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RANDOMIZATION IN THE TWO-ARMED BANDIT PROBLEM

BY ROBERT C. DALANG

University of California, Berkeley

We give a short new proof of the existence of optimal solutions to a continuous time formulation of the two-armed bandit problem, using a new topological embedding of the set of randomized optional increasing paths. We do not make any hypothesis on the two-parameter filtration, other than completeness and right-continuity.

1. Introduction. The objective of this note is to give an elementary solution to a general form of the continuous time two-armed bandit problem. This form of the problem was suggested and solved under rather weak regularity assumptions by Mazziotto and Millet [11]. The theory of multiparameter processes had already been used in the two-armed bandit problem by Mandelbaum [10], who studied the case of diffusion bandits. A different formulation of the problem had initially been solved by Karatzas [9]. References to the discrete version of this problem can be found in [2] and [13].

The approach used in this note is that of [11]. The two-armed bandit problem is expressed as the maximization of a function ϕ defined on the set of optional increasing paths by

$$(Z_u)_{u\in\overline{\mathbb{R}}_+} \mapsto E\left(\int_{\overline{\mathbb{R}}_+} X_{Z_u}\,dV_u\right),$$

where $(X_t)_{t\in\overline{\mathbb{R}}_+^2}$ is a (sufficiently regular) random process and $(V_u)_{u\in\overline{\mathbb{R}}_+}$ is a bounded process with nondecreasing sample paths. The proof of the existence of an optional increasing path (o.i.p.) at which ϕ is maximum uses the notion of randomization, as developed by Baxter and Chacon [1] and Meyer [12] and used for optimal stopping problems by Ghoussoub [8] and Dalang [4, 7] as well as by Mazziotto and Millet [11]. The set of o.i.p.'s is embedded into a compact convex set $\mathscr{Z}_r$ on which ϕ is extended to a (regular) affine functional Φ. This extended functional will attain its maximum at an extremal element of $\mathscr{Z}_r$. It turns out that this element is, in fact, an o.i.p. and is thus a solution to the problem.

Many of the ideas in this note can already be found in [11], to which reference could be made throughout. However, the topological embedding of the set of randomized optional increasing paths is new and enables a resolution of the problem that requires only straightforward topological arguments. Furthermore, we carry out the proofs rather carefully and this allows us to remove an unnecessary hypothesis on the two-parameter filtration (termed F5 in [11]). So the theorem of Mazziotto and Millet becomes one of the first nontrivial results in the theory of two-parameter processes which does not require any extra condition on the two-parameter filtration.

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2. Formulation of the problem. We first introduce some notation. The set $\mathbb{R}_+$ is endowed with its usual total order $\leq$, whereas on $\mathbb{R}^2_+$ it is natural to consider the two orders $\leq$ and $\underline{\wedge}$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \Leftrightarrow s_1 \leq t_1 \text{ and } s_2 \leq t_2,$$

$$s = (s_1, s_2) \underline{\wedge}\, t = (t_1, t_2) \Leftrightarrow s_1 \leq t_1 \text{ and } s_2 \geq t_2.$$

For $t \in \mathbb{R}^2_+$, we set $|t| = t_1 + t_2$.

We shall add to $\mathbb{R}_+$ and $\mathbb{R}^2_+$ an extra element, denoted in both cases ∞, and will set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$, $\overline{\mathbb{R}}^2_+ = \mathbb{R}^2_+ \cup \{\infty\}$. These sets will be equipped with their usual metric topologies making them compact. We will also suppose that $t \leq \infty$, for all t in either $\mathbb{R}_+$ or $\mathbb{R}^2_+$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. A *two-parameter filtration* is a family $(\mathscr{F}_t)_{t \in \mathbb{R}^2_+}$ of sub-σ-algebras of $\mathscr{F}$ which is increasing for $\leq$ ($s \leq t \Rightarrow \mathscr{F}_s \subset \mathscr{F}_t$), complete and *right-continuous*, i.e., $\mathscr{F}_s = \bigcap_{t \gg s} \mathscr{F}_t$, $\forall\, s \in \mathbb{R}^2_+$ ($t \gg s$ means $t_1 > s_1$ and $t_2 > s_2$). A random variable $T\colon \Omega \to \overline{\mathbb{R}}^2_+$ is a *stopping point* provided $\{T \leq t\} \in \mathscr{F}_t$, $\forall\, t \in \mathbb{R}^2_+$.

