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SOME EXTENSIONS OF THE LIL VIA SELF-NORMALIZATIONS

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We study some generalizations of the LIL when self-normalizations are used. Two particular results proved are: (1) an extension of the Kolmogorov–Erdös test for partial sums of symmetric i.i.d. random variables having finite second moments; this result eliminates distinctions required when nonrandom normalizers are used and $E(X^2 I(|X| > t))$ is not $O((L_2 t)^{-1})$, and (2) an extension of a universal bounded LIL of Marcinkiewicz to nonsymmetric random variables. An interesting corollary of this work is a short new proof of the classical LIL avoiding truncation methods.

1. Introduction. The aim of this paper is to provide some extensions of the law of the iterated logarithm (LIL) for sums of independent random variables. We will begin by considering the i.i.d. case. Thus let $X, X_1, X_2, \dots$ be a sequence of nondegenerate i.i.d. random variables and set $S_n = X_1 + \cdots + X_n$. The classical law of the iterated logarithm states that

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2\sigma^2 n L_2 n)^{1/2}} = 1 \quad \text{iff} \quad \operatorname{var}(X) = \sigma^2 < \infty, \tag{1.1}$$

where $Lx = \max(1, \log_e x)$ and $L_2 x = L(Lx)$. Observe that this is a two-sided result since replacing X by $-X$ immediately gives a lim inf of -1 in (1.1).

The normalizer in (1.1) is intimately related to the normalizer in the central limit theorem when $\operatorname{var}(X) < \infty$. However, asymptotic normality holds more generally than under finite variance, the necessary and sufficient condition being

$$\exists\, \delta_n \text{ and } \gamma_n \text{ such that} \quad (S_n - \delta_n)\gamma_n^{-1} \to N(0,1) \tag{1.2}$$

if and only if

$$\lim_{x\to\infty} \frac{x^2 P(|X| > x)}{E(X^2 I(|X| \le x))} = 0. \tag{1.3}$$

Thus it is reasonable to ask whether an analogous result, based on the related central limit normalizers, holds for the LIL under the assumption (1.3). The answer in general is no, because the large values of X_i cannot be controlled by these normalizers. There is, however, a generalization of (1.1) to (1.3) with

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nice central limit normalizers if we allow self-normalization. For this consider the sample variance

$$\hat{\sigma}_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n, \tag{1.4}$$

where $\bar{X} = S_n/n$. By the strong law, if $E(X^2) < \infty$, then $\hat{\sigma}_n \to \sigma$ with probability 1. Thus

$$\frac{S_n - nE(X)}{(\hat{\sigma}_n^2 n)^{1/2}} \to N(0,1) \tag{1.5}$$

and

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2\hat{\sigma}_n^2 n L_2 n)^{1/2}} = 1 \quad \text{a.s.} \tag{1.6}$$

It is known that (1.5) remains valid under (1.3), and in [8] we showed this is also the case for (1.6). Thus without having to change the normalizer, both (1.5) and (1.6) remain valid for all distributions in the domain of attraction of a Gaussian law. The reason the normalizer in (1.6) works when classical normalizers fail is that large values of X_i play no role in (1.6) since they appear in both numerator and denominator which effectively negates their influence.

The actual result proved in [8] uses the normalizer

$$(2V_n^2 L_2 n)^{1/2}, \tag{1.7}$$

where $V_n^2 = X_1^2 + \cdots + X_n^2$. This is technically easier to deal with than $\hat{\sigma}_n^2$ and if, as we assume from now on, $E(X) = 0$ when $E(X^2) < \infty$, then no matter what the distribution of X,

$$V_n^2/(\hat{\sigma}_n^2 n) \to 1 \quad \text{a.s.}$$

Thus (1.7) is equivalent to the normalizer in (1.6). In what follows below, we will also use the normalizer $(2V_n^2 L_2 V_n^2)^{1/2}$, which is easily seen to be equivalent to (1.7) under the conditions imposed in [8].

The use of self-normalization in the LIL is not new, it goes back to Marcinkiewicz who observed that for any symmetric distribution

$$\limsup_{n\to\infty} \frac{S_n}{(2V_n^2 L_2 V_n^2)^{1/2}} \le 1 \quad \text{a.s.} \tag{1.8}$$

For this result the random variables need not be identically distributed, just independent and $V_n^2 \to \infty$ w.p.1. Observe that this result and the strong law gives the upper bound in (1.1) for i.i.d. variables provided X is symmetric. Further, the proof of (1.8) is much easier than the classical one which involves truncation and Kolmogorov's exponential bounds. Section 2 contains a sketch of this proof together with a refinement of (1.8) which we present as

THEOREM 1. *Let $X_1, X_2, \ldots$ be independent, symmetric random variables with $V_n^2 \to \infty$ w.p.1. If ϕ is nondecreasing and positive eventually and $J(\phi) < \infty$*

where

$$J(\phi) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} e^{-\phi^2(n)/2}, \tag{1.9}$$

then

$$P\big(S_n > V_n \phi(V_n^2)\ i.o.\big) = 0. \tag{1.10}$$

REMARK 1. When (1.10) holds, ϕ is called an upper function relative to V_n and when it fails ϕ is called a lower function.

