The Annals of Probability
1990, Vol. 18, No. 1, 92–128

ON EXTREMAL THEORY FOR STATIONARY PROCESSES[1]

BY J. M. P. ALBIN

University of Lund and Lund Institute of Technology, Sweden

Let $\{\xi(t)\}_{t\geq 0}$ be a stationary stochastic process, with one-dimensional distribution function G. We develop a method to determine an asymptotic expression for $\Pr\{\sup_{0\leq t\leq h}\xi(t) > u\}$, when $u \uparrow \sup\{v\colon G(v) < 1\}$, applicable when G belongs to a domain of attraction of extremes, and we show that if G belongs to such a domain, then so does the distribution function of $\sup_{0\leq t\leq h}\xi(t)$. Applications are given to hitting probabilities for small sets for $\mathbb{R}^m$-valued Gaussian processes and to extrema of Rayleigh processes. Further, we prove the Gumbel, Fréchet and Weibull laws, for maxima over increasing intervals, when G is type I-, type II- and type III-attracted, respectively, and we establish the asymptotic Poisson character of ε-upcrossings and local ε-maxima.

1. Introduction. Throughout this presentation, we let $\{\xi(t)\}_{t\geq 0}$ denote a strictly stationary, real valued, separable stochastic process, with one-dimensional distribution function G. We shall also assume that our basic probability space (Ω, F, P) is complete and write $\to_D$ and $\hat{u}$ for weak convergence and $\sup\{u\colon G(u) < 1\}$, respectively. Further, we define $U = \hat{u}^-$, when $\hat{u} < \infty$, and $U = \infty$, when $\hat{u} = \infty$ (the symbol U is to be used in "$\lim_{u\to U}$-like" operations). A quantity of constant interest will be the maximum $M(B)$ of $\xi(t)$ over a Borel set B, i.e., $M(B) = \sup_{t\in B}\xi(t)$, and we shall use the abbreviation $M(T)$ for $M((0, T])$, when $T > 0$ is a real number.

Following such pioneers as Rice (1939, 1945), Kac and Slepian (1959), Volkonskii and Rozanov (1959, 1961) and Slepian (1961, 1962), the present view of extremal theory for stationary stochastic processes began with the papers by Cramér (1965, 1966) [see also Cramér and Leadbetter (1967)], in the case with finite upcrossing intensity, and by Pickands (1969a, b), when the upcrossing intensity is infinite. Both Cramér's and Pickands' papers concerned Gaussian processes, but their ideas are of much broader applicability.

Improvements of Cramér's papers were furnished by Belayev (1966, 1967a, b), Qualls (1968) and others, while, e.g., Berman (1971), Qualls and Watanabe (1972), Bickel and Rosenblatt (1973), Lindgren, de Maré and Rootzén (1975), Cuzick (1981) and Leadbetter, Lindgren and Rootzén (1983) provided improvements of Pickands' results.

There has been more recent work on extremes for processes with finite upcrossing intensities, directed away from the Gaussian case. See, e.g., Sharpe (1978), Lindgren (1980, 1984, 1989), Leadbetter and Rootzén (1982, 1988) and Aronowich and Adler (1985, 1986). Recent studies of processes with infinite

Received January 1988; revised September 1988.
[1]Partially supported by Swedish Natural Science Research Council Grant F-FU 9207-300.
AMS 1980 *subject classifications*. Primary 60G10, 60G15, 60G17, 60G55.
Key words and phrases. Extremal value theory, crossings, local maxima, Poisson processes, Gaussian processes, Rayleigh processes, star-shaped sets.

upcrossing intensities can be found in Lindgren, de Maré and Rootzén (1975), Berman (1982, 1983, 1988), Leadbetter and Rootzén (1982, 1988) and Leadbetter, Lindgren and Rootzén (1983).

In this treatment we consider extrema of (not necessarily Gaussian) processes with, possibly, infinite upcrossing intensities. Our concept is to apply the methods of Pickands (1969a), Lindgren, de Maré and Rootzén (1975), Leadbetter and Rootzén (1982) and Leadbetter, Lindgren and Rootzén (1983) to ideas of Berman (1982).

We apply our results to hitting probabilities for small sets for multidimensional Gaussian processes and to extremes of Rayleigh processes. In Albin (1987), they are applied to Γ- and $\sqrt{F}$-like processes (the latter being the quotient of two processes of the kind introduced in Section 3). In two publications in preparation, we give an application to a general class of differentiable processes and [using ideas of Berman (1985)] extend our results to nonstationary processes.

Berman (1982) [see also Berman (1983)] studied maxima over finite intervals of stationary processes whose one-dimensional distribution function (d.f.) G belongs to the type I domain of attraction of extremes, with $\hat{u} = \infty$. We shall extend his setting by studying processes for which G belongs to any of the three domains of attraction of extremes. More specifically, we shall assume that there exist functions w and F and constants $x_L < 0 < x_H$, such that

$$(1.1)\qquad \lim_{u \to U} \frac{1 - G(u + xw(u))}{1 - G(u)} = 1 - F(x) \quad \text{for all } x_L < x < x_H.$$

Hence, as was shown by Gnedenko (1943) [see also de Haan (1970) and Resnick (1987)], if G belongs to the type I domain of attraction of extremes, then (1.1) holds with $F(x) = 1 - e^{-x}$, $x_L = -\infty$, and $x_H = \infty$, and with $\lim_{u\to\infty} w(u)/u = 0$ when $\hat{u} = \infty$ and $\lim_{u \to U} w(u)/(\hat{u} - u) = 0$ when $\hat{u} < \infty$. Further, if G belongs to the type II or the type III domain of attraction, then (1.1) holds with $\hat{u} = \infty$, $F(x) = 1 - (1 + x)^{-b}$ (for some constant $b > 0$), $x_L = -1$, $x_H = \infty$ and $w(u) = u$, and with $\hat{u} < \infty$, $F(x) = 1 - (1 - x)^{b}$ (for some constant $b > 0$), $x_L = -\infty$, $x_H = 1$ and $w(u) = \hat{u} - u$, respectively.

In Section 2 we derive an asymptotic expression for the probability $\Pr\{M(h) > u\}$ when $u \to U$. Our approach is based on ideas of Pickands (1969a), Berman (1982) and Leadbetter, Lindgren and Rootzén (1983). We show that if there exist functions w and F such that (1.1) holds and a function $q(u)$ such that the finite-dimensional distributions (f.d.d.'s) of $\{((\xi(qt) - u)/w(u) | (\xi(0) - u)/w(u) > 0)\}_{t>0}$ converge weakly to those of some process $\{\zeta(t)\}_{t>0}$ when $u \to U$, then

$$(1.2)\qquad \Pr\Big\{\sup_{0\le t\le h} \xi(t) > u\Big\} \sim h\frac{H(1 - G(u))}{q(u)} \quad \text{when } u \to U,$$

where $H = \lim_{a\downarrow 0}(1/a)\Pr\{\sup_{k\ge 1}\zeta(ak) \le 0\}$. In order to make this result valid, we have to impose two extra conditions on $\xi(t)$. The first one guarantees that, as expressed by Berman (1982), "after going above a high level u, the sample function tends to fall quickly to some point below u." The second condition ensures that our discrete approximation becomes sufficiently accurate when the step length tends to zero.

In (1.2) it is crucial that H is strictly positive, since if $H = 0$, (1.2) merely yields an upper estimate for $\Pr\{M(h) > u\}$. We shall prove that $H > 0$, while, in analogues of (1.2) in the literature, it seems unclear if the inequality $H > 0$ follows from the invoked arguments [see, e.g., Berman (1982)].

We also prove, rather than conjecture, the fact that G and the d.f. of $M(h)$ belong to the same domain of attraction.

In extreme value theory one often has to impose certain conditions on the asymptotic accuracy of the discrete approximation. We shall provide a simple and general criterion, formulated in terms of two-dimensional probabilities only, which ensures sufficient accuracy of the discrete approximation.

We also show that

$$\liminf_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} > 0$$

and

$$\limsup_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} < \infty,$$

provided that simple two-dimensional conditions hold.

Berman (1982) uses the "sojourn" approach, that is, he derives results for the time $L_h(u)$ spent above a level u, by the process $\{\xi(t)\}_{0 \le t \le h}$, and then converts his results to extrema. Our approach shall be different and shall rely on the method of discrete approximation of Pickands (1969a).

Let $\omega_1, \dots, \omega_m$ be stationary Gaussian processes. In Section 3 we use the results of Section 2 to derive an asymptotic expression for the probability that the process $\{u(\omega_1(t), \dots, \omega_m(t))\}_{0 \le t \le h}$ visits A when $u \to \infty$, where A is an open star-shaped set with Lipschitzian boundary. Our result is valid under certain conditions on the covariance structure of $\omega_1, \dots, \omega_m$.

Let $\nu(t)$ be a Rayleigh process, (i.e., the root of a sum of squares of Gaussian processes). In Section 4 we apply the results of Section 2 to derive an asymptotic estimate of the probability $\Pr\{\sup_{0 \le t \le h} \nu(t) > u\}$, when $u \to \infty$, under certain conditions on the covariance structure, as in Section 3.

In Section 5 we combine our results from Section 2 with results from Leadbetter and Rootzén (1982), to obtain the Gumbel, Fréchet and Weibull laws, for maxima over increasing intervals, for processes with G type I-, type II- and type III-attracted, respectively. The conditions we impose on $\xi(t)$ are those of Section 2 and Conditions D and D′ of Leadbetter and Rootzén (1982).

Our last task is to study asymptotic Poisson process characteristics of ε-upcrossings and local ε-maxima. The ε prefix refers to a method, given by Pickands (1969a), for handling clusters of upcrossings. Our approach resembles those in Lindgren, de Maré and Rootzén (1975) and Leadbetter, Lindgren and Rootzén (1983). Now, writing $t_1, t_2, \dots$ for the ε-upcrossings of the level u by $\xi(t)$ and $T(u) = q(u)/(H(1 - G(u)))$, we show that the random collection

$\{t_i/T(u),\ i = 1,2,\dots\}$ converges weakly to a Poisson process on $(0,\infty)$, when $u \to U$. Further, letting $t_1, t_2, \dots$ denote the local ε-maxima for $\xi(t)$, we prove that $\{(t_i/T(u),\ (\xi(t_i) - u)/w(u)),\ i = 1,2,\dots\}$ converges to a Poisson process on $(0,\infty) \times (x_L, x_H)$, with intensity measure equal to the product of Lebesgue measure and that defined by the increasing function $-(1 - F(x))^{1-c}$. Here $c \in [0,1]$ is a constant such that $q(u + xw(u))/q(u) \to (1 - F(x))^c$, when $u \to U$, for $x_L < x < x_H$. The conditions required are essentially the same as those required to establish the limit laws for maxima over increasing intervals.

2. Maxima over finite intervals. In this section we derive an asymptotic expression for $\Pr\{M(h) > u\}$ when $u \to U$.

In order to formulate our conditions, let Λ be a subset of $\mathbb{R}$.

CONDITION A(Λ). This condition is said to hold if there exist a function $F(x)$, continuous at $x = 0$, a strictly positive function w and constants $x_L < 0 < x_H$, such that

$$\lim_{u \to U} \frac{1 - G(u + xw(u))}{1 - G(u)} = 1 - F(x) \quad \text{for all } x_L < x < x_H \tag{2.1}$$

and if there exist a random sequence $\{\zeta_{a,y}(k)\}_{k=1}^{\infty}$ and a strictly positive function q, with $\lim_{u \to U} q(u) = 0$, such that, for all $a > 0$, all N and all $y \in \Lambda$,

$$\left(\frac{\xi(aq) - u}{w(u)}, \dots, \frac{\xi(aqN) - u}{w(u)} \,\middle|\, \frac{\xi(0) - u}{w(u)} > y \right) \to_D \left(\zeta_{a,y}(1), \dots, \zeta_{a,y}(N) \right) \tag{2.2}$$

when $u \to U$.

Berman (1982) has a similar condition in his type I setting, although he requires continuous f.d.d.'s for the sequence $\{\zeta_{a,y}(k)\}_{k=1}^{\infty}$. If G belongs to a domain of attraction, then (2.1) holds (cf. Section 1). Further, (2.1) implies that $\Pr\{(\xi(0) - u)/w(u) \le x \mid (\xi(0) - u)/w(u) > 0\}$ converges, and it therefore seems natural to assume the existence of q and $\{\zeta_{a,0}(k)\}_{k=1}^{\infty}$, with the listed properties.

We now come to the "short-lasting-exceedance" assumption.

CONDITION B. This condition is said to hold if

$$\limsup_{u \to U} \sum_{k=N}^{[h/(aq)]} \Pr\{\xi(aqk) > u \mid \xi(0) > u\} \to 0$$

when $N \to \infty$, for all fixed $a > 0$.

Condition B is a discrete time analogue of a condition used by Berman (1982). In Albin (1987), Chapter 5, we exemplify that the asymptotic behaviour of $\Pr\{M(h) > u\}$ changes when Condition B is only "slightly violated."

The following conditions ensure accuracy of the discrete approximation.

CONDITION C. This condition is said to hold if the function $F(x)$ is strictly increasing in a neighbourhood of $x = 0$, if the function q is nonincreasing and if

$$\limsup_{a\downarrow 0}\limsup_{u\to U}\frac{\Pr\{M(aq) > u + \delta w(u),\ \xi(0) \le u\}}{a(1 - G(u))} = 0 \tag{2.3}$$

for all $\delta > 0$.

CONDITION $C^0(\Lambda)$. This condition is said to hold if

$$\limsup_{a\downarrow 0}\limsup_{u\to U}\frac{q(u)}{1 - G(u)}\Pr\Big\{M(h) > u + yw(u),\ \max_{0\le aqk\le h}\xi(aqk) \le u + yw(u)\Big\} = 0 \quad \text{for all } y \in \Lambda.$$

Here and in the sequel, the parameter k runs over the integers, i.e., $\max_{0\le aqk\le h}\xi(aqk) = \max\{\xi(aqk)\colon\ k \in \mathbb{Z},\ 0 \le aqk \le h\}$. In Theorem 6 we give a simple sufficient criterion for (2.3) to hold, formulated in terms of two-dimensional probabilities only [see also Theorems 2(c), 5 and 7].

Observe that the function $q(u)$ is the same in Conditions A, B, C and C^0 and that the function $w(u)$ is the same in Conditions A, C and C^0.

