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RANDOMIZED STOPPING POINTS AND OPTIMAL STOPPING ON THE PLANE[1]

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We prove that in continuous time, the extremal elements of the set of adapted random measures on $\mathbb{R}_+^2$ are Dirac measures, assuming the underlying filtration satisfies the conditional qualitative independence property. This result is deduced from a theorem in discrete time which provides a correspondence between adapted random measures on $\mathbb{N}^2$ and two-parameter randomized stopping points in the sense of Baxter and Chacon. As an application we show the existence of optimal stopping points for upper semicontinuous two-parameter processes in continuous time.

1. Introduction. The notion of randomized stopping point was first introduced by Baxter and Chacon [1] for one-parameter processes. Roughly speaking, a randomized stopping point is a stopping point $T(\omega, \lambda)$ which depends on an additional random parameter $\lambda \in [0, 1]$. A randomized stopping point T induces an adapted random measure μ on the parameter space J (usually J is $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{N}^2$ or $\mathbb{R}_+^2$) by means of the formula

$$\mu(\omega, B) = |\{\lambda : T(\omega, \lambda) \in B\}|, \tag{1.1}$$

where B is a Borel subset of J and $|\cdot|$ denotes the Lebesgue measure on $[0, 1]$. In the one-parameter case, it is easy to show that (1.1) establishes a bijective correspondence between randomized stopping points which are nondecreasing and left-continuous in the second variable and adapted random measures such that $\mu(\omega, J) = 1$ for all ω. Let us denote by $\mathscr{U}$ the convex set of random adapted measures on J with total mass equal to 1. The set $\mathscr{U}$ is compact with respect to a suitable topology introduced by Baxter and Chacon. Furthermore, the set $\mathscr{T}$ of ordinary stopping points can be embedded into $\mathscr{U}$ and, in the one parameter case, the representation (1.1) implies that the set of extremal elements of $\mathscr{U}$ is exactly the set of stopping points (see Edgar, Millet and Sucheston [10]). This property has been used (see, e.g., Bismut [2], Edgar, Millet and Sucheston [10] and Ghoussoub [12]) to study the optimal stopping problem for one-parameter processes.

When trying to extend these results to processes parametrized by $\mathbb{N}^2$ or $\mathbb{R}_+^2$, some differences appear. Simple examples (see Mazziotto and Millet [17]) show that there may be extremal elements of $\mathscr{U}$ which are not stopping points. Furthermore, the representation of randomized stopping points as random

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adapted measures given by the expression (1.1) is not bijective in general. That is, if $\mathscr{U}_r$ denotes the subset of $\mathscr{U}$ formed by the measures of the form (1.1), then the inclusion $\mathscr{U}_r \subset \mathscr{U}$ is in general strict.

As it is usual in the theory of two-parameter processes, one can try to recover the one-parameter results when the two-parameter filtration satisfies certain classical conditions such as Hypothesis F4 of Cairoli and Walsh [4] or Hypothesis CQI of Krengel and Sucheston. For instance, assuming CQI, Dalang, Trotter and de Werra [9] have proved the equality $\mathscr{T} = \text{ext}(\mathscr{U})$ on finite probability spaces and Dalang [6] has proved this property in discrete time. On the other hand, the property $\mathscr{T} = \text{ext}(\mathscr{U})$ has been proved by Dalang [7] in continuous time, assuming CQI, and provided the underlying probability space is a nonstandard (Loeb) space. Note that for the optimal control problem considered by Dalang in [8] one can show the existence of an optimal solution without the hypothesis CQI.

The main results of this paper are the following:

1. Under CQI and with a discrete parameter space (i.e., $\mathbb{N}^2$), the equality $\mathscr{U}_r = \mathscr{U}$ holds. That means, any adapted random measure on $\mathbb{N}^2$ of total mass equal to 1 can be represented in the form (1.1) for some randomized stopping point T.
2. In continuous time (i.e., the parameter space is $\mathbb{R}_+^2$) and assuming CQI, every extremal point of $\mathscr{U}$ is a stopping point.

These results can be applied to show the existence of optimal stopping points. The optimal stopping problem for two-parameter processes has been recently investigated by several authors. Starting with the paper of Cairoli and Gabriel [3], the problem was solved under different conditions by Krengel and Sucheston [13] and Mandelbaum and Vanderbei [14] using the notion of tactic. In discrete time, Mazziotto and Szpirglas proved the existence of an optimal stopping point without the CQI assumption and using Snell's envelope.

Following the approach used by Dalang in [5] we can apply property 2 to deduce the existence of optimal stopping points for upper-semicontinuous processes parametrized by $\mathbb{R}_+^2$. This result was stated in [17] and [19] but the proofs contained in these two papers are not complete. For nonstandard (Loeb) probability spaces, the result has been recently proved by Dalang in [7]. On the other hand, in discrete time and without the CQI assumption, one can still show the existence of an optimal stopping point provided the set $\mathscr{U}_r$ (which in general is strictly contained in $\mathscr{U}$) is closed for the Baxter–Chacon topology. To find sufficient conditions for $\mathscr{U}_r$ to be closed is an open problem. We remark that in some particular cases, like in the example introduced by Mazziotto and Millet in [17], one can easily show that $\mathscr{U}_r$ is closed and deduce the existence of optimal stopping points.

The paper is organized as follows. Section 2 contains some preliminary notations and results. Sections 3 and 4 are devoted to proving the above results 1 and 2, respectively. In Section 5 we discuss the problem of optimal stopping for two-parameter processes.

2. Definitions and notation. We will consider stochastic processes parametrized by I^2, where I is one of the sets $\mathbb{N}$, $\mathbb{D}_n = \{i2^{-n},\ i \in \mathbb{N}\}$ or $\mathbb{R}_+ = [0, +\infty)$. It will be convenient to add to I^2 an extra element denoted by ∞ and we will set $\overline{I^2} = I^2 \cup \{\infty\}$. With the usual topology $\overline{I^2}$ is a compact space. The notations $\mathscr{B}(I^2)$ and $\mathscr{B}(\overline{I^2})$ will denote the corresponding Borel σ-field.

We define on I^2 the usual order $s = (s_1, s_2) \le t = (t_1, t_2)$ if $s_1 \le t_1$ and $s_2 \le t_2$. We will use the notation $s < t$ to express that $s \le t$ and $s \ne t$. We can also consider the order $s = (s_1, s_2) \underline{\wedge}\, t = (t_1, t_2)$ if $s_1 \le t_1$ and $s_2 \ge t_2$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. A *two-parameter filtration* $(\mathscr{F}_t)_{t \in I^2}$ is a family of sub-σ-algebras of $\mathscr{F}$ verifying the following properties:

F1. $\mathscr{F}_{0,0}$ contains all P-null sets.

F2. $s \le t$ implies $\mathscr{F}_s \subset \mathscr{F}_t$.

F3. When $I = \mathbb{R}_+$, $\mathscr{F}_s = \bigcap_{t > s} \mathscr{F}_t$ for all $s \in I^2$.

Definition 2.1 (cf. [13]). Given three σ-algebras $\mathscr{F}^1$, $\mathscr{F}^2$ and $\mathscr{G}$ contained in $\mathscr{F}$, we will say that $\mathscr{F}^1$ and $\mathscr{F}^2$ are conditionally qualitatively independent given $\mathscr{G}$ if

$$\{P(A|\mathscr{G}) > 0\} \cap \{P(B|\mathscr{G}) > 0\} \subset \{P(A \cap B|\mathscr{G}) > 0\}$$

for all $A \in \mathscr{F}^1$, $B \in \mathscr{F}^2$. In the sequel this property will be denoted by $\mathrm{CQI}(\mathscr{F}^1, \mathscr{F}^2, \mathscr{G})$.

