^{n} \beta(m,i) \to c_m \quad \text{a.s. as } n \to \infty.$$

Furthermore,

$$c_m \to 0 \quad \text{as } m \to \infty.$$

Now observe that

$$\sum_{i=1}^{N(m,k)} \xi_0(m,i) = \sum_{i=1}^{I'(m,k)} \beta(m,i),$$

where $I'(m,k) = \sum_{i=1}^{N(m,k)} 1_{\{\eta(m,i-1)=0,\, \sigma'(m,i)<\sigma(m,i)\}}$. Thus for each m we have, by Theorem 2.4,

$$\frac{1}{I'(m,k)}\sum_{i=1}^{N(m,k)} \xi_0(m,i) \to_P c_m \quad \text{as } k \to \infty.$$

Furthermore, if m_k increases to ∞ slowly enough, then

$$\frac{1}{I'(m_k,k)}\sum_{i=1}^{N(m_k,k)} \xi_0(m_k,i) \to_P 0 \quad \text{as } k \to \infty.$$

Since $0 \le I'(m, k) \le I(m, k)$, part (a) of Theorem 2.1 combined with part (a) of Proposition 2.2 imply that the family $((1/k)I'(m_k, k);\ k \ge 1)$ is tight. Thus we get

PROPOSITION 2.5. *Assume that conditions* A, B *and* C *hold. If* m_k *increases to* ∞ *slowly enough, then*

$$\frac{1}{k} \sum_{i=0}^{N(m_k, k)} \xi_0(m_k, i) \to_P 0 \quad \text{as } k \to \infty.$$

Now choose integers $0 = m_0 < m_1 \le m_2 \le m_3 \le \cdots$ such that

$$\text{(2.2)} \quad \frac{1}{k}\phi\big(T(m_k e^k)\big) \to_d W \quad \text{as } k \to \infty \quad \text{and} \quad \log m_k = o(k) \quad \text{as } k \to \infty.$$

This is possible in view of (2.1) and Propositions 2.3 and 2.5. For $n = 1, 2, 3, \dots$ let $k(n)$ be the integer defined by

$$m_{k(n)} e^{k(n)} \le \sqrt{n} < m_{k(n)+1} e^{k(n)+1}$$

and write

$$\text{(2.3)} \quad \frac{2\phi(n)}{\log n} = \frac{\phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big)}{k(n)} \frac{2k(n)}{\log n} + 2\left[\frac{\phi(n) - \phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big)}{\log n}\right].$$

From (2.2) we have

$$\frac{1}{k(n)}\phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big) \to_d W \quad \text{as } n \to \infty.$$

From the definition of $k(n)$ we have

$$\frac{2k(n)}{\log n} \to 1 \quad \text{as } n \to \infty.$$

We now show that the last term on the right-hand side of (2.3) converges to 0 in probability as $n \to \infty$. Fix $\varepsilon > 0$. For every positive integer i we can write

$$\begin{aligned} &\mathbf{P}\left[\left|\frac{\phi(n) - \phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big)}{\log n}\right| > \varepsilon\right] \\ \text{(2.4)} \quad &\quad \le \mathbf{P}\left[\max_{T_{n,i}^- \le j \le T_{n,i}^+} \left|\phi(j) - \phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big)\right| > \varepsilon \log n\right] \\ &\quad\quad + \mathbf{P}[T_{n,i}^- > n] + \mathbf{P}[T_{n,i}^+ < n], \end{aligned}$$

where $T_{n,i}^- = T(m_{k(n)} e^{k(n)-i})$ and $T_{n,i}^+ = T(m_{k(n)} e^{k(n)+i})$. Donsker's invariance principle implies that

$$\max_{T_{n,i}^- \le j \le T_{n,i}^+} \left|\phi(j) - \phi\big(T\big(m_{k(n)} e^{k(n)}\big)\big)\right|$$

converges in distribution as $n \to \infty$. Thus the first term on the right-hand side of

(2.4) converges to 0 as $n \to \infty$, even with i replaced by a sequence $i(n)$ increasing to ∞ as $n \to \infty$, provided it increases slowly enough. Donsker's invariance principle also implies that

$$\frac{T\left(m_{k(n)}e^{k(n)-i(n)}\right)}{m_{k(n)}^2 e^{2(k(n)-i(n))}} \quad \text{and} \quad \frac{T\left(m_{k(n)}e^{k(n)+i(n)}\right)}{m_{k(n)}^2 e^{2(k(n)+i(n))}}$$

both converge in distribution, for every sequence $i(n)$ increasing to ∞ slowly enough. This implies that the second and third terms on the right-hand side of (2.4), with i replaced by $i(n)$, both converge to 0 as $n \to \infty$, provided $i(n)$ increases to ∞ slowly enough. Thus the left-hand side of (2.4) goes to 0 as $n \to \infty$.

In order to complete the proof of Theorem 1.1, it remains only to prove Theorems 2.1 and 2.4. Theorem 2.4 will be proved in the next section. Theorem 2.1 will be proved in Section 6, after we approximate the sequences $(\eta(m, i);\ i \geq 0)$ and $(\xi(m, i);\ i \geq 1)$ in Sections 4 and 5.

3. Proof of Theorem 2.4. For every m, the sequence $(\beta(m, i), S(V(m, i));\ i = 1, 2, 3, \dots)$ is an L^1-bounded Harris recurrent Markov chain. [Markovness follows from the strong Markov property of S. Harris recurrence is easily proved using condition C. The L^1-boundedness follows easily from the fact that $\sup_{\|x\| \leq m/e} \mathbf{E}_x[T(me)] < \infty$; here $\mathbf{E}_x[\cdot]$ denotes conditional expectation given $S(0) = x$.] The ergodic theorem implies that

$$\frac{1}{n}\sum_{i=1}^{n} \beta(m, i) \to c_m \quad \text{a.s. as } n \to \infty$$

for some finite constant c_m. Furthermore, c_m does not depend on the starting point x_0. [In the lattice case an elementary proof follows easily from the strong law of large numbers and Theorem 78 of Chapter 2 of Freedman (1983); in the continuous case one can use Theorem 3.6 of Chapter 4 of Revuz (1975).] This completes the proof of the first part of Theorem 2.4.

Now suppose that for some integers $0 < l(1) < l(2) < \cdots$ and for some constant c (possibly infinite)

$$\lim_{j \to \infty} c_{l(j)} = c. \tag{3.1}$$

Let G be a bounded Borel subset of $\mathbb{R}^2$ with $\mu(G) > 0$. If condition (C.1) holds, $\mu(G)$ denotes the Lebesgue measure of G. If condition (C.2) holds, $\mu(G)$ is d^2 times the cardinality of $G \cap \mathscr{L}_d$. Let $N_G(n)$ be the time of the nth return to G,

$$N_G(n) = \min\left\{ j \geq 1:\ \sum_{i=1}^{j} 1_G(S_i) = n \right\}.$$

Let $J(m, 0) = 0$ and, for every positive integer k, let

$$J(m, k) = \min\{ j > J(m, k-1):\ \eta(m, j) = 0 \}.$$

Finally, let $K_G(m, n)$ be the integer defined by

$$\sigma(m, J(m, K_G(m, n))) \le N_G(n) < \sigma(m, J(m, K_G(m, n) + 1)).$$

Thus $K_G(m, n)$ is the number of excursions from outside the disk of radius me to the disk of radius m completed at the time of the nth return to G. Then

$$\begin{aligned} \frac{\mu(G)\phi(N_G(n))}{\pi n} &= \frac{\mu(G)}{\pi n} \sum_{i=1}^{J(m, K_G(m, n))} \xi(m, i) \\ &\quad + \frac{\mu(G)}{\pi n} \sum_{i=0}^{J(m, K_G(m, n))} \xi_0(m, i) + \frac{\mu(G)}{\pi n} \Delta_{m, n}, \end{aligned} \tag{3.2}$$

where

$$\Delta_{m, n} = \phi(N_G(n)) - \phi(\sigma(m, J(m, K_G(m, n)))).$$

We will show that there exists a sequence of integers m_n, increasing to ∞ as $n \to \infty$, such that if we replace m by m_n on the right-hand side of (3.2), then the first term converges in distribution to a standard Cauchy random variable, the second term converges in probability to $c/2$, with c as in (3.1), and the third term converges in probability to 0; thus the left-hand side of (3.2) converges in distribution to a Cauchy distribution centered at $c/2$ (with the obvious interpretation if c is infinite). Then we will show that for a suitable G and a suitable initial distribution of S, the left-hand side of (3.2) has, for every n, a symmetric distribution. This will imply that $c = 0$, thus proving part (b).