THEOREM 1. *If Conditions* A($\{0\}$) *and* B *hold, and if either Condition* C *or Condition* $C^0(\{0\})$ *holds, then the limit*

$$\lim_{a\downarrow 0}\frac{1}{a}\Pr\Big\{\sup_{k\ge 1}\zeta_{a,0}(k) \le 0\Big\} \equiv H$$

exists with $0 < H < \infty$, *and*

$$\Pr\{M(h) > u\} \sim h\frac{H(1 - G(u))}{q(u)} \quad \textit{when } u \to U.$$

PROOF. Let

$$H(N, a) = 1 + \Pr\Big\{\max_{1\le k<2}\zeta_{a,0}(k) \le 0\Big\} + \cdots + \Pr\Big\{\max_{1\le k<N}\zeta_{a,0}(k) \le 0\Big\}$$

and let $0 < \delta < \max\{-x_L, x_H\}$. Using induction over N, we shall prove the inequalities

$$\begin{aligned}&\limsup_{u\to U}\frac{1}{1 - G(u)}\Pr\Big\{\max_{0\le k<N}\xi(aqk) > u - \delta w(u)\Big\}\\ &\qquad\le H(N, a) - NF(-\delta),\\ &\liminf_{u\to U}\frac{1}{1 - G(u)}\Pr\Big\{\max_{0\le k<N}\xi(aqk) > u + \delta w(u)\Big\}\\ &\qquad\ge H(N, a) - NF(\delta),\end{aligned} \tag{2.4}$$

for $N \ge 2$. To that end, write $A(i, j) = \{\max_{i\le k<j}\xi(aqk) > u - \delta w(u)\}$ and $B(i, j) = \{\max_{i\le k<j}\xi(aqk) > u + \delta w(u)\}$. We then have, by stationarity, (2.1), (2.2) and the fact that the d.f. of $\max_{1\le k<N}\zeta_{a,0}(k)$ has at most countably many

points of discontinuity,

$$
\begin{aligned}
\limsup_{u\to U}\frac{\Pr\{A(0,N)\}}{1-G(u)} &\le \limsup_{u\to U}\frac{\Pr\{A(1,N)\}}{1-G(u)} \\
&\quad + \limsup_{u\to U}\Pr\left\{A(1,N)^c \,\middle|\, \frac{\xi(0)-u}{w(u)} > 0\right\} \\
&\quad + \limsup_{u\to U}\frac{\Pr\{u-\delta w(u) < \xi(0) \le u\}}{1-G(u)} \\
&\le \limsup_{u\to U}\frac{\Pr\{A(0,N-1)\}}{1-G(u)} \\
&\quad + \Pr\left\{\max_{1\le k<N}\zeta_{a,0}(k)\le 0\right\} - F(-\delta) \quad \text{for } N\ge 2
\end{aligned} \tag{2.5}
$$

and similarly

$$
\begin{aligned}
\liminf_{u\to U}\frac{\Pr\{B(0,N)\}}{1-G(u)} &\ge \liminf_{u\to U}\frac{\Pr\{B(1,N)\}}{1-G(u)} \\
&\quad + \liminf_{u\to U}\Pr\left\{B(1,N)^c \,\middle|\, \frac{\xi(0)-u}{w(u)} > 0\right\} \\
&\quad - \limsup_{u\to U}\frac{\Pr\{u < \xi(0) \le u+\delta w(u)\}}{1-G(u)} \\
&\ge \liminf_{u\to U}\frac{\Pr\{B(0,N-1)\}}{1-G(u)} \\
&\quad + \Pr\left\{\max_{1\le k<N}\zeta_{a,0}(k)\le 0\right\} - F(\delta) \quad \text{for } N\ge 2.
\end{aligned} \tag{2.6}
$$

Now, (2.5) and (2.6), by (2.1) and a simple identification, yield that (2.4) holds for $N = 2$. If, further, (2.4) holds for $N = M$, then (2.5) and (2.6) show that (2.4) holds also for $N = M + 1$. Using induction over N, we conclude that (2.4) holds for all $N \ge 2$. Letting $\delta \downarrow 0$ in (2.4), we readily obtain

$$
\lim_{u\to U}\frac{1}{1-G(u)}\Pr\left\{\max_{0\le k<N}\xi(aqk) > u\right\} = H(N,a). \tag{2.7}
$$

Now, writing $C(i,j) = \{\max_{i\le k<j}\xi(aqk) > u\}$, we have, by (2.7), Boole's and Bonferroni's inequalities and stationarity

$$
\begin{aligned}
h\frac{H(N,a)}{aN} &= \limsup_{u\to U}\frac{h}{aN}\frac{\Pr\{C(0,N)\}}{1-G(u)} \\
&\ge \limsup_{u\to U}\frac{q(u)}{1-G(u)}\Pr\left\{\max_{0\le aqk\le h}\xi(aqk) > u\right\} \\
&\ge \liminf_{u\to U}\frac{q(u)}{1-G(u)}\Pr\left\{\max_{0\le aqk\le h}\xi(aqk) > u\right\} \\
&\ge h\frac{H(N,a)}{aN} - 2\rho_N,
\end{aligned} \tag{2.8}
$$

where, letting $m = [h/(aqN)]$,

$$\begin{aligned}
\rho_N &= \limsup_{u\to U} \frac{q(u)}{1-G(u)} \sum_{0\le r<s<m} \Pr\{C(rN,(r+1)N), C(sN,(s+1)N)\}\\
&\le \limsup_{u\to U} \frac{h}{aN} \frac{\Pr\{C(0,N), C(N,2N)\}}{1-G(u)}\\
&\quad + \limsup_{u\to U} \frac{h}{aN} \sum_{r=2}^{m-1} \frac{\Pr\{C(0,N), C(rN,(r+1)N)\}}{1-G(u)}\\
&\le \frac{h}{a}\left(\frac{2H(N,a)}{N} - \frac{H(2N,a)}{N}\right)\\
&\quad + \limsup_{u\to U} \frac{h}{a} \sum_{k=N+1}^{[h/(aq)]} \frac{\Pr\{\xi(0)>u, \xi(aqk)>u\}}{1-G(u)}.
\end{aligned}$$

Here the first term in the last inequality follows from (2.7) and the fact that $\Pr\{A\cap B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A\cup B\}$. Now, we have

$$\lim_{N\to\infty} \frac{H(N,a)}{Na} = \frac{1}{a}\Pr\Big\{\sup_{k\ge 1} \zeta_{a,0}(k) \le 0\Big\} \equiv H_a,$$

so that $\lim_{N\to\infty}(2H(N,a)/N - H(2N,a)/N) = 0$. Hence we obtain, by Condition B, $\lim_{N\to\infty}\rho_N = 0$. Letting $N\to\infty$ in (2.8), we thereby conclude

$$\lim_{u\to U} \frac{q(u)}{1-G(u)}\Pr\Big\{\max_{0\le aqk\le h} \xi(aqk) > u\Big\} = hH_a = \frac{h}{a}\Pr\Big\{\sup_{k\ge 1}\zeta_{a,0}(k)\le 0\Big\}. \tag{2.9}$$

Now, let $\tilde{q} = q(u-\delta w(u))$ for $0<\delta<-x_L$ and assume that Condition C holds. Then we have, by (2.1), (2.9) and Boole's inequality, and since $\tilde{q}/q \ge 1$,

$$\begin{aligned}
hH_a &= \lim_{u\to U} \frac{q(u)}{1-G(u)}\Pr\Big\{\max_{0\le aqk\le h}\xi(aqk)>u\Big\}\\
&\le \liminf_{u\to U} \frac{q(u)}{1-G(u)}\Pr\{M(h)>u\}\\
&\le \limsup_{u\to U} \frac{q(u)}{1-G(u)}\Pr\{M(h)>u\}\\
&\le \limsup_{u\to U} \frac{q(u)}{1-G(u)}\Pr\Big\{\max_{0\le a\tilde{q}k\le h}\xi(a\tilde{q}k) > u-\delta w(u)\Big\}\\
&\quad + \limsup_{u\to U} \frac{q(u)}{1-G(u)}\Pr\Big\{M(h)>u, \max_{0\le a\tilde{q}k\le h}\xi(a\tilde{q}k)\le u-\delta w(u)\Big\}\\
&\le h(1-F(-\delta))H_a + \limsup_{u\to U} h\frac{\Pr\{M(a\tilde{q})>u, \xi(0)\le u-\delta w(u)\}}{a(1-G(u))}.
\end{aligned} \tag{2.10}$$

Let $m \equiv \liminf_{u\to U} w(u)/w(u-\delta w(u))$ and assume that $m=0$. Then, to each constant $\alpha>0$, there exists a sequence $\{u_n\}_{n=1}^{\infty}$, such that $u_n\to U$ when

$n \to \infty$ and such that $\alpha w(u_n - \delta w(u_n)) - \delta w(u_n) > 0$ for all n. Clearly, (2.1) yields that $u_n - \delta w(u_n) \to U$ when $n \to \infty$ and hence it follows, again using (2.1), that

$$\frac{1}{1 - F(-\delta)} \leftarrow \frac{1 - G(u_n)}{1 - G(u_n - \delta w(u_n))}$$

$$\geq \frac{1 - G(u_n - \delta w(u_n) + \alpha w(u_n - \delta w(u_n)))}{1 - G(u_n - \delta w(u_n))} \to 1 - F(\alpha)$$

when $n \to \infty$. Here $1 - F(-\delta) > 1$, since F is strictly increasing, and letting $\alpha \downarrow 0$ we obtain $1 = \lim_{\alpha \downarrow 0}(1 - F(\alpha)) \leq 1/(1 - F(-\delta)) < 1$, which is a contradiction. Thus we have $m > 0$, and letting $\tilde{u} = u - \delta w(u)$, we therefore obtain, by a change of variable,

$$\limsup_{u \to U} \frac{\Pr\{M(a\tilde{q}) > u, \xi(0) \leq \tilde{u}\}}{a(1 - G(u))}$$

$$\text{(2.11)} \qquad \leq \limsup_{u \to U} (1 - F(-\delta)) \frac{\Pr\{M(a\tilde{q}) > \tilde{u} + \frac{1}{2}\delta m w(\tilde{u}), \xi(0) \leq \tilde{u}\}}{a(1 - G(\tilde{u}))}$$

$$\leq \limsup_{u \to U} (1 - F(-\delta)) \frac{\Pr\{M(aq) > u + \frac{1}{2}\delta m w(u), \xi(0) \leq u\}}{a(1 - G(u))}.$$

Combining (2.3), (2.10) and (2.11), we conclude

$$\limsup_{a \downarrow 0} hH_a \leq \liminf_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} \leq \limsup_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)}$$

$$\leq \liminf_{a \downarrow 0} h(1 - F(-\delta))H_a$$

for $0 < \delta < -x_L$. Here the middle limits are independent of a and, by (2.3), (2.9), (2.10) and (2.11), finite. Hence we have $\limsup_{a \downarrow 0} H_a < \infty$ and it follows, by letting $\delta \downarrow 0$, that the limit $\lim_{a \downarrow 0} H_a \equiv H$ exists and is finite, with

$$\lim_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} = hH.$$

On the other hand, if Condition $C^0(\{0\})$ holds, then the fact that

$$\lim_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} = \lim_{a \downarrow 0} hH_a < \infty$$

readily follows from taking $\delta = 0$ in (2.10).

It remains to show that $H > 0$. By Bonferroni's inequality, we have

$$hH_a = \lim_{u\to U} \frac{\Pr\{\max_{0\le aqk\le h} \xi(aqk) > u\}}{(1-G(u))/q(u)}$$

$$\ge \frac{h}{a} - \limsup_{u\to U} 2\frac{h}{a} \sum_{k=1}^{[h/(aq)]} \frac{\Pr\{\xi(0) > u, \xi(aqk) > u\}}{1-G(u)}$$

and since, by Condition B,

$$\limsup_{u\to U} \sum_{k=1}^{[h/(aNq)]} \frac{\Pr\{\xi(0) > u, \xi(aNqk) > u\}}{1-G(u)}$$

$$\le \limsup_{u\to U} \sum_{k=N}^{[h/(aq)]} \frac{\Pr\{\xi(0) > u, \xi(aqk) > u\}}{1-G(u)} \to 0$$

when $N \to \infty$, we have $H_{\hat{a}} > 0$ for $\hat{a}$ sufficiently large. Now,

$$\Pr\Big\{\max_{0\le k<N} \xi(\hat{a}qk) > u\Big\} \le \Pr\Big\{\max_{0\le k<Nn} \xi((\hat{a}/n)qk) > u\Big\},$$

so that, by (2.7), $H(N, \hat{a}) \le H(Nn, \hat{a}/n)$. This yields

$$H = \lim_{n\to\infty} H_{\hat{a}/n} = \lim_{n\to\infty}\lim_{N\to\infty} \frac{H(Nn, \hat{a}/n)}{Nn(\hat{a}/n)} \ge \lim_{N\to\infty} \frac{H(N, \hat{a})}{N\hat{a}} = H_{\hat{a}} > 0. \quad \square$$

As a simple consequence of Theorem 1, we have the following theorem.

THEOREM 2. (a) *If* (2.1) *and Condition* C *hold with* $\lim_{u\to U} q(u) = 0$, *then*

$$\limsup_{u\to U} \frac{\Pr\{M(h) > u\}}{(1-G(u))/q(u)} < \infty.$$

(b) *If Condition* B *holds with* $\lim_{u\to U} q(u) = 0$, *then*

$$\liminf_{u\to U} \frac{\Pr\{M(h) > u\}}{(1-G(u))/q(u)} > 0.$$

(c) *If Conditions* A({0}), B *and* C *hold, then Condition* $C^0(\{0\})$ *holds.*

PROOF. (a) Since

$$\limsup_{u\to U} \frac{q(u)}{1-G(u)} \Pr\Big\{\max_{0\le a\tilde{q}k\le h} \xi(a\tilde{q}k) > \tilde{u}\Big\} \le \limsup_{u\to U} \frac{h}{a}\frac{q}{\tilde{q}}\frac{1-G(\tilde{u})}{1-G(u)},$$

we obtain, by (2.1), (2.10) and (2.11), and in view of the fact that q is nonincreasing,

$$\limsup_{u\to U}\frac{\Pr\{M(h)>u\}}{(1-G(u))/q(u)}$$

$$\leq \frac{h}{a}(1-F(-\delta))\left(1+\limsup_{u\to U}\frac{\Pr\{M(aq)>u+\frac{1}{2}\delta m w(u),\ \xi(0)\leq u\}}{1-G(u)}\right)$$

for $0<\delta<-x_L$, where, by (2.3), the right-hand side is finite for small a.

(b) An easy analysis of the proof of Theorem 1 yields that, for large N,

$$\liminf_{u\to U}\frac{\Pr\{\max_{0\leq aNqk\leq h}\xi(aNqk)>u\}}{(1-G(u))/q(u)}$$

$$\geq \frac{h}{aN}-\limsup_{u\to U}\frac{2h}{aN}\sum_{k=N}^{[h/(aq)]}\frac{\Pr\{\xi(0)>u,\ \xi(aqk)>u\}}{1-G(u)}>0.$$

(c) The fact that Condition $C^0(\{0\})$ holds follows from letting first $a\downarrow 0$ and then $\delta\downarrow 0$ on the right-hand side of the readily established inequality

$$\limsup_{u\to U}\frac{q(u)}{1-G(u)}\Pr\left\{M(h)>u,\ \max_{0\leq aqk\leq h}\xi(aqk)\leq u\right\}$$

$$\leq \limsup_{u\to U}\frac{q(u)}{1-G(u)}\Pr\left\{M(h)>u+\delta w(u),\ \max_{0\leq aqk\leq h}\xi(aqk)\leq u\right\}$$

$$+\limsup_{u\to U}\frac{\Pr\{u<M(h)\leq u+\delta w(u)\}}{(1-G(u))/q(u)}$$

$$\leq \limsup_{u\to U} h\frac{\Pr\{M(aq)>u+\delta w(u),\ \xi(0)\leq u\}}{a(1-G(u))}$$

$$+hHF(\delta)\quad \text{for } 0<\delta<x_H.$$ □

Theorem 2(c) will be of interest in Section 5. The following result can be useful when verifying (2.2) [cf. Section 4 and Berman (1982)].

THEOREM 3. *If G has a density g and if there exist constants $x_L<0<x_H$, a strictly positive function w and a function $F(x)$, continuous at $x=0$, such that* (2.1) *holds and such that F: $(0,x_H)\to\mathbb{R}$ is a proper d.f. possessing a density f such that, writing $x_F=\sup\{x<x_H\colon F(x)<1\}$,*

$$\text{(2.12)}\quad \lim_{u\to U}\frac{w(u)g(u+xw(u))}{1-G(u)}=\begin{cases} f(x) & \text{for almost all } 0<x<x_F,\\ 0 & \text{for almost all } x_F<x,\end{cases}$$

and if there exist a random sequence $\{\eta_{a,x}(k)\}_{k=1}^{\infty}$ and a strictly positive *function* q, *with* $\lim_{u\to U} q(u) = 0$, *such that, for all* $a > 0$ *and all* N,

$$\left(\frac{\xi(aq)-u}{w(u)},\dots,\frac{\xi(aqN)-u}{w(u)}\,\middle|\,\frac{\xi(0)-u}{w(u)}=x\right) \tag{2.13}$$

$$\to_D \left(\eta_{a,x}(1),\dots,\eta_{a,x}(N)\right) \quad \textit{when } u \to U,$$

for almost all $0 < x < x_F$, *then Condition* $A([0, x_F))$ *holds with*

$$\Pr\left\{\bigcap_{k=1}^{N}\{\zeta_{a,y}(k)\le z_k\}\right\} = \int_y^{x_F}\Pr\left\{\bigcap_{k=1}^{N}\{\eta_{a,x}(k)\le z_k\}\right\}\frac{f(x)}{1-F(y)}\,dx.$$

PROOF. Write S, S_x, $G(u;\cdot)$ and $G_x(u;\cdot)$ for the d.f.'s of

$$\left(\zeta_{a,y}(1),\dots,\zeta_{a,y}(N)\right), \left(\eta_{a,x}(1),\dots,\eta_{a,x}(N)\right),$$

$$\left(\frac{\xi(aq)-u}{w(u)},\dots,\frac{\xi(aqN)-u}{w(u)}\,\middle|\,\frac{\xi(0)-u}{w(u)}>y\right),$$

and

$$\left(\frac{\xi(aq)-u}{w(u)},\dots,\frac{\xi(aqN)-u}{w(u)}\,\middle|\,\frac{\xi(0)-u}{w(u)}=x\right),$$

respectively, where $y \in [0, x_F)$. Let $\bar{\delta} = (\delta,\dots,\delta)$ for $\delta > 0$ and let

$$p_u(x) = \frac{w(u)g(u+xw(u))}{1-G(u+yw(u))}.$$

Since S_x has at least one point of continuity in each nonempty rectangle and since, by (2.1) and (2.12), $p_u(x) \to f(x))/(1 - F(y)$, we have, by (2.13) and Fatou's lemma,

$$S(\bar{x}-\bar{\delta}) = \int_y^{x_F} S_x(\bar{x}-\bar{\delta})\frac{f(x)}{1-F(y)}\,dx$$

$$\le \int_y^{x_F}\liminf_{u\to U}\left(G_x(u;\bar{x})p_u(x)\right)dx \le \liminf_{u\to U} G(u;\bar{x}).$$

On the other hand, since $p_u(x)$, $x > y$, and $f(x)/(1 - F(y))$, $y < x < x_F$, are densities, we get, by Scheffé's theorem [cf. Berman (1982) and Billingsley (1968), page 224],

$$S(\bar{x}+\bar{\delta}) \ge \int_y^{x_F}\limsup_{u\to U}\left(G_x(u;\bar{x})p_u(x)\right)dx \ge \limsup_{u\to U} G(u;\bar{x}).$$

Hence we have $\lim_{u\to U} G(u;\bar{x}) = S(\bar{x})$, for continuity points $\bar{x}$ of S. □

The idea to use Scheffé's theorem is taken from Berman (1982).