Let $(\mathscr{F}_t)_{t \in I^2}$ be a two-parameter filtration and for any $(\sigma, \tau) \in I^2$, define $\mathscr{F}_{\sigma,\infty} = \bigvee_{\tau \in I} \mathscr{F}_{\sigma,\tau}$ and $\mathscr{F}_{\infty,\tau} = \bigvee_{\sigma \in I} \mathscr{F}_{\sigma,\tau}$. Many results in the theory of two-parameter processes require a supplementary hypothesis on the two-parameter filtration. Usually one assumes hypothesis F4 of Cairoli and Walsh [4] which says that given any $z = (\sigma, \tau) \in I^2$, $\mathscr{F}_{\sigma,\infty}$ and $\mathscr{F}_{\infty,\tau}$ are conditionally independent given $\mathscr{F}_z$. In this paper we will make use of the weaker condition of *qualitative conditional independence* which can be formulated as follows:

(CQI): For all $(\sigma, \tau) \in I^2$, $\mathrm{CQI}(\mathscr{F}_{\sigma,m}, \mathscr{F}_{n,\tau}, \mathscr{F}_{\sigma,\tau})$ holds for each $n, m \in I$, $n \ge \sigma$, $m \ge \tau$.

One of the main ingredients in the proof of the main result of the next section is the notion of conditional supremum operator, which was introduced by Dalang in [6].

Definition 2.2. Given a bounded random variable Y and a σ-algebra $\mathscr{G} \subset \mathscr{F}$, the conditional supremum of Y given $\mathscr{G}$ is the random variable [denoted $S(Y|\mathscr{G})$] defined by

$$S(Y|\mathscr{G}) = \operatorname{ess} \inf_{\substack{Z \ge Y \\ Z\ \mathscr{G}\text{-measurable}}} Z.$$

The following result (cf. Proposition 4.6 in [6]) establishes a fundamental relationship between the conditional supremum operator and conditional probability.

PROPOSITION 2 3. *For any $k \in \mathbb{R}$ and for any bounded random variable Y, we have*

$$\{S(Y|\mathscr{G}) > k\} = \{P(Y > k|\mathscr{G}) > 0\}. \tag{2.1}$$

Equality (2.1) implies that for any nonnegative and bounded random variable Y, the conditional supremum operator $S(Y|\mathscr{G})$ coincides with the infinity norm of Y with respect to a regular version of the conditional probability given $\mathscr{G}$. That is, we have $S(Y|\mathscr{G}) = \lim_{p\to\infty}[E(Y|\mathscr{G})]^{1/p}$. For this reason, the conditioned supremum operator has many properties similar to those of the conditional expectation. We refer to Dalang [6] for the statement and proofs of these properties. We will just recall the properties of the conditional supremum operator that will be relevant here.

PROPOSITION 2.4. *The conditional supremum operator verifies the following properties*:

(a) *Monotonicity*: $\mathscr{G}_1 \subset \mathscr{G}_2 \Rightarrow S(Y|\mathscr{G}_1) \geq S(Y|\mathscr{G}_2)$.
(b) *Iteration*: $\mathscr{G}_1 \subset \mathscr{G}_2 \Rightarrow S(S(Y|\mathscr{G}_2)|\mathscr{G}_1) = S(Y|\mathscr{G}_1)$.
(c) *$\mathscr{G}$-additivity*: *If Y is $\mathscr{G}$-measurable, then*

$$S(X+Y|\mathscr{G}) = Y + S(X|\mathscr{G}).$$

(d) *Monotone convergence*:

$$Y_n \uparrow Y \Rightarrow S(Y_n|\mathscr{G}) \uparrow S(Y|\mathscr{G}),$$
$$\mathscr{G}_n \downarrow \mathscr{G} \Rightarrow S(Y|\mathscr{G}_n) \uparrow S(Y|\mathscr{G}).$$

PROPOSITION 2.5. *Let $\mathscr{F}^1$, $\mathscr{F}^2$ and $\mathscr{G}$ be three σ-algebras such that $CQI(\mathscr{F}^1, \mathscr{F}^2, \mathscr{G})$ holds. Then if X^1 and X^2 are two bounded random variables which are $\mathscr{F}^1$ and $\mathscr{F}^2$-measurable, respectively, we have*

$$S(X^1 + X^2|\mathscr{G}) = S(X^1|\mathscr{G}) + S(X^2|\mathscr{G}). \tag{2.2}$$

Property (2.2) will play a basic role in the sequel. Its heuristic interpretation is clear from the following fact. If X_1 and X_2 are bounded and independent random variables, then $\|X_1 + X_2\|_\infty = \|X_1\|_\infty + \|X_2\|_\infty$ a.s.

PROPOSITION 2.6. *Let $\mathscr{F}^1$, $\mathscr{F}^2$ and $\mathscr{G}$ be three σ-algebras such that $CQI(\mathscr{F}^1, \mathscr{F}^2, \mathscr{G})$ holds. Let Y be a bounded $\mathscr{F}^1$-measurable random variable. Then*

$$S(Y|\mathscr{G}) \leq S(Y|\mathscr{F}^2). \tag{2.3}$$

We will also need the notion of optional increasing path.

DEFINITION 2.7. A family $Z = \{Z(u),\ u \in I\}$ of stopping points is an *optional increasing path* provided $Z(0) = (0,0)$ a.s., $u \leq v$ implies $Z(u) \leq Z(v)$ a.s. and $Z_1(u) + Z_2(u) = u$ a.s. for all $u \in I$.

3. Randomized stopping points in the plane. A *stopping point* is a random variable $T: \Omega \to \overline{I^2}$ such that $\{T \le t\} \in \mathscr{F}_t$ for every $t \in I^2$. We will denote by $\mathscr{T}$ the set of all stopping points.

The notion of randomized stopping point was introduced by Baxter and Chacon [1] for one-parameter processes. A mapping $T: \Omega \times [0,1] \to \overline{I^2}$ will be called a *randomized stopping point* if it is a stopping point for the two-parameter filtration $\mathscr{M}_t = \mathscr{F}_t \otimes \mathscr{B}([0,1])$, that is, if

$$\{T \le t\} \in \mathscr{F}_t \otimes \mathscr{B}([0,1]), \qquad \forall\, t \in I^2.$$

Notice that if T is a randomized stopping point, then $T(\cdot, \lambda)$ is a stopping point for any $\lambda \in [0,1]$.

Another way to randomize the set $\mathscr{T}$ of stopping points is the following. Consider the convex set $\mathscr{U}$ of random probability measures $\mu(\omega, B)$, $\omega \in \Omega$, $B \in \mathscr{B}(\overline{I^2})$ such that $\mu(\cdot, [0, t])$ is $\mathscr{F}_t$-measurable for all $t \in I^2$. Then each randomized stopping point $T: \Omega \times [0,1] \to \overline{I^2}$ determines the random probability measure $\mu_T \in \mathscr{U}$ defined by

$$\mu_T(\omega, B) = |\{T(\omega, \cdot) \in B\}|, \tag{3.1}$$

where $|\cdot|$ denotes the Lebesgue measure.

Let $\mathscr{U}_r$ denote the set of random probability measures of the form (3.1) for some randomized stopping time T. Notice that $\mathscr{T}$ can be embedded into $\mathscr{U}_r$. The set $\mathscr{U}_r$ has the following properties.