An *optional increasing path* is a family $Z = (Z_u)_{u \in \overline{\mathbb{R}}_+}$ of stopping points such that $u \mapsto Z_u(\cdot)$ is increasing (for $\leq$), $|Z_u| = u$ a.s., $\forall\, u \in \mathbb{R}_+$ and $Z_\infty = \infty$ a.s. [note that $Z_0 \equiv (0,0)$ a.s.]. The set of all o.i.p.'s will be denoted $\mathscr{Z}$.

The following lemma is the crucial observation that allows removing Hypothesis F5 of [11]. Its short proof, which can also be found in [6], Lemma 2.2, is included for completeness.

2.1 LEMMA. *Let $Z \in \mathscr{Z}$. Then for $0 \leq v \leq u$,*

$$\{Z_u \underline{\wedge} (v, u - v)\} \in \mathscr{F}_{v, u-v}.$$

PROOF. Set $Z_u = (Z^1_u, Z^2_u)$, and observe that

$$\{Z_u \underline{\wedge} (v, u - v)\} = \bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcup_{r \in \mathbb{Q}_+} \{Z^1_r \leq v + \varepsilon,\ u - v \leq Z^2_r \leq u - v + \varepsilon\}.$$

Since Z_r is a stopping point and $(\mathscr{F}_t)_{t \in \mathbb{R}^2_+}$ is right-continuous, this event belongs to $\mathscr{F}_{v, u-v}$. □

Let $X = (X_t)_{t \in \overline{\mathbb{R}}^2_+}$ be a real-valued two-parameter process defined on $(\Omega, \mathscr{F}, P)$ with upper-semicontinuous (u.s.c.) sample paths such that $E(\sup_{t \in \overline{\mathbb{R}}^2_+} X_t) < \infty$ and let $V = (V_u)_{u \in \overline{\mathbb{R}}_+}$ be a bounded nonnegative right-continuous process with nondecreasing sample paths. The main theorem of this note is the following.

2.2. THEOREM. *For each $Z \in \mathscr{Z}$, set $R(Z) = E(\int_{\overline{\mathbb{R}}_+} X_{Z_u}\, dV_u)$. Then there is an o.i.p. $Z^* \in \mathscr{Z}$ such that $R(Z^*) = \sup_{Z \in \mathscr{Z}} R(Z)$, i.e., Z^* is an optimal o.i.p.*

2.3. REMARKS. (a) We do not assume any supplementary condition on the two-parameter filtration. In particular, Hypothesis F5 of [11] is removed.

(b) We do not assume separability of the process X, as was the case in [11].

(c) In Theorem 2.2 of [11], the random measure dV_u is not assumed to be positive. However, if dV_u is for instance negative, the map $Z \mapsto R(Z)$ will be lower-semicontinuous instead of u.s.c. and it is not difficult to construct examples where there is no optimal o.i.p.

3. Randomized optional increasing paths. Let $\mathscr{C}$ denote the space of continuous processes $Y = (Y_v)_{v \in \overline{\mathbb{R}}_+}$ such that $E(\sup_{v \in \overline{\mathbb{R}}_+} |Y_v|) < \infty$. $\mathscr{C}$ equipped with the norm $\|Y\| = E(\sup_v |Y_v|)$ is a Banach space (processes that differ on an evanescent set are identified).