REMARK 2. Suppose $V_n^2 \le n^{1+\gamma}$ eventually with probability 1 for some $\gamma > 0$, and the conditions of Theorem 1 hold. Then by the argument given in Step 1 of the proof of Theorem 2, we have $J(\phi) < \infty$ implying $J(\psi) < \infty$, where $\psi(t) = \phi(t^{1/(1+\gamma)})$. Hence by Theorem 1 we have

$$P\big(S_n > V_n \psi(V_n^2) \text{ i.o.}\big) = 0.$$

Now ψ eventually nondecreasing implies for all n sufficiently large that

$$\psi(V_n^2) \le \psi(n^{1+\gamma}) = \phi(n),$$

and hence

$$P(S_n > V_n \phi(n) \text{ i.o.}) = 0.$$

Thus when $P(V_n^2 \le n^{1+\gamma}$ eventually$) = 1$ for some $\gamma > 0$, we can replace $\phi(V_n^2)$ by $\phi(n)$ in (1.10). Hence it is easy to see that this more traditional form of the Kolmogorov–Erdös test for upper functions applies when $X, X_1, X_2, \ldots$ are i.i.d. and in the Feller class as in [8].

Our next result covers both upper and lower functions for the i.i.d. situation when self-normalizations are employed, and X is assumed symmetric. The theorem we prove generalizes some work of Feller on the Kolmogorov–Erdös test (see [1] and [6] for recent background). It differs from the classical results in that when $E(X^2) < \infty$, but nothing else is assumed, one need not use normalizers which differ from the natural one used when higher moments are given. More precisely, for nonrandom normalizers, ϕ nondecreasing and positive eventually and $J(\phi)$ as in (1.9), the following results are known (see [1] and [6]).

RESULT A. If $X, X_1, X_2, \ldots$ are i.i.d., $E(X) = 0$, $E(X^2) = 1$ and

$$\limsup_{t \to \infty} E\big(X^2 I(|X| > t)\big) L_2 t < \infty,$$

then

$$P\big(S_n > \sqrt{n}\,\phi(n) \text{ i.o.}\big) = 0 \qquad (= 1)$$

according as

$$J(\phi) < \infty \qquad (= \infty).$$

If just $E(X^2) = 1$ and $E(X) = 0$ is assumed, then this result is false. In this case $\sqrt{n}$ needs to be replaced by the less natural B_n as given below.

RESULT B. If $X, X_1, X_2, \dots$ are i.i.d., $E(X) = 0$, $E(X^2) = 1$ and

$$B_n^2 = \sum_{j=1}^{n} E\Big(X^2 I\big(|X| \le j^{1/2}/(L_2 j)^2\big)\Big),$$

then

$$P(S_n > B_n \phi(n) \text{ i.o.}) = 0 \qquad (= 1)$$

according as

$$J(\phi) < \infty \qquad (= \infty).$$

What we show, in case X is symmetric, is that Results A and B merge into one result if the random normalizer V_n is used. This makes the situation particularly pleasing when contrasted with the need for replacing $\sqrt{n}$ by B_n in the above. We prove

THEOREM 2. *Let $X, X_1, X_2, \dots$ be i.i.d. symmetric with $0 < E(X^2) < \infty$. If ϕ is eventually nondecreasing and positive, then*

$$P(S_n > V_n \phi(n) \ i.o.) = 0 \qquad (= 1) \tag{1.11}$$

according as

$$J(\phi) < \infty \qquad (= \infty), \tag{1.12}$$

where $J(\phi)$ is as in (1.9).

The proof of Theorem 2 (given in Section 3) involves the general results in [7] used in a way suggested by [1] and the proof of Lemma 4.2 in [4]. Of course, a number of Feller's papers dealt with the extension of the Kolmogorov–Erdös test to various settings, and interesting references are contained in [1], [6] and [7].

In Section 4 we turn to the problem of determining what happens when the symmetry assumption in (1.8) is dropped. The only results that we are aware of are those in [8] which deal with the i.i.d. case. If just independence is assumed then it seems to be a difficult problem to find anything positive to say. The natural first step is to try to prove (1.8) under the assumption $E(X_k) = 0$ for all k. This, however, is false as the following example of M. Weiss [12] shows. Let X_k be independent with

$$P\big(X_k = -(k^2 - 1)\big) = \frac{1}{k^2} \quad \text{and} \quad P(X_k = 1) = \frac{k^2 - 1}{k^2}.$$

Then by Borel–Cantelli,

$$P(X_k = 1 \text{ eventually}) = 1.$$

Hence $S_n/n \to 1$ and $V_n^2/n \to 1$, from which it is immediate that (1.8) fails. Thus to obtain a result like (1.8) under these conditions we must change the normalizer. The following curious mixture of classical and self-normalizers appeared previously in [2]:

$$W_n^2 = \sum_{k=1}^{n} \left(X_k^2 I(X_k > 0) + E\left(X_k^2 I(X_k < 0) \right) \right). \tag{1.13}$$

Using this we are able to prove

THEOREM 3. *Assume X_k are independent, $E(X_k) = 0$, $E(X_k^2 I(X_k < 0)) < \infty$ for all k and $W_n^2 \to \infty$, where W_n^2 is given by* (1.13). *Then*

$$\limsup_{n\to\infty} \frac{S_n}{\left(2W_n^2 L_2 W_n^2\right)^{1/2}} \le 1 \quad a.s. \tag{1.14}$$