PROPOSITION 3.1. *The embedding in $\mathscr{U}$ of the set $\mathscr{U}_r$ is convex and its extremal points are the stopping points; hence, $\mathscr{T} = \operatorname{ext}(\mathscr{U}_r)$.*

PROOF. Let us first show the convexity of $\mathscr{U}_r$. Fix $\mu_1, \mu_2 \in \mathscr{U}_r$ and $0 < \alpha < 1$. Denote by T_1 and T_2 two randomized stopping points associated with μ_1 and μ_2, respectively. Define

$$T(\omega, \lambda) = T_1(\omega, \alpha^{-1}\lambda)\mathbf{1}_{[0,\alpha)}(\lambda) + T_2\big(\omega, (1-\alpha)^{-1}\lambda\big)\mathbf{1}_{[\alpha,1]}(\lambda).$$

Then T is a randomized stopping point and for any Borel subset B of $\overline{I^2}$, we have

$$\begin{aligned} |\{\lambda: T(\omega, \lambda) \in B\}| &= |\{\lambda: T_1(\omega, \alpha^{-1}\lambda) \in B\} \cap [0, \alpha)| \\ &\quad + \Big|\Big\{\lambda: T_2\big(\omega, (1-\alpha)^{-1}\lambda\big) \in B\Big\} \cap [\alpha, 1]\Big| \\ &= \alpha\mu_1(\omega, B) + (1-\alpha)\mu_2(\omega, B). \end{aligned}$$

The stopping points, considered as elements of $\mathscr{U}_r$, are random Dirac measures and they are clearly extremal elements. Conversely suppose that $\mu \in \mathscr{U}_r$ is an extremal element. Let T be an associate randomized stopping point. We want to show that there exists a stopping point τ such that for any $\omega \in \Omega$,

$T(\omega, \lambda) = \tau(\omega)$ for almost all $\lambda \in [0,1]$. Fix $\alpha \in (0,1)$ and define

$$T_{1,\alpha}(\omega, \lambda) = T(\omega, \alpha\lambda), \qquad 0 \le \lambda \le 1,$$
$$T_{2,\alpha}(\omega, \lambda) = T(\omega, \alpha + \lambda(1-\alpha)), \qquad 0 \le \lambda \le 1,$$
$$\mu_{1,\alpha}(\omega, B) = |\{\lambda: T_{1,\alpha}(\omega, \lambda) \in B\}|,$$
$$\mu_{2,\alpha}(\omega, B) = |\{\lambda: T_{2,\alpha}(\omega, \lambda) \in B\}|, \qquad B \in \mathscr{B}(\overline{I^2}).$$

Then $\mu_{1,\alpha}, \mu_{2,\alpha} \in \mathscr{U}_r$ and $\mu = \alpha\mu_{1,\alpha} + (1-\alpha)\mu_{2,\alpha}$. Consequently, $\mu = \mu_{1,\alpha} = \mu_{2,\alpha}$ which implies

$$\alpha^{-1}|\{\lambda: T(\omega,\lambda) \in B\} \cap [0,\alpha]| = |\{\lambda: T(\omega,\lambda) \in B\}|$$

for any $B \in \mathscr{B}(\overline{I^2})$. Therefore, $|\{\lambda: T(\omega,\lambda)\} \in B|$ is equal to 0 or 1 for all B, which implies the result. □

In the one-parameter case, one has $\mathscr{U}_r = \mathscr{U}$ (cf. [1]), and therefore, $\mathscr{T}$ is the set of extremal points of $\mathscr{U}$. A simple example due to Mazziotto and Millet (see [17]) shows that the inclusion $\mathscr{U}_r \subset \mathscr{U}$ is generally strict in the two-parameter case. Furthermore (see [17]), this example shows that $\mathscr{T}$ is not the set of extremal points of $\mathscr{U}$. We postpone to Section 5 a detailed discussion of this example. On finite probability spaces, the class of filtrations for which $\mathscr{T}$ is the set of extremal elements of $\mathscr{U}$ has been characterized in [7].

Our purpose in this section is to show the equality $\mathscr{U}_r = \mathscr{U}$ assuming that the parameter space is discrete, that is, $\overline{I^2} = \overline{\mathbb{N}^2}$ and the conditional qualitative independence property holds. As a corollary we have $\mathscr{T} = \text{ext}(\mathscr{U})$ in this situation. This property was proved by Dalang in [6] using a direct approach.

For the rest of this section we will assume that the parameter space is $\overline{I^2} = \overline{\mathbb{N}^2}$. First observe that in this case a random probability measure $\mu \in \mathscr{U}$ can be identified with a *positive weight process* $(a_t)_{t \in \overline{\mathbb{N}^2}}$ defined by $a_t(\omega) = \mu(\omega, \{t\})$. This weight process satisfies the following conditions: (i) $a_t \ge 0$ a.s.; (ii) a_t is $\mathscr{F}_t$-measurable for all $t \in \mathbb{N}^2$; (iii) $\sum_{t \in \overline{\mathbb{N}^2}} a_t = 1$.

Before stating the main result, let us introduce some preliminary technical results.

LEMMA 3.2. *Let $\mu \in \mathscr{U}$ and assume the CQI property. Then, for any $t \in I^2$ we have*

$$\mu([0,t]) + S(\mu([0,t_1] \times (t_2,\infty))|\mathscr{F}_t) + S(\mu((t_1,\infty) \times [0,t_2])|\mathscr{F}_t) \le 1 \quad a.s.$$

PROOF. Using Proposition 2.5 and the $\mathscr{S}$-additivity property [see Proposition 2.4(c)] we obtain for all $M \ge t$,

$$\begin{aligned}\mu([0,t]) &+ S(\mu([0,t_1] \times (t_2,M_2))|\mathscr{F}_t) + S(\mu((t_1,M_1) \times [0,t_2])|\mathscr{F}_t)\\ &= \mu([0,t]) + S(\mu([0,t_1] \times (t_2,M_2) \cup (t_1,M_1) \times [0,t_2])|\mathscr{F}_t)\\ &= S(\mu([0,t] \cup [0,t_1] \times (t_2,M_2) \cup (t_1,M_1) \times [0,t_2])|\mathscr{F}_t) \le 1.\end{aligned}$$

Then the result follows by letting M_1 and M_2 tend to infinity and using the monotone convergence property of the conditional supremum operator [see Proposition 2.4(d)]. □

We denote by $\mathscr{R}$ the family of all finite disjoint unions of intervals of the form $(a, b]$, $0 < a < b \le 1$, and $[0, b]$, $0 < b \le 1$. For any set $B \in \mathscr{R}$ with $|B| < 1$ and any real number $\beta \le 1 - |B|$, we define

$$\wedge(B, \beta) = B^c \cap [0, \beta_B],$$

where $\beta_B = \inf\{\lambda: |B^c \cap [0, \lambda]| \ge \beta\}$. Then $\wedge(B, \beta) \in \mathscr{R}$ and $|\wedge(B, \beta)| = \beta$. The proof of the following two lemmas is immediate from the definition of $\wedge$:

LEMMA 3.3. *Set $A \subset B$, $A, B \in \mathscr{R}$, $0 \le \alpha \le 1 - |A|$, $0 \le \beta \le 1 - |B|$ and $\alpha \le \beta$. Then*

$$A \cup \wedge(A, \alpha) \subset B \cup \wedge(B, \beta) \quad \text{and} \quad \sup \wedge(A, \alpha) \le \sup \wedge(B, \beta).$$

LEMMA 3.4. *Let $B \in \mathscr{R}$ and $\beta + \alpha \le 1 - |B|$, $\alpha, \beta \ge 0$. Then*

$$\wedge(B \cup \wedge(B, \alpha), \beta) = \wedge(B, \alpha + \beta).$$

Given two sets $A, B \subset [0, 1]$ we will write $A < B$ if $x < y$ for each $x \in A$, $y \in B$.