Let $J(0) = 0$ and, for $k = 1, 2, 3, \dots$,

$$J(k) = \min\{j > J(k-1) \colon \eta(j) = 0\}.$$

Then, as in Section 2, Donsker's invariance principle implies that for each k we have $\sum_{i=1}^{J(m, k)} \xi(m, i) \to_d \sum_{i=1}^{J(k)} \xi(i)$ and

$$\max_{1 \le j \le k} \left| \sum_{i=1}^{J(m, j)} \xi(m, i) \right| \to_d \max_{1 \le j \le k} \left| \sum_{i=1}^{J(j)} \xi(i) \right|, \quad \text{as } m \to \infty.$$

In fact, the following stronger result holds. Its proof is omitted. It is essentially identical to the proof of Theorem 2.1 of Section 2.

THEOREM 3.1. *If conditions* A *and* B *hold, then*

(a) $$\frac{1}{k} \sum_{i=1}^{J(m, k)} \xi(m, i) \to_d \frac{1}{k} \sum_{i=1}^{J(k)} \xi(i)$$

as $m \to \infty$, uniformly in k;

(b) $$\frac{1}{k} \max_{1 \le j \le k} \left| \sum_{i=J(m, l)+1}^{J(m, l+j)} \xi(m, i) \right| \to_d \frac{1}{k} \max_{1 \le j \le k} \left| \sum_{i=J(l)+1}^{J(l+j)} \xi(i) \right|$$

as $m \to \infty$, uniformly in k and l.

Straightforward computations analogous to those leading to Proposition 2.2 yield

PROPOSITION 3.2.

(a) $$\frac{1}{k}\sum_{i=1}^{J(k)} \xi(i) \to_d C(1) \qquad \textit{as } k \to \infty;$$

(b) $$\frac{1}{k}\max_{1\le j\le k}\left|\sum_{i=J(l)+1}^{J(l+j)} \xi(i)\right| \to_d \sup_{0\le t\le 1}|C(t)| \quad \textit{as } k\to\infty, \textit{ uniformly in } l,$$

where $(C(t)\colon t \ge 0)$ *is a standard Cauchy process starting at* 0.

The next result is an ergodic theorem for $K_G(m, n)$.

PROPOSITION 3.3. *For each* m *there is a positive constant* a_m *such that*

(a) $$(1/\pi n)\mu(G)K_G(m,n) \to a_m \quad \textit{a.s. as } n \to \infty;$$

(b) $$a_m \to 1 \quad \textit{as } m \to \infty.$$

PROOF. Let D_m denote the open disk of radius m centered at the origin and write

$$\frac{k\mu(G)}{\pi\sum_{i=0}^{\sigma(m,\,J(m,k))}1_G(S_i)} = \frac{\mu(G)\sum_{i=0}^{\sigma(m,\,J(m,k))}1_{D_m}(S_i)}{\mu(D_m)\sum_{i=0}^{\sigma(m,\,J(m,k))}1_G(S_i)}\,\frac{k\mu(D_m)}{\pi\sum_{i=0}^{\sigma(m,\,J(m,k))}1_{D_m}(S_i)}.$$

Using the a.s. ergodic theorem for Markov chains [Revuz (1975), Chapter 4, Theorem 3.6] we get

$$\frac{k\mu(G)}{\pi\sum_{i=0}^{\sigma(m,\,J(m,k))}1_G(S_i)} \to a_m \quad \text{a.s. as } k\to\infty, \tag{3.3}$$

where a_m is defined by

$$a_m = \frac{\mu(D_m)}{\pi \mathbf{E}_{\nu_m}\left[\sum_{i=0}^{T(me)}1_{D_m}(S_i)\right]}.$$

Here ν_m denotes the limit distribution of $S(\sigma(m, J(m,k)))$, as $k \to \infty$, and $T(me)$ denotes the first exit time of the open disk of radius me centered at the origin. The functional central limit theorem yields (after verifying uniform integrability)

$$\frac{1}{m^2}\mathbf{E}_{\nu_m}\left[\sum_{i=0}^{T(me)}1_{D_m}(S_i)\right] \to 1 \quad \text{as } m\to\infty.$$

Combined with the fact that $\mu(D_m) \sim \pi m^2$, this yields $\lim_{m\to\infty} a_m = 1$. To complete the proof, replace k by $K_G(m, n)$ in (3.3) and note that

$$\frac{1}{n}\sum_{i=0}^{\sigma(m,\,J(m,\,K_G(m,n)))} 1_G(S_i) \to 1 \quad \text{a.s. as } n\to\infty. \qquad \square$$

Now consider the m_n-big winding at the time of the nth return to G, that is, the first term on the right-hand side of (3.2), with m replaced by m_n,

$$
\begin{aligned}
&\frac{\mu(G)}{\pi n}\sum_{i=1}^{J(m_n,\,K_G(m_n,\,n))}\xi(m_n,i)\\
(3.4)\qquad &=\frac{\mu(G)}{\pi n}\sum_{i=1}^{J(m_n,[\pi n/\mu(G)])}\xi(m_n,i)\\
&\quad+\frac{\mu(G)}{\pi n}\left(\sum_{i=1}^{J(m_n,\,K_G(m_n,\,n))}\xi(m_n,i)-\sum_{i=1}^{J(m_n,[\pi n/\mu(G)])}\xi(m_n,i)\right).
\end{aligned}
$$

Here $[\cdot]$ denotes the integer part. Theorem 3.1(a) and Proposition 3.2(a) imply that for every sequence m_n increasing to ∞, the first term on the right-hand side of (3.4) converges in distribution, as $n \to \infty$, to a standard Cauchy random variable. By Proposition 3.3,

$$\frac{\mu(G)}{\pi n}K_G(m_n,n) \to 1 \quad \text{a.s. as } n \to \infty,$$

for every sequence m_n increasing to ∞ slowly enough. Choose such a sequence m_n. Then for each $\varepsilon > 0$,

$$1_{[1-\varepsilon,1+\varepsilon]}\left(\frac{\mu(G)K_G(m_n,n)}{\pi n}\right) \to 1 \quad \text{a.s. as } n \to \infty.$$

Thus there exists a sequence ε_n (which may depend on our choice of m_n) decreasing to 0 and such that

$$(3.5)\qquad 1_{[1-\varepsilon_n,1+\varepsilon_n]}\left(\frac{\mu(G)K_G(m_n,n)}{\pi n}\right) \to 1 \quad \text{a.s. as } n \to \infty.$$

Equipped with these sequences m_n and ε_n, we now consider the second term on the right-hand side of (3.4). Let

$$\Delta_n = \sum_{i=1}^{J(m_n,\,K_G(m_n,\,n))}\xi(m_n,i)-\sum_{i=1}^{J(m_n,[\pi n/\mu(G)])}\xi(m_n,i).$$

Then

$$
\begin{aligned}
\left|\frac{\Delta_n}{\pi n}\right| &= \left|\frac{\Delta_n}{\pi n}\right| 1_{[1-\varepsilon_n,1+\varepsilon_n]}\left(\frac{\mu(G)K_G(m_n,n)}{\pi n}\right)\\
&\quad+\left|\frac{\Delta_n}{\pi n}\right| 1_{\mathbf{R}\setminus[1-\varepsilon_n,1+\varepsilon_n]}\left(\frac{\mu(G)K_G(m_n,n)}{\pi n}\right).
\end{aligned}
$$

In view of (3.5), the second term on the right-hand side of the last equation goes to 0 a.s. as $n \to \infty$. The first term is bounded above by

$$\frac{1}{\pi n}\max_{(\pi n/\mu(G))(1-\varepsilon_n)\le j\le(\pi n/\mu(G))(1+\varepsilon_n)}\left|\sum_{i=1}^{J(m_n,\,j)}\xi(m_n,i)-\sum_{i=1}^{J(m_n,[\pi n/\mu(G)])}\xi(m_n,i)\right|$$

and therefore converges to 0 in probability as $n \to \infty$, in view of Theorem 3.1(b) and Proposition 3.2(b). Thus for every sequence m_n increasing to ∞ slowly enough, the first term on the right-hand side of (3.2), with m replaced by m_n, converges in distribution to a standard Cauchy random variable. This is true in particular for every sequence m_n taking values in the set $\{l(1), l(2), l(3), \dots\}$ and increasing to ∞ slowly enough.