Following Meyer [12], a *randomized random variable* on $\overline{\mathbb{R}}_+$ is a nonnegative process $(A_v)_{v \in \overline{\mathbb{R}}_+}$ with nondecreasing sample paths such that $A_\infty \equiv 1$ a.s. Each random variable $S\colon \Omega \to \overline{\mathbb{R}}_+$ can be identified with the randomized random variable defined by $A_v = I_{\{S \le v\}}$ and so the set of all random variables identifies with a subset of the set $\mathscr{U}$ of all randomized random variables. It is convenient to set $A_v = 0$ when $v \le 0$.

The set $\mathscr{U}$ is clearly convex and it is well-known that $\mathscr{U}$ is compact for the *Baxter–Chacon topology* ([1], [12], [8]), that is, the smallest topology on $\mathscr{U}$ for which the maps $(A_v)_{v \in \overline{\mathbb{R}}_+} \mapsto E(\int_{\overline{\mathbb{R}}_+} Y_v \, dA_v)$ are continuous, for all $(Y_v)_{v \in \overline{\mathbb{R}}_+} \in \mathscr{C}$.

Observe that if $(Z_u)_{u \in \overline{\mathbb{R}}_+}$ is an o.i.p., then Z_u takes its values in the set $\{s \in \mathbb{R}_+^2\colon |s| = u\}$ and so we can identify Z_u with the element $A^u = (A_v^u)_{v \in \overline{\mathbb{R}}_+}$ of $\mathscr{U}$ defined by

$$A_v^u = I_{\{Z_u^1 \le v\}} = I_{\{Z_u \triangle (v, u-v)\}}. \tag{3.1}$$

By Lemma 2.1, A^u satisfies

$$A_v^u \text{ is } \mathscr{F}_{v, u-v}\text{-measurable for all } v \le u \text{ and } A_u^u \equiv 1 \quad \text{a.s.} \tag{3.2}$$

For $u \in \mathbb{R}_+$ let $\mathscr{U}^u$ be the subset of elements of $\mathscr{U}$ for which (3.2) holds and let $\mathscr{U}^\infty$ consist of the single probability measure $\delta_{\{\infty\}}$.

3.3. LEMMA. *$\mathscr{U}^u$ is a closed subset of $\mathscr{U}$ and thus is compact.*

PROOF. If $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, then an integrable random variable S is $\mathscr{G}$-measurable if and only if $E(\tilde{S}S) = E(E(\tilde{S}|\mathscr{G})S)$, for all (bounded) random variables $\tilde{S}$. Now by the definition of the Baxter–Chacon topology on $\mathscr{U}$, the conclusion of the lemma will follow from the observation that (3.2) is equivalent to the following (since the filtration is right-continuous):

for all $u \in \mathbb{R}_+$, $v \le u$, $\varepsilon > 0$, for all bounded random variables S and for all continuous functions f on $\overline{\mathbb{R}}_+$ such that $f = 1$ on $[0, v]$ and $f = 0$ on $[v + \varepsilon, \infty]$,

$$E\left(S \int_{\overline{\mathbb{R}}_+} f(v) \, dA_v^u\right) = E\left(E(S|\mathscr{F}_{v+\varepsilon, u-v}) \int_{\overline{\mathbb{R}}_+} f(v) \, dA_v^u\right), \tag{3.4}$$

and, for all continuous functions g with support in $[u, \infty]$,

$$E\left(S \int_{\overline{\mathbb{R}}_+} g(v) \, dA_v^u\right) = 0. \qquad \square$$

We can now embed the set $\mathscr{Z}$ of o.i.p.'s into the convex set $\prod_{u \in \overline{\mathbb{R}}_+} \mathscr{U}^u$, which is compact for the product topology. However, this set is too large to be of interest, since there is nothing in the definition of this set that corresponds to the fact that o.i.p.'s are increasing families of stopping points. Now if $(Z_u)_{u \in \overline{\mathbb{R}}_+}$ is an o.i.p. and A_v^u is defined as in (3.1), then the fact that $u \mapsto Z_u(\cdot)$ is increasing translates into the condition

$$A_{\tilde{u}-u+v}^{\tilde{u}} \le A_v^u \le A_v^{\tilde{u}}, \qquad \forall\, \tilde{u} \le u, \forall\, v \in [0, u], \tag{3.5}$$

so it is natural to define the set $\mathscr{Z}_r$ of randomized optional increasing paths by

$$\mathscr{Z}_r = \left\{ A = (A^u)_{u \in \overline{\mathbb{R}}_+} \in \prod_{u \in \overline{\mathbb{R}}_+} \mathscr{U}^u \text{: (3.5) holds} \right\}.$$

This definition is equivalent to that of [11].