Our proof of (1.14) also does not require truncation or the Kolmogorov exponential bounds. Further, observe that unlike the results discussed above, this is a one-sided LIL result. However, in the finite variance case it is two-sided and gives the upper bound in (1.1). That is, if $X, X_1, X_2, \dots$ are i.i.d. with $E(X) = 0$ and $0 < E(X^2) = \sigma^2 < \infty$, then $W_n^2 \sim n\sigma^2$ by the strong law of large numbers, and (1.14) implies

COROLLARY 1. *If $X, X_1, X_2, \dots$ are i.i.d. with $E(X) = 0$ and $0 < E(X^2) = \sigma^2 < \infty$, then*

$$\limsup_{n\to\infty} \frac{S_n}{\left(2\sigma^2 n L_2 n\right)^{1/2}} \le 1 \quad a.s. \tag{1.15}$$

The shortest proof of equality in (1.15) is based on a blocking argument and the central limit theorem. This approach is known to a number of workers in the area, but does not seem to be anywhere in the literature in its simplest form, so we give it in Section 5. A more refined, though less elementary version of this argument, is in [3]. For those just interested in the classical LIL the paper is written so that Sections 4 and 5 are self-contained. Here the reader will find a proof of (1.1) when $\sigma^2 < \infty$. Our hope is that these results will convince the reader that the classical LIL may be presented in a highly efficient way.

2. Proof of (1.8) and Theorem 1. To establish Marcinkiewicz's result (1.8) we assume without loss of generality that $X_1, X_2, \dots$ are defined on a probability space $(\Omega, \mathscr{F}, P)$ which also supports a sequence of independent Rademacher random variables $\{\varepsilon_j: j \ge 1\}$ independent of the initial sequence $\{X_j: j \ge 1\}$. If $\mathscr{G}$ is the minimal σ-field generated by $\{X_j: j \ge 1\}$, then to prove

(1.8) it suffices to show

$$P\left(\limsup_n \frac{\sum_{j=1}^n \varepsilon_j X_j}{\left(2V_n^2 L_2 V_n^2\right)^{1/2}} \le 1 \text{ a.s. } |\mathscr{G}\right) = 1.$$

But by independence this reduces to showing

$$\limsup_n \sum_{i=1}^n \frac{\varepsilon_i a_i}{\left(2A_n^2 L_2 A_n^2\right)^{1/2}} \le 1 \tag{2.1}$$

for all sequences $\{a_i: i \ge 1\}$, where $A_n^2 = \sum_{i=1}^n a_i^2 \to \infty$.

LEMMA 2.1. *For all $n \ge 1$ and all $\alpha > 0$,*

$$P(T_n > \alpha A_n) \le \exp\{-\alpha^2/2\}, \tag{2.2}$$

where $T_n = \sum_{i=1}^n \varepsilon_i a_i$.

PROOF. For any $u > 0$,

$$\begin{aligned} P(T_n > \alpha A_n) &\le \exp\{-u\alpha A_n\} E(\exp(uT_n)) \\ &= \exp\{-u\alpha A_n\} \prod_{i=1}^n \cosh(u a_i) \\ &\le \exp\{-u\alpha A_n\} \prod_{i=1}^n \exp\left(u^2 a_i^2/2\right) \\ &= \exp\left\{-u\alpha A_n + u^2 A_n^2/2\right\}. \end{aligned}$$

Set $u = \alpha A_n^{-1}$ to obtain (2.2).

The proof of (2.1) is now a standard Borel–Cantelli argument. Fix $\beta > 1$ and let

$$n_k = \sup\left\{n: A_n \le \beta^k\right\}.$$

Then $n_k \nearrow \infty$, since $A_n^2 \to \infty$. Next we define

$$E_k = \left\{T_n > \beta^2 \left(2A_n^2 L_2 A_n^2\right)^{1/2} \text{ for some } n \in (n_k, n_{k+1}]\right\}.$$

Then

$$\begin{aligned} P(E_k) &\le P\left(T_n > \beta^2\left(2\beta^{2k} L_2 \beta^{2k}\right)^{1/2} \text{ for some } n \in (n_k, n_{k+1}]\right) \\ &\le 2P\left(T_{n_{k+1}} > \beta^2\left(2\beta^{2k} L_2 \beta^{2k}\right)^{1/2}\right) \qquad \text{(by Lévy's inequality)} \\ &\le 2P\left(T_{n_{k+1}} > \beta\left(2A_{n_{k+1}}^2 L_2 \beta^{2k}\right)^{1/2}\right). \end{aligned}$$

Thus by (2.2),

$$\sum_k P(E_k) \le 2\sum_k \exp\left\{-\beta^2 L_2 \beta^{2k}\right\} < \infty.$$

Since $\beta > 1$ was arbitrary, this proves (2.1) and the proof of (1.8) is complete. □

REMARK. By again conditioning on $\mathscr{G}$, it follows from Kolmogorov's LIL that

$$\limsup_n \frac{S_n}{(2V_n^2 L_2 V_n^2)^{1/2}} = 1 \quad \text{a.s.} \tag{2.3}$$

if $V_n^2 \to \infty$ a.s. and

$$\frac{|X_n|(L_2 V_n^2)^{1/2}}{V_n} \to 0 \quad \text{a.s.} \tag{2.4}$$

The example of Marcinkiewicz and Zygmund [10] shows that if (2.4) is relaxed, then (2.3) may fail.