In Theorem 4 we shall see how (2.1) and (2.12) relate.

THEOREM 4. (I) *If G is type* I-*attracted and if G has a density g such that $g(u)$ is nonincreasing for all $u > u_0$, for some constant $u_0 < \hat{u}$, then* (2.1) *and* (2.12) *hold with $w(u) = (1 - G(u))/g(u)$, $x_L = -\infty$, $x_H = \infty$ and $F(x) = 1 - e^{-x}$.*

(II) *If $\hat{u} = \infty$ and G has a density g such that*

$$\lim_{u\to\infty} \frac{ug(u)}{1 - G(u)} = b > 0,$$

then (2.1) *and* (2.12) *hold with $w(u) = u$, $x_L = -1$, $x_H = \infty$ and $F(x) = 1 - (1 + x)^{-b}$.*

(III) *If $\hat{u} < \infty$ and G has a density g such that*

$$\lim_{u\to U} \frac{(\hat{u} - u)g(u)}{1 - G(u)} = b > 0,$$

then (2.1) *and* (2.12) *hold with $w(u) = \hat{u} - u$, $x_L = -\infty$, $x_H = 1$ and $F(x) = 1 - (1 - x)^b$.*

PROOF. (I) If (2.1) holds with $F(x) = 1 - e^{-x}$, $x_L = -\infty$ and $x_H = \infty$, then we have

$$\text{(2.14)} \qquad \lim_{u\to U} \frac{w(u + xw(u))}{w(u)} = 1 \quad \text{for all real } x.$$

In order to prove this, assume that $\lim_{u\to U} w(u + xw(u))/w(u) > \alpha > 1$, for some $x \in \mathbb{R}$. Then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ with $u_n \to U$ when $n \to \infty$, such that, for all n, $w(u_n + xw(u_n)) - \alpha w(u_n) > 0$. Letting $\tilde{u}_n = u_n + xw(u_n)$, we therefore get, by (2.1),

$$e^{-\alpha x} \leftarrow \frac{1 - G(u_n + (1 + \alpha)xw(u_n))}{1 - G(u_n + xw(u_n))}$$

$$\begin{cases} \geq \dfrac{1 - G(\tilde{u}_n + xw(\tilde{u}_n))}{1 - G(\tilde{u}_n)} \to e^{-x} & \text{if } x > 0, \\[2ex] \leq \dfrac{1 - G(\tilde{u}_n + xw(\tilde{u}_n))}{1 - G(\tilde{u}_n)} \to e^{-x} & \text{if } x < 0, \end{cases}$$

when $n \to \infty$, which is a contradiction. Thus we have

$$\limsup_{u\to U} \frac{w(u + xw(u))}{w(u)} \leq 1$$

and, similarly, it can be seen that $\liminf_{u\to U} w(u + xw(u))/w(u) \geq 1$, so that (2.14) holds.

Now, according to de Haan [(1970), page 88], in view of the fact that g is nonincreasing, (2.1) holds with $w(u) = (1 - G(u))/g(u)$, $x_L = -\infty$, $x_H = \infty$

and $F(x) = 1 - e^{-x}$ [cf. Lemma 5.1 in Berman (1982)]. Using (2.14), this yields

$$\frac{w(u)g(u + xw(u))}{1 - G(u)} = \frac{w(u)(1 - G(u + xw(u)))}{w(u + xw(u))(1 - G(u))} \to 1 - F(x) = F'(x).$$

(II) Since $\lim_{u \to \infty} ug(u)/(1 - G(u)) = b$, Leadbetter, Lindgren and Rootzén [(1983), pages 16 and 17] show that (2.1) holds with $w(u) = u$, $x_L = -1$, $x_H = \infty$ and $F(x) = 1 - (1 + x)^{-b}$, so that

$$\frac{ug((1 + x)u)}{1 - G(u)} = \frac{1}{1 + x} \frac{(1 + x)ug((1 + x)u)}{1 - G((1 + x)u)} \frac{1 - G((1 + x)u)}{1 - G(u)}$$
$$\to b(1 + x)^{-(b+1)}.$$

(III) The proof of (II) carries over with only obvious modifications. □

In Theorem 5, which will be of interest in Section 5, we relate Conditions C and C^0 and show that G and the d.f. of $M(h)$ belong to the same domain of attraction, that is, that there exists a constant $c \in [0, 1]$ such that

$$\text{(2.15)} \qquad \lim_{u \to U} \frac{q(u + xw(u))}{q(u)} = (1 - F(x))^c \quad \text{for } x_L < x < x_H.$$

THEOREM 5. (I) *If Conditions* A({0}), B *and* C *hold, with* $w(u)$ *continuous for all* $u > u_0$, *for some constant* $u_0 < \hat{u}$, $x_L = -\infty$, $x_H = \infty$ *and* $F(x) = 1 - e^{-x}$, *then Condition* $C^0((-\infty, \infty))$ *holds. If, in addition, Condition* A([0, ∞)) *holds, then there exists a constant* $c \in [0, 1]$ *such that* (2.15) *holds.*

(II) *If Conditions* A({0}), B *and* C *hold, with* $\hat{u} = \infty$, $w(u) = u$, $x_L = -1$, $x_H = \infty$ *and* $F(x) = 1 - (1 + x)^{-b}$, *for some constant* $b > 0$, *then Condition* $C^0((-1, \infty))$ *holds. If, in addition, Condition* A([0, ∞)) *holds, then there exists a constant* $c \in [0, 1]$ *such that* (2.15) *holds.*

(III) *If Conditions* A({0}), B *and* C *hold, with* $\hat{u} < \infty$, $w(u) = \hat{u} - u$, $x_L = -\infty$, $x_H = 1$ *and* $F(x) = 1 - (1 - x)^b$, *for some constant* $b > 0$, *then Condition* $C^0((-\infty, 1))$ *holds. If, in addition, Condition* A([0, 1)) *holds, then there exists a constant* $c \in [0, 1]$ *such that* (2.15) *holds.*

PROOF. Suppose we can prove that, for all sufficiently small $\delta > 0$,

$$\limsup_{a \downarrow 0} \limsup_{u \to U} \frac{\Pr\{M(aq) > u + (y + \delta)w(u), \xi(0) \le u + yw(u)\}}{a(1 - G(u))} = 0$$

$$\text{(2.16)} \qquad \text{for fixed } y \in (x_L, x_H).$$

Letting $\tilde{u} = u + yw(u)$, for $y \in (x_L, x_H)$, we have

$$\limsup_{u \to U} \frac{(1 - G(\tilde{u} + \delta w(u)))q(\tilde{u})}{(1 - G(\tilde{u}))q(\tilde{u} + \delta w(u))} \ge \frac{1 - F(y + \delta)}{1 - F(y)} \quad \text{for } \delta > 0 \text{ small},$$

and

$$\limsup_{u\to U}\frac{(1-G(\tilde{u}))q(u)}{(1-G(u))q(\tilde{u})}\le 1-F(\min\{0,y\})$$

(the latter inequality follows from Theorem 1 when $y\ge 0$). Hence Theorem 1 yields

$$\limsup_{u\to U}\frac{q(u)}{1-G(u)}\Pr\Big\{M(h)>u+yw(u),\ \max_{0\le aqk\le h}\xi(aqk)\le u+yw(u)\Big\}$$

$$\le \limsup_{u\to U} h\frac{\Pr\{M(aq)>\tilde{u}+\delta w(u),\ \xi(0)\le\tilde{u}\}}{a(1-G(u))}$$

$$+\limsup_{u\to U}\frac{\Pr\{\tilde{u}<M(h)\le\tilde{u}+\delta w(u)\}}{(1-G(u))/q(u)}$$

$$\le \limsup_{u\to U} h\frac{\Pr\{M(aq)>\tilde{u}+\delta w(u),\ \xi(0)\le\tilde{u}\}}{a(1-G(u))}$$

$$+(1-F(\min\{0,y\}))hH\left(1-\frac{1-F(y+\delta)}{1-F(y)}\right).$$

Letting first $a\downarrow 0$ and using (2.16) and then $\delta\downarrow 0$, it follows that Condition $C^0((x_L,x_H))$ holds. Thus, in order to show that Condition $C^0((x_L,x_H))$ holds, it is sufficient to prove (2.16) in cases (I)–(III).

(I) Assume that $y>0$. Clearly, the arguments used in (2.10) and (2.11) also yield

$$hH_a\le \liminf_{u\to U}\frac{q(u)}{1-G(u)}\Pr\{M(h)>u\}$$

$$\le h(1-F(-\delta)) \tag{2.17}$$

$$\times\left(\liminf_{u\to U}H_a\frac{q}{\tilde{q}}+\limsup_{u\to U}\frac{\Pr\{M(aq)>u+\frac{1}{2}\delta m w(u),\ \xi(0)\le u\}}{a(1-G(u))}\right)$$

for $\delta>0$,

where $\tilde{q}=q(u-\delta w(u))$. Letting $a\downarrow 0$, we get $\liminf_{u\to U}(q/\tilde{q})\ge 1/(1-F(-\delta))$. Hence we have

$$\limsup_{u\to U}\frac{q(u-\rho w(u))}{q(u)}<2\quad\text{for some constant } \rho\in(0,1). \tag{2.18}$$

Now we have, by (2.14), $u-2zw(u)+zw(u-2zw(u))\le u\le u+zw(u)$ for large $u<\hat{u}$ for fixed $z>0$. Hence, since w is continuous, the functions $v_1(u)=u$ and $v_2(u)=u+zw(u)$ must run over the same set of values for large $u<\hat{u}$. Thus the change of variable $u\to u+zw(u)$ is allowed in limit operations. Choosing a $\delta>0$, letting $N=[(2/\rho)(y+\delta/2)]+1$ and using (2.14) and (2.18), we therefore get

$$
\begin{aligned}
&\limsup_{u\to U}\frac{q\big(u-(y+\frac{1}{2}\delta)w(u)\big)}{q(u)}\\
&\quad\le \limsup_{u\to U}\frac{q\big(u-N\frac{1}{2}\rho w(u)\big)}{q(u)}\\
&\quad\le \prod_{n=0}^{N-1}\limsup_{u\to U}\frac{q\big(u-(n+1)\frac{1}{2}\rho w(u)\big)}{q\big(u-n\frac{1}{2}\rho w(u)\big)} \qquad (2.19)\\
&\quad= \prod_{n=0}^{N-1}\limsup_{u\to U}\frac{q\Big(u+\big(n-\frac{1}{2}\big)\frac{1}{2}\rho w(u)-(n+1)\frac{1}{2}\rho w\big(u+\big(n-\frac{1}{2}\big)\frac{1}{2}\rho w(u)\big)\Big)}{q\Big(u+\big(n-\frac{1}{2}\big)\frac{1}{2}\rho w(u)-n\frac{1}{2}\rho w\big(u+\big(n-\frac{1}{2}\big)\frac{1}{2}\rho w(u)\big)\Big)}\\
&\quad\le \prod_{n=0}^{N-1}\limsup_{u\to U}\frac{q(u-\rho w(u))}{q(u)} < 2^N.
\end{aligned}
$$

Writing $\tilde{u}=u-(y+\frac{1}{2}\delta)w(u)$ and $\tilde{q}=q(\tilde{u})$, we conclude, by (2.19) and (2.3),

$$
\begin{aligned}
&\limsup_{u\to U}\frac{\Pr\{M(aq)>u+(y+\delta)w(u),\ \xi(0)\le u+yw(u)\}}{a(1-G(u))}\\
&\quad\le \limsup_{u\to U}\frac{\Pr\{M(a\tilde{q})>\tilde{u}+(y+\delta)w(\tilde{u}),\ \xi(0)\le \tilde{u}+yw(\tilde{u})\}}{a\big(1-F\big(-y-\frac{1}{2}\delta\big)\big)(1-G(u))}\\
&\quad\le \limsup_{u\to U}\frac{\Pr\big\{M\big(a2^Nq\big)>u+(\delta/4)w(u),\ \xi(0)\le u\big\}}{a\big(1-F\big(-y-\frac{1}{2}\delta\big)\big)(1-G(u))}\to 0 \quad \text{when } a\downarrow 0,
\end{aligned}
$$

so that (2.16) holds for $y>0$. The case when $y<0$ is treated in a similar way. [Actually the proof is simpler when $y<0$, since (2.19) automatically holds.]

(II) First assume that $y>0$. Clearly, (2.17) and (2.18) still hold, and choosing an $\varepsilon\in(0,\frac{1}{2}\rho/(1+y))$ and an integer N with $1-N\varepsilon\in(\frac{1}{2}/(1+y),1/(1+y))$, we have, by (2.18),

$$
\begin{aligned}
\limsup_{u\to\infty} q(u)^{-1}q\left(\frac{u}{1+y}\right) &\le \limsup_{u\to\infty}\frac{q(u-N\varepsilon u)}{q(u)}\\
&\le \prod_{n=0}^{N-1}\limsup_{u\to\infty}\frac{q(u-(n+1)\varepsilon u)}{q(u-n\varepsilon u)}\\
&= \prod_{n=0}^{N-1}\limsup_{u\to\infty} q(u)^{-1}q\left(u-\frac{\varepsilon}{1-n\varepsilon}u\right) < 2^N.
\end{aligned}
$$

Arguing as in the proof of (I), we conclude that (2.16) holds for all $y\in(x_L,x_H)$.

(III) Since the d.f. of $(\hat{u}-\xi(t))^{-1}$ belongs to the type II domain of attraction [cf. Albin (1987), Chapter 9] and since

$$
\left(\frac{1}{\hat{u}-u}\right)^{-1}\left(\frac{1}{\hat{u}-\xi}-\frac{1}{\hat{u}-u}\right)\le x \quad \text{iff} \quad \frac{\xi-u}{\hat{u}-u}\le 1-\frac{1}{1+x}
$$

for all $x>-1$, it is a routine matter to derive (III) from (II).

It remains to prove (2.15) in cases (I)–(III).