Now consider the m_n-small winding at the time of the nth return to G, that is, the second term on the right-hand side of (3.2) with m replaced by m_n. Observe that

$$\sum_{i=1}^{J(m, K_G(m,n))} \xi_0(m,i) = \sum_{i=1}^{K'_G(m,n)} \beta(m,i),$$

where

$$K'_G(m,n) = \sum_{i=1}^{J(m, K_G(m,n))} 1_{\{\eta(m,i-1)=0,\ \sigma'(m,i)<\sigma(m,i)\}}.$$

The following result is an ergodic theorem for $K'_G(m,n)$. Its proof is omitted. It is essentially the same as the proof of Proposition 3.3.

PROPOSITION 3.4. *For each m there is a positive constant a'_m such that*

(a) $$(1/\pi n)\mu(G)K'_G(m,n) \to a'_m \quad a.s.\ as\ n \to \infty;$$

(b) $$a'_m \to 1/2 \quad as\ m \to \infty.$$

Combined with the first part of Theorem 2.4 and our choice of $l(1), l(2), l(3), \dots$, Proposition 3.4 implies that if m_n is a sequence of positive integers taking values in the set $\{l(1), l(2), l(3), \dots\}$ and increasing to ∞ slowly enough, then the second term on the right-hand side of (3.2), with m replaced by m_n, converges in probability to $c/2$ as $n \to \infty$.

Finally, consider the last term on the right-hand side of (3.2). If we let

$$X_{m,k} = \max_j |\phi(j) - \phi(\sigma(m, J(m,k)))|,$$

with the maximum taken over j's satisfying

$$\sigma(m, J(m,k)) \le j \le \sigma(m, J(m,k+1)),$$

and if m is large enough, then $|\Delta_{m,n}| \le X_{m, K_G(m,n)}$. Thus for every $\varepsilon > 0$,

$$\begin{aligned}
\mathbf{P}\left[\left|\frac{\Delta_{m,n}}{n}\right| > \varepsilon\right] &\le \mathbf{P}\left[X_{m,K_G(m,n)} > n\varepsilon\right] \\
&= \mathbf{P}\left[X_{m,K_G(m,n)} > n\varepsilon, \left|\frac{\mu(G)K_G(m,n)}{\pi n} - a_m\right| > 1\right] \\
&\quad + \mathbf{P}\left[X_{m,K_G(m,n)} > n\varepsilon, \left|\frac{\mu(G)K_G(m,n)}{\pi n} - a_m\right| \le 1\right] \\
&\le \mathbf{P}\left[\left|\frac{\mu(G)K_G(m,n)}{\pi n} - a_m\right| > 1\right] + \frac{2\pi(1+a_m)}{n\mu(G)\varepsilon^2}\max_{k\ge 1}\mathbf{E}\left[X_{m,k}^2\right].
\end{aligned}$$

The first term goes to 0 as $n \to \infty$, by Proposition 3.3. A straightforward argument (based on the fact that $\sup_{\|x\| \le m} \mathbf{E}_x[(T(me))^2] < \infty$) shows that $\sup_{k \ge 1} \mathbf{E}[X_{m,k}^2] < \infty$. Hence the second term also goes to 0 as $n \to \infty$. Thus for every sequence m_n increasing to ∞ slowly enough, the last term on the right-hand side of (3.2), with m replaces by m_n, converges to 0 in probability as $n \to \infty$. This is true in particular for every sequence m_n taking values in the set $\{l(1), l(2), l(3), \dots\}$ and increasing to ∞ slowly enough.

Thus we have shown that for every bounded Borel set G with $\mu(G) > 0$ and for every starting point x_0,

$$\frac{\mu(G)\phi(N_G(n))}{\pi n} \to_d C + c/2 \quad \text{as } n \to \infty,$$

where C is standard Cauchy and c is as in (3.1).

If condition (C.2) (the lattice case) holds, take $G = \{(0,0)\}$ and let $x_0 = (0,0)$. Define

$$\Pi = \big(S(0), S(1), \dots, S\big(N_{\{(0,0)\}}(n)\big)\big).$$

Then Π is a random variable taking values in the countable set

$$\bigcup_{k=n}^{\infty} \left\{ (s_0, s_1, \dots, s_k) \in (\mathscr{L}_d)^{k+1} \colon s_0 = s_k = (0,0) \text{ and } \sum_{i=1}^{k-1} 1_{\{s_i = (0,0)\}} = n - 1 \right\}.$$

If $p = (s_0, s_1, \dots, s_k)$ is an element of that set, let $\tilde{p} = (-s_k, -s_{k-1}, \dots, -s_1, -s_0)$ and observe that

(a) $\mathbf{P}[\Pi = p] = \mathbf{P}[\Pi = \tilde{p}]$;

(b) the windings of the paths p and $\tilde{p}$ are equal in magnitude but opposite in sign.

This implies that if S starts at the origin, then $\phi(N_{\{(0,0)\}}(n))$ has, for every n, a symmetric distribution centered at 0. Thus $c = 0$. If condition (C.1) holds (the absolutely continuous case), take $G = D$, the unit disk centered at the origin. An appropriate modification of the above argument can be used to show that if S starts with uniform distribution on D, then $\phi(N_D(n))$ has, for every n, a symmetric distribution centered at 0. Thus $c = 0$. This completes the proof of Theorem 2.4.

REMARK 1. In the process of proving Theorem 2.4 we have obtained the following result.

THEOREM 3.5. *If conditions* A, B *and* C *are satisfied and if G is a bounded Borel set with $\mu(G) > 0$, then*

$$\mu(G)\phi(N_G(n))/\pi n \to_d C \quad \text{as } n \to \infty,$$

where C is standard Cauchy.

The Brownian analogue of Theorem 3.5 says that

$$\mu(G)\theta(T_G(t))/2\pi t \to_d C \quad \text{as } t \to \infty, \tag{3.6}$$

where C is standard Cauchy, μ is Lebesgue measure and $T_G(t) = \inf\{s > 0: \int_0^s 1_{\{Z(u) \in G\}}\, du \geq t\}$ [Lyons and McKean (1984) and Pitman and Yor (1984, 1986)]. As in Section 1, the difference between the random walk result and the Brownian motion result can be explained in terms of small windings and big windings: A refinement of (3.6) says that

$$\left(\frac{\mu(G)\theta_{\text{small}}(T_G(t))}{\pi t}, \frac{\mu(G)\theta_{\text{big}}(T_G(t))}{\pi t} \right) \to_d (C_1, C_2) \quad \text{as } t \to \infty,$$

where C_1 and C_2 are independent standard Cauchy [Pitman and Yor (1986)].

REMARK 2. If S is the simple symmetric random walk on the integer lattice and if γ_i denotes the winding of S between the $(i-1)$th and ith visits to the origin, then the γ_i's are iid random variables with symmetric distribution having support $\{k\pi/2: k \in \mathbb{Z}^2\}$. Using Theorem 3.5, one can check that $(2/\pi n)\Sigma_{i=1}^n \gamma_i$ converges in distribution to a standard Cauchy random variable.

4. The first approximation. Throughout the rest of this paper we assume that x_0 is the origin. The general case can be handled in the same way.