To prove Theorem 1 we assume without loss of generality that ϕ is nondecreasing and positive, and also that $\{X_j\}$ is defined on a probability space $(\Omega, \mathscr{F}, P)$ which supports an independent sequence of i.i.d. Rademacher random variables $\{\varepsilon_j: j \geq 1\}$. It is also possible to show by a standard argument that if $J(\phi) < \infty$ implies (1.11) when $\phi \nearrow \infty$ and

$$L_2 n \leq \phi^2(n) \leq 3L_2 n, \tag{2.5}$$

then Theorem 1 can be proved without the restriction (2.5). Hence (2.5) is assumed throughout the remainder of the section.

If $J(\phi) < \infty$, we define

$$t_k = \exp\{k/Lk\},$$

$$\sigma_k = \sup\{n \in \mathbf{Z}^+ : V_n^2 \leq t_k\}.$$

Then $t_k \nearrow \infty$ implies $\sigma_k \nearrow \infty$ with probability 1. Further, if

$$F_k = \{S_m > V_m \phi(V_m^2) \text{ for some } m \in (\sigma_k, \sigma_{k+1}]\},$$

then (1.10) holds if $P(F_k \text{ i.o.}) = 0$.

Letting $\tau_k = \sigma_k + 1$, $I(k) = (\sigma_k, \sigma_{k+1}]$ and denoting the sigma field generated by $\{X_j: j \geq 1\}$ by $\mathscr{G}$ we have

$$\begin{aligned}
P(F_k) &= P\left(\sup_{m \in I(k)} \frac{S_m}{V_m \phi(V_m^2)} > 1 \right) \\
&= \int_\Omega P\left(\sup_{m \in I(k)} \sum_{j=1}^m \frac{\varepsilon_j X_j}{V_m \phi(V_m^2)} > 1 | \mathscr{G} \right) dP \\
&\leq \int_{\{\sigma_k < \sigma_{k+1}\}} P\left(\sup_{m \in I(k)} \sum_{j=1}^m \varepsilon_j X_j > V_{\tau_k} \phi(V_{\tau_k}^2) | \mathscr{G} \right) dP \\
&\leq 2 \int_{\{\sigma_k < \sigma_{k+1}\}} P\left(\sum_{j=1}^{\sigma_{k+1}} \varepsilon_j X_j > V_{\tau_k} \phi(V_{\tau_k}^2) | \mathscr{G} \right) dP
\end{aligned} \tag{2.6}$$

(by Lévy's inequality and the independence of $\{\varepsilon_j: j \geq 1\}$ and $\{X_j: j \geq 1\}$)

$$= 2\int_{\{\sigma_k < \sigma_{k+1}\}} P\left(\sum_{j=1}^{\sigma_{k+1}} \varepsilon_j X_j / V_{\sigma_{k+1}} > V_{\tau_k}\phi\big(V_{\tau_k}^2\big)/V_{\sigma_{k+1}} | \mathscr{G} \right) dP$$

$$\leq 2\Lambda \int_{\{\sigma_k < \sigma_{k+1}\}} \Big(V_{\tau_k}\phi\big(V_{\tau_k}^2\big)/V_{\sigma_{k+1}}\Big)^{-1} \exp\Big\{-\tfrac{1}{2}V_{\tau_k}^2\phi^2\big(V_{\tau_k}^2\big)/V_{\sigma_{k+1}}^2\Big\}\, dP,$$

where Λ is an absolute constant uniform in k provided $V_{\tau_k}\phi(V_{\tau_k}^2)/V_{\sigma_{k+1}} \geq 3$ by applying Corollary 2 of [5]. But on the set $\{\sigma_k < \sigma_{k+1}\}$, we have

$$V_{\tau_k}\phi\big(V_{\tau_k}^2\big)/V_{\sigma_{k+1}} \geq (t_k/t_{k+1})^{1/2}\phi(t_k) \geq 3$$

for all k sufficiently large, so (2.6) implies for k large that

$$P(F_k) \leq 2\Lambda\Big((t_k/t_{k+1})^{1/2}\phi(t_k)\Big)^{-1}\exp\big\{-\tfrac{1}{2}t_k\phi^2(t_k)/t_{k+1}\big\}. \tag{2.7}$$

Now $t_k/t_{k+1} = e^{k/Lk - ((k+1)/L(k+1))}$, so for large k (since $e^{-x} \geq 1 - \frac{3}{2}x$ for x small)

$$\frac{t_k}{t_{k+1}} \geq 1 - \frac{3}{2}\left\{\frac{(k+1)}{L(k+1)} - \frac{k}{Lk}\right\} \geq 1 - \frac{2}{Lk}.$$

Thus

$$\sum_k P(F_k) < \infty \tag{2.8}$$

if

$$\sum_k (\phi(t_k))^{-1}\exp\left\{-\frac{1}{2}\phi^2(t_k)\left(1 - \frac{2}{Lk}\right)\right\} < \infty.$$

Now $\phi^2(t_k)/Lk$ is bounded under (2.5) as $k \to \infty$ and hence we need only show

$$\sum_k (\phi(t_k))^{-1}\exp\left\{-\frac{1}{2}\phi^2(t_k)\right\} < \infty. \tag{2.9}$$

Now $J(\phi) < \infty$ implies

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} \frac{\phi(n)}{n} e^{-\phi^2(n)/2} \\
&= \sum_{k=1}^{\infty} \sum_{n \in (t_{k-1}, t_k)]} \frac{\phi(n)}{n} e^{-\phi^2(n)/2} \\
&\geq \sum_{k=1}^{\infty} \frac{\phi(t_k)}{t_k} e^{-\phi^2(t_k)/2}(t_k - t_{k-1}),
\end{aligned}$$

and since

$$t_k - t_{k-1} \sim t_k/Lk \approx t_k/\phi^2(t_k),$$

the above implies (2.9) (we are using the notation $a_k \approx b_k$ to mean $a_k b_k^{-1}$ is bounded above and below by strictly positive constants for large k). Thus (2.8) holds and $P(F_k \text{ i.o.}) = 0$ as required. Thus $J(\phi) < \infty$ implies (1.10) and the theorem is proved.