(I) Let $G(y;\cdot)$ denote the d.f. of $\xi(t) - yw(u)$ and write $\tilde{u} = u + yw(u)$ for $y \geq 0$. Then we have $(1 - G(y; u + xw(u)))/(1 - G(y;u)) \to 1 - F(x)$ and, by Condition A($[0, x_H)$),

$$\left(\frac{\xi(aq) - \tilde{u}}{w(u)}, \dots, \frac{\xi(aqN) - \tilde{u}}{w(u)} \,\middle|\, \frac{\xi(0) - \tilde{u}}{w(u)} > 0\right) \to_D \left(\zeta_{a,y}(1), \dots, \zeta_{a,y}(N)\right) - y. \tag{2.20}$$

Now, an analysis of the proof of Theorem 1 shows that if Conditions A($\{0\}$), B and $C^0(\{0\})$ hold for a "u-dependent" process $\xi_u(t)$, (stationary for each fixed u), then the conclusion of Theorem 1 still holds. Consequently, we obtain

$$\begin{aligned} \Pr\{M(h) > u + yw(u)\} &\sim hH(y)\frac{1 - G(y;u)}{q(u)} \\ &= hH(y)\frac{1 - G(u + yw(u))}{q(u)} \quad \text{when } u \to U, \end{aligned} \tag{2.21}$$

where $H(y) = \lim_{a\downarrow 0}(1/a)\Pr\{\sup_{k\geq 1}\zeta_{a,y}(k) \leq y\}$. On the other hand, we have

$$\Pr\{M(h) > u + yw(u)\} \sim hH(0)\frac{1 - G(u + yw(u))}{q(u + yw(u))} \quad \text{when } u \to U. \tag{2.22}$$

Hence we deduce $\lim_{u\to U} q(u + yw(u))/q(u) = H(0)/H(y) \equiv v(y)$ for $y \geq 0$. Now, by (2.14),

$$\begin{aligned} v(x + y) &= \lim_{u\to U} \frac{q(u + (x + y)w(u))}{q(u)} \\ &= v(x) \lim_{u\to U} \frac{q(u + (x + y)w(u))}{q(u + xw(u))} \\ &\leq v(x) \lim_{u\to U} \frac{q(u + xw(u) + (y - \delta)w(u + xw(u)))}{q(u + xw(u))} \\ &= v(x)v(y - \delta) \quad \text{for } \delta > 0. \end{aligned}$$

Letting $\delta \downarrow 0$, we infer $v(x + y) \leq v(x)v(y^-)$ for $x \geq 0,\ y > 0$. Similarly, we get $v(x + y) \geq v(x)v(y^+)$ for $x \geq 0,\ y \geq 0$. Writing $\tilde{v}(x) = \ln(v(x))$, we thus conclude

$$\tilde{v}(x) + \tilde{v}(y^+) \leq \tilde{v}(x + y) \leq \tilde{v}(x) + \tilde{v}(y^-) \quad \text{for } x \geq 0,\ y > 0.$$

Since $\tilde{v}$ is nonincreasing, $\tilde{v}$ has a continuity point $x_1 > 0$. Hence we have

$$0 \leq \tilde{v}(x - \varepsilon) - \tilde{v}(x) \leq -\tilde{v}(\varepsilon^+) \leq -\tilde{v}(2\varepsilon) \leq \tilde{v}(x_1^-) - \tilde{v}(x_1 + 2\varepsilon) \downarrow 0$$

when $\varepsilon \downarrow 0$ for $x > 0$ and, similarly, $0 \leq \tilde{v}(x) - \tilde{v}(x + \varepsilon) \leq -\tilde{v}(\varepsilon^+) \downarrow 0$ when $\varepsilon \downarrow 0$. Hence $\tilde{v}(x)$ is continuous for $x > 0$ with $\tilde{v}(x + y) = \tilde{v}(x) + \tilde{v}(y)$ for $x, y > 0$, which yields $\tilde{v}(x) = -cx$ for $x \geq 0$ for some constant $c \geq 0$. We conclude that $v(x) = e^{-cx}$ for $x \geq 0$, where, in view of Theorem 1 and the definition of v, we have $c \leq 1$.

In order to treat the case when $x = -y < 0$, we observe that, by (2.14),

$$\liminf_{u \to U} \frac{q(u - yw(u))}{q(u)} = \liminf_{u \to U} \frac{q(u + (y - \varepsilon)w(u) - yw(u + (y - \varepsilon)w(u)))}{q(u + (y - \varepsilon)w(u))}$$
$$\geq \lim_{u \to U} \frac{q(u)}{q(u + (y - \varepsilon)w(u))} = e^{c(y-\varepsilon)} \uparrow e^{cy} \quad \text{when } \varepsilon \downarrow 0,$$

and, similarly, $\limsup_{u \to U} q(u - yw(u))/q(u) \leq e^{c(y+\varepsilon)} \downarrow e^{cy}$ when $\varepsilon \downarrow 0$.

(II) Writing $G(y; \cdot\,)$ for the d.f. of $\xi(t) - yw(u)$, for $y \geq 0$, we have, by (2.1), $(1 - G(y; u + xw(u)))/(1 - G(y; u)) \to 1 - F(x/(1 + y))$. Further, (2.20) still holds. We conclude that (2.21) and (2.22) hold, so that by arguing as in the proof of (I), the limit $\lim_{u \to \infty} q(u + xw(u))/q(u) \equiv v(x)$ exists for $x > -1$. This yields

$$v(xy - 1) = \lim_{u \to \infty} \frac{q(xyu)}{q(u)} = \lim_{u \to \infty} \frac{q(xyu)}{q(yu)} \frac{q(yu)}{q(u)}$$
$$= v(x - 1)v(y - 1)$$

for $x, y > 0$. Now, v (being nonincreasing) is measurable, and hence we have by Aczél [(1966), page 41], $v(x) = (1 + x)^{-bc}$, for some constant $c \geq 0$, where, by Theorem 1, $c \leq 1$.

(III) This follows from (II) in the way indicated above. □

Observe that although we do not try to treat processes other than those for which G belongs to a domain of attraction, the application of Theorem 1 in the proof of Theorem 5(II) makes it necessary to require (2.1) with a general F, rather than simply require that G belong to a domain of attraction.

We now give a criterion for (2.3) to hold. The idea to impose restrictions on the increases of ξ is taken from Berman (1982), although his criterion is three-dimensional while ours is two-dimensional.

THEOREM 6. *If there exist constants* $\lambda_0, \rho, e, C, \delta_0 > 0$, $u_0 < \hat{u}$ *and* $d > 1$ *such that*

$$\Pr\{\xi(qt) - \xi(0) > \lambda w(u), \xi(0) \leq u + \delta_0 w(u) \mid \xi(qt) > u\} \leq Ct^d \lambda^{-e} \tag{2.23}$$

for all $0 < t^\rho < \lambda < \lambda_0$ *and* $u_0 < u < \hat{u}$, *then* (2.3) *holds.*

PROOF. Let Y denote the set of dyadic numbers in $[0, 1)$, choose a $u > u_0$ and let $0 < \delta < 2\min\{\delta_0, \lambda_0\}$. In order to show that

$$\Pr\Big\{\sup_{0 \leq t < aq} \xi(t) > u + \delta w(u), \xi(0) \leq u\Big\} \leq \Pr\Big\{\sup_{t \in Yaq} \xi(t) > u + \frac{\delta}{2} w(u), \xi(0) \leq u\Big\}, \tag{2.24}$$

let $D = \{d_k\}_{k=1}^{\infty}$ be a countable separating set for $\{\xi(t)\}_{0 \le t < aq}$ [see, e.g., Billingsley (1979), page 468], with $0 \in D$, and define $D(n) = \{d_1, \dots, d_n\}$. Clearly, $\max_{t \in D(n)} \xi(t) \uparrow \sup_{0 \le t < aq} \xi(t)$ when $n \to \infty$, for almost all $\omega \in \Omega$, and hence

$$\lim_{n \to \infty} \Pr\left\{\left\{\sup_{0 \le t < aq} \xi(t) > u + \delta w(u)\right\} - \left\{\max_{t \in D(n)} \xi(t) > u + \delta w(u)\right\}\right\} = 0. \tag{2.25}$$

Now, to each pair of integers n, k satisfying $1 \le k \le n$, choose a $y_k^n \in [0, d_k] \cap Yaq$ such that $d_k - y_k^n < (\delta/2)^{1/\rho} q/n$. Then $((d_k - y_k^n)/q)^{\rho} < \delta/2 < \lambda_0$, so that by (2.23),

$$\Pr\left\{\xi(d_k) - \xi(y_k^n) > \frac{\delta}{2} w(u), \xi(d_k) > u, \xi(y_k^n) \le u + \frac{\delta}{2} w(u)\right\}$$
$$\le C\left(\frac{\delta}{2}\right)^{d/\rho - e} n^{-d}.$$

Letting $Y(n) = \{y_1^n, \dots, y_n^n\}$, this yields

$$\begin{aligned} &\Pr\left\{\left\{\max_{t \in D(n)} \xi(t) > u + \delta w(u)\right\} \cap \left\{\max_{t \in Y(n)} \xi(t) \le u + \frac{\delta}{2} w(u)\right\}\right\} \\ &\le \sum_{k=1}^{n} \Pr\left\{\xi(d_k) - \xi(y_k^n) > \frac{\delta}{2} w(u), \xi(d_k) > u + \delta w(u), \right. \\ &\qquad\qquad \left. \xi(y_k^n) \le u + \frac{\delta}{2} w(u)\right\} \\ &\le C\left(\frac{\delta}{2}\right)^{d/\rho - e} n^{-(d-1)} \to 0 \quad \text{when } n \to \infty. \end{aligned} \tag{2.26}$$

Combining (2.25) and (2.26), we conclude

$$\begin{aligned} &\Pr\left\{\left\{\sup_{0 \le t < aq} \xi(t) > u + \delta w(u)\right\} - \left\{\sup_{t \in Yaq} \xi(t) > u + \frac{\delta}{2} w(u)\right\}\right\} \\ &\le \limsup_{n \to \infty} \Pr\left\{\left\{\sup_{0 \le t < aq} \xi(t) > u + \delta w(u)\right\} - \left\{\max_{t \in Y(n)} \xi(t) > u + \frac{\delta}{2} w(u)\right\}\right\} \\ &\le \limsup_{n \to \infty} \Pr\left\{\left\{\sup_{0 \le t < aq} \xi(t) > u + \delta w(u)\right\} - \left\{\max_{t \in D(n)} \xi(t) > u + \delta w(u)\right\}\right\} \\ &\quad + \limsup_{n \to \infty} \Pr\left\{\left\{\max_{t \in D(n)} \xi(t) > u + \delta w(u)\right\} \cap \left\{\max_{t \in Y(n)} \xi(t) \le u + \frac{\delta}{2} w(u)\right\}\right\} \\ &= 0 \end{aligned}$$

and hence (2.24) holds. It is therefore sufficient to prove that

$$\limsup_{a\downarrow 0}\limsup_{u\to U}\frac{1}{a(1-G(u))}\times\Pr\Big\{\sup_{t\in Yaq}\xi(t)>u+\delta w(u),\,\xi(0)\le u\Big\}=0. \tag{2.27}$$

In order to prove (2.27), let $\varepsilon\in(0,\rho)$ with $e\varepsilon<d-1$, let $a_0=((1-2^{-\varepsilon})\delta)^{1/\rho}$ and $p_n=2^{-n\varepsilon}(1-2^{-\varepsilon})\delta$. Then we have

$$\xi(aqt)-\xi(0)=\sum_{n=0}^{N}\left(\xi\left(aq\sum_{k=0}^{n+1}2^{-k}b_k\right)-\xi\left(aq\sum_{k=0}^{n}2^{-k}b_k\right)\right)\quad\text{for } t\in Y,$$

for some nonnegative integer N, where $b_0=0$ and $b_n\in\{0,1\}$. Clearly, $\Sigma p_n=\delta$, so that, writing $c_k=2^{-k}b_k$ and $C_n=b_0+2^{-1}b_1+\cdots+2^{-n}b_n$, we obtain

$$\begin{aligned}&\{\xi(aqt)>u+\delta w(u),\,\xi(0)\le u\}\\&\subset\left\{\bigcup_{n=0}^{N}\{\xi(aq(c_{n+1}+C_n))-\xi(aqC_n)>p_nw(u)\}\right\}\\&\qquad\cap\{\xi(0)\le u\}\cap\{\xi(aqt)>u+\delta w(u)\}.\end{aligned}\tag{2.28}$$

Now, if $\omega\in\Omega$ is a member of the event on the right-hand side of (2.28), then there must exist a largest integer $n=n(\omega)$ such that

$$\begin{aligned}\omega\in&\{\xi(aq(c_{n+1}+C_n))-\xi(aqC_n)>p_nw(u)\}\\&\cap\{\xi(aqt)>u+\delta w(u)\}\cap\{\xi(0)\le u\}.\end{aligned}$$

Hence $\xi(aq(c_{k+1}+C_k))-\xi(aqC_k)\le p_kw(u)$ for $k=n+1,\ldots,N$, so that by summing these differences and observing that $\xi(aqt)>u+\delta w(u)$,

$$\omega\in\{\xi(aq(c_{n+1}+C_n))-\xi(aqC_n)>p_nw(u),\,\xi(aq(c_{n+1}+C_n))>u\}.$$

This yields

$$\begin{aligned}&\{\xi(aqt)>u+\delta w(u),\,\xi(0)\le u\}\\&\subset\left\{\bigcup_{n=0}^{N}\{\xi(aq(c_{n+1}+C_n))-\xi(aqC_n)>p_nw(u),\right.\\&\qquad\left.\xi(aq(c_{n+1}+C_n))>u\}\right\}\cap\{\xi(0)\le u\}.\end{aligned}\tag{2.29}$$

Now, if $\omega\in\Omega$ is a member of the event on the right-hand side of (2.29), then there must exist a smallest integer $n=n(\omega)$ such that

$$\begin{aligned}\omega\in&\{\xi(aq(c_{n+1}+C_n))-\xi(aqC_n)>p_nw(u),\,\xi(aq(c_{n+1}+C_n))>u\}\\&\cap\{\xi(0)\le u\}.\end{aligned}$$

Letting $C_{-1}=0$, and since $\xi(0)\le u$, there further exists a largest integer

$l \in \{-1, \dots, n-1\}$ such that $\xi(aq(c_{l+1} + C_l)) \le u$. Hence

$$\xi(aq(c_{k+1} + C_k)) - \xi(aqC_k) \le p_k w(u) \quad \text{for } k = l+1, \dots, n-1,$$

so that by summing these differences,

$$\omega \in \{\xi(aq(c_{n+1} + C_n)) - \xi(aqC_n) > p_n w(u), \xi(aq(c_{n+1} + C_n)) > u, \\ \xi(aqC_n) \le u + \delta w(u)\}.$$

We conclude that

$$\{\xi(aqt) > u + \delta w(u), \xi(0) \le u\} \\ \subset \bigcup_{n=0}^{N} \{\xi(aq(c_{n+1} + C_n)) - \xi(aqC_n) > p_n w(u), \xi(aq(c_{n+1} + C_n)) > u, \\ \xi(aqC_n) \le u + \delta w(u)\}.$$

Using (2.23), we therefore obtain

$$\frac{1}{a(1 - G(u))} \Pr\Big\{\sup_{t \in Yaq} \xi(t) > u + \delta w(u), \xi(0) \le u\Big\} \\ \le \sum_{n=0}^{\infty} \sum_{k=0}^{2^n - 1} \frac{1}{a(1 - G(u))} \Pr\{\xi(aq2^{-n}(k + \tfrac{1}{2})) - \xi(aq2^{-n}k) > p_n w(u), \\ \xi(aq2^{-n}(k + \tfrac{1}{2})) > u, \xi(aq2^{-n}k) \le u + \delta w(u)\} \\ \le \frac{C}{a} \sum_{n=0}^{\infty} 2^n (a2^{-(n+1)})^d (p_n)^{-e}$$

for all $0 < a < a_0$, $0 < \delta < \min\{\delta_0, \lambda_0\}$ and $u_0 < u < \hat{u}$.

Here the right-hand side tends to zero when $a \downarrow 0$ and hence (2.27) holds. □

We shall now study the case when $\xi(t)$ has finite upcrossing intensity $\mu(u)$ of each level u by applying a criterion due to Leadbetter and Rootzén (1982) [cf. (2.30) below]. To that end, define $J_s(u) = (1/s)\Pr\{\xi(0) < u < \xi(s)\}$ for $s > 0$, so that (under very mild restrictions) $J_s(u) \to \mu(u)$ when $s \downarrow 0$.

