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THE CRITICAL CONTACT PROCESS DIES OUT

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By producing a finite-box criterion for survival in a slab, we show that the critical contact process dies out and that, in two and more dimensions, the critical parameter value coincides with the limit of slab thresholds. Using the techniques developed in this paper, one may obtain the complete convergence and shape theorems for contact processes in all dimensions and for all infection rates. Our results apply also to the discrete-time analogue of the contact process, viz., oriented percolation.

1. Introduction. This paper is concerned with the contact process in all dimensions; our results apply also to oriented percolation, which is the discrete-time analogue of the contact process.

The contact process was introduced by Harris (1974). It is a Markov process whose state space is the set of all subsets of $\mathbb{Z}^d$, and it may be thought of as a model for the spread of infection in a d-dimensional orchard. Here is a heuristic description. At each instant of (continuous) time, each site on the d-dimensional lattice $\mathbb{Z}^d$ is in one of two states: *infected* or *healthy*. (Sometimes the terms *occupied* and *vacant* are used, depending on the physical interpretation of the model.) Infected sites recover (become healthy) at rate 1 and a healthy site becomes infected at a rate proportional to the number of its infected neighbours on the lattice. The model is parametrized by the proportionality constant, usually denoted by λ.

We shall discuss briefly the history of the problems solved in this paper. Since there are several accessible expositions on the subject of contact processes, we shall discuss only those aspects of the subject which are relevant here. For more general surveys of the theory we refer the reader to the papers by Durrett (1984) and Griffeath (1981) and to the books of Durrett (1988) and Liggett (1985).

In a sense, the contact process is a variant of a continuous-time branching process. Let ξ_t be the number of individuals alive at time t in such a branching process in which the offspring distribution has mass function $f(0) = 1/(1 + 2d\lambda)$ and $f(2) = 2d\lambda/(1 + 2d\lambda)$. Since the mean of this offspring distribution is equal to $4d\lambda/(1 + 2d\lambda)$, it follows from well known (and easily established) results that $\xi_t \to 0$ with probability 1 if and only if $4d\lambda \leq 1 + 2d\lambda$, that is, if and only if $\lambda \leq 1/(2d)$. Thus in this model there is a

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critical value λ_c (namely $\lambda_c = 1/(2d)$) of λ so that the system dies out with probability 1 if λ is below λ_c and survives with positive probability if λ is above this threshold. Moreover, in this model *the system dies out when* $\lambda = \lambda_c$.

The contact process is similar to the model just described, except that individuals are constrained to live on a lattice ($\mathbb{Z}^d$ with $d \geq 1$). When an individual reproduces, it gives birth onto a site chosen randomly from among its $2d$ nearest neighbours on the lattice. If an individual tries to give birth onto a site which is already occupied, the birth is disallowed. The spatial structure thus imposed makes it harder for the system to survive. However, the system has a certain monotonicity property which ensures that if there is a positive probability of survival for a given value of the parameter λ, then the same is true for any larger parameter value. This monotonicity makes it possible to define a critical value $\lambda_c = \lambda_c(d)$ such that, if $\lambda < \lambda_c$, the contact process dies out with probability 1, whereas, if $\lambda > \lambda_c$, there is a positive probability of survival. A comparison with the branching process described above shows that the critical value is strictly positive and in fact provides the lower bound $\lambda_c(d) \geq 1/(2d)$. It is harder, but nevertheless possible, to prove that λ_c is finite [Harris (1974), Griffeath (1981), et al.]. However, these crude arguments give no information about the behaviour of the system *at* its critical value and indeed the question of the survival or nonsurvival of the critical contact process has remained unanswered until now. The main result of this paper is a proof that, as generally believed, the d-dimensional contact process dies out when $\lambda = \lambda_c(d)$, for all values of d.

Nobody has yet established rigorously the numerical value of λ_c. Some rigorous upper and lower bounds have been established and heuristic arguments suggest that for the case $d = 1$, λ_c is about 1.65. Holley and Liggett (1978) proved that $P_\lambda(\xi^0 \text{ survives}) \geq \frac{1}{2}$ when $\lambda = 2$ and $d = 1$; it then follows from the main result of the present paper that $\lambda_c(1) < 2$, a result established earlier by Durrett but unpublished. For the best known upper and lower bounds on $\lambda_c(d)$, see Durrett (1988) and Liggett (1985).

In his 1978 paper, Harris introduced the so-called graphical representation of the contact process. This representation, which we discuss at more length in Section 3, enables one to think of the contact process as a type of percolation process. This observation has been exploited fruitfully by many authors; see Durrett (1988), Griffeath (1979) and Liggett (1985) for more details. In this paper, we make heavy use of Harris's graphical representation. By such means, we are able to adapt proofs of similar results for (unoriented, discrete) percolation to the setting of the contact process (oriented with continuous time). Such adaptations to oriented percolation are somewhat easier, since one does not have to worry about the fact that the time parameter is continuously varying.

Suppose for the moment that $d \geq 2$. In proving results about a contact process with $\lambda > \lambda_c$, it has been customary and convenient to assume the extra hypothesis that $\lambda > \lambda_c^K$ for some K, where λ_c^K is the critical value of λ for the process when restricted to the two-dimensional space-time slab $[-K, K]^{d-1} \times \mathbb{Z} \times [0, \infty)$ (here and later, the component $[-K, K]^{d-1} \times \mathbb{Z}$

($\subseteq \mathbb{Z}^d$) refers to the space in which the process lives and $[0, \infty)$ to the time variable). For example, Durrett and Griffeath (1982) assume that $\lambda > \lambda_c^0$ ($= \lambda_c(1)$), and Durrett and Schonmann (1987) assume effectively that $\lambda > \lambda_c^\infty$ ($= \lim_{K \to \infty} \lambda_c^K$); actually the latter authors assume the corresponding hypothesis for oriented percolation. Our second main result is that, as expected, $\lambda_c = \lambda_c^\infty$, which is to say that, whenever the contact process survives with positive probability in the whole of $\mathbb{Z}^d \times [0, \infty)$, then it does so also in a sufficiently deep slab (see Theorem 2). The analogous result for unoriented percolation has recently been proved by Barsky, Grimmett and Newman (1989) and Grimmett and Marstrand (1989). Our technique is an adaptation to the directed setting of that of the first of these two papers.

The contact process is attractive, which is to say that the presence of extra particles (infected sites) makes it no harder for the process to survive, i.e., for infection to persist forever. It follows easily from this fact that the Markov property that, if the initial state of the process is $\mathbb{Z}^d$ (all sites infected), the process converges weakly to a certain measure ν on the state space, called the upper invariant measure.

The contact process has a property of reversibility (sometimes called self-duality) in that, if time is run backwards and all arrows, representing infections of one site by another, are reversed, then the new graphical representation has the same law as the original. It is an easy consequence of this fact that ν is nontrivial (i.e., $\nu \neq \delta_\varnothing$, the point mass on the empty set) if and only if the contact process *with a single initial infected site* survives forever with positive probability. It is known [Durrett (1980)] that, for $d = 1$, if the system is started with the infection on a subset A of $\mathbb{Z}^d$, then it converges weakly to a convex combination of ν and $\delta_\varnothing$, the weight assigned to ν being the probability of survival starting from A. This result is known as complete convergence. The analogous result for $d \geq 2$ was proved for $\lambda > \lambda_c^0$ by Durrett and Griffeath (1982) (it is clearly true for λ below λ_c) and by Durrett and Schonmann (1987) for $\lambda > \lambda_c^\infty$ (actually the latter result was proved for the discrete-time process). The construction given in this paper may be combined with the techniques of those two papers to prove complete convergence for all values of λ. We give a rigorous statement of this in (4).

In one dimension, Harris's graphical representation provides a coupling of the contact process started with a single infected site and the process started with all sites infected in which the two systems always agree on the interval between the extreme infected sites in the former system. (This is a simple consequence of the fact that infection can move only one space unit at a time, together with certain two-dimensional geometric constraints.) In his 1980 paper, Durrett showed by means of a subadditive ergodic theorem which has been generalized and appears in Liggett [(1985), page 277] that, conditioned on survival, this interval grows linearly almost surely. The analogue of this statement in higher dimensions is that the two systems may be coupled in such a way that, conditioned on the survival of the former system, almost surely they agree on a region which is asymptotically convex and whose diameter increases approximately linearly in time. This result is known as the shape theorem and was proved in the same papers and under the same restrictions as the complete convergence theorem. Again, our construction may be combined with the techniques of those papers to yield the shape theorem for all values of λ and in all dimensions strictly greater than 1. See (5) for a precise statement.