3. Proof of Theorem 2. We assume without loss of generality that $\phi(n)$ is nondecreasing and positive and that $\{X_j: j \geq 1\}$ is defined on a probability space $(\Omega, \mathscr{F}, P)$ which also supports an independent sequence of i.i.d. Rademacher random variables $\{\varepsilon_j: j \geq 1\}$. As before it is a standard argument to show that if Theorem 2 holds for $\phi(n)$ nondecreasing, positive and such that

$$L_2 n \leq \phi^2(n) \leq 3L_2 n \tag{3.1}$$

for large n, then Theorem 2 holds without the restriction (3.1). Hence we will assume that (3.1) holds.

STEP 1. Proof that $J(\phi) < \infty$ implies the probability in (1.11) is zero. Extend ϕ to be continuous on $[0, \infty)$ by setting $\phi(0) = \phi(1)$ and then linearly interpolating between the integers. Since $J(\phi) < \infty$ iff $\int_1^\infty (\phi(t)/t) e^{-\phi^2(t)/2}\, dt < \infty$, a simple change of variables yields that

$$J(\phi) < \infty \quad \text{iff} \quad J(\psi) < \infty,$$

where

$$\psi(t) = \phi(t/2\sigma^2), \qquad 0 \leq t < \infty.$$

Hence $J(\phi) < \infty$ implies $J(\psi) < \infty$, and by Theorem 1,

$$P\big(S_n > V_n \psi(V_n^2) \text{ i.o.}\big) = 0.$$

By the strong law of large numbers $V_n^2 \sim \sigma^2 n$, and thus ψ eventually nondecreasing implies $\psi(V_n^2) \leq \psi(2\sigma^2 n)$, so

$$P\big(S_n > V_n \psi(2\sigma^2 n) \text{ i.o.}\big) = 0.$$

Now $\psi(2\sigma^2 n) = \phi(n)$ and hence $J(\phi) < \infty$ implies the probability in (1.11) is zero.

Now we turn to the proof that $J(\phi) = \infty$ implies

$$P(S_n > V_n \phi(n) \text{ i.o.}) = 1.$$

STEP 2. Setting the problem up. Write

$$S_n = T_n + U_n, \qquad n \geq 1,$$

where

$$T_n = \sum_{j=1}^n X_j I\big(|X_j| \leq j^{1/2}\big),$$

$$U_n = \sum_{j=1}^n X_j I\big(|X_j| > j^{1/2}\big).$$

Now $P(|X_j| > j^{1/2} \text{ i.o.}) = 0$, by Borel–Cantelli since $E(X^2) < \infty$; thus $U_n = O(1)$. Further, by the strong law of large numbers, $\lim_{n \to \infty} V_n^2/n = E(X^2)$, and hence

$$\frac{|U_n|}{V_n} < \frac{C}{\phi(n)},$$

eventually for every $C > 0$. Thus for each $C > 0$,

$$\{S_n > V_n\phi(n) \text{ i.o.}\} \supseteq \left\{T_n > V_n\left(\phi(n) + \frac{C}{\phi(n)}\right) \text{ i.o.}\right\}. \tag{3.2}$$

Next let $B_n^2 = \sum_{j=1}^n X_j^2 I(|X_j| \le j^{1/2})$; then as above $E(X^2) < \infty$ implies

$$V_n^2 - B_n^2 = O(1).$$

Hence for any $C > 0$,

$$\{T_n > V_n(\phi(n) + C/\phi(n)) \text{ i.o.}\} \supseteq \{T_n > B_n(\phi(n) + 2C/\phi(n)) \text{ i.o.}\}. \tag{3.3}$$

To see this we only need show

$$(V_n - B_n)(\phi(n) + C/\phi(n)) \le CB_n/\phi(n)$$

for large n. Since $V_n^2 \sim nE(X^2)$ and $V_n - B_n = O(1)$, this follows from the bounds on $\phi(n)$ in (3.1).

Combining (3.2) and (3.3) we have $J(\phi) = \infty$, implying

$$P(S_n > V_n\phi(n) \text{ i.o.}) = 1, \tag{3.4}$$

provided we show $J(\phi) = \infty$ implies

$$P(T_n > B_n(\phi(n) + d/\phi(n)) \text{ i.o.}) = 1 \tag{3.5}$$

for arbitrary $d > 0$.