3.6. LEMMA. *$\mathscr{Z}_r$ is a closed subset of $\prod_{u \in \overline{\mathbb{R}}_+} \mathscr{U}^u$ and thus is compact.*

PROOF. By the definition of the Baxter–Chacon topology, it is sufficient to observe that condition (3.5) is equivalent to the following:

(3.7) for all $\tilde{u} \le u$, for all bounded nonnegative random variables S and for all nonnegative nonincreasing continuous functions f on $\overline{\mathbb{R}}_+$,

$$E\left(S \int_{\overline{\mathbb{R}}_+} f(u - \tilde{u} + v)\, dA_v^{\tilde{u}} \right) \le E\left(S \int_{\overline{\mathbb{R}}_+} f(v)\, dA_v^u \right) \le E\left(S \int_{\overline{\mathbb{R}}_+} f(v)\, dA_v^{\tilde{u}} \right). \quad \square$$

3.8. LEMMA. *$\mathscr{Z}_r$ is a convex set. The extremal elements of $\mathscr{Z}_r$ are exactly those which correspond through* (3.1) *with elements of $\mathscr{Z}$.*

PROOF. Convexity of $\mathscr{Z}_r$ is clear. From (3.1), it is also clear that each element of $\mathscr{Z}$ defines an extremal element of $\mathscr{Z}_r$. Now if $A = (A^u)_{u \in \overline{\mathbb{R}}_+} \in \mathscr{Z}_r$, we can define two elements ${}^1A$ and ${}^2A$ of $\mathscr{Z}_r$ by the formulas ${}^1A_v^u = \min(2A_v^u, 1)$ and ${}^2A_v^u = \max(2A_v^u - 1, 0)$. Then ${}^1A$ and ${}^2A$ satisfy $A = \frac{1}{2}{}^1A + \frac{1}{2}{}^2A$ and so A can be an extremal element of $\mathscr{Z}_r$ if and only if ${}^1A = A = {}^2A$. These equalities imply that $A_v^u \in \{0, 1\}$, for all u, v. So A identifies through (3.1) with the o.i.p. $(Z_u)_{u \in \overline{\mathbb{R}}_+}$, where

$$Z_u = (Z_u^1, u - Z_u^1) \quad \text{and} \quad Z_u^1 = \inf\{v \colon A_v^u = 1\}. \qquad \square$$

4. Existence of an optimal optional increasing path. Lemmas 3.6 and 3.8 show that $\mathscr{Z}_r$ satisfies appropriate convexity and compactness properties. We shall now show that the function $\phi_X\colon \mathscr{Z} \to \mathbb{R}$ defined by

$$\phi_X(Z) = R(Z) = E\left(\int_{\overline{\mathbb{R}}_+} X_{Z_u}\, dV_u \right)$$

extends, for regular processes X, to a regular affine function on $\mathscr{Z}_r$. A natural affine extension of ϕ_X is the function Φ_X: $\mathscr{X}_r \to \mathbb{R}$ defined by

$$\Phi_X(A) = E\left(\int_{\overline{\mathbb{R}}_+} dV_u \int_{\overline{\mathbb{R}}_+} dA_v^u X_{v,u-v}\right).$$

4.1. LEMMA. *Suppose $X \in \mathscr{C}$. Then Φ_X is continuous on $\mathscr{X}_r$.*

The (technical) proof of this lemma will be given below. We now use this lemma to prove Theorem 2.2.