Theorem 7. *Assume that $\xi(t)$ has finite upcrossing intensity $\mu(u)$ of all levels $u > u_0$ for some constant $u_0 < \hat{u}$ and that $\xi(t)$ possesses continuous sample paths with probability* 1. *If Conditions* A({0}) *and* B *hold and if*

$$\liminf_{a \downarrow 0} \liminf_{u \to U} \mu(u)^{-1} J_{aq}(u) \ge 1, \tag{2.30}$$

then Condition $C^0(\{0\})$ *holds and the limits*

$$\lim_{a \downarrow 0} \frac{1}{a} \Pr\{\zeta_{a,0}(1) \le 0\} \equiv E \quad \text{and} \quad \lim_{a \downarrow 0} \frac{1}{a} \Pr\Big\{\sup_{k \ge 1} \zeta_{a,0}(k) \le 0\Big\} \equiv H$$

exist with $0 < H \le E < \infty$ *and*

$$\lim_{u \to U} \frac{\mu(u) q(u)}{1 - G(u)} = E \quad \text{and} \quad \lim_{u \to U} \frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} = hH.$$

Proof. Clearly, by (2.1), $\lim_{u \to U}(\Pr\{\xi(0) = u\}/(1 - G(u))) = 0$, so that $\Pr\{\xi(0) < u < \xi(aq)\}/(1 - G(u))$ and $\Pr\{\xi(0) \le u < \xi(aq)\}/(1 - G(u))$ are asymptotically equivalent. Hence we obtain, by (2.7),

$$\begin{aligned}\frac{q(u)}{1-G(u)}J_{aq}(u) &\sim \frac{1}{a(1-G(u))}\Pr\Big\{\max_{0\le k<2}\xi(aqk)>u\Big\}-\frac{1}{a}\\ &\to \frac{1}{a}\Pr\{\zeta_{a,0}(1)\le 0\}.\end{aligned}\tag{2.31}$$

Now, an analysis of the proof of Theorem 1 shows that Conditions A($\{0\}$) and B imply $\limsup_{a\downarrow 0} H_a > 0$, so that $\Pr\{\zeta_{\hat{a},0}(1)\le 0\}>0$ for some $\hat{a}>0$. In view of (2.31) and since $J_s(u)\le\mu(u)$ [cf. Leadbetter and Rootzén (1982)], we therefore get

$$\begin{aligned}\limsup_{u\to U}\mu(u)^{-1}\frac{1-G(u)}{q(u)} &\le \limsup_{u\to U}J_{\hat{a}q}(u)^{-1}\frac{1-G(u)}{q(u)}\\ &= \frac{\hat{a}}{\Pr\{\zeta_{\hat{a},0}(1)\le 0\}}<\infty.\end{aligned}\tag{2.32}$$

Again using (2.31) and the fact that $J_s(u)\le\mu(u)$, (2.32) yields

$$\begin{aligned}1\ge\limsup_{a\downarrow 0}\limsup_{u\to U}\frac{J_{aq}(u)}{\mu(u)} &= \limsup_{a\downarrow 0}\limsup_{u\to U}\left(\frac{q(u)J_{aq}(u)}{1-G(u)}\,\frac{1-G(u)}{\mu(u)q(u)}\right)\\ &= \limsup_{a\downarrow 0}\left(\frac{\Pr\{\zeta_{a,0}(1)\le 0\}}{a}\limsup_{u\to U}\frac{1-G(u)}{\mu(u)q(u)}\right),\end{aligned}$$

while (2.30), (2.31) and (2.32) combine to show

$$1\le\liminf_{a\downarrow 0}\liminf_{u\to U}\frac{J_{aq}(u)}{\mu(u)}=\liminf_{a\downarrow 0}\left(\frac{\Pr\{\zeta_{a,0}(1)\le 0\}}{a}\liminf_{u\to U}\frac{1-G(u)}{\mu(u)q(u)}\right).$$

It readily follows that the limits $\lim_{a\downarrow 0}(1/a)\Pr\{\zeta_{a,0}(1)\le 0\}\equiv E$ and $\lim_{u\to U}(1-G(u))/(\mu(u)q(u))$ exist, with strictly positive and finite values, and with $\lim_{u\to U}\mu(u)q(u)/(1-G(u))=E$.

Further, as observed by Leadbetter and Rootzén [(1982), Lemma 5.1(ii)], (2.30) implies that

$$\limsup_{u\to U}\frac{\Pr\{M(aq)>u,\ \xi(0)<u,\ \xi(aq)<u\}}{aq(u)\mu(u)}\to 0\quad\text{when } a\downarrow 0,$$

which yields

$$\begin{aligned}&\limsup_{u\to U}\frac{q(u)}{1-G(u)}\Pr\Big\{M(h)>u,\ \max_{0\le aqk\le h}\xi(aqk)\le u\Big\}\\ &\quad\le\limsup_{u\to U}\frac{Eh}{aq(u)\mu(u)}\Pr\{M(aq)>u,\ \xi(0)<u,\ \xi(aq)<u\}\\ &\qquad+\limsup_{u\to U}\frac{2h}{a}\,\frac{\Pr\{\xi(0)=u\}}{1-G(u)}\to 0\quad\text{when } a\downarrow 0.\end{aligned}$$

Hence Condition $C^0(\{0\})$ holds, so that, by Theorem 1,

$$\frac{\Pr\{M(h) > u\}}{(1 - G(u))/q(u)} \to hH. \qquad \square$$

As an easy consequence of Theorem 7, assuming that Conditions $A(\{0\})$ and B hold, (2.30) is equivalent with requiring that the limits $\lim_{u \to U} \mu(u)q(u)/(1 - G(u))$ and $\lim_{a \downarrow 0}(1/a)\Pr\{\zeta_{a,0}(1) \le 0\}$ exist, with a common value E satisfying $0 < E < \infty$.

We remark here that one can replace Condition C with the requirements that q is nonincreasing, that $\xi(t)$ possesses continuous sample paths, that μ is finite and that

$$\liminf_{a \downarrow 0} \liminf_{u \to U} \mu(u + yw(u))^{-1} J_{aq(u)}(u + yw(u)) \ge 1 \quad \text{for all } x_L < y < x_H$$

in the hypothesis of Theorem 5, without violating the conclusion that Condition $C^0((x_L, x_H))$ and (2.15) hold. We omit the details.

In extremal theory, a (2.30)-like condition is needed in order to couple Condition D (cf. Section 5), which is formulated in discrete time, to the behaviour of continuous-time extrema. In the classical approach to extremes, in order to show that $\Pr\{M(h) > u\} \sim \mu(u)$, one further has to verify that

$$\text{(2.33)} \qquad E\{N_u(N_u - 1)\} = o(\mu(u)) \quad \text{when } u \to U,$$

where N_u is the number of upcrossings of the level u by $\xi(t)$ for $0 \le t \le 1$.

As indicated by Theorem 7, one cannot in general expect that (2.33) holds, since, in general, $E \ne H$ [see, e.g., Albin (1987) for examples].

Even when (2.33) holds, it is frequently more complicated to verify (2.33) than to verify Conditions $A(\{0\})$ and B. The reason for this is that estimation of $E\{N_u(N_u - 1)\}$ requires quantitative knowledge concerning four-dimensional densities [cf. Belayev (1968) and Marcus (1977)], while Condition B is formulated in terms of two-dimensional probabilities. Observe that even if (2.33) can be proven readily, one must still estimate such two-dimensional probabilities in order to verify Condition D′ of Section 5. Further, if (2.33) holds, then

$$\lim_{a \downarrow 0} \frac{1}{a} \Pr\{\zeta_{a,0}(1) \le 0\} = \lim_{a \downarrow 0} \frac{1}{a} \Pr\Big\{\sup_{k \ge 1} \zeta_{a,0}(k) \le 0\Big\}.$$

Thus the sequence $\{\zeta_{a,0}(k)\}_{k=1}^{\infty}$ must be degenerate in some sense, which should make it particularly easy to verify Condition $A(\{0\})$.

3. Asymptotic hitting probabilities for small sets for Gaussian processes. Let $\{\bar{\omega}(t)\}_{t \ge 0}$ be a separable $\mathbb{R}^m$-valued stationary Gaussian process, with independent component processes $\omega_1, \ldots, \omega_m$ possessing covariance functions $R_1, \ldots, R_m$ such that, for some constants $0 < \alpha \le 2$ and $C_1, \ldots, C_\eta > 0$,

$$\text{(3.1)} \quad \begin{aligned} R_i(t) &= \operatorname{Var}\{\omega_i(0)\} - C_i|t|^\alpha + o(|t|^\alpha) && \text{when } t \to 0, \text{for } i = 1, \ldots, \eta, \\ R_i(t) &= \operatorname{Var}\{\omega_i(0)\} - o(|t|^\alpha) && \text{when } t \to 0, \text{for } i = \eta + 1, \ldots, m. \end{aligned}$$

We shall derive an asymptotic expression for the probability that the process $\{u\bar{\omega}(t)\}_{0 \le t \le h}$ visits a fixed set A, when $u \to \infty$. Our problem relates to those of

Berman (1983, 1984, 1985), although he considers "sojourns" rather than extremes.

DEFINITION 1. An open nonempty subset A of $\mathbb{R}^m$ is an open star-shaped Lipschitzian domain (o.s.L.d.) if the following three conditions hold.

(a) $M_A \equiv \sup\{|\bar{x}|: \bar{x} \in A\} < \infty$.

(b) $\lambda\bar{x} \in A$ for all $\lambda \in [0, \rho_A(\bar{x}))$ and for all $\bar{x} \in \mathbb{R}^m$, where $\rho_A(\bar{x}) \equiv \sup\{\lambda > 0: \lambda\bar{x} \in A\}$.

(c) There exists a constant K_A such that $|r_A(\bar{x}) - r_A(\bar{y})| \leq K_A|\bar{x} - \bar{y}|$, for all $\bar{x}, \bar{y} \in \mathbb{R}^m$, where $r_A(\bar{x}) = \sup\{\lambda > 0: (1/\lambda)\bar{x} \notin A\} = \rho_A(\bar{x})^{-1}$.

THEOREM 8. *If A is an o.s.L.d. and if $\eta > 2/\alpha$, then there exists a constant $L_\alpha^m(A; C_1, \dots, C_\eta)$ with $0 < L_\alpha^m(A; C_1, \dots, C_\eta) < \infty$, such that for each $h > 0$ satisfying* $\max_{1 \leq i \leq \eta} \operatorname{Var}\{\omega_i(0)\}^{-1} R_i(t) < 1$ *for all $0 < t \leq h$, we have*

$$\lim_{u \to \infty} u^{m-2/\alpha} \Pr\{u\bar{\omega}(t) \in A, \text{ for some } t \in [0, h]\}$$
$$= h\left(\prod_{i=1}^m \operatorname{Var}\{\omega_i(0)\}^{-1/2}\right) \exp\left(-\sum_{i=1}^m \frac{E\{\omega_i(0)\}^2}{2\operatorname{Var}\{\omega_i(0)\}}\right) L_\alpha^m(A; C_1, \dots, C_\eta).$$

PROOF. Let G be the d.f. of $\rho_A(\bar{\omega}(t))$. Clearly, since $u\bar{\omega}(t) \in A$ iff $\rho_A(\bar{\omega}(t)) > u$, since, by routine calculations,

$$\begin{aligned} 1 - G(u) \sim (2\pi)^{-m/2} &\left(\prod_{i=1}^m \operatorname{Var}\{\omega_i(0)\}^{-1/2}\right) \\ &\times \exp\left(-\sum_{i=1}^m \frac{E\{\omega_i(0)\}^2}{2\operatorname{Var}\{\omega_i(0)\}}\right) \lambda^m(A) u^{-m} \end{aligned} \tag{3.2}$$

when $u \to \infty$, where λ^m is the Lebesgue measure over $\mathbb{R}^m$, and in view of Theorems 1 and 6, it suffices to show that the process $\{\rho_A(\bar{\omega}(t))\}_{t \geq 0}$ satisfies (2.2) with $w(u) = u$ and $q(u) = u^{-2/\alpha}$ (and with the limit process depending on α, m, A and $C_1, \dots, C_\eta$ only) and that Condition B and (2.23) hold.

In order to prove (2.2), let $\bar{x}_{\bar{r}} = (r_1(qt)x_1, \dots, r_m(qt)x_m)$ and $\bar{\omega}^u(t) = u(\omega_1(qt) - r_1(qt)\omega_1(0), \dots, \omega_m(qt) - r_m(qt)\omega_m(0))$, where $r_i(t) = V_i^{-1}R_i(t)$ and $V_i = \operatorname{Var}\{\omega_i(0)\}$. Writing f_u for the density of $u\bar{\omega}(t)$ and observing that $\bar{\omega}(0)$ and $\bar{\omega}^u(t)$ are independent, we obtain, by routine calculations,

$$\Pr\left\{\bigcap_{k=1}^N \left\{\frac{1}{u}(\rho_A(\bar{\omega}(qt_k)) - u) \leq z_k\right\} \middle| \frac{\rho_A(\bar{\omega}(0)) - u}{u} > 0\right\}$$
$$= \int_{\bar{x} \in A} \Pr\left\{\bigcap_{k=1}^N \{\rho_A(u\bar{\omega}(qt_k)) \leq z_k + 1\} \middle| u\bar{\omega}(0) = \bar{x}\right\} \frac{f_u(\bar{x})}{1 - G(u)}\, d\bar{x}$$
$$\sim \frac{1}{\lambda^m(A)} \int_{\bar{x} \in A} \Pr\left\{\bigcap_{k=1}^N \{\rho_A(\bar{\omega}^u(t_k) + \bar{x}_{\bar{r}}) \leq z_k + 1\}\right\} d\bar{x}.$$

Clearly, letting $\zeta_1(t), \dots, \zeta_m(t)$ be independent zero-mean Gaussian processes, with $\operatorname{Cov}\{\zeta_i(s), \zeta_i(t)\} = C_i(t^\alpha + s^\alpha - |t - s|^\alpha)$ for $i = 1, \dots, \eta$ and with $\zeta_i(t) \equiv 0$ for $i = \eta + 1, \dots, m$, we have, by (3.1), $E\{\omega_i^u(t)\} \to E\{\zeta_i(t)\}$ for $i = 1, \dots, m$

and

$$\text{(3.3)}\qquad \operatorname{Cov}\{\omega_i^u(s),\omega_i^u(t)\} = u^2V_i(r_i(q(t-s)) - r_i(qs)r_i(qt)) \to \operatorname{Cov}\{\zeta_i(s),\zeta_i(t)\}$$

for $i = 1,\dots,m$. Hence the f.d.d.'s of

$$\{((1/u)(\rho_A(\bar\omega(qt)) - u)\,|\,(1/u)(\rho_A(\bar\omega(0)) - u) > 0)\}_{t>0}$$

converge weakly to those of $\{\rho_A(\bar\zeta(t) + \bar X) - 1\}_{t>0}$, where $\bar X$ is an $\mathbb{R}^m$-valued random variable, independent of the process $\bar\zeta(t)$ and uniformly distributed over A.

Now, the density g_u of $u(\omega_1(0),\omega_1(t),\dots,\omega_\eta(0),\omega_\eta(t))$ satisfies

$$g_u(x_1,y_1,\dots,x_\eta,y_\eta) \le Ku^{-2\eta}\left(1 - r_1(t)^2\right)^{-1/2}\cdot\dots\cdot\left(1 - r_\eta(t)^2\right)^{-1/2},$$

where (here and in the sequel) K denotes a positive generic constant. Using the easily established fact that $\rho_A(\bar x) \le M_A|\bar x|^{-1}$, we therefore deduce

$$\begin{aligned}
&\Pr\{\rho_A(\bar\omega(0)) > u, \rho_A(\bar\omega(t)) > u\}\\
&\quad\le \Pr\{|u\bar\omega(0)| < M_A, |u\bar\omega(t)| < M_A\}\\
&\quad\le \prod_{i=1}^{\eta}\Pr\{|u\omega_i(0)| < M_A, |u\omega_i(t)| < M_A\}\prod_{i=\eta+1}^{m}\Pr\{|u\omega_i(0)| < M_A\}\\
&\quad\le Ku^{-(m+\eta)}\prod_{i=1}^{\eta}\left(1 - r_i(t)^2\right)^{-1/2}\quad \text{for } 0 < t \le h.
\end{aligned}$$

Choosing constants A, $\varepsilon > 0$ such that $1 - r_i(t)^2 \ge At^\alpha$ for $0 < t \le \varepsilon$, for $1 \le i \le \eta$, we obtain, by (3.2) and since $1 - r_i(t)^2$ is bounded away from zero for $\varepsilon < t \le h$,

$$\begin{aligned}
&\Pr\{\rho_A(\bar\omega(0)) > u, \rho_A(\bar\omega(qt)) > u\} \le K(1 - G(u))t^{-\eta\alpha/2} &&\text{for } 0 < qt \le \varepsilon,\\
&\Pr\{\rho_A(\bar\omega(0)) > u, \rho_A(\bar\omega(qt)) > u\} \le K(1 - G(u))u^{-\eta} &&\text{for } \varepsilon < qt \le h.
\end{aligned}$$

Using the fact that $\eta > 2/\alpha$, it is now readily seen that Condition B holds.