LEMMA 3.5. *Consider a family of subsets of $[0, 1]$, belonging to $\mathscr{R}$, $B_1, \dots, B_k$, such that $B_i \cap B_j = \varnothing$ if $i \ne j$. Let α_i, β_i, $1 \le i \le k$, be nonnegative real numbers such that*

$$\sum_{i=1}^{j} \alpha_i \le \sum_{i=1}^{j} \beta_i, \qquad 1 \le j \le k,$$

and $\sum_{i=1}^{k} \beta_i \le 1 - |B_1| - \cdots - |B_k|$. Define the following sets:

$$C_1 = \wedge(B_1, \alpha_1), \qquad D_1 = \wedge(B_1, \beta_1),$$
$$C_2 = \wedge(C_1 \cup B_1 \cup B_2, \alpha_2), \qquad D_2 = \wedge(D_1 \cup B_1 \cup B_2, \beta_2),$$
$$\vdots$$
$$C_k = \wedge(C_1 \cup \cdots \cup C_{k-1} \cup B_1 \cup \cdots \cup B_k, \alpha_k), \qquad D_k = \wedge(D_1 \cup \cdots \cup D_{k-1} \cup B_1 \cup \cdots \cup B_k, \beta_k).$$

Then

$$C_1 \cup C_2 \cup \cdots \cup C_k \cup B_1 \cup B_2 \cup \dots \cup B_k \subset D_1 \cup \cdots \cup D_k \cup B_1 \cup \cdots \cup B_k \tag{3.2}$$

and

$$\sup C_k \le \sup D_k. \tag{3.3}$$

PROOF. We will prove this lemma by induction on k. Clearly the lemma holds for $k = 1$. Suppose it holds for k. Set

$$F_k = \bigcup_{i=1}^{k} (C_i \cup B_i), \qquad G_k = \bigcup_{i=1}^{k} (D_i \cup B_i).$$

From (3.2) we get

$$F_k \cup B_{k+1} \subset G_k \cup B_{k+1}.$$

Now, by Lemma 3.3, we obtain

$$\bigcup_{i=1}^{k+1} (C_i \cup B_i) \subset \bigcup_{i=1}^{k+1} (D_i \cup B_i)$$

and

$$\sup C_{k+1} \leq \sup D_{k+1}. \qquad \square$$

The following is the main result of this section:

THEOREM 3.6. *Let $\{a_t,\ t \in \overline{\mathbb{N}^2}\}$ be a positive weight process. Then there exists a randomized stopping point $T: \Omega \times [0,1] \to \overline{\mathbb{N}^2}$ such that $|\{T = t\}| = a_t$ for any $t \in \overline{\mathbb{N}^2}$. Moreover T can be chosen in such a way that the following property holds*:

$$\{T = (t_1, t_2)\} < \{T = (s_1, s_2)\} \quad \text{if } t_1 < s_1 \text{ and } t_2 \geq s_2. \tag{3.4}$$

PROOF. Our aim is to construct a countable collection of random subsets of $[0,1]$: $\{I_t, t \in \overline{\mathbb{N}^2}\}$ such that the following properties hold:

(i) For every $\omega \in \Omega$, the sets $I_t(\omega)$ form a partition of $[0,1]$.
(ii) For every $t \in \mathbb{N}^2$, $\{(\omega, \lambda): \lambda \in I_t(\omega)\} \in \mathscr{F}_t \otimes \mathscr{B}([0,1])$.
(iii) For every $t \in \mathbb{N}^2$, we have $|I_t| = a_t$.

If we define $T(\omega, \lambda) = t$ whenever $\lambda \in I_t(\omega)$, then T will be a randomized stopping point such that $|\{T = t\}| = a_t$ for any $t \in \overline{\mathbb{N}^2}$. Let us first describe the procedure we will use to define the sets I_t.

Consider first the points $(0,0)$, $(1,0)$ and $(0,1)$. Observe that it is not convenient to set $I_{0,0} = [0, a_{0,0}]$, $I_{1,0} = (a_{0,0}, a_{0,0} + a_{1,0}]$ and $I_{0,1} = (a_{0,0} + a_{1,0}, a_{0,0} + a_{1,0} + a_{0,1}]$ because the interval $I_{0,1}$ does not satisfy the measurability property (ii). In order to avoid this difficulty we will set $I_{0,0} = [0, a_{0,0}]$ and $I_{1,0} = (a_{0,0} + K, a_{0,0} + a_{1,0} + K]$, where K is a random variable which is $\mathscr{F}_{0,0}$-measurable and larger than or equal to $\sum_{j=1}^{\infty} a_{0,j}$. That means, K is the length of a free space between the intervals $I_{0,0}$ and $I_{1,0}$ that will contain the random sets $I_{0,t}$, $t \geq 1$. The right choice for the random variable K will be $S(\sum_{j=1}^{\infty} a_{0,j} | \mathscr{F}_{0,0})$, where S denotes the conditional supremum operator introduced in Section 2.

Following these ideas we will construct the sets I_t in a recursive way. Set $Q_t = [0, t_1] \times (t_2, \infty)$. With this notation we have $\sum_{j=1}^{\infty} a_{0,j} = \mu(Q_{0,0})$, where μ

denotes the random probability measure in $\mathscr{U}$ corresponding to $\{a_t\}$. First we will define I_t for the points t of the form $(t_1, 0)$:

$$I_{0,0} = [0, a_{0,0}],$$
$$J_{0,0} = (a_{0,0}, a_{0,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,0})],$$
$$I_{1,0} = (a_{0,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,0}), a_{0,0} + a_{1,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,0})],$$
$$J_{1,0} = (a_{0,0} + a_{1,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,0}), a_{0,0} + a_{1,0} + S(\mu(Q_{1,0})|\mathscr{F}_{1,0})],$$

and, for $i \geq 2$,

$$I_{i,0} = (a_{0,0} + \cdots + a_{i-1,0} + S(\mu(Q_{i-1,0})|\mathscr{F}_{i-1,0}), \quad a_{0,0} + \cdots + a_{i,0} + S(\mu(Q_{i-1,0})|\mathscr{F}_{i-1,0})]$$

and

$$J_{i,0} = (a_{0,0} + \cdots + a_{i,0} + S(\mu(Q_{i-1,0})|\mathscr{F}_{i-1,0}), \quad a_{0,0} + \cdots + a_{i,0} + S(\mu(Q_{i-1,0})|\mathscr{F}_{i,0})].$$

Notice that the sets $J_{i,0}$ are well defined because by Proposition 2.6 and the CQI hypothesis, we have

$$S(\mu(Q_{i-1,0})|\mathscr{F}_{i-1,0}) \leq S(\mu(Q_{i-1,0})|\mathscr{F}_{i,0}) \leq S(\mu(Q_{i,0})|\mathscr{F}_{i,0}).$$

Furthermore, from Lemma 3.2, we have

$$a_{0,0} + \cdots + a_{i,0} + a_{i+1,0} + S(\mu(Q_{i,0})|\mathscr{F}_{i,0}) \leq 1, \quad \text{a.s.},$$

for any $i \geq 1$. Consequently the intervals $\{I_{i,0}, J_{i,0}, i \geq 0\}$ are disjoint and included in $[0, 1]$. The intervals $I_{i,0}$ verify properties (ii) and (iii) and $I_{i,0} < I_{i+1,0}$. On the other hand, $J_{i,0}$ is $\mathscr{F}_{i,0}$-measurable and

$$|J_{0,0}| + |J_{1,0}| + \cdots + |J_{i,0}| = S(\mu(Q_{i,0})|\mathscr{F}_{i,0}).$$

In order to define the sets $I_{i,1}$ we proceed as follows. Define

$$I_{0,1} = (a_{0,0}, a_{0,0} + a_{0,1}],$$
$$J_{0,1} = (a_{0,0} + a_{0,1}, a_{0,0} + a_{0,1} + S(\mu(Q_{0,1})|\mathscr{F}_{0,1})].$$

We have $I_{0,1} \cup J_{0,1} \subset J_{0,0}$. In fact, using the $\mathscr{G}$-additivity and monotonicity properties of the conditional supremum operator we obtain

$$a_{0,0} + a_{0,1} + S(\mu(Q_{0,1})|\mathscr{F}_{0,1}) = a_{0,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,1}) \leq a_{0,0} + S(\mu(Q_{0,0})|\mathscr{F}_{0,0}).$$