PROOF OF THEOREM 2.2. This proof is similar to that of [4], Proposition 3.2; [11], Theorem 2.2 and [7], Theorem 7.2. By the proof of Proposition 7.1 of [7], there is a nonincreasing sequence $(X^n)_{n \in \mathbb{N}}$ of elements of $\mathscr{C}$ such that

$$X_t(\omega) = \lim_{n\to\infty} \downarrow X_t^n(\omega), \qquad \forall\, t \in \overline{\mathbb{R}}_+^2, \forall\, \omega \in \Omega$$

(a similar statement for bounded separable processes was proved in [4]). Since dV_u is a nonnegative measure, we obtain

$$\lim_{n\to\infty} \downarrow \Phi_{X^n}(A) = \Phi_X(A), \qquad \forall\, A \in \mathscr{X}_r.$$

By Lemma 4.1, this shows that Φ is u.s.c. on $\mathscr{X}_r$. Hence Φ attains its maximum on $\mathscr{X}_r$ and since Φ is affine, this maximum is attained at an extremal element of $\mathscr{X}_r$ ([3], II.58, Proposition 1); by Lemma 3.8, this extremal element is an o.i.p., which is clearly optimal. □

We now turn to the proof of Lemma 4.1. Let $\mathscr{M}$ denote the deterministic elements of $\mathscr{X}_r$, i.e., $a = (a^u)_{u \in \overline{\mathbb{R}}_+} \in \mathscr{M}$ provided each a^u is a nonnegative nondecreasing function such that $a^u(u) = 1$ and

$$a^{\tilde{u}}(\tilde{u} - u + v) \le a^u(v) \le a^{\tilde{u}}(v), \qquad 0 \le \tilde{u} \le u,\, v \in [0, u].$$

Let x be an integrable function on $\overline{\mathbb{R}}_+^2$. For $a \in \mathscr{M}$, let $a(x)$: $\overline{\mathbb{R}}_+ \to \mathbb{R}$ be defined by

$$a(x)(u) = \int_{\overline{\mathbb{R}}_+} x(v, u - v)\, da_v^u, \quad u \in \mathbb{R}_+, \quad a(x)(\infty) = x(\infty)$$

(note that if $v > u$, $da_v^u = 0$ and so we can define x on $\mathbb{R}^2 \setminus \mathbb{R}_+^2$ arbitrarily).

4.2. LEMMA. *If x is continuous on $\overline{\mathbb{R}}_+^2$, then $\{a(x)\colon a \in \mathscr{M}\}$ is an equicontinuous set of functions on $\overline{\mathbb{R}}_+$, i.e., for all $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|u - \tilde{u}| < \delta,\, a \in \mathscr{M} \Rightarrow |a(x)(u) - a(x)(\tilde{u})| < \varepsilon. \tag{4.3}$$

PROOF. Since $\overline{\mathbb{R}}_+$ is compact, a standard argument shows that we only need to prove that for all $\varepsilon > 0$ and for each fixed $u \in \overline{\mathbb{R}}_+$, there is $\delta > 0$ such that (4.3) holds. The case $u = \infty$ follows immediately from the continuity of x at ∞, so we assume $u \in \mathbb{R}_+$. Now

$$|a(x)(u) - a(x)(\tilde{u})| \leq \left| \int_{\overline{\mathbb{R}}_+} x(v, u - v)\, da_v^u - \int_{\overline{\mathbb{R}}_+} x(v, u - v)\, da_v^{\tilde{u}} \right| + \left| \int_{\overline{\mathbb{R}}_+} x(v, u - v)\, da_v^{\tilde{u}} - \int_{\overline{\mathbb{R}}_+} x(v, \tilde{u} - v)\, da_v^{\tilde{u}} \right|.$$