PROPOSITION 4.1. *Suppose that the sequence of iid $\mathbb{R}^2$-valued random variables $X = (X_1, X_2, X_3, \dots)$ satisfies conditions* A *and* B. *Then, without changing its distribution, one can redefine the sequence X on a richer probability space together with a standard two-dimensional Brownian motion Z, starting at the origin, in such a way that for some positive constants α and κ (depending only on the distribution of X_1)*

$$\mathbf{P}\left[\frac{\sup_{0 \leq u \leq t} \|S(u) - Z(u)\|}{t^{1/2}} > \frac{1}{t^\alpha} \right] \leq \frac{\kappa}{t^\alpha} \quad \textit{for all } t > 0, \tag{4.1}$$

where

$$S(u) = \sum_{i=1}^{[u]} X_i + (u - [u]) X_{[u]+1}.$$

($[u]$ denotes the integer part of u.)

PROOF. If (4.1) were replaced by $\|S(t) - Z(t)\| = O(t^{1/2 - \delta})$ a.s. (for some constant $\delta > 0$ depending only on the distribution of X_1), then the proposition would be a special case of Theorem 3 in Berkes and Philipp (1979). The final step of their proof can be adapted to prove (4.1) [see the appendix of Bélisle (1986)]. □

PROPOSITION 4.2. *Suppose that conditions* A *and* B *hold. Let X, Z, α and κ be as in Proposition* 4.1. *There exist positive constants δ, r_0 and c (depending* *only on the distribution of* X_1) *such that the following approximations hold for every* $r \geq r_0$.

Let A *be the open annulus of radii* re^{-1} *and* re *centered at the origin. Let* X_0 *be an* $\mathbb{R}^2$*-valued random variable independent of* (X, Z) *and satisfying, with* b *as in condition* B,

$$\mathbf{P}[r - b < \|X_0\| < r + b] = 1. \tag{4.2}$$

Let S' *be the random walk defined by* $S'(n) = X_0 + \sum_{i=1}^{n} X_i$, *let* Z' *be the Brownian motion defined by* $Z'(t) = rX_0/\|X_0\| + Z(t)$, *and let*

$$\tau = \min\{j \geq 0: S'(j) \notin A\},$$

$$\tau_* = \inf\{t \geq 0: Z'(t) \notin A\},$$

$\xi =$ *the total angle wound by* S' *around the origin up to time* τ,

$\xi_* =$ *the total angle wound by* Z' *around the origin up to time* τ_*,

$$\rho = \begin{cases} 1 & \text{if } \|S'(\tau)\| \geq re, \\ -1 & \text{if } \|S'(\tau)\| \leq re^{-1}, \end{cases}$$

$$\rho_* = \begin{cases} 1 & \text{if } \|Z'(\tau_*)\| = re, \\ -1 & \text{if } \|Z'(\tau_*)\| = re^{-1}. \end{cases}$$

Then

(a) $$1/2 - c/r^{\delta} \leq \mathbf{P}[\rho = \rho_* = 1] \leq 1/2;$$

(b) $$1/2 - c/r^{\delta} \leq \mathbf{P}[\rho = \rho_* = -1] \leq 1/2;$$

(c) $$\mathbf{E}[|\xi - \xi_*|] \leq c/r^{\delta}.$$

PROOF. Take $\delta = \alpha/3$ and $r_0 = b^{1/(1-\alpha)}$. Fix $r \geq r_0$. Let A^0 be the open annulus of radii $re^{-1} + 2r^{1-\alpha}$ and $re + 2r^{1-\alpha}$, let

$$\tau_*^0 = \inf\{t \geq 0: Z'(t) \notin A^0\},$$

consider the events

$$E_1 = \{\omega \in \Omega: \|Z'(\tau_*^0)\| = re + 2r^{1-\alpha}\},$$

$$E_2 = \{\omega \in \Omega: \tau_*^0 \leq r^{2+\alpha}\},$$

$$E_3 = \left\{\omega \in \Omega: \sup_{0 \leq s \leq r^{2+\alpha}} \|S'(s) - Z'(s)\| \leq 2r^{1-\alpha}\right\}$$

and observe that $E_1 \cap E_2 \cap E_3 \subset \{\omega \in \Omega: \rho = \rho_* = 1\}$. Thus

$$\mathbf{P}[\rho = \rho_* = 1] \geq \mathbf{P}[E_1] + \mathbf{P}[E_2] + \mathbf{P}[E_3] - 2.$$

Here Ω is the probability space on which X and Z are defined. Standard computations yield

$$\mathbf{P}[E_1] \geq 1 - \frac{c}{r^{\alpha}} \quad \text{and} \quad \mathbf{P}[E_2] \geq 1 - \frac{c}{r^{\alpha}}$$

for some constant c (which may now change from line to line) and from (4.1) and

(4.2) we get

$$\mathbf{P}[E_3] \geq 1 - \frac{c}{r^\alpha}.$$

Hence we have

$$\mathbf{P}[\rho = \rho_* = 1] \geq \frac{1}{2} - \frac{c}{r^\alpha} \geq \frac{1}{2} - \frac{c}{r^\delta}.$$

Since $\mathbf{P}[\rho_* = 1] = \frac{1}{2}$, we get (a). The proof of (b) is similar. Now consider the winding difference $|\xi - \xi_*|$. This time let A^+ be the annulus of radii $re^{-1} - 2r^{1-\alpha}$ and $re + 2r^{1-\alpha}$ and let A^- be the annulus of radii $re^{-1} + 2r^{1-\alpha}$ and $re - 2r^{1-\alpha}$. Let

$$\tau_*^- = \inf\{t \geq 0: Z'(t) \notin A^-\},$$
$$\tau_*^+ = \inf\{t \geq 0: Z'(t) \notin A^+\}$$

and consider the events

$$E_4 = \left\{\omega \in \Omega: \sup_{\tau_*^- \leq t \leq \tau_*^+} |\theta(t) - \theta(\tau_*^-)| \leq \frac{1}{r^\delta}\right\},$$
$$E_5 = \{\omega \in \Omega: \tau_*^+ \leq r^{2+\alpha}\},$$
$$E_6 = \left\{\omega \in \Omega: \sup_{0 \leq s \leq r^{2+\alpha}} \|S'(s) - Z'(s)\| \leq 2r^{1-\alpha}\right\},$$

where $\theta(t)$ is the total angle wound by Z' around the origin up to time t. Write

$$\mathbf{E}[|\xi - \xi_*|] = \int_{E_4 \cap E_5 \cap E_6} |\xi - \xi_*|\, dP + \int_{(E_4 \cap E_5 \cap E_6)^c} |\xi - \xi_*|\, dP. \tag{4.3}$$

If $\omega \in E_4 \cap E_5 \cap E_6$, then

$$|\xi(\omega) - \xi_*(\omega)| \leq \frac{c}{r^\delta}.$$

Hence the first term on the right-hand side of (4.3) is bounded above by c/r^δ. For the second term,

$$\begin{aligned}
&\int_{(E_4 \cap E_5 \cap E_6)^c} |\xi - \xi_*|\, dP \\
&\quad\leq \sqrt{\mathbf{E}\left[(\xi - \xi_*)^2\right]}\sqrt{\mathbf{P}\left[(E_4 \cap E_5 \cap E_6)^c\right]} \\
&\quad\leq \left(\sqrt{\mathbf{E}[\xi^2]} + \sqrt{\mathbf{E}[\xi_*^2]}\right)\sqrt{\mathbf{P}[E_4^c] + \mathbf{P}[E_5^c] + \mathbf{P}[E_6^c]}.
\end{aligned} \tag{4.4}$$

From Section 2 we know that for every $r > 0$, $\mathbf{E}[\xi_*^2] = 1$. Now let Q_1 and Q_3 be the first and third quadrants of $\mathbb{R}^2$. For $|\xi(\omega)|$ to be larger than $j^{1/2}$, the path $S'(\omega)$ has to wind around the origin at least $[j^{1/2}/2\pi]$ loops before it exits the annulus A. Hence there has to be at least $[j^{1/2}/2\pi] - 1$ excursions from the set $Q_1 \cap A$ to the set $Q_2 \cap A$ before exiting A. Thus for every $r > 0$,

$$\mathbf{E}[\xi^2] \leq \sum_{j=0}^{\infty} \mathbf{P}[\xi^2 \geq j] = \sum_{j=0}^{\infty} \mathbf{P}[|\xi| \geq j^{1/2}] \leq \sum_{j=0}^{\infty} p^{[j^{1/2}/2\pi]-1},$$

with $p = \sup_{r>0} \sup_{x \in Q_1 \cap A} p(x, r)$, where $p(x, r)$ is the probability, starting at x, that S will hit $Q_2 \cap A$ before it exits A. It is easy to check that $p < 1$. Thus $\sup_{r>0} \mathbf{E}[\xi^2] < \infty$.