The next random set $I_{1,1}$ will no longer be an interval. We define it as follows:

$$I_{1,1} = \Lambda(I_{0,0} \cup I_{0,1} \cup I_{1,0} \cup J_{0,1}, a_{1,1}).$$

Then $I_{1,1}$ is an $\mathscr{F}_{1,1}$-measurable random subset of $[0, 1]$ such that $|I_{1,1}| = a_{1,1}$. Moreover we have that $I_{1,1} \subset I_{0,0} \cup J_{0,0} \cup I_{1,0} \cup J_{1,0}$. In fact, notice that the supremum of $I_{1,1}$ is bounded by

$$a_{0,0} + a_{0,1} + S(\mu(Q_{0,1})|\mathscr{F}_{0,1}) + a_{1,0} + a_{1,1},$$

and the union $I_{0,0} \cup J_{0,0} \cup I_{1,0} \cup J_{1,0}$ is equal to the interval

$$[0, a_{0,0} + a_{1,0} + S(\mu(Q_{1,0})|\mathscr{F}_{1,0})].$$

Consequently, the above inclusion follows from the following inequalities:

$$\begin{aligned} a_{0,1} + a_{1,1} + S(\mu(Q_{0,1})|\mathscr{F}_{0,1}) &= a_{1,1} + S(\mu(Q_{0,0})|\mathscr{F}_{0,1}) \\ &\le a_{1,1} + S(\mu(Q_{0,0})|\mathscr{F}_{1,1}) \\ &= S(a_{1,1} + \mu(Q_{0,0})|\mathscr{F}_{1,1}) \\ &\le S(\mu(Q_{1,0})|\mathscr{F}_{1,1}) \le S(\mu(Q_{1,0})|\mathscr{F}_{1,0}). \end{aligned}$$

Then we define

$$\begin{aligned} J_{1,1} = \wedge\big(I_{0,0} \cup I_{0,1} \cup I_{1,0} \cup I_{1,1} \cup J_{0,1}, S(\mu(Q_{1,1})|\mathscr{F}_{1,1}) \\ - S(\mu(Q_{0,1})|\mathscr{F}_{0,1})\big). \end{aligned}$$

The sets $I_{i,1}$, $J_{i,1}$, $i \ge 2$, will be introduced in a similar way. Having in mind the preceding construction, we can introduce now the general definition of the sets $I_{i,j}$ and $J_{i,j}$:

$$I_{i,j} = \wedge\left(\left(\bigcup_{t<(i,j)} I_t\right) \cup J_{0,j} \cup J_{1,j} \cup \cdots \cup J_{i-1,j}, a_{i,j}\right), \tag{3.5}$$

$$J_{i,j} = \wedge\left(\left(\bigcup_{t\le(i,j)} I_t\right) \cup J_{0,j} \cup J_{1,j} \cup \cdots \cup J_{i-1,j}, |J_{i,j}|\right), \tag{3.6}$$

where

$$|J_{i,j}| = S(\mu(Q_{i,j})|\mathscr{F}_{i,j}) - S(\mu(Q_{i-1,j})|\mathscr{F}_{i-1,j}). \tag{3.7}$$

Notice first that for any fixed $j \ge 0$, we have that the sets $\{I_{i,j}, J_{i,j}, i \ge 0\}$ verify

$$I_{0,j} < J_{0,j} < I_{1,j} < J_{1,j} < \cdots < I_{i,j} < J_{i,j} < \cdots. \tag{3.8}$$

It is possible to define these sets because by Lemma 3.2, we have

$$\mu([0,t]) + S(\mu(Q_{i,j})|\mathscr{F}_{i,j}) \le 1.$$

The sets $I_{i,j}$ and $J_{i,j}$ are $\mathscr{F}_{i,j}$-measurable and $|I_{i,j}| = a_{i,j}$. So, it remains to show that the sets $\{I_{i,j}, (i,j) \in \mathbb{N}^2\}$ are disjoint. Clearly $I_{i,j} \cap I_{k,l} = \varnothing$ if $(k,l) \le (i,j)$, $(k,l) \ne (i,j)$. Suppose now that $k > i$ and $l < j$. We want to show that $I_{i,j} < I_{k,l}$ for $k > i$ and $l < j$. To do this we will show the following relations:

$$(3.9)\quad \left(\bigcup_{t\le(i,j)} I_t\right)\cup J_{0,j}\cup\cdots\cup J_{i,j}\subset\left(\bigcup_{t\le(i,j-1)} I_t\right)\cup J_{0,j-1}\cup\cdots\cup J_{i,j-1},$$

$$(3.10)\quad \sup\left[I_{i,j}\cup J_{i,j}\right]\le\sup J_{i,j-1}$$

for $j\ge 1$, $i\ge 0$. Then the inequality $I_{i,j}<I_{k,l}$ will be an immediate consequence of (3.10) because by iteration we have

$$\sup(I_{i,j})\le\sup J_{i,l}\quad\text{and}\quad J_{i,l}<I_{k,l},$$

where the second inequality follows from (3.8).

Proof of the relations (3.9) and (3.10). The proof will be done by induction on i. Suppose first that $i=0$. We want to show that

$$\bigcup_{k=0}^{j} I_{0,k}\cup J_{0,j}\subset\bigcup_{k=0}^{j-1} I_{0,k}\cup J_{0,j-1}$$

and

$$\sup\left[I_{0,j}\cup J_{0,j}\right]\le\sup J_{0,j-1}\quad\text{for all } j\ge 1.$$

Notice that the sets $I_{0,j}\cup J_{0,j}$ and $J_{0,j-1}$ are intervals with the same left boundary. In fact,

$$I_{0,j}\cup J_{0,j}$$
$$=\left(a_{0,0}+\cdots+a_{0,j-1},a_{0,0}+\cdots+a_{0,j-1}+a_{0,j}+S\left(\sum_{k=j+1}^{\infty}a_{0,k}|\mathscr{F}_{0,j}\right)\right]$$

and

$$J_{0,j-1}=\left(a_{0,0}+\cdots+a_{0,j-1},a_{0,0}+\cdots+a_{0,j-1}+S\left(\sum_{k=j}^{\infty}a_{0,k}|\mathscr{F}_{0,j-1}\right)\right].$$

We have

$$a_{0,j}+S\left(\sum_{k=j+1}^{\infty}a_{0,k}|\mathscr{F}_{0,j}\right)\le S\left(\sum_{k=j}^{\infty}a_{0,k}|\mathscr{F}_{0,j-1}\right)$$

and this implies

$$I_{0,j}\cup J_{0,j}\subset J_{0,j-1}.$$

Now suppose that (3.9) and (3.10) hold for all the indexes (i',j) with $0\le i'\le i$ and $j\ge 0$ and let us show these properties for $(i+1,j)$, $j\ge 0$. First notice that by Lemma 3.4, we have

$$I_{i,j}\cup J_{i,j}=\Lambda\left(\left(\bigcup_{t<(i,j)} I_t\right)\cup J_{0,j}\cup\cdots\cup J_{i-1,j},a_{i,j}+|J_{i,j}|\right)$$
$$=\Lambda\left(\bigcup_{l=0}^{i-1}\left[I_{l,j}\cup J_{l,j}\right]\cup\bigcup_{l=0}^{i}\left[\bigcup_{k=0}^{j-1}I_{l,k}\right],a_{i,j}+|J_{i,j}|\right).$$

On the other hand we have

$$J_{i,j-1} = \wedge\left(\bigcup_{l=0}^{i-1}(J_{l,j-1}) \cup \bigcup_{l=0}^{i}\left[\bigcup_{k=0}^{j-1} I_{l,k}\right], |J_{i,j-1}|\right).$$