Our basic strategy is to adapt the arguments of Barsky, Grimmett and Newman (1989) to the contact process. Under the hypothesis that the process survives with positive probability from a single initial infected site, we shall show the existence of a large bounded region of space-time which contains, with large probability, many chains of infection of certain sorts. By a certain progressive positioning of copies of this region, we shall build unbounded chains of infection within a sufficiently deep slab of space-time. Thus our argument is a type of dynamic block technique.

More primitive but similar techniques have been used successfully in the past. For example, Russo (1978, 1981), Seymour and Welsh (1978) and Kesten (1981) used related approaches in studying ordinary percolation; in using path-intersection arguments, their constructions were often two-dimensional and in addition were static rather than dynamic in the sense that the blocks or regions were fixed in advance in space rather than having their positions determined by the (random) past history of the construction. Such arguments inspired Durrett and Griffeath (1983) in their work on the one-dimensional contact process. It is largely the *dynamic* aspect of our approach which enables us to obtain results in higher dimensions. Whereas in two-dimensional space-time one may use path-intersection properties of the plane, such properties are not so readily available in higher dimensions.

2. Statement of results. In this section we give a formal statement of our results. We outline the proofs in Section 3 and give details in Section 4.

We study the contact process in d (≥ 1) dimensions with infection rate λ. Our results apply equally to oriented percolation, the discrete-time analogue of the contact process; for this process, the proofs are essentially the same, but slightly easier. Our principal observation is that, if the process continues forever having started from a single infected site, then there exists within it an infinite discrete-time process growing in a slab of sufficiently large depth. As consequences of this observation one obtains resolutions of four of the main problems of the area: (i) if $\lambda = \lambda_c(d)$, the critical infection rate, then the process dies out, (ii) λ_c equals the limit of the critical values of slabs, (iii) the complete convergence theorem is valid for all values of λ and (iv) the shape result is valid for all values of λ.

Let ξ denote the contact process on $\mathbb{Z}^d \times [0, \infty)$. The first component $\mathbb{Z}^d$ is the spatial component of the process and $[0, \infty)$ is the time component; sometimes we shall suppress reference to the time component, referring to the contact process on $\mathbb{Z}^d$. Thus ξ_t^A is the set of sites infected at time t when the infection is initially on A ($\subseteq \mathbb{Z}^d$). Let P_λ denote the associated measure when the total rate of infection emanating from each site is $2d\lambda$, and the death rate is 1. We say that ξ^A *survives* if $\xi_t^A \neq \varnothing$ for all t and ξ^A *dies out* otherwise.

Let λ_c denote the critical infection rate: $\lambda_c = \inf\{\lambda > 0\colon P_\lambda(\xi^0 \text{ survives}) > 0\}$. If $K \in \mathbb{N}$ $(= \{1, 2, \dots\})$, define *slab thresholds* λ_c^K as follows. Consider the contact process ${}^K\xi$ on the two-dimensional slab $[-K, K]^{d-1} \times \mathbb{Z} \times [0, \infty)$ with free boundary conditions, i.e., if x is in the boundary of the slab, attempts by x to infect sites outside the slab are overlooked. Let $\lambda_c^K = \inf\{\lambda > 0\colon P_\lambda({}^K\xi^0 \text{ survives}) > 0\}$. As K increases, λ_c^K decreases. Define $\lambda_c^\infty = \lim_{K \to \infty} \lambda_c^K$. Our main results are:

(1) THEOREM. *For $d \geq 1$, $P_{\lambda_c}(\xi^0$ survives$) = 0$.*

(2) THEOREM. *For $d \geq 2$, $\lambda_c^\infty = \lambda_c$.*

We prove these results at one stroke by showing that if $P_\lambda(\xi^0 \text{ survives}) > 0$, then there exist a finite disc D $(\subset \mathbb{Z}^d)$, a number $\delta > 0$ and an integer L such that

$$P_{\lambda-\delta}\big(\xi^D \text{ survives in } [-2L, 2L]^{d-1} \times \mathbb{Z} \times [0, \infty)\big) > 0. \tag{3}$$

It is not difficult to see why this suffices. It implies that $\lambda - \delta \geq \lambda_c^{2L}$, and hence $\lambda > \lambda_c^{2L} \geq \lambda_c^\infty \geq \lambda_c$ whenever $P_\lambda(\xi^0 \text{ survives}) > 0$. Therefore $P_{\lambda_c}(\xi^0 \text{ survives}) = 0$ and $\lambda_c^\infty = \lambda_c$.

Our principal observation, referred to above, is obtained by an adaptation of the argument of Barsky, Grimmett and Newman (1989) [see also Grimmett and Marstrand (1989)] who have proved similar results for percolation. The full limitations of the technique are not yet clear; certainly it may be adapted to a certain class of attractive spin systems including all additive systems [Bezuidenhout and Gray (1990)].

The argument of this paper may be used without substantial difficulty to settle two related matters. First, let ν denote the upper invariant measure, i.e., the weak limit of the distribution of $\xi_t^{\mathbb{Z}^d}$ as $t \to \infty$, and let $\delta_\varnothing$ be the probability measure which puts mass one on the empty set.

(4) COMPLETE CONVERGENCE THEOREM. *Let $A \subset \mathbb{Z}^d$. Then*

$$\xi_t^A \Rightarrow \nu \cdot P_\lambda(\xi^A \text{ survives}) + \delta_\varnothing \cdot P_\lambda(\xi^A \text{ dies out}),$$

where $\Rightarrow$ refers to weak convergence as $t \to \infty$.

Next we turn to the question of shape. For $x \in \mathbb{Z}^d$, we write $t(x) = \inf\{t\colon x \in \xi_t^0\}$ for the first infection time of x and

$$H_t = \{y \in \mathbb{R}^d\colon \exists\, x \in \mathbb{Z}^d \text{ with } \|x - y\| \leq \tfrac{1}{2} \text{ and } t(x) \leq t\},$$

where $\|\cdot\|$ denotes the L^∞ norm on $\mathbb{R}^d$. We may couple together contact processes with all possible starting sets and measures in the usual way, so that $\xi_t^0 \subseteq \xi_t^{\mathbb{Z}^d}$ for all t. We write K_t for the coupled region of $\mathbb{R}^d$,

$$K_t = \{y \in \mathbb{R}^d\colon \exists\, x \in \mathbb{Z}^d \text{ with } \|x - y\| \leq \tfrac{1}{2} \text{ and } \xi_t^0(x) = \xi_t^{\mathbb{Z}^d}(x)\}.$$

(5) SHAPE THEOREM. *There exists a convex subset U of $\mathbb{R}^d$ such that, for any $\varepsilon > 0$,*

$$(1-\varepsilon)U \subset \frac{1}{t}H_t \subset (1+\varepsilon)U \quad \textit{eventually},$$

almost surely on the event that ξ^0 survives. Furthermore,

$$(1-\varepsilon)U \subset \frac{1}{t}(H_t \cap K_t) \subset (1+\varepsilon)U \quad \textit{eventually},$$

almost surely on the event that ξ^0 survives.

We prove Theorems 1 and 2 in Section 4. Some of the geometrical complications in the proofs may be avoided if $d = 1$, using path-intersection properties of two-dimensional space-time; we do not go into this here. We omit formal proofs of Theorems 4 and 5, since these are very close to those of Durrett and Griffeath (1982) and Durrett and Schonmann (1987); instead we make some remarks in Section 5.

In a companion paper [Bezuidenhout and Grimmett (1989)] we study the subcritical phase ($\lambda < \lambda_c$) of the contact process, showing exponential-decay theorems for the size of the region infected from a single initial site.