Since x is uniformly continuous on $\overline{\mathbb{R}}_+^2$, the second term on the right-hand side will be small for all a whenever $\tilde{u}$ is close enough to u. So we only need to prove, given a continuous function $g\colon \overline{\mathbb{R}}_+ \to \mathbb{R}$, that for all $n \in \mathbb{N}$ and for each fixed $u \in \mathbb{R}_+$, there is $k \in \mathbb{N}$ such that

$$(4.4) \qquad |u - \tilde{u}| < 2^{-k},\ a \in \mathscr{M} \quad \Rightarrow \quad \left| \int_{\overline{\mathbb{R}}_+} g(v)\left(da_v^u - da_v^{\tilde{u}}\right) \right| < 2^{-n}.$$

Fix $n \in \mathbb{N}$ and $u \in \mathbb{R}_+$. Set $M = \sup_v |g(v)|$ and choose $k' \in \mathbb{N}$ such that $|g(v) - g(\tilde{v})| < 2^{-n}/3$ whenever $|v - \tilde{v}| < 2^{-k'+1}$. Let $l \in \mathbb{N}$ be such that $l\, 2^{-k'} \geq u + 1$.

Fix $k \in \mathbb{N}$ such that $2^{k-1} > lM2^{n+2k'+3}$ and let $a \in \mathscr{M}$. Since $0 \leq a^u(v) \leq 1$ and $a^u(\cdot)$ is nondecreasing, each interval $](j-1)2^{-k'}, j2^{-k'}]$ contains at least one dyadic point v_j of order k (v_j depends on a and on u) such that

$$a_{v_j+2^{-k}}^u - a_{v_j-2^{-k}}^u < 2^{-n-k'-3}/(lM).$$

Indeed, the interval $](j-1)2^{-k'}, j2^{-k'}]$ contains at least $2^{-k'}/2^{-k+1} = 2^{k-1-k'} > lM2^{n+k'+3}$ disjoint intervals of length 2^{k+1}. Set $v_0 = 0$. Observe that for $|\tilde{u} - u| < 1$, the definition of k' implies that

$$\left| \int_{\overline{\mathbb{R}}_+} g(v)\, da_v^{\tilde{u}} - \sum_{j=1}^{l2^{k'}} g(v_j)\left(a_{v_j}^{\tilde{u}} - a_{v_{j-1}}^{\tilde{u}}\right) \right| < 2^{-n}/3.$$

Now if $|u - \tilde{u}| < 2^{-k}$, then

$$(4.5) \qquad \begin{aligned} \left| \int_{\overline{\mathbb{R}}_+} g(v)\left(da_v^u - da_v^{\tilde{u}}\right) \right| &\leq 2.2^{-n}/3 \\ &\quad + \left| \sum_{j=1}^{l2^{k'}} g(v_j)\left(a_{v_j}^u - a_{v_{j-1}}^u\right) - \sum_{j=1}^{l2^{k'}} g(v_j)\left(a_{v_j}^{\tilde{u}} - a_{v_{j-1}}^{\tilde{u}}\right) \right| \\ &\leq 2.2^{-n}/3 + M \sum_{j=1}^{l2^{k'}} \left(|a_{v_j}^u - a_{v_j}^{\tilde{u}}| + |a_{v_{j-1}}^u - a_{v_{j-1}}^{\tilde{u}}|\right). \end{aligned}$$

Using (3.5), it is not difficult to see that

$$|a_{v_j}^u - a_{v_j}^{\tilde{u}}| \leq a_{v_j+2^{-k}}^u - a_{v_j-2^{-k}}^u \leq 2^{-n-k'-3}/(lM),$$

both when $\tilde{u} \leq u$ and when $\tilde{u} \geq u$. So the last expression in (4.5) is not greater than $2^{-n}/4$. This completes the proof. □

PROOF OF LEMMA 4.1. Let V^k denote the increasing process defined by

$$V_u^k = \sum_{j=1}^{k2^k} \Delta V_j^k I_{\{j2^{-k} \le u\}}, \quad \text{where} \quad \Delta V_j^k = V_{j2^{-k}} - V_{(j-1)2^{-k}}.$$