The sets $B_0 = \bigcup_{k=0}^{j-1} I_{0,k}$, $B_1 = \bigcup_{k=0}^{j-1} I_{1,k}, \dots,$ $B_i = \bigcup_{k=0}^{j-1} I_{i,k}$ are disjoint by our induction hypothesis. Take

$$\alpha_k = a_{k,j} + |J_{k,j}|,$$

$$\beta_k = |J_{k,j-1}|.$$

Then $\sum_{k=0}^{M} \alpha_k \le \sum_{k=0}^{M} \beta_k$, because

$$\sum_{k=0}^{M} a_{k,j} + S\big(\mu(Q_{M,j})|\mathscr{F}_{M,j}\big) \le S\big(\mu(Q_{M,j-1})|\mathscr{F}_{M,j-1}\big),$$

and $\sum_{k=0}^{i} \beta_k + \sum_{m=0}^{i}|B_m| \le 1$, because

$$S\big(\mu(Q_{i,j-1})|\mathscr{F}_{i,j-1}\big) + \sum_{l=0}^{i}\sum_{m=0}^{j-1} a_{l,m} \le 1.$$

Now the properties (3.9) and (3.10) follow from Lemma 3.5 applied to the sets $B_0, B_1, B_2, \dots, B_i$ and the real numbers α_k, β_k.

In order to complete the definition of the randomized stopping point T we set $T = \infty$ on $[0,1] - \bigcup_{t\in\mathbb{N}^2} I_t$. □

Remark. Consider the randomized stopping point T constructed in Theorem 3.6. For every $j \ge 0$, we denote by $\Gamma_\alpha(j)$ the first index i such that $\alpha \in [0, \sup I_{i,j}]$, where $\alpha \in (0,1)$. Observe that $\Gamma_\alpha(j)$ is a stopping time with respect to the one-parameter filtration $(\mathscr{F}_{i,j})_{i\ge 0}$. Set $\Gamma_\alpha(j) = \infty$ if there is no such index i.

We can define the measures $\mu_{\alpha,1}$ and $\mu_{\alpha,2}$ by

$$\mu_{\alpha,1} = \sum_{t\in\overline{I^2}} \alpha^{-1}|\{T = t\} \cap [0,\alpha]|\delta_t,$$

$$\mu_{\alpha,2} = \sum_{t\in\overline{I^2}} (1-\alpha)^{-1}|\{T = t\} \cap (\alpha,1]|\delta_t.$$

Suppose $\Gamma_\alpha(j) < \infty$. Then

$$\mu_{\alpha,2}(\{(k,j)\}) = 0 \quad \text{for all } k = 0,\dots,\Gamma_\alpha(j) - 1,$$

in fact

$$\mu_{\alpha,2}(\{(k,j)\}) = (1-\alpha)^{-1}|I_{k,j} \cap (\alpha,1]| = 0$$

because $I_{k,j} \subset [0,\alpha]$; and

$$\mu_{\alpha,1}(\{(k,j)\}) = 0 \quad \text{for all } k > \Gamma(j),$$

in fact

$$\mu_{\alpha,1}(\{(k,j)\}) = \alpha^{-1}|I_{k,j} \cap [0,\alpha]| = 0$$

because $I_{k,j} \subset (\alpha, 1]$.

4. Extremal elements of the set $\mathscr{U}$ of random adapted measures. The purpose of this section is to show the property $\mathscr{T} = \operatorname{ext}(\mathscr{U})$ when the parameter space is $\overline{I^2} = \overline{\mathbb{R}_+^2}$ and we assume the conditional qualitative independence property. The proof will make use of Theorem 3.6 which provides the equality $\mathscr{U} = \mathscr{U}_r$ in discrete time.

Let $\mathscr{C}$ denote the set of continuous real-valued processes $(X_t)_{t \in \overline{I^2}}$ such that $E(\sup_{t \in \overline{I^2}} |X_t|) < \infty$. The space $\mathscr{C}$ equipped with the norm $\|X\| = E(\sup_{t \in \overline{I^2}} |X_t|)$ is a Banach space. We will denote by $\sigma(\mathscr{C}^*, \mathscr{C})$ the weak topology on the dual $\mathscr{C}^*$ of $\mathscr{C}$. Observe that $\mathscr{U}$ is a closed subset of the unit ball in $\mathscr{C}^*$ and, therefore, is compact in the weak topology.

THEOREM 4.1. *Let $(\mathscr{F}_t)_{t \in \mathbb{R}_+^2}$ be a filtration satisfying conditions* F1 *to* F3. *Suppose also that the qualitative independence property holds. Then the extremal elements of the set $\mathscr{U}$ of adapted random measures on $\overline{\mathbb{R}_+^2}$ are exactly the stopping points.*

PROOF. Suppose that $\mu \in \mathscr{U}$ is an extremal element of $\mathscr{U}$. We denote by $A_t = \mu([0,t])$ the increasing process associated with μ. For every $n \geq 1$, we define the countable set of points $t_{i,j}^n = (i/2^n, j/2^n)$, $i, j \geq 0$. Set $\Delta_{i,j}^n = (t_{i-1,j-1}^n, t_{i,j}^n]$, $i, j \geq 1$, $\Delta_{0,j}^n = \{0\} \times ((j-1)/2^n, j/2^n]$, $\Delta_{i,0}^n = ((i-1)/2^n, i/2^n] \times \{0\}$, $\Delta_{0,0}^n = \{(0,0)\}$ and

$$\mu^n = \sum_{i,j \geq 0} \mu(\Delta_{i,j}^n)\, \delta_{t_{i,j}^n} + \mu(\{\infty\})\, \delta_\infty.$$

Then $\mu^n \in \mathscr{U}$ and the sequence μ^n converges to μ in the weak topology $\sigma(\mathscr{C}^*, \mathscr{C})$. By Theorem 3.6, for each $n \geq 1$, there exists a measurable mapping $T^n \colon \Omega \times [0,1] \to \overline{\mathbb{D}^n}$ ($\mathbb{D}^n = \{(i/2^n, j/2^n), i, j \geq 0\}$) such that:

(i) $\{T^n(\cdot, \lambda) = t\} \in \mathscr{F}_t$ for all $t \in \overline{\mathbb{D}^n}$, $\lambda \in [0,1]$;
(ii) $|\{T^n = t\}| = \mu^n(\{t\})$ for all $t \in \overline{\mathbb{D}^n}$.

That means T^n is a randomized discrete stopping point associated with the random measure μ^n. We fix $\alpha \in (0,1)$, and define

$$T_{\alpha,1}^n(\lambda) = T^n(\alpha\lambda), \qquad 0 \leq \lambda \leq 1,$$

$$T_{\alpha,2}^n(\lambda) = T^n(\alpha + \lambda(1-\alpha)), \qquad 0 < \lambda \leq 1,$$

$$\mu_{\alpha,1}^n = \sum_{i,j} \left|\left\{T_{\alpha,1}^n = t_{i,j}^n\right\}\right| \delta_{t_{i,j}^n} + \left|\left\{T_{\alpha,1}^n = \infty\right\}\right| \delta_\infty,$$

$$\mu_{\alpha,2}^n = \sum_{i,j} \left|\left\{T_{\alpha,2}^n = t_{ij}^n\right\}\right| \delta_{t_{i,j}^n} + \left|\left\{T_{\alpha,2}^n = \infty\right\}\right| \delta_\infty.$$