3. Outline of proof. We shall make abundant use of the graphical representation of the contact process due to Harris (1978); see the introduction for references. Thus we think of the process as being imbedded in space-time. Along each 'time-line' $x \times [0, \infty)$ are positioned 'deaths' at the points of a Poisson process with intensity 1 and between each ordered pair $x_1 \times [0, \infty)$, $x_2 \times [0, \infty)$ of adjacent time-lines are positioned edges directed from the first to the second having centres forming a Poisson process of intensity λ on the set $\frac{1}{2}(x_1 + x_2) \times [0, \infty)$. These Poisson processes are taken to be independent of one another. The random graph obtained from $\mathbb{Z}^d \times [0, \infty)$ by deleting all points at which a death occurs and adding in all directed edges can be used as a percolation superstructure on which a realization of the contact process is built. We shall make free use of the language of percolation. For example, for $A, B \subset \mathbb{Z}^d \times [0, \infty)$, we say that A is joined to B if there exist $a \in A$ and $b \in B$ such that there exists a path from a to b traversing time-lines in the direction of increasing time (but crossing no death) and directed edges between such lines; for $C \subset \mathbb{Z}^d \times [0, \infty)$, we say that A is joined to B in C if such a path exists using segments of time-lines lying entirely in C. Suppose $A \subset \mathbb{Z}^d$. If we define ξ_t^A to be the set of x in $\mathbb{Z}^d$ for which $A \times \{0\}$ is joined to (x, t), then $\{\xi_t^A: t \geq 0\}$ is a realization of the contact process with initial state A. Notice that this construction gives a coupling of the process with all possible initial states.

Suppose that λ is such that $P_\lambda(\xi^0$ survives$) > 0$. We first find a finite disc D ($\subseteq \mathbb{Z}^d$) such that $P_\lambda(\xi^D$ survives$)$ is close to 1. Next we show that, for any large integer N, there exists a box $B = [-L, L]^d \times [0, t]$ in space-time such that, with large probability, some point of $D \times 0$ is joined within B to at least N

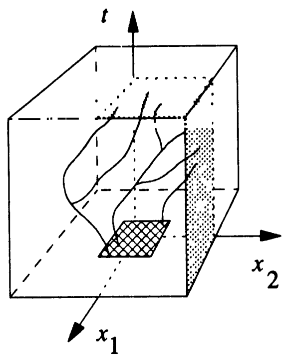

FIG. 1. *With large probability, the disc D, at the centre of the bottom of the box* $B_d(L) \times [0, T]$, *is joined to many points on the top and sides of the box; furthermore, the number of such points lying in any specified orthant of any side is large.*

points on each of the other $2d + 1$ faces of B. The number N is chosen to be large compared with $P_\lambda(\xi_h^0 \supseteq D)^{-1}$ for a given fixed positive h; this is in order that the probability that at least one of N independent attempts to grow an infected copy of the disc D in time h, starting from a single infected point, be close to 1. With this choice of N and hence of the box B, we know that there is a large probability that, on each of the top and sides of B, there exists some point [say $(x, t) \in \mathbb{Z}^d \times [0, \infty)$] which is (i) infected from $D \times 0$ and (ii) infects the entire region $(x, t) + (D \times h) = D_{x,t}$, say. This completes the basic step of the proof; see Figure 1. Certain complications arise in making rigorous the argument so far, largely arising from the fact that N given points on a side of B may lie arbitrarily close to each other in time. We note that, in the proof proper, we shall define $D_{x,t}$ slightly differently.

Having found the disc $D_{x,t}$, we may use it to restart the process: consider the box $(x, t + h) + B$ and especially the points on its top and sides which are infected from $D_{x,t}$. We may restart the process again from one of these new points, if they exist, and so on; see Figure 2. At any stage in the repetition of this process, there is a small but positive probability that the construction fails.

At each stage in the iteration of this process, the active point (x, t) may be chosen from either the top or the sides of the copy of B in question. This freedom of choice allows us some control over the shape of the infected region which we build. Further control is obtained from the fact that, by an argument using symmetry and the FKG inequality, we may assume that there are large numbers of potential active points within any specified orthant of either the top or any of the sides of the box. It is a consequence of this latter fact that we may build our infected region within a sufficiently deep slab of space-time.

If the block construction sketched above is *linear* only, then it is bound to fail after some finite time and little of value will have been achieved. An extra essential ingredient is that the construction may be progressively spread out in

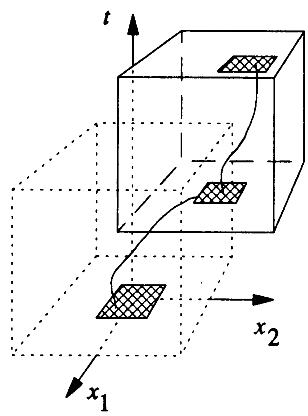

FIG. 2. *With large probability, the disc D is joined to some point $(y, L) \times \tau$ such that* (i) *$(y, L) \times \tau$ is joined to every point in a nearby translate of D and* (ii) *this translate is joined to every point in some disc on the top of a cube based upon this translate.*

such a way that it resembles a process in *two* dimensions rather than in one. To achieve this, we exploit the fact that active points may be found (with large probability) on either a given side of the box in question, or on the *opposite* side.

With the aid of some rather tedious geometrical arguments, we shall deduce that the infected region of the contact process contains a certain two-dimensional discrete-time infection process not dissimilar to oriented percolation. If the probability of success at each stage is sufficiently close to 1, then this discrete-time process is supercritical, and therefore is unbounded with positive probability.

This will prove that

$$P_\lambda\big(\xi^D \text{ survives in } [-2L, 2L]^{d-1} \times \mathbb{Z} \times [0, \infty)\big) > 0.$$

That the corresponding inequality is valid with λ replaced by $\lambda - \delta$ for some $\delta > 0$ follows from the fact that B is bounded. Any event E defined on the interior of B is such that $P_\lambda(E)$ is a continuous function of λ. Hence, if the P_λ-probability that B contains many paths of infection exceeds $1 - \varepsilon$, then so does the corresponding $P_{\lambda-\delta}$-probability for some $\delta > 0$. Hence, if $P_\lambda(\xi^0$ survives$) > 0$, then there exists $\delta > 0$ such that (3) is valid.

4. Proofs of (1) and (2). As outlined above, our first task will be to show that a sufficiently large disc (called D below) is connected to many points on various parts of the surface of a large space-time box. In order to give a precise statement of this result, we introduce notation to specify space-time boxes and orthants of their various faces and random variables to count the number of points on parts of boxes to which $D \times 0$ is connected.

For $L \in \mathbb{N}$, define the following subsets of $\mathbb{Z}^{d-1}$:

$$B_{d-1}(L) = \{x \in \mathbb{Z}^{d-1}: |x_i| \le L \text{ for } 1 \le i \le d-1\},$$
$$\partial B_{d-1}(L) = \{x \in B_{d-1}(L): |x_i| = L \text{ for some } i\},$$

and define subsets $B_d(L)$ and $\partial B_d(L)$ of $\mathbb{Z}^d$ similarly.

The space-time boxes mentioned above will be boxes of the form $B_d(l) \times [0,t]$ contained in $\mathbb{Z}^d \times [0,\infty)$. The top $B_d(l) \times t$ of such a space-time box is itself a d-dimensional box and can therefore be partitioned into 2^d orthants, since each of the d spatial coordinates of a point in such a box can be either positive or negative.

For $\mathbf{u} \in \{+1, -1\}^d$, let $\mathscr{O}_d^{\mathbf{u}}$ be the (open) orthant in $\mathbb{R}^d$ determined by $\mathbf{u}$. That is,

$$\mathscr{O}_d^{\mathbf{u}} = \{x \in \mathbb{R}^d: \operatorname{sgn} x_i = u_i\},$$

and let $\overline{\mathscr{O}}_d^{\mathbf{u}}$ be the closure in $\mathbb{R}^d$ of $\mathscr{O}_d^{\mathbf{u}}$. We also use the notation $\mathscr{O}_{d-1}^{\mathbf{u}}$ and $\overline{\mathscr{O}}_{d-1}^{\mathbf{u}}$ later for similarly defined orthants in $\mathbb{R}^{d-1}$. Then $B_d(l) \times t$ may be expressed as the union of the orthants $[\overline{\mathscr{O}}_d^{\mathbf{u}} \cap B_d(l)] \times t$.