As above, we have $\mathbf{P}[E_5^c] \le c/r^\alpha$ and $\mathbf{P}[E_6^c] \le c/r^\alpha$. Now, using conformal invariance, stationarity, scaling and symmetry, we get

$$\mathbf{P}[E_4^c] \le \mathbf{P}\left[\sup_{0 \le t \le \sigma} \beta(t) > cr^{2\delta}\right],$$

where $((\alpha(t), \beta(t));\ t \ge 0)$ is a standard two-dimensional Brownian motion starting at the origin and where $\sigma = \inf\{t \ge 0;\ \alpha(t) = 1\}$. A standard computation yields $\mathbf{P}[E_4^c] \le c/r^{2\delta}$. Thus the right-hand side of (4.4), and hence (4.3), is bounded above by c/r^δ. This completes the proof of Proposition 4.2. □

THEOREM 4.3. *Let $X = (X_1, X_2, X_3, \dots)$ be a sequence of iid $\mathbb{R}^2$-valued random variables. Assume that conditions* A *and* B *are satisfied. Then there exists positive constants δ and c (depending only on the distribution of X_1) such that for every positive integer m we can, without changing its distribution, redefine the sequence X, and hence the sequences $(\eta(m, i);\ i = 0, 1, 2, \dots)$ and $(\xi(m, i);\ i = 1, 2, \dots)$, on an appropriate probability space, together with sequences $(\eta_*(m, i);\ i = 0, 1, 2, \dots)$ and $(\xi_*(m, i);\ i = 1, 2, \dots)$, in such a way that*

(a) *$((\eta_*(m, i);\ i \ge 0), (\xi_*(m, i);\ i \ge 1))$ is equal in distribution to the pair $((\eta(i);\ i \ge 0), (\xi(i);\ i \ge 1))$ of Section* 2, *and*

(b) *$((\eta_*(m, i);\ i \ge 0), (\xi_*(m, i);\ i \ge 1))$ is close to $((\eta(m, i);\ i \ge 0), (\xi(m, i);\ i \ge 1))$ in the sense that for every positive integer j, the following approximations hold: Let $(k_0, k_1, k_2, \dots, k_j)$ and $(l_0, l_1, l_2, \dots, l_j)$ be nonnegative integers such that $\mathbf{P}[F \cap F_*] > 0$, where $F = \bigcap_{i=0}^j \{\eta(m, i) = k_i\}$ and where $F_* = \bigcap_{i=0}^j \{\eta_*(m, i) = l_i\}$. Then:*

(i) *If $k_j > 0$ and $l_j > 0$,*

$$\frac{1}{2} - \frac{c}{(me^{k_j})^\delta} \le \mathbf{P}\big[\eta(m, j+1) = k_j + 1,\ \eta_*(m, j+1) = l_j + 1 | F \cap F_*\big] \le \frac{1}{2},$$

$$\frac{1}{2} - \frac{c}{(me^{k_j})^\delta} \le \mathbf{P}\big[\eta(m, j+1) = k_j - 1,\ \eta_*(m, j+1) = l_j - 1 | F \cap F_*\big] \le \frac{1}{2}.$$

(ii) *If $k_j = 0$ and $l_j > 0$,*

$$\mathbf{P}\big[\eta(m, j+1) = 1,\ \eta_*(m, j+1) = l_j + 1 | F \cap F_*\big] = \frac{1}{2},$$

$$\mathbf{P}\big[\eta(m, j+1) = 1,\ \eta_*(m, j+1) = l_j - 1 | F \cap F_*\big] = \frac{1}{2}.$$

(iii) *If* $k_j > 0$ *and* $l_j = 0$,

$$\frac{1}{2} - \frac{c}{(me^{k_j})^{\delta}} \le \mathbf{P}\big[\eta(m, j+1) = k_j + 1, \eta_*(m, j+1) = 1 | F \cap F_*\big] \le \frac{1}{2} + \frac{c}{(me^{k_j})^{\delta}},$$

$$\frac{1}{2} - \frac{c}{(me^{k_j})^{\delta}} \le \mathbf{P}\big[\eta(m, j+1) = k_j - 1, \eta_*(m, j+1) = 1 | F \cap F_*\big] \le \frac{1}{2} + \frac{c}{(me^{k_j})^{\delta}}.$$

(iv) *If* $k_j = 0$ *and* $l_j = 0$,

$$\mathbf{P}[\eta(m, j+1) = 1, \eta_*(m, j+1) = 1 | F \cap F_*] = 1.$$

(v)

$$\mathbf{E}[|\xi(m, j+1) - \xi_*(m, j+1)| \, | F \cap F_*] \le \frac{c}{(me^{k_j})^{\delta}}.$$

PROOF. Let X, Z, δ, c and r_0 be as in Proposition 4.2. Fix $m \ge 1$. Consider a sequence of iid copies of (X, Z),

$$(X_{m,i}, Z_{m,i}) = ((X_{m,i}(n);\ n \ge 1), (Z_{m,i}(t);\ t \ge 0)), \qquad i = 0, 1, 2, \dots .$$

For each i, let $S_{m,i} = (S_{m,i}(n);\ n \ge 0)$ be the random walk defined by $S_{m,i}(n) = \sum_{j=1}^{n} X_{m,i}(j)$. Proceed with the following construction:

Step 0. Let $\tau(m,0) = \min\{n \ge 0;\ \|S_{m,0}(n)\| \ge m\}$, $\eta(m,0) = \eta_*(m,0) = 0$.

Step 1. Consider the pair $(X_{m,1}, Z_{m,1})$. Let

$$S'_{m,1}(n) = S_{m,1}(n) + S'_{m,0}(\tau(m,0)), \qquad n \ge 0,$$

$$Z'_{m,1}(t) = Z_{m,1}(t) + m\frac{S'_{m,0}(\tau(m,0))}{\|S'_{m,0}(\tau(m,0))\|}, \qquad t \ge 0,$$

$$\tau(m,1) = \min\{n \ge 0:\ \|S'_{m,1}(n)\| \ge me\},$$

$$\tau'(m,1) = \min\{n \ge 0:\ \|S'_{m,1}(n)\| \le me^{-1} \text{ or } \ge me\},$$

$$\tau_*(m,1) = \inf\{t \ge 0:\ \|Z'_{m,1}(t)\| = me^{-1} \text{ or } = me\},$$

$$\eta(m,1) = 1,$$

$$\eta_*(m,1) = 1,$$

$$\xi(m,1) = \text{the total angle wound by } S'_{m,1} \text{ around the origin at time } \tau'(m,1),$$

$$\xi(m,1) = \text{the total angle wound by } Z'_{m,1} \text{ around the origin at time } \tau_*(m,1).$$

For $j \ge 2$, proceed recursively with:

Step j. Consider the pair $(X_{m,j}, Z_{m,j})$. Let

$$S'_{m,j}(n) = S_{m,j}(n) + S'_{m,j-1}(\tau(m, j-1)), \quad n \geq 0,$$

$$Z'_{m,j}(t) = Z_{m,j}(t) + me^{\eta(m,j-1)} \frac{S'_{m,j-1}(\tau(m, j-1))}{\|S'_{m,j-1}(\tau(m, j-1))\|}, \qquad t \geq 0$$