Now, let $\delta \in (0,1)$ and choose constants B, $\varepsilon > 0$ such that $1 - r_i(t) \le Bt^\alpha$ for $0 < t \le \varepsilon$, for $1 \le i \le m$, and let $r = \min_{1\le i\le m} r_i(qt)$, $\bar E = (E\{\omega_1(0)\},\dots,E\{\omega_m(0)\})$, $V = \max_{1\le i\le m} V_i$,

$$\lambda_0 = \min\left\{\varepsilon^{\alpha/2}, \frac{1}{1-\delta}, \frac{(1-\delta)^2}{4BK_AM_A}, \frac{(1-\delta)^2}{8BK_A|\bar E|\sqrt{m}}\right\}$$

and $\bar\omega_{\bar r}(qt) = (r_1(qt)\omega_1(qt),\dots,r_m(qt)\omega_m(qt))$. For $r_A(\bar\omega(qt)) < 1/u$, we then have

$$|\bar\omega(qt) - \bar\omega_{\bar r}(qt)| \le (1 - r)|\bar\omega(qt)| \le B(qt)^\alpha M_A\frac{1}{u} \le \frac{\lambda(1-\delta)^2}{4K_Au}$$

and

$$|(1 - r_i(qt))E_i| \le \frac{\lambda(1-\delta)^2}{8K_A\sqrt{m}\,u}$$

for $0 < t^{\alpha/2} < \lambda < \lambda_0$ and $u > u_0 \equiv 1$. Writing Φ for the standardized Gaussian distribution function, we therefore get, by Definition 1(c), the triangle inequality and Boole's inequality, and using the fact that $\bar{\omega}(0) - \bar{\omega}_{\bar{r}}(qt)$ and $\bar{\omega}(qt)$ are independent,

$$\Pr\left\{\rho_A(\bar{\omega}(qt)) - \rho_A(\bar{\omega}(0)) > \lambda u, \rho_A(\bar{\omega}(0)) \le \frac{u}{1-\delta}, \rho_A(\bar{\omega}(qt)) > u\right\}$$

$$\le \Pr\left\{r_A(\bar{\omega}(0)) - r_A(\bar{\omega}(qt)) > \lambda(1-\delta) r_A(\bar{\omega}(qt)), r_A(\bar{\omega}(0)) \ge \frac{1-\delta}{u},\right.$$

$$\left. r_A(\bar{\omega}(qt)) < \frac{1}{u}\right\}$$

$$\le \Pr\left\{K_A|\bar{\omega}(0) - \bar{\omega}(qt)| > \lambda(1-\delta)\left(\frac{1-\delta}{u} - K_A|\bar{\omega}(0) - \bar{\omega}(qt)|\right),\right.$$

$$\left. r_A(\bar{\omega}(qt)) < \frac{1}{u}\right\}$$

$$= \Pr\left\{|\bar{\omega}(0) - \bar{\omega}(qt)| > \frac{\lambda(1-\delta)^2}{K_A(1+\lambda(1-\delta))}\frac{1}{u}, r_A(\bar{\omega}(qt)) < \frac{1}{u}\right\}$$

$$\le \Pr\left\{|\bar{\omega}(0) - \bar{\omega}_{\bar{r}}(qt)| > \frac{\lambda(1-\delta)^2}{4K_A}\frac{1}{u}, r_A(\bar{\omega}(qt)) < \frac{1}{u}\right\}$$

$$\le (1 - G(u)) \sum_{i=1}^{m} \Pr\left\{|\omega_i(0) - r_i(qt)\omega_i(qt) - (1 - r_i(qt))E_i|\right.$$

$$\left. > \frac{\lambda(1-\delta)^2}{8K_A\sqrt{m}}\frac{1}{u}\right\}$$

$$\le 2m(1 - G(u))\left\{1 - \Phi\left(\frac{(1-\delta)^2\lambda}{8\sqrt{2mBV}K_A t^{\alpha/2}}\right)\right\},$$

$$\text{for } 0 < t^{\alpha/2} < \lambda < \lambda_0 \text{ and } u > u_0.$$

Clearly for each constant $p \ge 1$, there exists a corresponding constant $K_p > 0$ such that $2m(2\pi)^{-1/2}(1/x)\exp\{-\frac{1}{2}x^2\} \le K_p x^{-p}$ for $x > 0$. Using the well-known inequality $1 - \Phi(x) \le (2\pi)^{-1/2}(1/x)\exp\{-\frac{1}{2}x^2\}$, for $x > 0$, we therefore obtain

$$\frac{1}{1-G(u)}\Pr\left\{\rho_A(\bar{\omega}(qt)) - \rho_A(\bar{\omega}(0)) > \lambda u, \rho_A(\bar{\omega}(0)) \le \frac{u}{1-\delta}, \rho_A(\bar{\omega}(qt)) > u\right\}$$

$$\le K_p\left(\frac{t^{\alpha/2}}{K\lambda}\right)^p.$$

Choosing $p > 2/\alpha$, we conclude that (2.23) holds. □

In Albin (1987) we give a formula for $L_2^m(A; C_1, \dots, C_m)$ and, in particular, show that, in the notation of Theorem 7, $E = H$ iff A is convex. Further, we show that $L_1^m(A; C_1, \dots, C_m) = Cm(m-2)$ when A is the unit ball in $\mathbb{R}^m$ $(m \geq 3)$ and $C_1 = \cdots = C_m = C$. In Albin (1987) it is also proven that Conditions D and D′ of Section 5 hold for the process $\rho_A(\overline{\omega}(t))$ if $\max_{1 \leq i \leq m} |R_i(t)| = o(t^{-\gamma})$ when $t \to \infty$, for some constant $\gamma > (m/2 - 1/\alpha)^{-1}$.

4. High level extremes of Rayleigh processes. Let $\{\overline{\omega}(t)\}_{t \geq 0}$ be a separable $\mathbb{R}^m$-valued stationary Gaussian process, with independent standardized component processes $\omega_1, \dots, \omega_m$ possessing covariance functions $r_1, \dots, r_m$ such that, for some constants $0 < \alpha \leq 2$ and $C_1, \dots, C_m > 0$,

$$(4.1) \qquad r_i(t) = 1 - C_i |t|^\alpha + o(|t|^\alpha) \quad \text{when } t \to 0, \text{ for } i = 1, \dots, m.$$

We shall study high level extremes of the Rayleigh process $|\overline{\omega}(t)| = (\omega_1(t)^2 + \cdots + \omega_m(t)^2)^{1/2}$ when (4.1) holds. We remark here that Sharpe (1978) and Lindgren (1980, 1984, 1989) treated the cases when $r_1 = \cdots = r_m$ and $\alpha = 2$, and when $\alpha = 2$, respectively, while Berman (1982) studied sojourns when $r_1 = \cdots = r_m$.

THEOREM 9. *There exists a constant* $H_\alpha^m(C_1, \dots, C_m)$ *with* $0 < H_\alpha^m(C_1, \dots, C_m) < \infty$, *such that for each* $h > 0$ *satisfying* $\max_{1 \leq i \leq m} r_i(t) < 1$ *for all* $0 < t \leq h$, *we have*

$$\lim_{u \to \infty} u^{2 - 2/\alpha - m} \exp\left\{\tfrac{1}{2}u^2\right\} \Pr\left\{\sup_{0 \leq t \leq h} |\overline{\omega}(t)| > u\right\}$$
$$= \frac{h}{\Gamma\left(\frac{1}{2}m\right)} 2^{-(m-2)/2} H_\alpha^m(C_1, \dots, C_m).$$

PROOF. Since $g(x) = 2^{-(m-2)/2}\Gamma(m/2)^{-1} x^{m-1} \exp\{-\frac{1}{2}x^2\}$, $x > 0$, the fact that (2.1) and (2.12) hold with $w(u) = 1/u$ and $F(x) = 1 - e^{-x}$ follows from observing that

$$1 - G(u) = \int_u^\infty g(x)\, dx$$
$$= \frac{\left(u/\sqrt{2}\right)^{m-2}}{\Gamma\left(\frac{1}{2}m\right)} \exp\left\{-\tfrac{1}{2}u^2\right\} \int_0^\infty \left(1 + yu^{-2}\right)^{m-1} \exp\left\{-y - \tfrac{1}{2}\left(\frac{y}{u}\right)^2\right\} dy$$
$$\sim \frac{\left(u/\sqrt{2}\right)^{m-2}}{\Gamma\left(\frac{1}{2}m\right)} \exp\left\{-\tfrac{1}{2}u^2\right\}, \quad \text{when } u \to \infty.$$

In view of Theorems 1, 3 and 6, it is therefore sufficient to show that (2.13) holds with $q(u) = u^{-2/\alpha}$ and that Condition B and (2.23) hold.

In order to prove (2.13), let $\omega_i^u(t) = u(\omega_i(qt) - r_i(qt)\omega_i(0))$ and $\Delta_i(t) = u(u + x/u)(1 - r_i(qt))$, and write κ^m for the $(m-1)$-dimensional Hausdorff measure over $\mathbb{R}^m$ [cf. Federer (1969), page 171]. Then we have, by a Taylor expansion,

$$\Pr\left\{\bigcap_{k=1}^{N}\{u(|\bar{\omega}(qt_k)|-u)\le z_k\}\,\middle|\,u(|\bar{\omega}(0)|-u)=x\right\}$$

$$=\int_{\{|\bar{x}|=1\}}\Pr\left\{\bigcap_{k=1}^{N}\{u(|\bar{\omega}(qt_k)|-|\bar{\omega}(0)|)+x\le z_k\}\,\middle|\,\bar{\omega}(0)=\left(u+\frac{x}{u}\right)\bar{x}\right\}$$
$$\times\frac{(2\pi)^{-m/2}(u+x/u)^{m-1}\,d\kappa^m(\bar{x})}{g(u+x/u)\exp\left\{\frac{1}{2}(u+x/u)^2\right\}}$$

$$=\frac{\Gamma\left(\frac{1}{2}m\right)}{2\pi^{m/2}}\int_{\{|\bar{x}|=1\}}\Pr\left\{\bigcap_{k=1}^{N}\left\{\sum_{i=1}^{m}ux_i(\omega_i(qt_k)-\omega_i(0))+x+o(1)\le z_k\right\}\,\middle|\,\bar{\omega}(0)=\left(u+\frac{x}{u}\right)\bar{x}\right\}d\kappa^m(\bar{x})$$

$$=\frac{\Gamma\left(\frac{1}{2}m\right)}{2\pi^{m/2}}\int_{\{|\bar{x}|=1\}}\Pr\left\{\bigcap_{k=1}^{N}\left\{\bar{x}\cdot\bar{\omega}^u(t_k)-\sum_{i=1}^{m}x_i^2\Delta_i(t_k)+x+o(1)\le z_k\right\}\right\}$$
$$\times d\kappa^m(\bar{x}).$$

Now, let $\zeta_1(t),\dots,\zeta_m(t)$ be independent zero-mean Gaussian processes with $\mathrm{Cov}\{\zeta_i(s),\zeta_i(t)\}=C_i(t^\alpha+s^\alpha-|t-s|^\alpha)$. Since, by (4.1), $\Delta_i(t)\to C_it^\alpha$, an application of (3.3) yields that the f.d.d.'s of $\{(u(|\bar{\omega}(qt)|-u)\mid u(|\bar{\omega}(0)|-u)=x)\}_{t>0}$ converge weakly to those of $\{\eta(t)\}_{t>0}$, where

$$\Pr\left\{\bigcap_{k=1}^{N}\{\eta(t_k)\le z_k\}\right\}=\frac{\Gamma\left(\frac{1}{2}m\right)}{2\pi^{m/2}}\int_{\{|\bar{x}|=1\}}\Pr\left\{\bigcap_{k=1}^{N}\left\{\bar{x}\cdot\bar{\zeta}(t_k)-\sum_{i=1}^{m}C_ix_i^2t_k^\alpha+x\le z_k\right\}\right\}d\kappa^m(\bar{x}).\tag{4.2}$$

Now, choose constants $A,B,\varepsilon>0$ such that $At^\alpha\le 1-r_i(t)\le Bt^\alpha$ for $0<t\le\varepsilon$, for $1\le i\le m$, and let $R=\max_{1\le i\le m}r_i(qt)$, $r=\min_{1\le i\le m}r_i(qt)$ and $\bar{\omega}_{\bar{r}}(qt)=(r_1(qt)\omega_1(0),\dots,r_m(qt)\omega_m(0))$. We then have, by the triangle inequality,

$$u(1-R)<|\bar{\omega}(qt)|-R|\bar{\omega}(0)|\le|\bar{\omega}(qt)|-|\bar{\omega}_{\bar{r}}(qt)|\le|\bar{\omega}(qt)-\bar{\omega}_{\bar{r}}(qt)|,$$

for $|\bar{\omega}(qt)|>|\bar{\omega}(0)|>u$. Using symmetry we therefore get, by Boole's inequality,

$$\begin{aligned}\Pr\{|\bar{\omega}(qt)|>u,|\bar{\omega}(0)|>u\}&=2\Pr\{|\bar{\omega}(qt)|>|\bar{\omega}(0)|>u\}\\&\le 2\Pr\{|\bar{\omega}(0)|>u,|\bar{\omega}(qt)-\bar{\omega}_{\bar{r}}(qt)|>u(1-R)\}\\&\le 2(1-G(u))\sum_{i=1}^{m}\Pr\{|\omega_i(qt)-r_i(qt)\omega_i(0)|>m^{-1/2}u(1-R)\}\\&\le 4m(1-G(u))\left\{1-\Phi\left(\frac{(1-R)u}{\sqrt{m}\,(1-r^2)^{1/2}}\right)\right\},\end{aligned}$$

where Φ denotes the standardized Gaussian distribution function. This yields

$$\frac{1}{1-G(u)}\Pr\{|\bar{\omega}(qt)|>u,\,|\bar{\omega}(0)|>u\}\le 4m\big(1-\Phi\big(A(2mB)^{-1/2}t^{\alpha/2}\big)\big)$$

$$\text{for } 0<qt\le\varepsilon,$$

$$\frac{1}{1-G(u)}\Pr\{|\bar{\omega}(qt)|>u,\,|\bar{\omega}(0)|>u\}\le 4m\big(1-\Phi(\lambda m^{-1/2}u)\big)$$

$$\text{for } \varepsilon<qt\le h,$$

where $\lambda=1-\sup_{\varepsilon<s\le h}\max_{1\le i\le m}r_i(s)$, from which it is seen that Condition B holds.