There are $2d$ sides of the space-time box $B_d(l) \times [0,t]$, each corresponding to the product of one of the faces of $B_d(l)$ with the time interval $[0,t]$. Writing $B = B_d(l)$, let $F_i^{\pm}(B)$ be the face of ∂B with outward normal $\pm e_i$, where e_i is the unit vector of $\mathbb{Z}^d$ in the ith coordinate direction; that is to say

$$F_i^{\pm}(B) = \{x \in \partial B: x_i = \pm l\}.$$

The sides of $B_d(l) \times [0,t]$ are the sets $F_i^{\pm}(B_d(l)) \times [0,t]$. Each side $F_i^{\pm} \times [0,t] = F_i^{\pm}(B_d(l)) \times [0,t]$ of $B_d(l) \times [0,t]$ may be divided into 2^{d-1} orthants. We require some extra notation for this. Let

$$\mathscr{O}_{d-1}^{\mathbf{u}}(i,l) = \{x \in \mathbb{R}^d: \operatorname{sgn} x_j = u_j \text{ for } j < i,\ x_i = l,\ \operatorname{sgn} x_j = u_{j-1} \text{ for } j > i\}$$

and write $\overline{\mathscr{O}}_{d-1}^{\mathbf{u}}(i,l)$ for the closure in $\mathbb{R}^d$ of this set. The orthants of $F_i^{\pm} \times [0,t]$ are the sets $[F_i^{\pm} \cap \overline{\mathscr{O}}_{d-1}^{\mathbf{u}}(i,l)] \times [0,t]$.

We now choose the disc D; copies of D will be used to restart the process. We write $D_r = \{z \in \mathbb{Z}^d: \forall\ i,\ |z_i| \le r\}$ for the disc of $\mathbb{Z}^d$ with radius r and centre at the origin. Suppose $P_\lambda(\xi^0 \text{ survives}) > 0$. Fix $\varepsilon > 0$. By a standard argument [see Liggett (1985), Theorem 1.10(d), page 267], there exists a disc $D = D_r$ such that

$$P_\lambda(\xi^D \text{ survives}) > 1 - \tfrac{1}{2}\varepsilon^{2^d + (2d)2^{d-1}}. \tag{6}$$

Our reason for choosing the complicated exponent is simple. There is one factor of ε for each of the 2^d orthants of the top $B_d(l) \times t$ of the space-time box $B_d(l) \times [0,t]$ and one for each of the 2^{d-1} orthants of each of the $2d$ sides $F_i^{\pm} \times [0,t]$ of the box.

We now introduce random variables to count the numbers of points on various parts of a space-time box which are infected from $D \times 0$. For $l \in \mathbb{N}$, $t > 0$ and $\mathbf{u} \in \{+1, -1\}^d$, denote by $N_T^{\mathbf{u}}(l,t)$ the number of points in $[\overline{\mathscr{O}}_d^{\mathbf{u}} \cap B_d(l)] \times t$ to which $D \times 0$ is joined inside the interior of $B_d(l) \times [0,t]$. Note that $N_T^{\mathbf{u}}(l,t)$ counts the number of points in a certain orthant of the *top* of the box $B_d(l) \times [0,t]$; the subscript T stands for top and is not a variable. Let

$$N_T(l,t) = \sum_{\mathbf{u} \in \{+1,-1\}^d} N_T^{\mathbf{u}}(l,t).$$

Note that $N_T(l,t)$ is a measure of the total number of points on the top $B_d(l) \times t$ of the space-time box $B_d(l) \times [0,t]$.

We now introduce notation to enable us to count the number of infected points on the sides of the space-time box $B_d(l) \times [0,t]$. Recall from the outline of the proof that these points may be used in later attempts to generate infected copies of D and that some fixed amount of time (h below) is required for this. Choose $h \in (0,(1+2d\lambda)^{-1})$. For $l \in \mathbb{N}$, $t > 0$, $i = 1,\dots,d$, and $\mathbf{u} \in \{+1,-1\}^{d-1}$, let $N_{F_i^\pm}^{\mathbf{u}}(l,t)$ denote the size of the maximal subset of the orthant $(F_i^\pm(B_d(l)) \cap \overline{\mathscr{O}}_{d-1}^{\mathbf{u}}(i, \pm l)) \times [0,t]$ of $F_i^\pm$ determined by $\mathbf{u}$ having the properties that (i) every pair of points in the set is separated by a distance (in the L^∞ norm on $\mathbb{Z}^d \times [0,\infty)$) of at least h and (ii) $D \times 0$ is connected inside the interior of $B_d(l) \times [0,t]$ to each point in the set (thus we count only points which are at the endpoints of arrows from the interior of the box). Let

$$N_{F_i^\pm}(l,t) = \sum_{\mathbf{u} \in \{+1,-1\}^{d-1}} N_{F_i^\pm}^{\mathbf{u}}(l,t)$$

and

$$N_S(l,t) = \sum_{i=1}^{d} \left[N_{F_i^+}(l,t) + N_{F_i^-}(l,t) \right].$$

Note that $N_S(l,t)$ is a measure of the total size of the union of certain sets of points on the sides of the space-time box $B_d(l) \times [0,t]$; the S in the subscript stands for side and is not a variable. Finally, if $l \in \mathbb{N}$ and $t > 0$, let

$$N(l,t) = N_T(l,t) + N_S(l,t).$$

With h in $(0,(1+2d\lambda)^{-1})$ fixed, let α (> 0) be the minimum of (i) the probability that 0 is connected inside $D \times [0,h]$ to every point in $D \times h$ and (ii) the probability that 0 is connected inside $(D \pm re_d) \times [0,h]$ to every point in $(D \pm re_d) \times h$. Let M be large enough to ensure that in M or more independent trials of an experiment with success probability α, the probability of obtaining at least one success exceeds $1 - \varepsilon$. Let N be large enough to ensure that, in any subset of $\mathbb{Z}^d$ or $\mathbb{Z}^{d-1}$ having size N or larger, there are at least M points all pairs of which are L^∞-distance at least $3r + 1$ apart.

The first step in our construction will be to prove a lemma illustrated in Figure 1. This corresponds closely to the main step of Barsky, Grimmett and Newman (1989).

(7) LEMMA. *There exist L in $\mathbb{N}$ and $T > 0$ such that, for every $\mathbf{v}$ in $\{+1,-1\}^d$,*

$$P_\lambda(N_T^{\mathbf{v}}(L,T) \geq N) \geq 1 - \varepsilon \tag{8}$$

and for each $i = 1, \dots, d$ *and* $\mathbf{u}$ *in* $\{+1, -1\}^{d-1}$,

$$P_\lambda\big(N_{F_i^\pm}^{\mathbf{u}}(L, T) \geq MN\big) \geq 1 - \varepsilon. \tag{9}$$

PROOF. Note that, for every $t \geq 0$,

$$P_\lambda\big(\xi^D \text{ dies out} \mid |\xi_t^D| < 2^d N\big) \geq (1 + 2d\lambda)^{-2^d N}. \tag{10}$$

It follows that $P_\lambda(\xi^D$ survives, $|\xi_t^D| < 2^d N$ for arbitrarily large times $t) = 0$. The justification for this is the usual one, proceeding roughly as follows. Suppose that $|\xi_t^D| < 2^d N$ for arbitrarily large values of t. At each random time t at which $|\xi_t^D|$ drops below $2^d N$, there is, by (10), a strictly positive probability that ξ^D dies out subsequently. By constructing an appropriate sequence of stopping times and using the strong Markov property, one obtains that ξ^D dies out almost surely on the event in question. Together with (6), this implies that there exists a positive number T_1 so that

$$P_\lambda\big(\forall\, t \geq T_1,\ N_T(\infty, t) \geq 2^d N\big) > 1 - \tfrac{3}{4}\varepsilon^{2^d}. \tag{11}$$