Since $X \in \mathscr{C}$, it is not difficult to see using 4.2 that

$$\sup_{A \in \mathscr{X}_r} \left| \int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u - \int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u^k \right| \underset{k \to \infty}{\to} 0 \quad \text{a.s.}$$

Fix $\varepsilon > 0$. Again since $X \in \mathscr{C}$, it follows that we may choose $k \in \mathbb{N}$ such that

$$E\left(\sup_{A \in \mathscr{X}_r} \left| \int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u - \int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u^k \right| \right) < \varepsilon/3. \tag{4.6}$$

Fix $A \in \mathscr{X}_r$, and define an open subset $\mathscr{O}$ of $\mathscr{X}_r$ by

$$\mathscr{O} = \Big\{ \tilde{A} \in \mathscr{X}_r : \left| E\big((\tilde{A}(X)(j2^{-k}) - A(X)(j2^{-k}))\, \Delta V_j^k \big) \right| < k^{-1}2^{-k}\varepsilon/3,\ j = 1, \dots, k2^k \Big\}.$$

Using (4.6), we see that for $\tilde{A} \in \mathscr{O}$,

$$\begin{aligned} &\left| E\left(\int_{\overline{\mathbb{R}}_+} \tilde{A}(X)(u)\, dV_u \right) - E\left(\int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u \right) \right| \\ &\quad \le 2\varepsilon/3 + \left| E\left(\int_{\overline{\mathbb{R}}_+} \tilde{A}(X)(u)\, dV_u^k \right) - E\left(\int_{\overline{\mathbb{R}}_+} A(X)(u)\, dV_u^k \right) \right| < \varepsilon \end{aligned}$$

by the definitions of $\mathscr{O}$ and V^k. This completes the proof. □

REFERENCES

[1] BAXTER, J. R. and CHACON, R. V. (1977). Compactness of stopping times. *Z. Wahrsch. verw. Gebiete* **40** 169–181.

[2] BERRY, D. A. and FRISTEDT, B. (1985). *Bandit Problems.* Chapman and Hall, London.

[3] BOURBAKI, N. (1981). *Eléments de Mathématique Espaces Vectoriels Topologiques*, Chapters I–V. Hermann, Paris.

[4] DALANG, R. C. (1984). Sur l'arrêt optimal de processus à temps multidimensional continu. *Séminaire de Probabilités XVIII. Lecture Notes in Math.* **1059** 379–390. Springer, Berlin.

[5] DALANG, R. C. (1988). On infinite perfect graphs and randomized stopping points on the plane. *Probab. Theory Related Fields* **78** 357–378.

[6] DALANG, R. C. (1988). On stopping points in the plane that lie on a unique optional increasing path. *Stochastics* **24** 245–268.

[7] DALANG, R. C. (1989). Optimal stopping of two-parameter processes on nonstandard probability spaces. *Trans. Amer. Math. Soc.* **313** 697–719.

[8] GHOUSSOUB, N. (1982). An integral representation of randomized probabilities and its applications. *Séminaire de Probabilités XVI. Lecture Notes in Math.* **920** 519–543. Springer, Berlin.

[9] KARATZAS, I. (1984). Gittins indices in the dynamic allocation problem for diffusion processes. *Ann. Probab.* **12** 173–192.

[10] MANDELBAUM, A. (1987). Continuous multi-armed bandits and multiparameter processes. *Ann. Probab.* **15** 1527–1556.

[11] MAZZIOTTO, G. and MILLET, A. (1987). Stochastic control of two-parameter processes application: The two-armed bandit problem. *Stochastics* **22** 251–288.

[12] MEYER, P.-A. (1978). Convergence faible et compacité des temps d'arrêt d'après Baxter et Chacon. *Séminaire de Probabilités XII. Lecture Notes in Math.* **850** 411–423. Springer, Berlin.

[13] WHITTLE, P. (1982). *Optimization over Time: Dynamic Programming and Stochastic Control.* Wiley, New York.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720