STEP 3. Proof that $J(\phi) = \infty$ implies (3.5) holds. Letting $\mathscr{G}$ denote the minimal σ-field generated by $\{X_j: j \ge 1\}$, we have (3.5) if

$$P\left(\sum_{j=1}^n Y_j > B_n(\phi(n) + d/\phi(n)) \text{ i.o.}|\mathscr{G}\right) = 1 \quad \text{a.s.}, \tag{3.6}$$

where

$$Y_j = \varepsilon_j X_j I(|X_j| \le j^{1/2}), \qquad j \ge 1.$$

To prove (3.6), we apply a result of Feller [7] to the sequence $\{Y_j: j \ge 1\}$ conditionally, i.e., for $X_1, X_2, \dots$ fixed. That is, for $X_1, X_2, \dots$ fixed, $\{Y_j: j \ge 1\}$ then consists of independent symmetric random variables with respect to $P(\cdot|\mathscr{G})$ such that

$$E((Y_1 + \cdots + Y_n)^2|\mathscr{G}) = B_n^2.$$

Using Feller's notation we define

$$\alpha_n = \phi(n) \quad \text{and} \quad \beta_n = 1/\phi(n)$$

with

$$a_n = B_n\phi(n) \quad \text{and} \quad b_n = B_n/\phi(n).$$

Then $\{\alpha_n\}$ increases and Condition A of Feller ([7], page 403) is easily satisfied. Condition B of Feller also holds since both B_n and $\phi(n)$ are nondecreasing. We omit the details as verification is elementary in this setting.

Hence Feller's assumptions are satisfied (conditionally) and Theorem 1(b) of [7] applies.

Now (3.6) holds if

$$P\left(\sum_{j=1}^{n} Y_j \in B_n\phi(n)\left(1+\frac{d}{\phi^2(n)},\,1+\frac{4d}{\phi^2(n)}\right)\text{ i.o.}|\mathscr{G}\right)=1, \tag{3.7}$$

and by Theorem 1(b) of [7], (3.7) follows if

$$\sum_{n=1}^{\infty}\min\left(1,\frac{(a_{n+1}-a_n)}{b_n}\right)P\left(\left(\sum_{j=1}^{n}Y_j-a_n\right)b_n^{-1}\in(2d,3d)|\mathscr{G}\right)=\infty \tag{3.8}$$

with probability 1.

Now

$$\left(\sum_{j=1}^{n}Y_j-a_n\right)b_n^{-1}\in(2d,3d)$$
$$\text{iff}\quad\sum_{j=1}^{n}Y_j\in B_n\phi(n)\left(1+\frac{2d}{\phi^2(n)},\,1+\frac{3d}{\phi^2(n)}\right),$$

and by applying the Berry–Esseen result in Petrov ([11], page 132), the divergence of (3.8) follows if

$$\begin{aligned}\sum_{n=1}^{\infty}\min\left(1,\frac{a_{n+1}-a_n}{b_n}\right)&\left(\Phi\left(\phi(n)+\frac{3d}{\phi(n)}\right)\right.\\ &\left.-\Phi\left(\phi(n)+\frac{2d}{\phi(n)}\right)\right)=\infty,\end{aligned} \tag{3.9}$$

where Φ is the distribution function of a standard Gaussian random variable and

$$\sum_{n=1}^{\infty}\min\left(1,\frac{a_{n+1}-a_n}{b_n}\right)\sum_{j=1}^{n}E\left(|Y_j|^3|\mathscr{G}\right)\left(B_n(1+\phi(n))\right)^{-3}<\infty \tag{3.10}$$

with probability 1. Since $B_n \approx \sqrt{n}$ and $\phi^2(n) \approx L_2 n$ as $n \to \infty$, it suffices to show (3.9) and

$$\sum_{n=1}^{\infty}\min\left(1,\frac{a_{n+1}-a_n}{b_n}\right)\sum_{j=1}^{n}E\left(|Y_j|^3|\mathscr{G}\right)(nL_2n)^{-3/2}<\infty, \tag{3.11}$$

with probability 1.

Now

$$\sum_{n=1}^{\infty} \min\left(1, \frac{a_{n+1} - a_n}{b_n}\right) \sum_{j=1}^{n} E\big(|Y_j|^3|\mathscr{G}\big)(nL_2n)^{-3/2}$$

$$\leq \sum_{n=1}^{\infty} \frac{(a_{n+1} - a_n)}{b_n} \sum_{j=1}^{n} |X_j|^3 I\big(|X_j| \leq j^{1/2}\big)(nL_2n)^{-3/2}$$

$$\leq \sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} \frac{(a_{n+1} - a_n)}{b_n} \sum_{j=1}^{2^{k+1}} |X_j|^3 I\big(|X_j| \leq j^{1/2}\big)(nL_2n)^{-3/2}.$$

By (3.1), $b_n^2 \approx n(L_2n)^{-1}$ and $a_n^2 \approx nL_2n$; hence (3.11) converges if

$$\sum_{k=1}^{\infty} \sum_{j=1}^{2^{k+1}} |X_j|^3 I\big(|X_j| \leq j^{1/2}\big) 2^{-3k/2}\big(L_2 2^k\big)^{-1/2} \tag{3.12}$$

converges with probability 1. The series (3.12) converges with probability 1 if it has finite expectation and taking expectations

$$\begin{aligned} &E\left(\sum_{k=1}^{\infty} \sum_{j=1}^{2^{k+1}} |X_j|^3 I\big(|X_j| \leq j^{1/2}\big) 2^{-3k/2}\big(L_2 2^k\big)^{-1/2}\right) \\ &\quad \leq \sum_{k=1}^{\infty} 2^{k+1} E\big(|X|^3 I(|X| \leq 2^{(k+1)/2})\big) 2^{-3k/2}\big(L_2 2^k\big)^{-1/2} < \infty, \end{aligned} \tag{3.13}$$

where the last inequality follows from an argument in Bai ([1], page 390). Hence (3.11) holds.