From (11) it follows that, if $t \geq T_1$, there exists $l(t)$ in $\mathbb{N}$ such that, if $l \geq l(t)$ and $s = t$, then

$$P_\lambda\big(N_T(l, s) \geq 2^d N\big) > 1 - \varepsilon^{2^d}. \tag{12}$$

It is a consequence of (12) and the FKG inequality [see Harris (1960), Fortuin, Kasteleyn, Ginibre (1971), Durrett (1988), Bezuidenhout and Grimmett (1989) for statements and proofs of this inequality] that (8) holds for every $\mathbf{v}$ in $\{+1, -1\}^d$ when (L, T) is taken to equal (l, s). This argument is well known. An argument of a similar type was used by Russo (1978) and Seymour and Welsh (1978): by (12),

$$\begin{aligned}
\varepsilon^{2^d} &\geq P_\lambda\big(N_T(l, s) < 2^d N\big)\\
&\geq P_\lambda\big(N_T^{\mathbf{v}}(l, s) < N \text{ for all } \mathbf{v} \in \{+1, -1\}^d\big)\\
&\geq \prod_{\mathbf{v} \in \{+1, -1\}^d} P_\lambda\big(N_T^{\mathbf{v}}(l, s) < N\big)\\
&= P_\lambda\big(N_T^{\mathbf{v}}(l, s) < N\big)^{2^d}
\end{aligned} \tag{13}$$

for any fixed $\mathbf{v} \in \{+1, -1\}^d$. This explains the term 2^d in the exponent in (6). Thus we have found arbitrarily large boxes so that, with large probability, $D \times 0$ is connected to many points in every orthant of the top of the box. We require also that the probability that $D \times 0$ is connected to lots of points in all orthants of the sides be large. To achieve this, we need to work a little harder. For l fixed, $l \geq l(t)$, (12) cannot be true for every $s \geq t$, otherwise the contact process would survive forever with positive probability inside the one-dimensional cylinder $B_d(l) \times [0, \infty)$. Let $s(l, t)$ be the infimum of those values $s \geq t$ for which (12) fails. The left-hand side of (12) is a continuous function of s and so, for $t \geq T_1$, $l \geq l(t)$,

$$P_\lambda\big(N_T(l, s(l,t)) \geq 2^d N\big) = 1 - \varepsilon^{2^d}.$$

We note that it is only here that the discrete-time process (oriented percolation) is harder to handle than the (continuous-time) contact process, since in discrete time one cannot necessarily achieve equality in the relation above. Instead, one finds the smallest height s for the top of the box with the property that inequality ($\leq$) holds here, but with N replaced by KN, where K is large enough to ensure that with large probability, at least N of the KN points of infection on the top of the box with height $s - 1$ retain their infection until the next instant of discrete time.

Construct a nested sequence of space-time boxes $B_k = B_d(l_k) \times [0, s_k]$ as follows. Let $l_1 = l(T_1)$ and $s_1 = s(l_1, T_2)$. Suppose $k \geq 1$ and suppose that $B_1, \dots, B_k$ have been constructed. Choose $T_{k+1} \geq s_k + 1$ and $l_{k+1} \geq (l_k + 1) \vee l(T_{k+1})$; $a \vee b$ and $a \wedge b$ denote $\max\{a, b\}$ and $\min\{a, b\}$, respectively. Let $s_{k+1} = s(l_{k+1}, T_{k+1})$. Note that, for $k \geq 1$,

$$P_\lambda\big(N_T(l_k, s_k) \geq 2^d N\big) = 1 - \varepsilon^{2^d}. \tag{14}$$

Let $N_k = N(l_k, s_k)$. We claim that there exists $k_0 \geq 1$ so that

$$P_\lambda\big(\forall\, k \geq k_0,\ N_k \geq 2^d N(dM + 1)\big) > 1 - \varepsilon^{2^d + d2^d}. \tag{15}$$

We defer the proof of (15) until the end of proof of the lemma. Note however that, in spirit, (15) should be no harder to prove than was (11).

From (14), (15) and the FKG inequality, we have that if $k \geq k_0$, then

$$\begin{aligned}
\varepsilon^{2^d + d2^d} &> P_\lambda\big(N_T(l_k, s_k) + N_S(l_k, s_k) < 2^d N(dM + 1)\big) \\
&\geq P_\lambda\big(N_T(l_k, s_k) < 2^d N \text{ and } N_S(l_k, s_k) < d2^d MN\big) \\
&\geq P_\lambda\big(N_T(l_k, s_k) < 2^d N\big) P_\lambda\big(N_S(l_k, s_k) < d2^d MN\big) \\
&= \varepsilon^{2^d} P_\lambda\big(N_S(l_k, s_k) < d2^d MN\big).
\end{aligned}$$

So, if $k \geq k_0$,

$$P_\lambda\big(N_S(l_k, s_k) \geq d2^d MN\big) \geq 1 - \varepsilon^{d2^d}. \tag{16}$$

This is a variation on the argument given in (13). Let $L = l_{k_0}$ and $T = s_{k_0}$. Then (8) and (9) follow from (14) and (16), respectively, as in (13).

It remains to prove (15). To this end, we shall show that, for $\nu \in \mathbb{N}$, $k \geq 1$,

$$P_\lambda(\xi^D \text{ dies out} \mid N_k < \nu) \geq \left(\frac{1 - 2dh\lambda}{1 + 2d\lambda} \right)^\nu. \tag{17}$$

This, together with (6) and an argument similar to the one which one would use to obtain (11) from (10), gives (15).

We prove (17) in essentially the same way as (10), with one major difference. The sides of the box are unions of continuous intervals (whereas the top was a subset of $\mathbb{Z}^d$), and furthermore we may count only points on the sides which are a certain distance apart from each other.

In order to prove (17), note that infection may be prevented from leaving B_k by suitable dispositions of deaths on the sides of B_k and along the time-lines leaving B_k. For any point $x \times s_k$ on the top of B_k, there is probability $(1 + 2d\lambda)^{-1} \geq (1 - 2dh\lambda)(1 + 2d\lambda)^{-1}$ that the earliest event on $x \times [s_k, \infty)$ is a death; such a death prevents the spread of infection upwards from $x \times s_k$.

We turn next to the sides of B_k. Suppose that the configuration inside $\mathring{B}_d(l_k) \times [0, s_k]$, including crossings from $(F_j^{\pm}(B_d(l_k)) \mp e_j) \times [0, s_k]$ to $F_j^{\pm}(B_d(l_k)) \times [0, s_k]$ for each j, has been constructed, but that deaths on $\partial B_d(l_k) \times [0, s_k]$ and arrows leading from $\partial B_d(l_k) \times [0, s_k]$ have yet to be added; here, $\mathring{B}$ denotes the interior of B. Let $x \in \partial B_d(l_k)$ and suppose for concreteness that $x \in F_i^+(B_d(l_k))$. Let $(x, p_1), \dots, (x, p_n)$ with $0 \leq p_1 < p_2 < \cdots < p_n \leq s_k$ be the points on $x \times [0, s_k]$ to which $D \times 0$ is connected inside the interior of the box B_k (if there are no such points then there is nothing to do).

We divide the points (x, p_i) into groups $\{(x, p_{i_k}), (x, p_{i_k+1}), \dots, (x, p_{j_k})\}$ so that (i) the distance $p_{j_k} - p_{i_k}$ between the highest and lowest point in each group is no more than h and (ii) the distance $p_{i_{k+1}} - p_{i_k}$ between the lowest point of the $(k+1)$th group and that of the kth group is larger than h (i.e., each group is as big as it can be without violating (i)). In the following estimate, we deal with the intervals between successive members of a single group in one bite.

Let $p_{n+1} = s_k$ and $y_i = p_{i+1} - p_i$ for $i = 1, \dots, n$. Let $i_1 = 1$. Let

$$j_1 = (n+1) \wedge \min\left\{ j \colon i_1 \leq j \leq n,\ \sum_{\tau = i_1}^{j} y_\tau > h \right\},$$

where $\min \varnothing = \infty$. Suppose $l \geq 1$ and suppose that $i_1, \dots, i_l$ and $j_1, \dots, j_l$ have been constructed. If $j_l < n$, let $i_{l+1} = j_l + 1$ and

$$j_{l+1} = (n+1) \wedge \min\left\{ j \colon i_{l+1} \leq j \leq n,\ \sum_{\tau = i_{l+1}}^{j} y_\tau > h \right\}.$$