Now, choose a $\delta\in(0,1)$, choose constants $A,\varepsilon>0$ such that $r_i(t)>\frac{1}{2}$ and $1-r_i(t)\le At^{\alpha}$ for $0<t\le\varepsilon$, for $1\le i\le m$, and let $\lambda_0=\min\{\varepsilon^{\alpha/2},1/(8A)\}$. Further, let $u_0=1$ and $\bar{\omega}_{\bar{r}}(qt)=(r_1(qt)^{-1}\omega_1(0),\dots,r_m(qt)^{-1}\omega_m(0))$. Then we have

$$|\bar{\omega}_{\bar{r}}(qt)|-|\bar{\omega}(0)|\le(r^{-1}-1)|\bar{\omega}(0)|\le 2A(qt)^{\alpha}u(1+\delta)\le 4A\frac{1}{u}t^{\alpha}\le\frac{\lambda}{2u},$$

for $0<t^{\alpha/2}<\lambda<\lambda_0$, $u>u_0$ and $|\bar{\omega}(0)|\le u+\delta/u$. Hence we obtain, by the triangle inequality and by arguing as in the verification of Condition B,

$$\Pr\left\{|\bar{\omega}(qt)|-|\bar{\omega}(0)|>\frac{\lambda}{u},|\bar{\omega}(0)|\le u+\frac{\delta}{u},|\bar{\omega}(qt)|>u\right\}$$
$$\le\Pr\left\{|\bar{\omega}(qt)|>u,|\bar{\omega}(qt)-\bar{\omega}_{\bar{r}}(qt)|>\frac{\lambda}{2u}\right\}$$
$$\le 2m(1-G(u))\left\{1-\Phi\left(\frac{\lambda}{4\sqrt{2mA}\,t^{\alpha/2}}\right)\right\}.$$

Arguing as in the verification of (2.23) in Section 3, we conclude that (2.23) holds also in the present context. □

Clearly, if $C_1=\cdots=C_m=C$, then the right-hand side of (4.2) is independent of m. Comparing Theorem 9 and Pickands (1969a), we concluded that $H_{\alpha}^{m}(C,\dots,C)=H_{\alpha}^{1}(C)=C^{1/\alpha}H_{\alpha}$, where H_{α} is the constant introduced by Pickands. It is known that $H_1=1$ and $H_2=\pi^{-1/2}$. Lindgren (1980) calculated $H_2^m(C_1,\dots,C_m)$. We also remark here that Conditions D and D′ of Section 5 hold for the process $|\bar{\omega}(t)|$, if $\max_{1\le i\le m}|r_i(t)|=o(1/\ln(t))$ when $t\to\infty$ [cf. Albin (1987)].

5. Maxima over increasing intervals and asymptotic sample path behavior. In this section we prove a limit law for maxima over increasing intervals and establish the asymptotic Poisson character for ε-upcrossings and local ε-maxima. Since our process under consideration, in general, has infinite upcrossing intensity, we consider Pickands' ε-upcrossings, i.e., clusters of ordinary upcrossings, and local ε-maxima, rather than ordinary upcrossings and local maxima. We also remark here that traditional approaches to Poisson-convergence results for processes with finite upcrossing intensity only apply when, in the notation of Theorem 7, $E = H$, while the present approach applies also when $E \neq H$, i.e., when clustering occurs. Our proofs use ideas of Lindgren, de Maré and Rootzén (1975), Leadbetter and Rootzén (1982) and Leadbetter, Lindgren and Rootzén (1983).

We have to impose two further conditions, first given by Leadbetter and Rootzén (1982).

CONDITION $\mathrm{D}(x_1, \dots, x_r)$. This condition is said to hold if for each fixed $\tau > 0$ there exist constants $a_0 > 0$ and $u_0 < \hat{u}$ and a function $\alpha_u(a, \cdot)$, such that

$$\begin{aligned}&\left|\Pr\{\xi(s_1) \le u_1, \dots, \xi(s_p) \le u_p, \xi(t_1) \le v_1, \dots, \xi(t_{p'}) \le v_{p'}\}\right.\\ &\quad\left. -\Pr\{\xi(s_1) \le u_1, \dots, \xi(s_p) \le u_p\}\Pr\{\xi(t_1) \le v_1, \dots, \xi(t_{p'}) \le v_{p'}\}\right| \le \alpha_u(a, \delta),\end{aligned}$$

for any levels $u_1, \dots, u_p, v_1, \dots, v_{p'}$ belonging to $\{u + x_1 w(u), \dots, u + x_r w(u)\}$ and any points $s_1 < \cdots < s_p < t_1 < \cdots < t_{p'}$ belonging to $\{aqk\colon k \in \mathbb{Z}, 0 \le aqk \le \tau q(u)/(1 - G(u))\}$ with $t_1 - s_p \ge \delta$, for all $\delta > 0$ and $u_0 < u \le \hat{u}$, where

$$\lim_{u \to U} \alpha_u\left(a, \frac{\lambda q(u)}{1 - G(u)}\right) = 0 \quad \text{for all } \lambda > 0, \text{for all fixed } 0 < a < a_0.$$

CONDITION D'. This condition is said to hold if for all fixed $a > 0$,

$$\limsup_{u \to U} \sum_{k=[h/(aq)]+1}^{[\lambda/(1-G(u))]} \Pr\{\xi(aqk) > u \mid \xi(0) > u\} \to 0 \quad \text{when } \lambda \downarrow 0.$$

Condition D is a mixing condition, while Condition D' bounds the probability for clustering of clusters of upcrossings of the level u. Clearly, Condition D' holds if, e.g., $\Pr\{\xi(0) > u, \xi(t) > u\} \le K(1 - G(u))^2$ for $t > h$.

Now, let $\varepsilon > 0$ be an arbitrarily chosen fixed number. We adopt the original definition of ε-upcrossing due to Pickands (1969a).

DEFINITION 2. The process $\xi(t)$ has an ε-upcrossing of the level u at the point t_0 if $\xi(t_0) = u$ and if $\xi(t) < u$ for all $t \in (t_0 - \varepsilon, t_0)$.

THEOREM 10. *Assume that Conditions* $\mathrm{A}(\{0\})$, B, $\mathrm{C}^0(\{0\})$, $\mathrm{D}(0)$ *and* D' *hold, with* F *continuous, and let* $T(u) \sim q(u)/(H(1 - G(u)))$ *and* $x_F = \sup\{x < x_H\colon F(x) < 1\}$. *If there exists a constant* $c \in [0,1]$ *such that* (2.15) *holds, then*

$$\begin{aligned}&\lim_{u \to U} \Pr\left\{\frac{1}{w(u)}(M(T(u)) - u) \le x\right\}\\ &\qquad = \exp\left\{-(1 - F(x))^{1-c}\right\} \quad \textit{for all } x_L < x < x_F.\end{aligned}$$

If, in addition, ξ *possesses continuous sample paths with probability* 1 *and*

$$N_u^x(A) = \#\{t \in \tau T(u)A\colon \xi \textit{ has an } \varepsilon\textit{-upcrossing of } u + xw(u) \textit{ at } t\}$$

for $A \subset (0, \infty)$, $\tau > 0$ *and* $x \in (x_L, x_F)$, *then* N_u^x *converges weakly to a Poisson process on* $(0, \infty)$ *with intensity* $\tau(1 - F(x))^{1-c}$.

PROOF. In view of Theorem 1, Theorem 4.3 of Leadbetter and Rootzén (1982) yields $\lim_{u \to U} \Pr\{M(\lambda T(u)) \le u\} = e^{-\lambda}$ for all $\lambda > 0$: Clearly, their proof only needs that their equations (3.7) and (4.1) hold for a single $h > 0$ [rather than for all $h \in (0, h_0)$] for which Condition D′ holds and for which $\Pr\{M(h) > u\} \sim \frac{1}{2}\Pr\{M(2h) > u\}$. In our setting, the latter condition on h holds, since Conditions B and $C^0(\{0\})$, in view of Condition D′ and Boole's inequality, also hold when h is replaced by $2h$.

Letting $\lambda = (1 - F(x))^{1-c}$, we therefore obtain, by a change of variable,

$$\lim_{u \to U} \Pr\Big\{M\big((1 - F(x))^{1-c}T(u + xw(u))\big) \le u + xw(u)\Big\}$$
$$= \exp\Big\{-(1 - F(x))^{1-c}\Big\}.$$

Here $T(u + xw(u)) \sim (1 - F(x))^{-(1-c)}T(u)$ and the limit law for maxima follows.

Next we observe that, letting $\tilde{u} = u + xw(u)$, by Theorem 1 and continuity of sample paths, and using the facts that $\Pr\{M(h) = u\} = o(\Pr\{M(h) > u\})$ (since F is continuous) and that $\Pr\{A^c \cap B\} = \Pr\{A \cup B\} - \Pr\{A\}$,

$$\frac{\varepsilon}{T(\tilde{u})} \sim \Pr\{M((0, \varepsilon]) < \tilde{u} \le M((\varepsilon, 2\varepsilon))\}$$
$$\le \Pr\left\{N_u^x\left(\left(\frac{\varepsilon}{\tau T}, \frac{2\varepsilon}{\tau T}\right)\right) = 1\right\} \le \Pr\{M((0, \varepsilon)) \ge \tilde{u}\} \sim \frac{\varepsilon}{T(\tilde{u})}$$

when $u \to U$. Thus we have, since $N_u^x(0, \varepsilon/(\tau T)) \le 1$ and $T(u) \sim (1 - F(x))^{1-c}T(\tilde{u})$,

$$E\{N_u^x((c, d])\} = \frac{(d - c)\tau T}{\varepsilon}\Pr\left\{N_u^x\left(\left(0, \frac{\varepsilon}{\tau T}\right)\right) = 1\right\} \tag{5.1}$$
$$\to \tau(1 - F(x))^{1-c}(d - c)$$

for $0 \le c < d < \infty$ [cf. Lemma 12.4.1 in Leadbetter, Lindgren and Rootzén (1983)]. Further, we have

$$0 \leftarrow -\Pr\left\{\bigcup_{p=1}^{k} \left\{M\big((\tau Tc_p, \tau Td_p]\big) = \tilde{u}\right\}\right\}$$
$$\le \Pr\left\{\bigcap_{p=1}^{k} \left\{N_u^x\big((c_p, d_p]\big) = 0\right\}\right\} - \Pr\left\{\bigcap_{p=1}^{k} \left\{M\big((\tau Tc_p, \tau Td_p]\big) \le \tilde{u}\right\}\right\}$$
$$\le \sum_{p=1}^{k} \Pr\left\{M\big((\tau Tc_p, \tau Tc_p + \varepsilon]\big) \ge \tilde{u}\right\} \to 0$$

for $0 \le c_1 < d_1 < \cdots < c_k < d_k < \infty$ [since $(1 - G(u))/q(u) \to 0$, when ξ is continuous]. In view of Theorem 4.7 in Kallenberg (1983), the Poisson result therefore follows if

$$\Pr\left\{ \bigcap_{p=1}^{k} \left\{ M\big((\tau T c_p, \tau T d_p]\big) \le u + x w(u) \right\} \right\} \tag{5.2}$$
$$\to \exp\left\{ - \sum_{p=1}^{k} \tau (1 - F(x))^{1-c} (d_p - c_p) \right\}$$

for $0 \le c_1 < d_1 < \cdots < c_k < d_k < \infty$. Here (5.2) follows from applying the inequality (5.9), proven below, and from choosing suitable λ's in the law $\Pr\{M(\lambda T(u)) \le u\} \to e^{-\lambda}$. □

Theorem 10 proves the Gumbel law

$$\lim_{u \to U} \Pr\left\{ \frac{1-c}{w(u)} (M(T(u)) - u) \le x \right\} = \exp\{-e^{-x}\} \quad \text{for } x \in \mathbb{R}, \text{ when } c < 1,$$

the Fréchet law

$$\lim_{u \to \infty} \Pr\left\{ \frac{1}{u} M(T(u)) \le x \right\} = \exp\{-x^{-(1-c)b}\} \quad \text{for } x > 0$$

and the Weibull law

$$\lim_{u \to \infty} \Pr\left\{ u\left(\hat{u} - M\left(T\left(\hat{u} - \frac{1}{u} \right) \right) \right) \le x \right\} = 1 - \exp\{-x^{(1-c)b}\} \quad \text{for } x > 0$$

when G is type I-, type II- and type III-attracted, respectively.

If G belongs to a domain of attraction and if Conditions A, B and C hold, then Condition C^0 and (2.15) hold (cf. Theorems 2 and 5). See the discussion following Theorem 7 for conditions in terms of requirements on μ.

The invoked Theorem 4.3 of Leadbetter and Rootzén (1982) requires Conditions C^0, D and D′ and that one can handle probabilities of the type occurring in Condition B (in order to verify Condition D′). Hence, the extra work introduced in order to get rid of their requirement that an asymptotic expression $\Psi(u)$ for $\Pr\{M(h) > u\}$ is known, virtually adds up to little more than the verification of Condition A.

O'Brien (1974, 1987), Mittal and Ylvisaker (1975), McCormick (1980), Leadbetter (1983), Hsing (1987) and Hsing, Hüsler and Leadbetter (1988) studied possible types of limit results in the sequence case when Condition D or D′ is violated.

Also our definition of local ε-maximum differs slightly from the one used by Lindgren, de Maré and Rootzén (1975) and Leadbetter, Lindgren and Rootzén (1983).

DEFINITION 3. The process $\xi(t)$ has a local ε-maximum at the point t_0 if $\xi(t) < \xi(t_0)$ for all $t \in (t_0 - \varepsilon, t_0)$ and if $\xi(t) \le \xi(t_0)$ for all $t \in (t_0, t_0 + \varepsilon)$.

Theorem 11. *Assume that Conditions* $A(\{0\})$, B, $C^0((x_L, x_F))$, $D(x_1, \dots, x_r)$ *and* D′ *hold, for all families of constants* $x_L < x_1, \dots, x_r < x_F$, *where* F: $(0, x_H) \to \mathbb{R}$ *is a continuous proper d.f. and* $x_F = \sup\{x < x_H\colon F(x) < 1\}$. *Further, let* $t_1, t_2, \dots$ *be the local* ε*-maxima for* $\xi(t)$ *and let* $T(u) \sim q(u)/(H(1 - G(u)))$. *If* ξ *possesses continuous sample paths with probability* 1, *if there exists a constant* $c \in [0,1]$ *such that* (2.15) *holds and if* $S(x) = (1 - F(x))^{1-c}$, *then the random collection*

$$\left\{\left(\frac{t_i}{T(u)}, \frac{\xi(t_i) - u}{w(u)}\right), i = 1, 2, \dots\right\}$$

converges weakly to a Poisson process on $(0,\infty) \times (x_L, x_F)$, *with intensity measure equal to the product of Lebesgue measure and the measure defined by the increasing function* $-S$.

Proof. Let $s_{1,x} = \inf\{s \in I\colon (\xi(sT) - u)/w \ge x\}$, $s_{i,x} = \inf\{s \ge s_{i-1,x} + \varepsilon\colon (\xi(sT) - u)/w \ge x\}$, for $i = 2, 3, \dots$ and $M_u^x(I) = \max\{i \in \mathbb{Z}\colon i = 0 \text{ or } s_{i,x} \in I\}$, for $I \subset (0,\infty)$. Further, let

$$N_{x,u}^n(A) = \#\left\{t \in TA\colon \xi \text{ has an } \frac{\varepsilon}{n}\text{-upcrossing of } u + xw \text{ at } t\right\} \qquad \text{for } A \subset (0,\infty),$$

$$N_u(B) = \#\left\{i \in \mathbb{Z}\colon \left(\frac{t_i}{T}, \frac{\xi(t_i) - u}{w}\right) \in B\right\} \quad \text{for } B \subset (0,\infty) \times (x_L, x_F)$$

and let $N_{x,u} \equiv N_{x,u}^1$. In view of Kallenberg (1983), it suffices to prove that

$$E\{N_u((c,d]) \times (\gamma,\delta])\} \to (d - c)(S(\gamma) - S(\delta)) \tag{5.3}$$

for $0 \le c < d < \infty$, $0 \le \gamma < \delta < \infty$, and that

$$\Pr\left\{N_u\left(\bigcup_{i=1}^{k} (c_i, d_i] \times \bigcup_{j=1}^{J(i)} \left(\gamma_j^i, \delta_j^i\right]\right) = 0\right\} \to \prod_{i=1}^{k} \prod_{j=1}^{J(i)} \exp\left\{-(d_i - c_i)\left(S\left(\gamma_j^i\right) - S\left(\delta_j^i\right)\right)\right\} \tag{5.4}$$

for $0 \le c_1 < d_1 \cdots < c_k < d_k < \infty$, $0 \le \gamma_1^1 < \delta_1^1 < \cdots < \gamma_{J(1)}^1 < \delta_{J(1)}^1 < \infty, \dots$ and $0 \le \gamma_1^k < \delta_1^k < \cdots < \gamma_{J(k)}^k < \delta_{J(k)}^k < \infty$.