Thus it suffices to prove (3.9) when $J(\phi) = \infty$ and $\phi(n) \nearrow \infty$. Now with probability 1, eventually for all n,

$$\begin{aligned} \frac{(a_{n+1} - a_n)}{b_n} &= \frac{B_{n+1}^2 \phi^2(n+1) - B_n^2 \phi^2(n)}{b_n(a_{n+1} + a_n)} \\ &\geq \frac{1}{4} \frac{\big(B_{n+1}^2 \phi^2(n+1) - B_n^2 \phi^2(n)\big)}{(nE(X^2))} \\ &\geq \frac{1}{4} \frac{\big(B_{n+1}^2 - B_n^2\big)\phi^2(n)}{(nE(X^2))} \qquad [\text{since } \phi(n) \nearrow] \\ &\geq \frac{1}{4} \frac{X_{n+1}^2}{n} \frac{I\big(|X_{n+1}| \leq (n+1)^{1/2}\big)\phi^2(n)}{E(X^2)}. \end{aligned}$$

Hence by (3.1), for each $C > 0$ and all n sufficiently large,

$$\min\left(1, \frac{a_{n+1} - a_n}{b_n}\right) \geq \frac{1}{4} \frac{X_{n+1}^2}{n} \frac{I(|X_{n+1}| \leq C)\phi^2(n)}{E(X^2)}.$$

Further, given any $M > 0$, there exists a constant $c > 0$ such that if $0 < a < b$ and $b^2 - a^2 \le M$, then

$$\Phi(b) - \Phi(a) \ge ce^{-a^2/2}(b^2 - a^2)/(2b\sqrt{2\pi}).$$

Hence the series in (3.9) diverges with probability 1 if for some $C > 0$,

$$\sum_{n=1}^{\infty} \frac{X_{n+1}^2}{n} I(|X_{n+1}| \le C) \frac{\phi^2(n)}{2\phi(n)} e^{-(\phi^2(n)/2)(1+2d/\phi^2(n))^2} \left(2d + \frac{5d^2}{\phi^2(n)} \right)$$

diverges with probability 1 for each $d > 0$.

Now the last series diverges with probability 1 for $d > 0$ and $\phi(n) \nearrow \infty$ if

$$\sum_{n=1}^{\infty} X_{n+1}^2 I(|X_{n+1}| \le C) \frac{\phi(n)}{n} e^{-\phi^2(n)/2} \tag{3.14}$$

diverges with probability 1. If (3.14) converges with positive probability, it converges with probability 1 (recall the X_j's are independent), and by Kolmogorov's three series theorem,

$$\sum_{n=1}^{\infty} E(X^2 I(|X| \le C)) \frac{\phi(n)}{n} e^{-\phi^2(n)/2} \tag{3.15}$$

must also converge for each $C > 0$. However, by choosing C such that $E(X^2 I(|X| \le C)) > 0$, the series (3.15) diverges because $J(\phi) = \infty$. Thus the series in (3.9) diverges and Step 3 is proved.

4. Proof of Theorem 3. Throughout this section we assume X_k are independent, $EX_k = 0$, $E(X_k^2 I(X_k < 0)) < \infty$ for all k, and $W_n^2 \to \infty$ a.s. where W_n^2 is as in (1.13). For $y \in \mathbf{R}$, let $y^+ = yI(y \ge 0)$ and $y^- = (-y)^+$. The following two lemmas are proved in a more general setting in [2]. Since the proofs are so much simpler in our situation they are given below.

LEMMA 4.1. *For any* $\theta > 0$,

$$M_n^\theta = \exp\{\theta S_n - \theta^2 W_n^2/2\}, \qquad n \ge 1, \tag{4.1}$$

is a supermartingale.

PROOF. The key observation is that for all y,

$$\exp\{y - \tfrac{1}{2}(y^+)^2\} \le 1 + y + \tfrac{1}{2}(y^-)^2.$$

Thus for all k,

$$E\left(\exp\{\theta X_k - \theta^2 (X_k^+)^2/2\}\right) \le 1 + (\theta^2/2)E\left((X_k^-)^2\right) \le \exp\left\{(\theta^2/2)E\left((X_k^-)^2\right)\right\}.$$

Hence by independence, M_n^θ is integrable, and letting $\mathscr{F}_n = \sigma\{X_i: 1 \le i \le n\}$,

$$E\left(M_{n+1}^\theta | \mathscr{F}_n\right) = M_n^\theta E\Big(\exp\Big\{\theta X_{n+1} - \theta^2\big((X_{n+1}^+)^2 + E((X_{n+1}^-)^2)\big)/2\Big\}\Big)$$
$$\le M_n^\theta,$$

so the lemma is proved. □

LEMMA 4.2. *If σ is a positive integer valued random variable possibly taking the value ∞ and $x > 0$, $y > 0$, then*

$$P(S_\sigma > x, W_\sigma < y, \sigma < \infty) \le \exp\left\{-\frac{1}{2}\left(\frac{x}{y}\right)^2\right\}. \tag{4.2}$$

PROOF. For any $\theta > 0$,

$$\begin{aligned} P(S_\sigma > x, W_\sigma < y, \sigma < \infty) &\le P\left(M_\sigma^\theta \ge \exp\{\theta x - \theta^2 y^2/2\}, \sigma < \infty\right) \\ &\le P\Big(\sup_n M_n^\theta \ge \exp\{\theta x - \theta^2 y^2/2\}\Big) \\ &\le E\left(M_1^\theta\right)\exp\{\theta^2 y^2/2 - \theta x\}, \end{aligned}$$

by Doob's inequality. Since $E(M_1^\theta) \le 1$, the result is established by setting $\theta = xy^{-2}$. □