Otherwise, we set $m = m(x) = l$ and we terminate the process. The probability that either there is no arrow leading away from $x \times (p_j, p_{j+1})$ or that all such arrows are preceded by a death equals

$$\frac{1}{1+2d\lambda}+\frac{2d\lambda}{1+2d\lambda}e^{-(1+2d\lambda)y_j}.$$

Therefore the probability that infection from $\mathring{B}_k$ is blocked by the line $x \times [0, s_k]$, given that the points described above are $(x, p_1), \dots, (x, p_n)$, is at least

$$\begin{aligned}
&\prod_{j=1}^{n}(1+2d\lambda)^{-1}(1+2d\lambda e^{-(1+2d\lambda)y_j})\\
&\quad\geq \prod_{l=1}^{m}\left\{\left[\prod_{\tau=i_l}^{j_l-1}(1+2d\lambda)^{-1}(1+2d\lambda e^{-(1+2d\lambda)y_\tau})\right](1+2d\lambda)^{-1}\right\}\\
&\quad\geq (1+2d\lambda)^{-m}\prod_{l=1}^{m}\left(1-2d\lambda\sum_{\tau=i_l}^{j_l-1}y_\tau\right)\\
&\quad\geq \left(\frac{1-2dh\lambda}{1+2d\lambda}\right)^m,
\end{aligned}$$

since $e^{-x}\geq 1-x$ and $h\leq(1+2d\lambda)^{-1}$. Now, there is probability at least

$$\prod_{x\in\partial B_d(l_k)}\left(\frac{1-2dh\lambda}{1+2d\lambda}\right)^{m(x)}\geq\left(\frac{1-2dh\lambda}{1+2d\lambda}\right)^{N_S(l_k,s_k)}$$

that infection cannot leak through the sides of B_k to the outside world. Inequality (17) now follows easily. This completes the proof of Lemma 7. □

The next step in the construction is to restart the process from one of the points on the side of B_k and the next lemma is the principal ingredient here; it is illustrated in Figure 2. Note that the symbol 0 represents both the origin (in $\mathbb{Z}$ and $\mathbb{Z}^d$ and time) and the number zero, depending on the context.

(18) LEMMA. *Suppose $P_\lambda(\xi^0$ survives$) > 0$ and $\varepsilon > 0$. There exist $r > 0$, $L \in \mathbb{N}$, $S > 0$ and $\delta > 0$ such that the following holds. For every $x \in B_{d-1}(L) \times 0$, with $P_{\lambda-\delta}$-probability greater than $1-\varepsilon$, there exists a translate $\Delta + (D_r \times 0)$ of the disc $D_r \times 0$ such that*

(i) *$\Delta \in B_{d-1}(L) \times [L, 2L] \times [S, 2S]$,*
(ii) *$(x + D_r) \times 0$ is connected inside $B_{d-1}(2L) \times [-L, 3L] \times [0, 2S]$ to every point in the disc $\Delta + (D_r \times 0)$.*

PROOF. Let r, $D = D_r$, L and T be as in the discussion culminating in Lemma 7 and let $S = T + h$, where $h \in (0, (1 + 2\lambda d)^{-1})$ is fixed as before.

Suppose $x \in B_{d-1}(L) \times 0$ and choose $\mathbf{u}$ such that $x \in [B_{d-1}(L) \cap \overline{\mathscr{O}}_{d-1}^{\mathbf{u}}] \times 0$, where $\mathbf{u} \in \{+1, -1\}^{d-1}$ (if there is more than one such $\mathbf{u}$, pick the one which is maximal in some fixed ordering of $\{+1, -1\}^{d-1}$). Then, by (9) (with $\mathbf{u}$ replaced by $-\mathbf{u}$ and $F_i^{\pm}$ chosen to be F_d^+) and translation invariance, with probability at least $1 - \varepsilon$, $(x + D) \times 0$ is connected inside $(x + B_d(L)) \times [0, T]$ $\{\subseteq B_{d-1}(2L) \times [-L, L] \times [0, T]\}$ to at least NM points on $(x + [F_d^+(B_d(L)) \cap (\overline{\mathscr{O}}_{d-1}^{-\mathbf{u}} \times 0 + Le_d)]) \times [0, T]$ any two of which are at least distance h apart; such points lie on $F_d^+(B_d(L)) \times [0, T]$. Either the projection of this set onto $x + [F_d^+(B_d(L)) \cap (\overline{\mathscr{O}}_{d-1}^{-\mathbf{u}} \times 0 + Le_d)]$ has size at least N or its intersection with some time-line $y \times [0, T]$ contains at least M points, any two of which are at least distance h apart. In either case the choice of M and N ensures that, with large probability, at least one of these points (y, t) is connected in $(y + re_d + D) \times [t, t + h]$ to every point in $(y + re_d + D) \times (t + h)$. Let τ be the smallest time t for which $(x + D) \times 0$ is connected inside $B_{d-1}(2L) \times [-L, L + 2r] \times [0, t]$ to every point in some translate of D centerd in $(F_d^+(B_d(L)) + re_d) \times t$. Then, by the preceding discussion, with probability at least $(1 - \varepsilon)^2$, it is the case that $\tau \in [0, T + h]$. If $\tau < \infty$, then $(x + D) \times 0$ is connected inside $B_{d-1}(2L) \times [-L, L + 2r] \times [0, \tau]$ to every point in a unique translate of D centred on $(F_d^+(B_d(L)) + re_d) \times \tau$. Suppose this translate of D is centred at $(y, L + r) \times \tau$ with $y \in \overline{\mathscr{O}}_{d-1}^{\mathbf{v}} \cap B_{d-1}(L)$, where $\mathbf{v} \in \{+1, -1\}^{d-1}$ is maximal as before. Conditional on the σ-field generated by the process up to time τ, we have by the strong Markov property and (8) that, with probability at least $1 - \varepsilon$, the disc $[(y, L + r) + D] \times \tau$ is connected inside $[(y, L + r) + B_d(L)] \times [\tau, \tau + T]$ $\{\subseteq B_{d-1}(2L) \times [0, 2L + r] \times [0, T + \tau]\}$ to at least N points on

$$\big((y, L + r) + \big[B_d(L) \cap \overline{\mathscr{O}}_d^{(-\mathbf{v}, 1)}\big]\big) \times (T + \tau)$$
$$\{\subseteq B_{d-1}(L) \times [L, 2L + r] \times (T + \tau)\},$$

where $(-\mathbf{v}, 1)$ is the d-vector obtained from $-\mathbf{v}$ by appending 1 in the dth coordinate.

Having found a set of N such points, we have from the definition of N that there exists a subset S of size at least M of which each pair of members is at least distance $3r + 1$ apart. We partition S into two, $S = S_1 \cup S_2$, depending respectively on whether the dth coordinate of any given point is greater than $2L$ or not. With points $z \in S_1$ we associate the cylinders $(z - re_d + D) \times [T + \tau, T + \tau + h]$ and with points $z \in S_2$ we associate $(z + D) \times [T + \tau, T + \tau + h]$. These cylinders are disjoint, and thus, with probability at least $1 - \varepsilon$, there exists a point $z \in S$ which is joined within the corresponding cylinder to every point on the top of this cylinder. On this event, $D \times 0$ is connected inside $B_{d-1}(2L) \times [-L, 3L] \times [0, 2(T + h)]$ to every point in some translate of D centred in $B_{d-1}(L) \times [L, 2L] \times [T + h, 2(T + h)]$.

The previous construction results in a point $\Delta = \Delta_x$ with the properties claimed in the lemma. The construction has P_λ-probability at least $1 - 4\varepsilon$ of succeeding and we replace ε by $\varepsilon/5$ to ensure that the probability of success under P_λ is strictly greater than $1 - \varepsilon$. Since the event in question depends

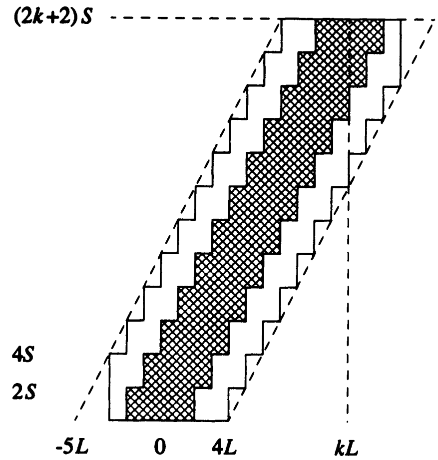

FIG. 3. *The basic step is iterated k times. At each step the target region moves upwards and to the right. The paths in question may lie in the outer region, but the intermediate discs are centred in the shaded regions.*

only on the configuration inside the finite space-time region $B_{d-1}(2L) \times [-L, 3L] \times [0, 2S]$, its probability depends continuously on λ. There are only finitely many choices for $x \in B_{d-1}(L) \times 0$, so there exists a positive number δ so that the probability of success remains greater than $1 - \varepsilon$ under $P_{\lambda-\delta}$ for every choice of x. □

The remainder of the proof consists of repeating this step many times in order to build paths across large portions of space-time. There are various ways of doing this, of which the following is one (see Figure 3).