$\big[$if $\|S'_{m,j-1}(\tau(m, j-1))\| = 0$, set

$S'_{m,j-1}(\tau(m, j-1))/\|S'_{m,j-1}(\tau(m, j-1))\| = (1,0)\big]$,

$$\tau'(m, j) = \min\{n \geq 0: \|S'_{m,j}(n)\| \leq me^{\eta(m,j-1)-1} \text{ or } \geq me^{\eta(m,j-1)+1}\},$$

$$\tau(m, j) = \begin{cases} \tau'(m, j) & \text{if } \eta(m, j-1) > 0, \\ \min\{n \geq 0: \|S'_{m,j}(n)\| \geq me\} & \text{if } \eta(m, j-1) = 0, \end{cases}$$

$$\tau_*(m, j) = \inf\{t \geq 0: \|Z'_{m,j}(t)\| = me^{\eta(m,j-1)-1} \text{ or } = me^{\eta(m,j-1)+1}\},$$

$$\eta(m, j) = \begin{cases} \eta(m, j-1) + 1 & \text{if } \|S'_{m,j}(\tau(m, j))\| \geq me^{\eta(m,j-1)+1}, \\ \eta(m, j-1) - 1 & \text{if } \|S'_{m,j}(\tau(m, j))\| \leq me^{\eta(m,j-1)-1}, \end{cases}$$

$$\eta_*(m, j) = \begin{cases} 1 & \text{if } \eta_*(m, j-1) = 0, \\ \eta_*(m, j-1) + 1 & \text{if } \eta_*(m, j-1) > 0 \\ & \text{and } \|Z'_{m,j}(\tau_*(m, j))\| = me^{\eta(m,j-1)+1}, \\ \eta_*(m, j-1) - 1 & \text{if } \eta_*(m, j-1) > 0 \\ & \text{and } \|Z'_{m,j}(\tau_*(m, j))\| = me^{\eta(m,j-1)-1}, \end{cases}$$

$\xi(m, j)$ = the total angle wound by $S'_{m,j}$ around the origin at time $\tau'(m, j)$,

$\xi_*(m, j)$ = the total angle wound by $Z'_{m,j}$ around the origin at time $\tau_*(m, j)$.

This completes the construction. Connecting the excursions $(S'_{m,j}(n);\ n = 0, 1, 2, \dots, \tau(m, j))$, $j = 0, 1, 2, \dots$, in the obvious way we obtain a random walk which is, in distribution, identical to the original random walk of Section 1. Hence, the sequences $(\eta(m, i);\ i = 0, 1, 2, \dots)$ and $(\xi(m, i);\ i = 0, 1, 2, \dots)$ just constructed are, in distribution, identical to those constructed in Section 2. Strong Markov property and conformal invariance yield (a). If $me^{k_j} \geq r_0$ the conclusion of (b) follows from Proposition 4.2. There are only finitely many pairs of integers $m \geq 1$ and $k \geq 0$ for which $me^k < r_0$. It follows easily that (b) holds if we take c large enough. □

5. The second approximation. Let m_0 be a positive integer large enough so that $c/m_0^\delta < 1/2$, where c and δ are as in Theorem 4.3. For integers $m \geq m_0$ and $k \geq 1$ let

$$\triangle(m, k) = c/(me^k)^\delta.$$

Now fix $m \geq m_0$ and let $V_2, V_3, V_4, \dots$ be a sequence of independent random variables with uniform distribution over the interval [0, 1]. Let

$$\eta_L(m,0) = \hat{\eta}(m,0) = \hat{\eta}_*(m,0) = \eta_U(m,0) = 0,$$
$$\eta_L(m,1) = \hat{\eta}(m,1) = \hat{\eta}_*(m,1) = \eta_U(m,1) = 1$$

and, for $j \geq 1$, proceed recursively with

$$\eta_L(m, j+1) = \begin{cases} 1 & \text{if } \eta_L(m, j) = 0, \\ \eta_L(m, j) - 1 & \text{if } \eta_L(m, j) > 0 \\ & \quad \text{and } V_{j+1} \leq 1/2 + \Delta(m, \eta_L(m, j)), \\ \eta_L(m, j) + 1 & \text{if } \eta_L(m, j) > 0 \\ & \quad \text{and } V_{j+1} > 1/2 + \Delta(m, \eta_L(m, j)), \end{cases}$$

$$\hat{\eta}_*(m, j+1) = \begin{cases} \hat{\eta}_*(m, j) - 1 & \text{if } \hat{\eta}_*(m, j) > 0 \text{ and } V_{j+1} \leq 1/2, \\ \hat{\eta}_*(m, j) + 1 & \text{if } \hat{\eta}_*(m, j) > 0 \text{ and } V_{j+1} > 1/2, \end{cases}$$

$$\hat{\eta}(m, j+1) = \begin{cases} 1 & \text{if } \hat{\eta}(m, j) = 0, \\ \hat{\eta}(m, j) - 1 & \text{if } \hat{\eta}(m, j) > 0,\ \hat{\eta}_*(m, j) = 0 \\ & \quad \text{and } V_{j+1} \in [0, p_0] \\ & \text{or if } \hat{\eta}(m, j) > 0,\ \hat{\eta}_*(m, j) > 0 \\ & \quad \text{and } V_{j+1} \in [0, p_1] \cup [1/2, p_2], \\ \hat{\eta}(m, j) + 1 & \text{if } \hat{\eta}(m, j) > 0,\ \hat{\eta}_*(m, j) = 0 \\ & \quad \text{and } V_{j+1} \in (p_0, 1] \\ & \text{or if } \hat{\eta}(m, j) > 0,\ \hat{\eta}_*(m, j) > 0 \\ & \quad \text{and } V_{j+1} \in (p_1, 1/2) \cup (p_2, 1], \end{cases}$$

$$\eta_U(m, j+1) = \begin{cases} 1 & \text{if } \eta_U(m, j) = 0, \\ \eta_U(m, j) - 1 & \text{if } \eta_U(m, j) > 0 \\ & \quad \text{and } V_{j+1} \leq 1/2 - \Delta(m, \eta_U(m, j)), \\ \eta_U(m, j) + 1 & \text{if } \eta_U(m, j) > 0 \\ & \quad \text{and } V_{j+1} > 1/2 - \Delta(m, \eta_U(m, j)), \end{cases}$$

where

$$p_0 = \mathbf{P}[\eta(m, j+1) = \eta(m, j) - 1 | \eta(m, i), \eta_*(m, i);\ i = 0, 1, 2, \dots, j],$$
$$p_1 = \mathbf{P}[\eta(m, j+1) = \eta(m, j) - 1, \eta_*(m, j+1)$$
$$= \eta_*(m, j) - 1 | \eta(m, i), \eta_*(m, i);\ i = 0, 1, 2, \dots, j],$$
$$p_2 = \mathbf{P}[\eta(m, j+1) = \eta(m, j) + 1, \eta_*(m, j+1)$$
$$= \eta_*(m, j) + 1 | \eta(m, i), \eta_*(m, i);\ i = 0, 1, 2, \dots, j].$$

The subscripts L and U stand for "lower" and "upper." The following result follows at once from Theorem 4.3 and the above construction.

PROPOSITION 5.1. *For* $m \geq m_0$,

(a) $(\hat{\eta}(m,j), \hat{\eta}_*(m,j);\ j \geq 0) =_d (\eta(m,j), \eta_*(m,j);\ j \geq 0)$;
(b) $\mathbf{P}[\eta_L(m,j) \leq \hat{\eta}(m,j) \leq \eta_U(m,j)$ *for every* $j \geq 0] = 1$;
(c) $\mathbf{P}[\eta_L(m,j) \leq \hat{\eta}_*(m,j) \leq \eta_U(m,j)$ *for every* $j \geq 0] = 1$;
(d) $(\eta_L(m,j);\ j \geq 0)$ *is a Markov chain on the nonnegative integers, starting at* 0, *with transition probabilities given by* $P_L(0,1) = 1$ *and, for* $k \geq 1$, $P_L(k, k-1) = 1/2 + \Delta(m,k)$ *and* $P_L(k, k+1) = 1/2 - \Delta(m,k)$;
(e) $(\eta_U(m,j);\ j \geq 0)$ *is a Markov chain on the nonnegative integers, starting at* 0, *with transition probabilities given by* $P_U(0,1) = 1$ *and, for* $k \geq 1$, $P_U(k, k-1) = 1/2 - \Delta(m,k)$ *and* $P_U(k, k+1) = 1/2 + \Delta(m,k)$.