Now we have, by (5.1), $E\{N_{x,u}^n((0,1])\} \sim E\{N_{x,u}((0,1])\}$ and in view of the readily established fact that $M_u^x((0,1]) \ge N_{x,u}((0,1])$, we obtain

$$\begin{aligned} 0 &\le E\{M_u^x((0,1]) - N_{x,u}((0,1])\} \\ &\sim E\{M_u^x((0,1]) - N_{x,u}^n((0,1])\} \\ &\le \sum_{k=0}^{[T/\varepsilon]+1} \Pr\left\{M\left(\left(k\varepsilon, \left(k + \frac{1}{n}\right)\varepsilon\right]\right) \ge u + xw\right\} \sim \frac{1}{n} S(x) \quad \text{when } u \to U. \end{aligned}$$

Letting $n \to \infty$, we deduce $E\{M_u^x((0,1]) - N_{x,u}((0,1])\} \to 0$ and, more generally,

$$\lim_{u\to U} E\{M_u^x((c,d]) - N_{x,u}((c,d])\} = 0 \quad \text{for all } 0 \le c < d < \infty. \tag{5.5}$$

Clearly, $N_u((c,d] \times [x,\infty)) \ge N_{x,u}((c,d]) - 1$, with strict inequality when $M((dT, dT+\varepsilon]) \le u + xw$. Since $M_u^x((c,d]) \ge N_u((c,d] \times [x,\infty))$, we obtain, by (5.5),

$$\begin{aligned} 0 &= \limsup_{u\to U} E\{M_u^x((c,d]) - N_{x,u}((c,d])\} \\ &\ge \limsup_{u\to U} E\{N_u((c,d] \times [x,\infty)) - N_{x,u}((c,d])\} \\ &\ge \liminf_{u\to U} E\{N_u((c,d] \times [x,\infty)) - N_{x,u}((c,d])\} \\ &\ge -\limsup_{u\to U} \Pr\{M((dT, dt+\varepsilon]) > u + xw\} = 0. \end{aligned} \tag{5.6}$$

Combining (5.6) and (5.1), (5.3) readily follows (since S is continuous).

In order to prove (5.4), choose constants $x_L < x_1 < \cdots < x_r < x_F$, let N_1 be a Poisson process on $(0,\infty)$, with "points" $\{\sigma_j\}_{j=1}^{\infty}$ and intensity $S(x_1)$ and let $\{\beta_j\}_{j=1}^{\infty}$ be $\{1,\dots,r\}$-valued i.i.d. variables, independent of N_1, with $\Pr\{\beta_j = s\} = (S(x_s) - S(x_{s+1}))/S(x_1)$ for $s = 1,\dots,r$, where $x_{r+1} = x_F^-$. Further, define $N_i(A) = \#\{\sigma_j \in A : \beta_j \ge i\}$ for $A \subset (0,\infty)$, for $i = 2,\dots,r$, and let $N_{\bar{x}}(B) = \sum_{i=1}^r N_i(B \cap \{x = x_i\})$ and $N_{\bar{x},u}(B) = \sum_{i=1}^r N_{x_i,u}(B \cap \{x = x_i\})$ for $B \subset (0,\infty) \times (x_L, x_F)$. In order to prove

$$\begin{aligned} &\Pr\left\{N_{\bar{x},u}\left(\bigcup_{i=1}^{k}(c_i,d_i] \times \bigcup_{j=1}^{J(i)}\left(\gamma_j^i, \delta_j^i\right]\right) = 0\right\} \\ &\quad \to \Pr\left\{N_{\bar{x}}\left(\bigcup_{i=1}^{k}(c_i,d_i] \times \bigcup_{j=1}^{J(i)}\left(\gamma_j^i, \delta_j^i\right]\right) = 0\right\}, \end{aligned} \tag{5.7}$$

we can assume that for each $i \in \{1,\dots,k\}$ there exists a smallest $I(i) \in \{1,\dots,r\}$ such that $x_{I(i)} \in (\delta_j^i, \gamma_j^i]$ for some $j \in \{1,\dots,J(i)\}$. Then, by routine calculations, the right-hand side of (5.7) is equal to $\prod_{i=1}^k \exp\{-(d_i - c_i)S(x_{I(i)})\}$. Further, we have, for $1 \le i < j \le r$, by Boole's inequality and by Theorems 1 and 10,

$$\begin{aligned} &\limsup_{u\to U} \Pr\left\{N_{x_j,u}((c,d]) > N_{x_i,u}((c,d])\right\} \\ &\quad \le \limsup_{u\to U} \Pr\left\{\bigcup_{k=0}^{K-1}\left\{M\left(\left(\frac{d-c}{K}kT, \frac{d-c}{K}kT + \varepsilon\right]\right) \ge u + x_i w\right\}\right\} \\ &\quad\quad + \limsup_{u\to U} \Pr\left\{\bigcup_{k=0}^{K-1}\left\{N_{x_j,u}\left(\left(\frac{d-c}{K}kT, \frac{d-c}{K}(k+1)T\right]\right) \ge 2\right\}\right\} \\ &\quad \le K\left(1 - \left(1 + S(x_j)\frac{d-c}{K}\right)\exp\left\{-S(x_j)\frac{d-c}{K}\right\}\right) \to 0 \end{aligned}$$

when $K \to \infty$ [cf. Lemma 9.3.1 in Leadbetter, Lindgren and Rootzén (1983)]. Hence (5.7) holds iff

$$\Pr\left\{ \bigcap_{i=1}^{k} \left\{ N_{x_{I(i)},u}((c_i, d_i]) = 0 \right\} \right\} \to \prod_{i=1}^{k} \exp\left\{ -(d_i - c_i) S(x_{I(i)}) \right\}.$$

Arguing as in the proof of Theorem 10, we deduce that it is sufficient to show

$$\text{(5.8)} \quad \Pr\left\{ \bigcap_{i=1}^{k} \left\{ M((Tc_i, Td_i]) \le u + x_{I(i)} w \right\} \right\} \to \prod_{i=1}^{k} \exp\left\{ -(d_i - c_i) S(x_{I(i)}) \right\}.$$

Now, it is relatively straightforward to establish the inequality

$$\begin{aligned} &\limsup_{u \to U} \left| \Pr\left\{ \bigcap_{i=1}^{k} \left\{ M((Tc_i, Td_i]) \le u + x_{I(i)} w \right\} \right\} \right. \\ &\qquad \left. - \prod_{i=1}^{k} \Pr\left\{ M((Tc_i, Td_i]) \le u + x_{I(i)} w \right\} \right| \\ &\le \limsup_{a \downarrow 0} \limsup_{u \to U} \frac{2T}{h} \sum_{i=1}^{k} (d_i - c_i) \\ &\qquad \times \Pr\left\{ M(h) > u + x_{I(i)} w, \max_{0 \le aqk \le h} \xi(aqk) \le u + x_{I(i)} w \right\} \\ &\quad + \limsup_{a \downarrow 0} \limsup_{u \to U} 2 \sum_{i=1}^{k} \Pr\left\{ M(h) > u + x_{I(i)} w \right\} \\ &\quad + \limsup_{a \downarrow 0} \limsup_{u \to U} \sum_{i=1}^{k-1} \alpha_u\left(a, (c_{i+1} - d_i) \frac{q(u)}{H(1 - G(u))} \right), \end{aligned} \tag{5.9}$$

where, in view of Conditions $C^0((x_L, x_F))$ and $D(x_1, \dots, x_r)$, the right-hand side equals zero. We conclude that in order to prove (5.8), it suffices to prove

$$\prod_{i=1}^{k} \Pr\left\{ M((Tc_i, Td_i]) \le u + x_{I(i)} w \right\} \to \prod_{i=1}^{k} \exp\left\{ -(d_i - c_i) S(x_{I(i)}) \right\}.$$

Choosing suitable λ's in the law $\Pr\{M(\lambda T(u)) \le u\} \to e^{-\lambda}$, as in the proof of Theorem 10, we therefore deduce that (5.8) holds and hence (5.7) holds. Since, by (5.1), $E\{N_{\bar{x},u}((c, d] \times (\gamma, \delta])\} \to E\{N_{\bar{x}}((c, d] \times (\gamma, \delta])\}$, it follows that $N_{\bar{x},u}$ converges to $N_{\bar{x}}$. Hence, by routine calculations, the probability

$$\Pr\left\{ \bigcap_{i=1}^{k} \bigcap_{j=1}^{J(i)} \left\{ N_{\gamma_j^i, u}((c_i, d_i]) = N_{\delta_j^i, u}((c_i, d_i]) \right\} \right\}$$

converges to the right-hand side of (5.4). Since the arguments leading to (5.6) also yield $\Pr\{N_u((c, d] \times [x, \infty)) \le N_{x,u}((c, d])\} \to 0$ and since S is continuous, we readily conclude that (5.4) holds. □

Results of the kind established in Theorem 11 have been given for sequences by Qualls (1969), Pickands (1971), Resnick (1975) and Adler (1978).

Recall that we analyzed Condition $C^0(x_L, x_F)$) and (2.15) in Theorem 5.

Acknowledgments. The present work is a development of a part of my thesis [Albin (1987)]. I want to express my sincere thanks to Georg Lindgren, my supervisor, for his infinite help, to Holger Rootzén for much valuable and constructive advice and to Gunnar Blom and Ross Leadbetter.

REFERENCES

ACZÉL, J. (1966). *Lectures on Functional Equations and Their Applications.* Academic, New York.

ADLER, R. J. (1978). Weak convergence results for extremal processes generated by dependent random variables. *Ann. Probab.* **6** 660–667.

ALBIN, J. M. P. (1987). On extremal theory for nondifferentiable stationary processes. Thesis, Dept. Mathematical Statistics, Univ. Lund and Lund Inst. Technology.

ARONOWICH, M. and ADLER, R. J. (1985). Behaviour of χ^2 processes at extrema. *Adv. in Appl. Probab.* **17** 280–297.

ARONOWICH, M. and ADLER, R. J. (1986). Extrema and level crossings of χ^2 processes. *Adv. in Appl. Probab.* **18** 901–920.

BELAYEV, YU. K. (1966). On the number of intersections of a level by a Gaussian stochastic process. I. *Theory Probab. Appl.* **11** 106–113.

BELAYEV, YU. K. (1967a). On the number of level crossings by a Gaussian stochastic process. II. *Theory Probab. Appl.* **12** 392–404.

BELAYEV, YU. K. (1967b). Limit theorem for the number of high level crossings by a stationary Gaussian process. *Soviet Math. Dokl.* **8** 449–450.

BELAYEV, YU. K. (1968). On the number of exits across the boundary of a region by a vector stochastic process. *Theory Probab. Appl.* **13** 320–324.

BERMAN, S. M. (1971). Maxima and high level excursions of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **160** 65–85.

BERMAN, S. M. (1982). Sojourns and extremes of stationary processes. *Ann. Probab.* **10** 1–46.

BERMAN, S. M. (1983). Sojourns of stationary processes in rare sets. *Ann. Probab.* **11** 847–866.

BERMAN, S. M. (1984). Sojourns of vector Gaussian processes inside and outside spheres. *Z. Wahrsch. verw. Gebiete* **66** 529–542.

BERMAN, S. M. (1985). Limit theorems for sojourns of stochastic processes. *Probability in Banach Spaces V. Lecture Notes in Math.* **1153** 40–71. Springer, New York.

BERMAN, S. M. (1988). Extreme sojourns of diffusion processes. *Ann. Probab.* **16** 361–374.

BICKEL, P. J. and ROSENBLATT, M. (1973). Two-dimensional random fields. In *Multivariate Analysis III* (P. R. Krishnaiah, ed.) 3–15. Academic, New York.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

BILLINGSLEY, P. (1979). *Probability and Measure.* Wiley, New York.

CRAMÉR, H. (1965). A limit theorem for the maximum values of certain stochastic processes. *Theory Probab. Appl.* **10** 126–128.

CRAMÉR, H. (1966). On the intersections between the trajectories of a normal stationary stochastic process and a high level. *Ark. Mat.* **6** 337–349.

CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes.* Wiley, New York.

CUZICK, J. M. (1981). Boundary crossing probabilities for stationary Gaussian processes and Brownian motion. *Trans. Amer. Math. Soc.* **263** 469–492.

DE HAAN, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes. Math. Centre Tracts* **32.** Mathematische Centrum, Amsterdam.

FEDERER, H. (1969). *Geometric Measure Theory*. Springer, New York.

GNEDENKO, B. V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. of Math.* (2) **44** 423–453.

HSING, T. (1987). On the characterization of certain point processes. *Stochastic Process. Appl.* **26** 297–316.

HSING, T., HÜSLER, J. and LEADBETTER, M. R. (1988). On the exceedance point process for a stationary sequence. *Probab. Theory Related Fields* **78** 97–112.

KAC, M. and SLEPIAN, D. (1959). Large excursions of Gaussian processes. *Ann. Math. Statist.* **30** 1215–1228.

KALLENBERG, O. (1983). *Random Measures*, 3rd ed. Academic, New York.

LEADBETTER, M. R. (1983). Extremes and local dependence in stationary sequences. *Z. Wahrsch. verw. Gebiete* **65** 291–306.

LEADBETTER, M. R. and ROOTZÉN, H. (1982). Extreme value theory for continuous parameter stationary processes. *Z. Wahrsch. verw. Gebiete* **60** 1–20.

LEADBETTER, M. R. and ROOTZÉN, H. (1988). Extremal theory for stochastic processes. *Ann. Probab.* **16** 431–478.

LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.

LINDGREN, G. (1980). Extreme values and crossings for the χ^2-process and other functions of multidimensional Gaussian processes, with reliability applications. *Adv. in Appl. Probab.* **12** 746–774.

LINDGREN, G. (1984). Extremal ranks and transformation of variables for extremes of functions of multivariate Gaussian processes. *Stochastic Process. Appl.* **17** 285–312.

LINDGREN, G. (1989). Slepian models for χ^2-processes with dependent components with application to envelope upcrossings. *J. Appl. Probab.* **26** 36–49.

LINDGREN, G., DE MARÉ, J. and ROOTZÉN, H. (1975). Weak convergence of high level crossings and maxima for one or more Gaussian processes. *Ann. Probab.* **3** 961–978.

MARCUS, M. B. (1977). Level crossings of a stochastic process with absolutely continuous sample paths. *Ann. Probab.* **5** 52–71.

MCCORMICK, W. P. (1980). Weak convergence for the maxima of stationary Gaussian processes using random normalization. *Ann. Probab.* **8** 483–497.

MITTAL, Y. and YLVISAKER, N. D. (1975). Limit distributions for the maxima of stationary Gaussian processes. *Stochastic Process. Appl.* **3** 1–18.

O'BRIEN, G. L. (1974). The maximum term of uniformly mixing stationary processes. *Z. Wahrsch. verw. Gebiete* **30** 57–63.

O'BRIEN, G. L. (1987). Extreme values for stationary and Markov sequences. *Ann. Probab.* **15** 281–291.

PICKANDS, J., III (1969a). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51–73.

PICKANDS, J., III (1969b). Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.* **145** 75–86.

PICKANDS, J., III (1971). The two-dimensional Poisson process and extremal processes. *J. Appl. Probab.* **8** 745–756.

QUALLS, C. (1968). On a limit distribution of high level crossings of a stationary Gaussian process. *Ann. Math. Statist.* **39** 2108–2113.

QUALLS, C. (1969). On the joint distribution of crossings of high multiple levels by a stationary Gaussian process. *Ark. Mat.* **8** 129–137.

QUALLS, C. and WATANABE, H. (1972). Asymptotic properties of Gaussian processes. *Ann. Math. Statist.* **43** 580–596.

RESNICK, S. I. (1975). Weak convergence to extremal processes. *Ann. Probab.* **3** 951–960.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

RICE, S. O. (1939). The distribution of the maxima of a random curve. *Amer. J. Math.* **61** 409–416.

RICE, S. O. (1945). Mathematical analysis of random noise. *Bell System Tech. J.* **24** 46–156.

SHARPE, K. (1978). Some properties of the crossings process generated by a stationary χ^2 process. *Adv. in Appl. Probab.* **10** 373–391.

SLEPIAN, D. (1961). First passage time for a particular Gaussian process. *Ann. Math. Statist.* **32** 610–612.

SLEPIAN, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.* **41** 463–501.

VOLKONSKII, V. A. and ROZANOV, YU. A. (1959). Some limit theorems for random functions. I. *Theory Probab. Appl.* **4** 178–197.

VOLKONSKII, V. A. and ROZANOV, YU. A. (1961). Some limit theorems for random functions. II. *Theory Probab. Appl.* **6** 186–198.

INSTITUTIONEN FÖR MATEMATISK STATISTIK
LUNDS UNIVERSITET
BOX 118
S-221 00 LUND
SWEDEN