(19) LEMMA. *Suppose $P_\lambda(\xi^0$ survives$) > 0$, $\varepsilon > 0$, and assume the notation of Lemma 18. Suppose $k \in \mathbb{N}$, $x \in B_{d-1}(L) \times [-2L, 2L]$ and $t \in [0, 2S]$. There exists, with $P_{\lambda-\delta}$-probability at least $(1-\varepsilon)^{2k}$, a translate $\Pi + (D_r \times 0)$ of the disc $D_r \times 0$ such that*

(i) $\Pi \in B_{d-1}(L) \times (kL + [-2L, 2L]) \times (2kS + [0, 2S])$,
(ii) $(x + D_r) \times t$ *is connected to every point in the disc* $\Pi + (D_r \times 0)$ *by paths lying inside the region*

$$(20) \qquad \mathscr{S} = \bigcup_{j=0}^{k-1} B_{d-1}(2L) \times (jL + [-3L, 4L]) \times (2jS + [0, 4S]).$$

PROOF. We show first that, if $x \in B_{d-1}(L) \times [-2L, 2L]$ and $t \in [0, 2S]$, then, with $P_{\lambda-\delta}$-probability at least $(1-\varepsilon)^2$, $(x + D) \times t$ is connected to every point of some translate of D centred in $B_{d-1}(L) \times [-L, 3L] \times [2S, 4S]$ by paths lying in $B_{d-1}(2L) \times [-3L, 4L] \times [0, 4S]$ (see Figure 4). In doing this,

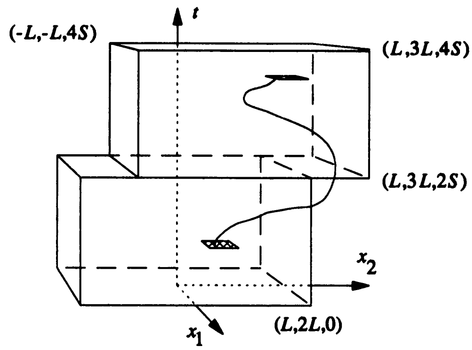

FIG. 4. *Every disc centred in the lower box may be joined, with large probability, to every point in some disc centred in the upper box.*

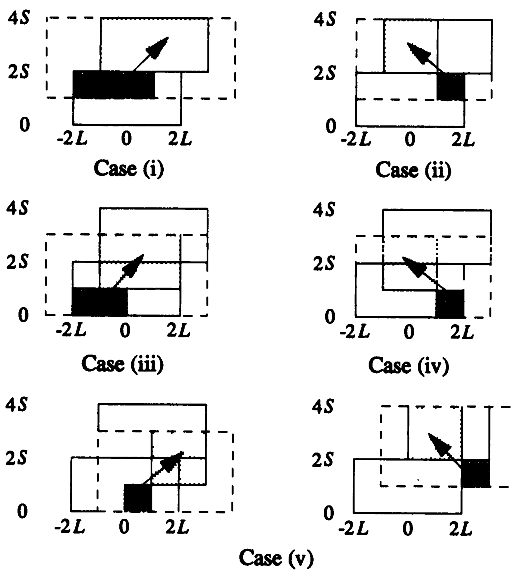

FIG. 5. *Illustrations of the five cases which arise in the proof of Lemma 19. The procedure to be followed depends upon which of the heavily shaded regions contains the starting point $x \times t$. The lightly shaded regions are the target regions.*

we shall make use of Lemma 18 and the strong Markov property. There are five cases, depending on the position of the point $x \times t$; these are illustrated in Figure 5.

CASE 1. Suppose $x \in B_{d-1}(L) \times [-2L, L]$ and $t \in [S, 2S]$. It follows from Lemma 18 that, with $P_{\lambda-\delta}$-probability at least $1 - \varepsilon$, $(x + D) \times t$ is connected inside $B_{d-1}(2L) \times [-L + x_d, 3L + x_d] \times [t, 2S + t]$ $\{\subseteq B_{d-1}(2L) \times [-3L, 4L] \times [S, 4S]\}$ to every point in some translate of D centred in $B_{d-1}(L) \times [x_d + L, x_d + 2L] \times [t + S, t + 2S]$ $\{\subseteq B_{d-1}(L) \times [-L, 3L] \times [2S, 4S]\}$.

CASE 2. Suppose $x \in B_{d-1}(L) \times [L, 2L]$ and $t \in [S, 2S]$. By Lemma 18 and symmetry, with $P_{\lambda-\delta}$-probability at least $1 - \varepsilon$, $(x + D) \times t$ is connected inside $B_{d-1}(2L) \times [-3L + x_d, L + x_d] \times [t, 2S + t]$ $\{\subseteq B_{d-1}(2L) \times [-2L, 3L] \times [S, 4S]\}$ to every point in some translate of D centred in $B_{d-1}(L) \times [-2L + x_d, -L + x_d] \times [t + S, t + 2S]$ $\{\subseteq B_{d-1}(L) \times [-L, L] \times [2S, 4S]\}$.

CASE 3. Suppose $x \in B_{d-1}(L) \times [-2L, 0]$ and $t \in [0, S]$. By Lemma 18, with $P_{\lambda-\delta}$-probability at least $1 - \varepsilon$, $(x + D) \times t$ is connected inside $B_{d-1}(2L) \times [-L + x_d, 3L + x_d] \times [t, 2S + t]$ $\{\subseteq B_{d-1}(2L) \times [-3L, 3L] \times [0, 3S]\}$ to every point in some translate of D centred in $B_{d-1}(L) \times [x_d + L, x_d + 2L] \times [t + S, t + 2S]$ $\{\subseteq B_{d-1}(L) \times [-L, 2L] \times [S, 3S]\}$. This translate of D may lie outside the required target region. If the earliest such translate is centred in $B_{d-1}(L) \times [-L, 2L] \times [S, 2S]$, then we repeat the process, using whichever of cases (i) and (ii) is appropriate. The composite step is successful with $P_{\lambda-\delta}$-probability at least $(1 - \varepsilon)^2$.

The other two cases are similar to case (iii) and the reader is referred to Figure 5 for the geometrical details. This proves the claim made at the beginning of the proof of the lemma.

One obtains the full result of the lemma by iterating this construction and using the strong Markov property. □

Using the notation of Lemma 18, we define, for $k \in \mathbb{N}$,

$$\mathscr{R}^{\pm} = B_{d-1}(2L) \times V^{\pm},$$

where

$$V^{\pm} = \left\{ (x_d, t) \in \mathbb{Z} \times \mathbb{R} : 0 \le t \le (2k + 2)S,\ -5L \pm \frac{L}{2S}t \le x_d \le 5L \pm \frac{L}{2S}t \right\}.$$

Note that $\mathscr{S}$, given in (20), is contained inside the region $\mathscr{R}^+$ and that $\mathscr{R}^-$ is a reflection of $\mathscr{R}^+$.

We use the construction above to compare the original contact process with a supercritical discrete-time process obtained from the configuration inside a two-dimensional slab. Suppose $P_\lambda(\xi^0 \text{ survives}) > 0$ and $\eta > 0$; later we shall take η to be small. Choose $k > 10$ and $\varepsilon > 0$ such that $(1 - \varepsilon)^{2k} > 1 - \eta$. With this value of ε, let r, $D = D_r$, L, S and T be as in the discussion culminating in Lemmas 18 and 19. For $x \in B_{d-1}(L) \times [-2L, 2L]$ and $t \in [0, 2S]$, let $G^{\pm}(x, t)$ be the event that $(x + D) \times t$ is connected inside the region $\mathscr{R}^{\pm}$ to every point in some translate of D centred in $B_{d-1}(L) \times [(\pm k - 2)L, (\pm k + 2)L] \times [2kS, 2(k + 1)S]$. By Lemma 19 and the choice of ε, uniformly for (x, t) in $B_{d-1}(L) \times [-2L, 2L] \times [0, 2S]$, it is the case that $P_{\lambda-\delta}(G^{\pm}(x, t)) > 1 - \eta$. For sufficiently small values of η, this fact alone may be used to prove (3), as will be done in Lemma 21.