Now let $a_L(m,0) = 0$, $a_L(m,1) = 1$ and for $k \geq 2$ define $a_L(m,k)$ recursively via

$$\begin{aligned}&\big(a_L(m,k-1) - a_L(m,k-2)\big)/\big(a_L(m,k) - a_L(m,k-2)\big)\\&\quad = 1/2 - \Delta(m,k-1).\end{aligned}$$

Let h_L be the inverse of a_L, defined by

$$h_L(m, a_L(m,k)) = k, \qquad k \geq 0.$$

Let $B_L = (B_L(t);\ t \geq 0)$ be a standard reflected Brownian motion starting at 0. Let $\tau_L(m,0) = 0$ and for $j \geq 1$ define $\tau_L(m,j)$ recursively via

$$\begin{aligned}\tau_L(m,j) = \inf\{t > \tau_L(m,j-1)\colon B_L(t) = a_L(m,k)\\ \text{for some } k \neq h_L(m, B_L(\tau_L(m,j-1)))\}.\end{aligned}$$

Then for each $m \geq m_0$ the sequence $(h_L(m, B_L(\tau_L(m,j)));\ j \geq 0)$ is equal in distribution to the sequence $(\eta_L(m,j);\ j \geq 0)$. A similar representation holds for the sequence $(\eta_U(m,j);\ j \geq 0)$. These Brownian embedding representations allow the following computations [where we define $N_L(m,k)$, $I_L(m,k)$ in terms of the sequence $(\eta_L(m,j),\ j \geq 0)$ and $N_U(m,k)$, $I_U(m,k)$ in terms of the sequence $(\eta_U(m,j);\ j \geq 0)$ just like we defined $N(m,k)$, $I(m,k)$ in terms of the sequence $(\eta(m,j);\ j \geq 0)$ in Section 2].

PROPOSITION 5.2.

(a) $$I_L(m,k)/k \to_d I(k)/k,\ I_U(m,k)/k \to_d I(k)/k$$

as $m \to \infty$, *uniformly in* k.

(b) $$N_L(m,k)/k^2 \to_d N(k)/k^2,\ N_U(m,k)/k^2 \to_d N(k)/k^2$$

as $m \to \infty$, *uniformly in* k.

PROOF. We will consider the sequences $I_L(m,k)$ and $N_L(m,k)$. The sequences $I_U(m,k)$ and $N_U(m,k)$ can be analyzed in the same way. From (d) of Proposition 5.1, we have $(\eta_L(m,j);\ j \geq 0) \to_d (\eta(j);\ j \geq 0)$ as $m \to \infty$ from which we get

$$\frac{I_L(m,k)}{k} \to_d \frac{I(k)}{k} \quad \text{as } m \to \infty, \text{ for each } k \geq 1, \tag{5.1}$$

and

$$\frac{N_L(m,k)}{k^2} \to_d \frac{N(k)}{k^2} \quad \text{as } m \to \infty, \text{ for each } k \geq 1. \tag{5.2}$$

A direct computation, using the Brownian embedding representation of η_L, shows that $I_L(m, k)$ has a geometric distribution with mean $a_L(m, k)$. Furthermore, for $m \geq m_0$, $a_L(m, k)/k$ converges to α_m as $k \to \infty$, uniformly in m, for some constant α_m such that $\alpha_m \to 1$ as $m \to \infty$. Thus, for $m \geq m_0$,

$$\frac{I_L(m,k)}{k} \to_d \alpha_m E \quad \text{as } k \to \infty, \text{ uniformly in } m,$$

where E is standard exponential. Combined with (5.1), this implies part (a) of the proposition. For part (b), observe that for $m \geq m_0$ and $k \geq 1$,

$$\frac{N(k)}{k^2} \leq_s \frac{N_L(m,k)}{k^2} \leq_s \frac{N_L(m_0,k)}{k^2}. \tag{5.3}$$

Here $\leq_s$ denotes stochastic ordering. From Proposition 2.2, $N(k)/k^2 \to_d T$ as $k \to \infty$, where T is the hitting time of 1 by a standard reflected Brownian motion starting at 0. The sequence of processes $\eta_{L, m_0, n}(t) = (1/\sqrt{n})\eta_L(m_0, [nt])$, converges weakly to a standard reflected Brownian motion starting at 0, as seen from Guttorp, Kulperger and Lockhart (1985). Thus $N_L(m_0, k)/k^2 \to_d T$ as $k \to \infty$. Thus from (5.3) we get, for $m \geq m_0$,

$$\frac{N_L(m,k)}{k^2} \to_d T \quad \text{as } k \to \infty, \text{ uniformly in } m.$$

Combined with (5.2), this implies part (b) of the proposition. □

Proposition 5.3. *For every nonnegative integer l,*

$$\sup_{m \geq m_0} \sup_{k \geq 1} \frac{1}{\sqrt{k}} \sum_{i=0}^{k} \mathbf{P}[\eta_L(m,i) \leq l] < \infty.$$

Proof. We consider the case $l = 0$. The general case can be analyzed in the same way. Recalling the definition of $a_L(m, k)$, we observe that for $k \geq 2$ and $m \geq m_0$,

$$a_L(m,k) - a_L(m,k-1) = \prod_{i=1}^{k-1} (1/2 + \Delta(m,i))/(1/2 - \Delta(m,i)).$$

Hence, recalling the definition of $\Delta(m, i)$, we observe that $(a_L(m, k) - a_L(m, k-1);\ m \geq m_0,\ k \geq 1)$ is decreasing in m and increasing in k. Thus

$$\sup_{m \geq m_0} \sup_{k \geq 1} (a_L(m,k) - a_L(m,k-1)) = \lim_{k \to \infty} (a_L(m_0,k) - a_L(m_0,k-1)).$$

Let a denote the limit on the right-hand side. Since $\sum_{k=1}^{\infty}\Delta(m_0, k) < \infty$, it is easy to show that $a < \infty$. Now fix $m \geq m_0$. Recalling the Brownian embedding representation of the sequence $(\eta_L(m, j);\ j \geq 0)$, let $\tau_L'(j) = 0$ and for $j \geq 1$ define $\tau_L'(j)$ recursively via

$$\tau_L'(j) = \inf\{t > \tau_L'(j-1) \colon B_L(t) = B_L(\tau_L'(j-1)) - 3a \text{ or } B_L(\tau_L'(j-1)) + 3a\}.$$

The differences $\tau_L'(j) - \tau_L'(j-1)$ are iid with mean $9a^2$ and variance $54a^4$. Thus

$$\mathbf{E}[\tau_L'(k)] = 9a^2 k \quad \text{and} \quad \operatorname{var}[\tau_L'(k)] = 54a^4 k. \tag{5.4}$$

Furthermore, each interval $[3ak, 3a(k+1)]$ contains at least two of the $a_L(m, j)$'s and this implies that $\tau_L(m, k+1) \leq \tau_L'(k)$. Therefore

$$\begin{aligned}
\sum_{i=0}^{k} \mathbf{P}[\eta_L(m, i) = 0] &= \sum_{i=0}^{k} \mathbf{E}[1_{\{\eta_L(m, i)=0\}}] \\
&\leq \mathbf{E}\left[\int_0^{\tau_L(m, k+1)} 1_{\{B_L(t) \leq 1\}}\, dt\right] \\
&\leq \mathbf{E}\left[\int_0^{\tau_L'(k)} 1_{\{B_L(t) \leq 1\}}\, dt\right] \\
&= \mathbf{E}\left[\int_0^{\mathbf{E}[\tau_L'(k)]} 1_{\{B_L(t) \leq 1\}}\, dt\right] \\
&\quad + \mathbf{E}\left[\int_0^{\tau_L'(k)} 1_{\{B_L(t) \leq 1\}}\, dt - \int_0^{\mathbf{E}[\tau_L'(k)]} 1_{\{B_L(t) \leq 1\}}\, dt\right] \\
&\leq 1 + \int_1^{\mathbf{E}[\tau_L'(k)]} \mathbf{P}[B_L(t) \leq 1]\, dt + \mathbf{E}[|\tau_L'(k) - \mathbf{E}[\tau_L'(k)]|] \\
&\leq 1 + \int_1^{\mathbf{E}[\tau_L'(k)]} \sqrt{\frac{2}{\pi t}}\, dt + \sqrt{\operatorname{var}[\tau_L'(k)]}\,.
\end{aligned}$$

Now, using (5.4), we obtain

$$\sup_{m \geq m_0} \sup_{k \geq 1} (1/\sqrt{k}) \sum_{i=0}^{k} \mathbf{P}[\eta_L(m, i) = 0] < \infty. \qquad \square$$

PROPOSITION 5.4.

$$\sup_{m \geq m_0} \sup_{k \geq 1} \frac{m^{\delta}}{\sqrt{k}} \sum_{i=1}^{k} \mathbf{E}[|\xi(m, i) - \xi_*(m, i)|] < \infty,$$

where δ is as in Theorem 4.3.