The random measures $\mu^n_{\alpha,1}$ and $\mu^n_{\alpha,2}$ belong to the set $\mathscr{U}$ and we have

$$\mu^n = \alpha\mu^n_{\alpha,1} + (1-\alpha)\mu^n_{\alpha,2}.$$

Indeed,

$$\begin{aligned}\mu^n(B) &= \sum_{t\in B\cap\overline{\mathbb{D}^n}} \mu^n(\{t\}) = \sum_{t\in B\cap\overline{\mathbb{D}^n}} |\{T^n = t\}| = \sum_{t\in B\cap\overline{\mathbb{D}^n}} |\{T^n = t\} \cap [0,\alpha]| \\ &\qquad + \sum_{t\in B\cap\overline{\mathbb{D}^n}} |\{T^n = t\} \cap (\alpha,1]| \\ &= \sum_{t\in B\cap\overline{\mathbb{D}^n}} \alpha|\{T^n_{\alpha,1} = t\}| + \sum_{t\in B\cap\overline{\mathbb{D}^n}} (1-\alpha)|\{T^n_{\alpha,2} = t\}| \\ &= \alpha\mu^n_{\alpha,1}(B) + (1-\alpha)\mu^n_{\alpha,2}(B).\end{aligned}$$

By the compactness of the set $\mathscr{U}$ in the weak topology $\sigma(\mathscr{C}^*, \mathscr{C})$ we can choose an increasing sequence of natural numbers $k_n \geq 1$ such that the sequences $\mu^{k_n}_{\alpha,1}$ and $\mu^{k_n}_{\alpha,2}$ converge weakly to some elements $\mu_{\alpha,1}, \mu_{\alpha,2} \in \mathscr{U}$. Consequently, we obtain

$$\mu = \alpha\mu_{\alpha,1} + (1-\alpha)\mu_{\alpha,2},$$

and this implies $\mu = \mu_{\alpha,1} = \mu_{\alpha,2}$. Actually, notice that all the limit points of the sequence $\mu^n_{\alpha,1}$ must be equal to μ, so $\lim_n \mu^n_{\alpha,1} = \mu$ and similarly $\lim_n \mu^n_{\alpha,2} = \mu$. Define

$$B_t = A_t + S(\mu(Q_t)|\mathscr{F}_t).$$

The process B_t is adapted and increasing in the coordinate $t_1$and decreasing in the coordinate t_2. In fact:

(i) Suppose $s_1 \leq s_2$. Then using Proposition 2.6, we obtain for all $\tau \geq 0$,

$$\begin{aligned}A_{s_1,\tau} + S(\mu(Q_{s_1,\tau})|\mathscr{F}_{s_1,\tau}) &\leq A_{s_2,\tau} + S(\mu(Q_{s_1,\tau})|\mathscr{F}_{s_2,\tau}) \\ &\leq A_{s_2,\tau} + S(\mu(Q_{s_2,\tau})|\mathscr{F}_{s_2,\tau}).\end{aligned}$$

(ii) Suppose $t_1 \leq t_2$. Then using the $\mathscr{G}$-additivity and the monotonicity of the conditional supremum operator we obtain for all $\sigma \geq 0$,

$$\begin{aligned}A_{\sigma,t_1} + S(\mu(Q_{\sigma,t_1})|\mathscr{F}_{\sigma,t_1}) &= S(A_{\sigma,t_1} + \mu(Q_{\sigma,t_1})|\mathscr{F}_{\sigma,t_1}) \\ &= S(A_{\sigma,t_2} + \mu(Q_{\sigma,t_2})|\mathscr{F}_{\sigma,t_1}) \\ &\geq S(A_{\sigma,t_2} + \mu(Q_{\sigma,t_2})|\mathscr{F}_{\sigma,t_2}) \\ &= A_{\sigma,t_2} + S(\mu(Q_{\sigma,t_2})|\mathscr{F}_{\sigma,t_2}).\end{aligned}$$

We also have $0 \leq B_t \leq 1$, due to Lemma 3.2. Fix $t \geq 0$. Define

$$T_{\alpha,\tau} = \inf\{\sigma \geq 0 : S(\mu(Q_{\sigma,\tau})|\mathscr{F}_{\sigma,\tau}) \geq \alpha\}.$$

$T_{\alpha,\tau}$ is a stopping time for the one-parameter filtration $\{\mathscr{F}_{\sigma,\tau}, \sigma \geq 0\}$.

CLAIM.

$$\mu((T_{\alpha,\tau},\infty)\times[0,\tau)) = 0 \quad \text{if } T_{\alpha,\tau} < \infty. \tag{4.1}$$

PROOF OF (4.1). It suffices to show that

$$\mu((T_{\alpha,\tau}+\varepsilon,\infty)\times[0,\tau-\varepsilon)) = 0, \qquad \forall\, \varepsilon > 0, \text{ if } T_{\alpha,\tau} < \infty.$$

We will show that

$$E\big(\mathbf{1}_{\{T_{\alpha,\tau}<\infty\}}\mu((T_{\alpha,\tau}+\varepsilon,\infty)\times[0,\tau-\varepsilon))\big) = 0.$$

We have, using the fact that the set $(T_{\alpha,\tau}+\varepsilon,\infty)\times[0,\tau)$ is open in $\mathbb{R}_+^2$, that the above quantity is the limit as $n\to\infty$ of

$$E\big(\mathbf{1}_{\{T_{\alpha,\tau}<\infty\}}\mu_{\alpha,2}^n((T_{\alpha,\tau}+\varepsilon,\infty)\times[0,\tau-\varepsilon))\big).$$

Suppose $2^{-n} < \varepsilon/2$. Then if $T_{\alpha,\tau} < \infty$, we have that $\mu_{\alpha,1}^n((T_{\alpha\tau}+\varepsilon,\infty)\times[0,\tau-\varepsilon)) = 0$. In fact, consider a dyadic point $t_{i,j}^n = (\alpha_i,\beta_j)$ such that $\alpha_{i-1} \le T_{\alpha,\tau} < \alpha_i < T_{\alpha,\tau}+\varepsilon$ and $\beta_{j-1} \le \tau-\varepsilon < \beta_j < \tau$. The definition of $T_{\alpha,\tau}$ implies

$$S\big(\mu(Q_{\alpha_i,\tau})|\mathscr{F}_{\alpha_i,\tau}\big) \ge \alpha,$$

so

$$S\left(\sum_{\substack{0\le l\le i\\ k\ge j+1}} \mu^n(t_{l,k}^n)|\mathscr{F}_{\alpha_i,\beta_j}\right) = S\big(\mu(Q_{\alpha_i,\beta_j})|\mathscr{F}_{\alpha_i,\beta_j}\big) \ge \alpha.$$

By the construction of the measure $\mu_{\alpha,1}^n$ (see the remark after the proof of Theorem 3.6), we have

$$\mu_{\alpha,1}^n\left(\bigcup_{\substack{l>i\\ k\le j}}\{t_{l,k}^n\}\right) = 0$$

and, consequently, $\mu_{\alpha,1}^n((\alpha_i,\infty)\times[0,\beta_j]) = 0$ which implies $\mu_{\alpha,1}^n((T_{\alpha,\tau}+\varepsilon,\infty)\times[0,\tau-\varepsilon)) = 0$.

Set $T_\tau = \inf_{\alpha>0} T_{\alpha,\tau}$. Then T_τ is a stopping time for the filtration $(\mathscr{F}_{\sigma,\tau})_{\sigma\ge 0}$ and we have

$$\mu((T_\tau,\infty)\times[0,\tau)) = 0 \tag{4.2}$$

and

$$\mu([0,T_\tau)\times[\tau,\infty)) = 0. \tag{4.3}$$

In fact, property (4.2) follows from the definition of T_τ, and to check (4.3) notice that the set $\{\sigma < T_\tau\}$ belongs to $\mathscr{F}_{\sigma,\tau}$ and on this set we have $S(\mu(Q_{\sigma,\tau})|\mathscr{F}_{\sigma,\tau}) = 0$. Consequently,

$$0 = \mathbf{1}_{\{\sigma<T_\tau\}}S\big(\mu(Q_{\sigma,\tau})|\mathscr{F}_{\sigma,\tau}\big) = S\big(\mathbf{1}_{\{\sigma<T_\tau\}}\mu(Q_{\sigma,\tau})|\mathscr{F}_{\sigma,\tau}\big) \ge \mathbf{1}_{\{\sigma<T_\tau\}}\mu(Q_{\sigma,\tau}).$$

If $t_1 \le t_2$, then $S(\mu(Q_{\sigma,t_1})|\mathscr{F}_{\sigma,t_1}) \ge S(\mu(Q_{\sigma,t_2})|\mathscr{F}_{\sigma,t_2})$ and, therefore, $T_{t_1} \le T_{t_2}$ a.s. Also, using the right-continuity of the filtration and the monotone convergence properties of the conditional supremum operator [see Proposition 2.4(d)], we have that $t_n \downarrow \tau$ implies $T_{t_n} \downarrow T_\tau$ a.s. Therefore, we can assume that $\{T_\tau\}$ is an increasing and right-continuous family of stopping times.