(21) LEMMA. *If λ is such that $P_\lambda(G^{\pm}(x, t)) > 1 - \eta$ for every $x \in B_{d-1}(L) \times [-2L, 2L]$ and $t \in [0, 2S]$ and η is sufficiently small, then*

$$P_\lambda\big(\xi^D \text{ survives inside } B_{d-1}(2L) \times \mathbb{Z} \times [0, \infty)\big) > 0.$$

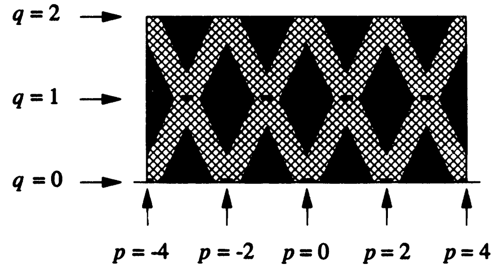

FIG. 6. *Part of the region $\mathscr{R}$ with the sets $v_{p,q}$ indicated in black.*

PROOF. We construct a process in discrete time as follows. For $p, q \in \mathbb{Z}$ with $q \geq 0$ and $p + q$ even, let

$$v_{p,q} = (B_{d-1}(L) \times [-2L, 2L] \times [0, 2S]) + w(p,q),$$

where $w(p,q) = pkLe_d \times 2qkS$ and let

$$\mathscr{R} = \bigcup_{\substack{q \geq 0 \\ p+q \text{ even}}} \{(\mathscr{R}^+ \cup \mathscr{R}^-) + w(p,q)\}.$$

The region $\mathscr{R}$ is illustrated in Figure 6.

We shall define random variables $\{\Xi_n(i) = (I_n(i), P_n(i)): n \geq 0, i \geq 0\}$, where $I_n(i)$ may take the value 0 or 1 and $P_n(i) \in \mathbb{Z}^d \times [0, \infty)$; note that each $P_n(i)$ will be undefined whenever the corresponding $I_n(i)$ equals 0. First we set $\Xi_0(0) = (1, 0)$, where the second component is the origin of space-time and $I_0(i) = 0$ for $i \neq 0$. Having defined $\{\Xi_n(i): i \geq 0, n \leq N\}$, we define $\Xi_{N+1}(i) = (I_{N+1}(i), P_{N+1}(i))$ as follows. The random variable $I_{N+1}(i)$ is the indicator function of the event that for either $j = i$ or $j = i - 1$, it is the case that $I_N(j) = 1$ and some point in $P_N(j) + D$ is joined to every point in some translate of D centred in $\nu_{2i-N-1, N+1}$ by a path lying entirely within $\mathscr{R}$. If $I_{N+1}(i) = 1$, then $P_{N+1}(i)$ is defined to be the centre of the earliest translate (earliest in time) of D which is thus joined. We write Ξ_n for the set of integers i for which $I_n(i) = 1$. Note that if $\Xi_n \neq \varnothing$ for all n, then ξ^D survives inside $B_{d-1}(2L) \times \mathbb{Z} \times [0, \infty)$.

Let $\mathscr{F}_n$ be the σ-field generated by $\{\Xi_m: 0 \leq m \leq n\}$. We have, by the strong Markov property and the hypothesis of the lemma, that

$$\begin{aligned} P_\lambda(I_{n+1}(i) = 1 | \mathscr{F}_n) &= P_\lambda(I_{n+1}(i) = 1 | \Xi_n(i), \Xi_n(i-1)) \\ &\geq \alpha(I_n(i), I_n(i-1)), \end{aligned}$$

where $\alpha(0,0) = 0$, $\alpha(0,1) = \alpha(1,0) = \alpha(1,1) = 1 - \eta$. The fact that $k > 10$ guarantees that two translates of $\mathscr{R}^+$ or $\mathscr{R}^-$ by vectors $w(p_1, q_1)$ and $w(p_2, q_2)$ do not overlap whenever $|p_1 - p_2| + |q_1 - q_2| > 2$. As a consequence of this, conditional on $\mathscr{F}_n$, the random variables $\Xi_{n+1}(i)$ and $\Xi_{n+1}(i + k)$ are dependent if and only if k equals 0 or ± 1. Durrett [(1988), page 85] has used a contour argument to show that the critical parameter value for a 1-dependent oriented site percolation process is strictly less than 1; his argument is easily adapted to show that $I_n(i) = 1$ for infinitely many pairs (n, i) with positive probability, for all sufficiently small positive values of η. Thus $\Xi_n \neq \emptyset$ for all n with positive probability and the claim of the lemma follows.

The argument via 1-dependent processes may be avoided at the price of some extra geometrical complication. In loose terms, this is achieved as follows. When growing the discretized process from one renormalized site to the next, we stop just short of the second site, so that we have no information about whether or not this site may be reached from the other possible source. We declare a renormalized site to be good if, having constructed all previous generations, the construction may be continued through this site to translates of D centred in designated regions close to both of the renormalized sites at the next level. The corresponding site indicator functions are (conditionally) independent and the result follows as before, by appealing now to standard results for ordinary oriented site percolation. □

5. Remarks on Theorems 4 and 5. Conditions sufficient for (4) and (5) are derived and discussed by Durrett and Griffeath (1982) and Durrett and Schonmann (1987). Durrett and Griffeath (1982) have proved versions of (4) and (5) by studying one-dimensional contact processes embedded in the d-dimensional process; corresponding results for discrete-time processes have been studied by Durrett and Schonmann (1987) using subprocesses embedded in slabs. Their arguments are easily adapted to prove (4) and (5), but the geometrical details are somewhat complicated to write out afresh and require almost no new ideas. We therefore omit the proofs, confining ourselves to making limited remarks.

To prove the complete convergence theorem, one follows Durrett and Schonmann [(1987), page 111ff]. The principal requirement is their equation (4) [which is the hypothesis in the lemma of Griffeath (1978)] and this may be proved either in their way or by utilizing the following idea. The error probability ε in Lemma 19 may be made sufficiently small that the *intersection of two independent copies* of the discrete-time process constructed in its proof is infinite with positive probability. Working forwards in time from the set A and backwards in time from the set B (see the original for an explanation of the notation), one uses the restart argument of Durrett and Schonmann (1987) until *both* the corresponding discrete-time processes take off. If these processes are defined in the correct way, then for large t, there is only small probability that they do not reach many discs paired off in a natural way so that each disc in the first process is close to the corresponding disc in the second. With large probability, some such pair of these discs is joined and (4) of Durrett and Schonmann (1987) follows (after some work).

To prove the shape theorem, one may follow Durrett and Griffeath [(1982), page 545ff.], working in slabs rather than strictly one-dimensional subsets. The only difficulty is in building chains of infection from the origin to a line of the form $x \times [0, t]$ $\{\subset \mathbb{Z}^d \times [0, \infty)\}$. One cannot be sure that infection within a suitable thick slab will hit the line exactly, but only that it will pass the line at a distance not exceeding the thickness of the slab. This problem may be overcome by using the restart technique once again.

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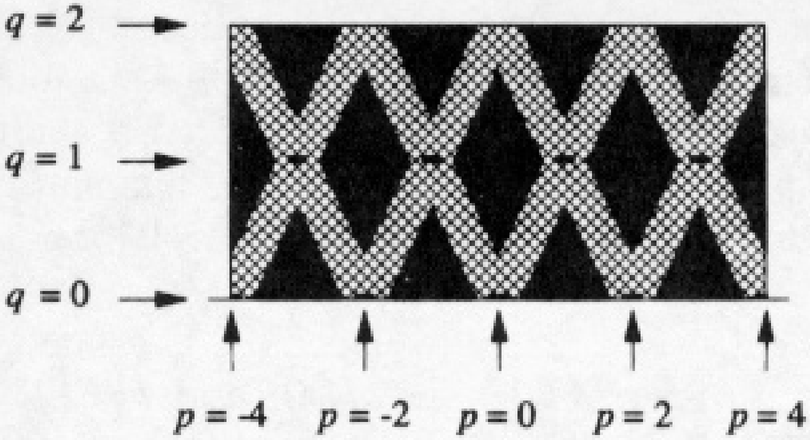

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q = 1
q = 0
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p = -2
p = 0
p = 2
p = 4