PROOF OF THEOREM 3. Fix $\beta > 1$ and define

$$\sigma_k = \sup\{n: W_n \le \beta^k\},$$
$$E_k = \left\{S_n > \beta^2\left(2W_n^2 L_2 W_n^2\right)^{1/2} \text{ for some } n \in (\sigma_k, \sigma_{k+1}]\right\}.$$

Since $\beta^k < W_n \le \beta^{k+1}$ if $\sigma_k < n \le \sigma_{k+1}$,

$$P(E_k) \le P\Big(S_n > \beta^2\left(2\beta^{2k} L_2 \beta^{2k}\right)^{1/2} \text{ and } W_n \le \beta^{k+1} \text{ for some } n\Big).$$

Now let

$$\sigma = \inf\Big\{n: S_n > \beta^2\left(2\beta^{2k} L_2 \beta^{2k}\right)^{1/2} \text{ and } W_n \le \beta^{k+1}\Big\}.$$

Then

$$\begin{aligned} P(E_k) &\le P(\sigma < \infty) \\ &= P\Big(S_\sigma > \beta^2\left(2\beta^{2k} L_2 \beta^{2k}\right)^{1/2}, W_\sigma \le \beta^{k+1}, \sigma < \infty\Big) \\ &\le \exp\{-\beta^2 L_2 \beta^{2k}\}, \end{aligned}$$

by (4.2). Hence $\sum_k P(E_k) < \infty$, and (1.14) follows by Borel–Cantelli since $\beta > 1$ was arbitrary. □

5. Proof of equality in (1.15). We assume without loss of generality that $EX = 0$ and $E(X^2) = 1$. To prove equality in (1.15) we fix $\alpha \in (0, 1)$ and choose $\eta > 0$ sufficiently large that

$$(5.1) \qquad P(G > \eta) \geq e^{-\eta^2/2\alpha},$$

where G is $N(0,1)$.

This is possible since $P(G > \eta) \sim e^{-\eta^2/2}/(2\pi\eta^2)^{1/2}$ as $\eta \to \infty$ and $0 < \alpha < 1$. Now let $a = \eta^2/(2\alpha^2)$ and define $b_j = j[an/L_2 n]$ for $j = 0, \dots, j_n$, where $j_n = [L_2 n/a]$ and $[\cdot]$ is the greatest integer function. Then

$$\left\{S_n > \alpha(2nL_2 n)^{1/2}\right\} \supseteq \left\{S_{b_j} - S_{b_{j-1}} > \alpha(2nL_2 n)^{1/2} a (L_2 n)^{-1}, \right.$$
$$\left. j = 1, \dots, j_n, S_n - S_{b_{j_n}} > 0\right\}.$$

By the central limit theorem,

$$\inf_k P(S_k > 0) \geq c > 0$$

and

$$P\left(S_{[an/L_2 n]}(an/L_2 n)^{-1/2} > \alpha(2a)^{1/2}\right) \geq (1 - \varepsilon)P(G > \eta),$$

for any $\varepsilon > 0$ provided n is sufficiently large (depending on ε). Now choose $\varepsilon > 0$ small enough that $\delta = \alpha - a^{-1}\log(1 - \varepsilon)$ satisfies $\delta < 1$. Then by independence of increments and (5.1) for large n,

$$\begin{aligned} P\left(S_n > \alpha(2nL_2 n)^{1/2}\right) &\geq c((1 - \varepsilon)P(G > \eta))^{L_2 n/a} \\ &\geq c\exp\left\{-L_2 n\left(\alpha - a^{-1}\log(1 - \varepsilon)\right)\right\} \\ &= c\exp\{-\delta L_2 n\}. \end{aligned}$$

Now let $n_k = k^k$. Then $n_{k+1}/n_k \to \infty$ and if $\beta < \alpha$, for large k,

$$\begin{aligned} &P\left(S_{n_{k+1}} - S_{n_k} > \beta(2n_{k+1}L_2 n_{k+1})^{1/2}\right) \\ &\quad \geq P\left(S_{n_{k+1}-n_k} > \alpha(2(n_{k+1} - n_k)L_2(n_{k+1} - n_k))^{1/2}\right) \\ &\quad \geq \exp\{-\delta L_2(n_{k+1} - n_k)\}. \end{aligned}$$

Since $\delta < 1$, this gives a divergent series, and hence

$$\limsup_{k\to\infty} \frac{S_{n_{k+1}} - S_{n_k}}{(2n_{k+1}L_2 n_{k+1})^{1/2}} \geq \beta.$$

But

$$\limsup_{k\to\infty} \frac{|S_{n_k}|}{(2n_{k+1}L_2 n_{k+1})^{1/2}} = 0$$

by the upper bound result in (1.15) (applied to X and $-X$) when $n_{k+1}/n_k \to \infty$. Since $\beta < \alpha < 1$ are arbitrary, this completes the proof of equality in (1.15).

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result applied Theorem 1(a) of [7], and was much more lengthy. A similar simplification was also suggested to us by Uwe Einmahl, and is applied in Remark 2 following the statement of Theorem 1.

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