Proof. For $m \geq m_0$ and $i \geq 1$,

$$
\begin{aligned}
&\mathbf{E}[|\xi(m,i) - \xi_*(m,i)|] \\
&\quad = \sum_{j=0}^{\infty} \mathbf{E}[|\xi(m,i) - \xi_*(m,i)| \,|\eta(m,i-1) = j]\mathbf{P}[\eta(m,i-1) = j] \\
&\quad \leq \frac{c}{m^\delta} \sum_{j=0}^{\infty} e^{-\delta j}\mathbf{P}[\eta(m,i-1) = j] \\
&\quad \leq \frac{c}{m^\delta} \sum_{j=0}^{\infty} e^{-\delta j}\mathbf{P}[\eta_L(m_0,i-1) = j] \\
&\quad \leq \frac{c}{m^\delta(1-e^{-\delta})} \max_{j \geq 0} \mathbf{P}[\eta_L(m_0,i-1) = j] \\
&\quad \leq \frac{c}{m^\delta(1-e^{-\delta})} \mathbf{P}[\eta_L(m_0,i-1) \leq 1].
\end{aligned} \tag{5.5}
$$

This first inequality follows from Theorem 4.3. Parts (a) and (b) of Proposition 5.1 imply that $\eta_L(m, i-1) \leq_s \eta(m, i-1)$ while part (d) implies that $\eta_L(m_0, i-1) \leq_s \eta_L(m, i-1)$. Thus $\mathbf{E}[e^{-\delta\eta(m,i-1)}] \leq \mathbf{E}[e^{-\delta\eta_L(m_0,i-1)}]$. This is the second inequality of (5.5). The last inequality of (5.5) follows from part (d) of Proposition 5.1. Proposition 5.4 now follows from Proposition 5.3. □

6. Proof of Theorem 2.1. From Proposition 5.1 we have, for $m \geq m_0$ and $k \geq 1$,

$$\frac{1}{k}I_U(m,k) \leq_s \frac{1}{k}I(m,k) \leq_s \frac{1}{k}I_L(m,k)$$

and

$$\frac{1}{k^2}N_U(m,k) \leq_s \frac{1}{k^2}N(m,k) \leq_s \frac{1}{k^2}N_L(m,k).$$

Thus (a) and (b) of Theorem 2.1 follow from (a) and (b) of Proposition 5.2. Now consider the right-hand side of

$$\frac{1}{k}\sum_{i=1}^{N(m,k)} \xi(m,i) = \frac{1}{k}\sum_{i=1}^{N_*(m,k)} \xi_*(m,i) + \frac{1}{k}\left(\sum_{i=1}^{N(m,k)} \xi(m,i) - \sum_{i=1}^{N_*(m,k)} \xi_*(m,i)\right)$$

[with $N_*(m,k) = \min\{j > 0;\ \eta_*(m,j) = k\}$]. For every m and every k the first term is equal in distribution to $(1/k)\sum_{i=1}^{N(k)}\xi(i)$ and the absolute value of the second term is bounded above by

$$\frac{1}{k}\sum_{i=1}^{N(m,k)} |\xi(m,i) - \xi_*(m,i)| + \frac{1}{k}\max_{i_* \leq l_1 \leq l_2 \leq i^*}\left|\sum_{i=l_1+1}^{l_2} \xi_*(m,i)\right|, \tag{6.1}$$

where $i_* = N(m,k) \wedge N_*(m,k)$ and $i^* = N(m,k) \vee N_*(m,k)$. Thus in order to prove part (c) of Theorem 2.1, it suffices to show that both terms in (6.1) converge in probability to 0 as $m \to \infty$, uniformly in k. Fix ε between 0 and 2δ.

Then

$$\mathbf{P}\left[\frac{1}{k}\sum_{i=1}^{N(m,k)}|\xi(m,i)-\xi_*(m,i)|>\varepsilon\right]$$
$$\leq \mathbf{P}\left[\frac{1}{k}\sum_{i=1}^{m^\varepsilon k^2}|\xi(m,i)-\xi_*(m,i)|>\varepsilon\right]+\mathbf{P}\left[N(m,k)>m^\varepsilon k^2\right]$$
$$\leq \frac{1}{k\varepsilon}\sum_{i=1}^{m^\varepsilon k^2}\mathbf{E}\left[|\xi(m,i)-\xi_*(m,i)|\right]+\mathbf{P}\left[N(m,k)>m^\varepsilon k^2\right].$$

By Proposition 5.4, the first term goes to 0, and $m \to \infty$, uniformly in k. By part (b), the second term goes to 0, as $m \to \infty$, uniformly in k. Now consider the second term on the right-hand side of (6.1). Fix $\varepsilon > 0$. For every $a > 0$ we can write

$$\mathbf{P}\left[\frac{1}{k}\max_{i_*\leq l_1\leq l_2\leq i^*}\left|\sum_{i=l_1+1}^{l_2}\xi_*(m,i)\right|>\varepsilon\right]$$
$$\leq \mathbf{P}\left[\frac{1}{k}\max_{i_*\leq l_1\leq l_2\leq i^*}\left|\sum_{i=l_1+1}^{l_2}\xi_*(m,i)\right|>\varepsilon \text{ and } |N(m,k)-N_*(m,k)|\leq ak^2\right]$$
$$+\mathbf{P}\left[|N(m,k)-N_*(m,k)|>ak^2\right]$$
$$\leq \mathbf{P}\left[\frac{1}{k}\max_{0\leq l_1\leq l_2\leq 2[ak^2]}\left|\sum_{i=l_1+1}^{l_2}\xi(i)\right|>\varepsilon\right]+\mathbf{P}\left[|N(m,k)-N_*(m,k)|>ak^2\right] \tag{6.2}$$
$$\leq \mathbf{P}\left[\frac{1}{k}\max_{0\leq l_1\leq l_2\leq 2[ak^2]}\left|\sum_{i=l_1+1}^{l_2}\xi(i)\right|>\varepsilon\right]$$
$$+\mathbf{P}\left[\left(N_L(m,k)-N_U(m,k)\right)>ak^2\right].$$

The second inequality follows from the fact that

$$|N(m,k)-N_*(m,k)|\leq ak^2$$
$$\Rightarrow N_*(m,k)-[ak^2]\leq i_*\leq i^*\leq N_*(m,k)+[ak^2]$$

and the fact that $\xi_*(m,1), \xi_*(m,2),\dots$ are iid independent of $N_*(m,k)$ and equal in distribution to $\xi(1), \xi(2),\dots$. The last inequality follows from Proposition 5.1. Now observe that the sequence

$$\left(\max_{0\leq l_1<l_2\leq l}\frac{1}{\sqrt{l}}\left|\sum_{i=l_1+1}^{l_2}\xi_*(i)\right|;\ l\geq 1\right)$$

is tight since, by Donsker's invariance principle, it converges in distribution as $l \to \infty$. This implies that for every sequence a_m decreasing to 0, the first term on the right-hand side of (6.2), with a replaces by a_m, goes to 0 as $m \to \infty$.

Furthermore, from Proposition 5.2 we get $(N_L(m, k) - N_U(m, k))/k^2 \to_P 0$ as $m \to \infty$, uniformly in k. Thus if a_m decreases to 0 slowly enough, then the second term on the right-hand side of (6.2), with a replaced by a_m, goes to 0 as $m \to \infty$, uniformly in k. This shows that the second term on the right-hand side of (6.1) goes to 0 in probability as $m \to \infty$, uniformly in k. The proof of Theorem 2.1 is complete.

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