Let us denote by D the closure of the set of points (σ, τ) such that $T_\tau \le \sigma$ or $\tau = 0$. The indicator function $\mathbf{1}_D$ is an adapted process because

$$\{z \in D\} = \left(\bigcap_n \{T_{\tau-(1/n)} \le \sigma\}\right) \cup \{\tau = 0\} \in \mathscr{F}_z$$

for all $z = (\sigma, \tau)$. Moreover $\mathbf{1}_D$ is increasing with respect to the order $\underline{\wedge}$ and contains the axis $\{(\sigma, 0), \sigma \ge 0\}$. Following Walsh [20] we introduce the upper left portion of the boundary of D defined as

$$\Gamma(D) = \{(s,t) \in D\text{: there exists no } (\sigma,\tau) \in D \text{ with } \sigma < s \text{ and } \tau > t\}.$$

By Theorem 2.7 of [20], $\Gamma(D)$ can be parametrized to be an optional increasing path $\{Z(u),\ u \ge 0\}$ (see Definition 2.7). Properties (4.2) and (4.3) imply that with probability 1, the support of μ is contained into the image of Z. Finally, Lemma 6.4 of [7] shows that under this condition μ must be a stopping point. □

REMARK. (i) Dalang pointed out to us that the proof of the property (4.1) in the above theorem can be done in a different way. More precisely, instead of using Theorem 3.6, one can apply the direct approach developed in [6] in order to show the discrete time analogue to Theorem 4.1.

(ii) We can replace $[0,1]^2$ by $\overline{\mathbb{R}_+^2}$ in the preceding theorem and the result is still true.

5. Application to the existence of optimal stopping points. Consider the case of processes parametrized by $[0,1]^2$ and let $(\mathscr{F}_t)_{t\in[0,1]^2}$ be a two-parameter filtration verifying the CQI hypothesis. The main result of this paper is the following theorem:

THEOREM 5.1. *Suppose that $(X_t)_{t\in[0,1]^2}$ is a measurable process with upper semicontinuous sample paths such that $E(\sup_{t\in[0,1]^2} X_t) < \infty$. Then there exists a stopping point T_0 such that*

$$E(X_{T_0}) = \sup_{T\in\mathscr{T}} E(X_T). \tag{5.1}$$

PROOF. The proof can be done as follows. Consider the mapping $\Phi\colon \mathscr{U} \to \mathbb{R}$ defined by $\Phi(\mu) = E(\int_{[0,1]^2} X_t(\omega)\mu(\omega, dt))$. By Proposition 7.1 of [7] (see also [12]), this functional is upper-semicontinuous on $\mathscr{U}$. Since $\mathscr{U}$ is convex and compact and Φ is affine, it attains its maximum on $\mathscr{U}$ at an extremal element $\mu^0 \in \operatorname{ext}(\mathscr{U})$. By Theorem 4.1, μ^0 must be a stopping point T_0 and (5.1) follows. □

The existence of an optimal stopping point still holds without requiring the CQI property if we suppose that $\mathscr{U}_r$ is a closed subset of $\mathscr{U}$. Indeed, it suffices to consider the restriction of the mapping Φ to $\mathscr{U}_r$. However, we do not have general conditions ensuring that $\mathscr{U}_r$ is a closed subset of $\mathscr{U}$.

We will close this section with the discussion of a particular example that has been taken from [16].

EXAMPLE. Let $\Omega = [0,1[$ equipped with the Borel σ-field $\mathscr{B}([0,1[)$ and the Lebesgue measure. Consider the sub-σ-fields of $\mathscr{B}([0,1[)$ given by $\mathscr{A}_1 = \sigma(\mathscr{B}([0,\frac{1}{3}[))$, $\mathscr{A}_2 = \sigma(\mathscr{B}([\frac{1}{3},\frac{2}{3}[))$ and $\mathscr{A}_3 = \sigma(\mathscr{B}([\frac{2}{3},1[))$. Now we introduce the two-parameter filtration:

$$\mathscr{F}_{0,0} = \mathscr{F}_{0,1} = \mathscr{F}_{1,0} = \{\phi,\Omega\}, \qquad \mathscr{F}_{0,2} = \mathscr{A}_1, \qquad \mathscr{F}_{1,1} = \mathscr{A}_2, \qquad \mathscr{F}_{2,0} = \mathscr{A}_3,$$

and $\mathscr{F}_t = \bigvee_{s<t}\mathscr{F}_s$ for all $t \in \mathbb{N}^2 - \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)\}$. In this example the inclusion $\mathscr{U}_r \subset \mathscr{U}$ is strict and CQI does not hold. Indeed, consider the element $\mu \in \mathscr{U}$ defined by

$$\begin{aligned}\mu(\omega) = {} & \mathbf{1}_{[0,\frac{1}{3}[}(\omega)\left(\tfrac{1}{2}\delta_{(1,1)} + \tfrac{1}{2}\delta_{(2,0)}\right)\\ & + \mathbf{1}_{[\frac{1}{3},\frac{2}{3}[}(\omega)\left(\tfrac{1}{2}\delta_{(0,2)} + \tfrac{1}{2}\delta_{(2,0)}\right)\\ & + \mathbf{1}_{[\frac{2}{3},1[}(\omega)\left(\tfrac{1}{2}\delta_{(0,2)} + \tfrac{1}{2}\delta_{(1,1)}\right).\end{aligned}$$

We claim that $\mu \notin \mathscr{U}_r$. Suppose that $\mu \in \mathscr{U}_r$ and let T be a randomized stopping point such that $|\{\lambda: T(\omega,\lambda) = t\}| = \mu(\omega,\{t\})$ for all $t \in \overline{\mathbb{N}}^2$. Consider the sets

$$\begin{aligned} I_1(\omega_1) &= \{\lambda \in [0,1]: T(\omega_1,\lambda) = (0,2)\},\\ I_2(\omega_2) &= \{\lambda \in [0,1]: T(\omega_2,\lambda) = (1,1)\},\\ I_3(\omega_3) &= \{\lambda \in [0,1]: T(\omega_3,\lambda) = (2,0)\},\end{aligned}$$

where $\omega_1 \in [\frac{1}{3},1[$, $\omega_2 \in [0,\frac{1}{3}[\cup[\frac{2}{3},1[$ and $\omega_3 \in [0,\frac{2}{3}[$. Then $|I_1(\omega_1)| = |I_2(\omega_2)| = |I_3(\omega_3)| = \frac{1}{2}$ but $I_1(\omega_1)\cap I_2(\omega_2) = I_1(\omega_1)\cap I_3(\omega_3) = I_2(\omega_2)\cap I_3(\omega_3) = \varnothing$ which is contradictory.

In this example the fact that a measure $\mu \in \mathscr{U}$ belongs to $\mathscr{U}_r$ can be characterized by an analytical condition. As a consequence, one can show that $\mathscr{U}_r$ is a closed subset of $\mathscr{U}$ and deduce the existence of optimal stopping points.

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