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# A GENERALIZATION OF HÖLDER'S INEQUALITY AND SOME PROBABILITY INEQUALITIES

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The main result of this article is a generalization of the generalized Hölder inequality for functions or random variables defined on lower-dimensional subspaces of $n$-dimensional product spaces. It will be seen that various other inequalities are included in this approach. For example, it allows the calculation of upper bounds for the product measure of $n$-dimensional sets with the help of product measures of lower-dimensional marginal sets. Furthermore, it yields an interesting inequality for various cumulative distribution functions depending on a parameter $n \in \mathbb{N}$.

**1. Introduction.** We first recall the generalized Hölder inequality in terms of a measure-theoretic approach. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $L_p(\Omega, \mathscr{A}, \mu)$ be the set of $p$-integrable $(1 \le p < \infty)$ measurable functions from $(\Omega, \mathscr{A})$ into $(\mathbb{R}, \mathscr{B})$, where $\mathscr{B}$ denotes the Borel $\sigma$-field over $\mathbb{R}$. Let $m \ge 2$, $p_j > 1$ with $\sum_{j=1}^m p_j^{-1} = 1$ and let $f_j \in L_{p_j}(\Omega, \mathscr{A}, \mu)$, $j = 1, \dots, m$. Then it is well known [see, e.g., Kufner, John and Fučik (1977), page 67] that $\prod_{j=1}^m |f_j| \in L_1(\Omega, \mathscr{A}, \mu)$ and

$$(1.1) \qquad \int \prod_{j=1}^m |f_j| \, d\mu \le \prod_{j=1}^m \left( \int |f_j|^{p_j} \, d\mu \right)^{1/p_j},$$

with equality iff either there exists at least one $j$ with $\int |f_j|^{p_j} \, d\mu = 0$ or there exist constants $A_j > 0$, $j = 1, \dots, m$, such that

$$A_1 |f_1|^{p_1} = \cdots = A_m |f_m|^{p_m} \, [\mu].$$

(1.1) is called the generalized Hölder inequality. If $0 < \mu(\Omega) < \infty$, then it can be shown with (1.1) that for $1 \le p < q$ and $f \in L_q(\Omega, \mathscr{A}, \mu)$,

$$(1.2) \qquad \mu(\Omega)^{-1/p} \left( \int |f|^p \, d\mu \right)^{1/p} \le \mu(\Omega)^{-1/q} \left( \int |f|^q \, d\mu \right)^{1/q},$$

with equality iff $|f| = \text{const} \, [\mu]$.

In terms of a probability measure $P$ and expectations of random variables $X_j$, $j = 1, \dots, m$, one may write (1.1) as

$$E \prod_{j=1}^m |X_j| \le \prod_{j=1}^m \left( E|X_j|^{p_j} \right)^{1/p_j},$$

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and for $1 \leq p \leq q < \infty$ it follows from (1.2) that

$$(E|X|^p)^{1/p} \leq (E|X|^q)^{1/q}.$$

In this article we are concerned with a generalization of (1.1) for measure and probability spaces $(\Omega, \mathscr{A}, \mu)$ being the product of spaces $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i = 1, \dots, n$, and measurable functions $f_j$, $j = 1, \dots, m$, defined on lower-dimensional subspaces of $(\Omega, \mathscr{A}, P)$. It will be seen in Section 2 that as the $f_j$ are defined on subspaces of the original space $(\Omega, \mathscr{A}, \mu)$ there is an improvement [if $\mu(\Omega) < \infty$] and generalization of the classic generalized Hölder inequality (1.1). Furthermore, this new inequality includes two other interesting variants of Hölder's inequality, the Gagliardo inequality [Gagliardo (1958)] and the Loomis–Whitney inequality [Loomis and Whitney (1949)]. Although these inequalities were only proved for Lebesgue measure, they hold true for arbitrary product measures. The generalized Loomis–Whitney inequality for (probability) measures especially allows some interesting applications in Section 3. For example, for various distribution functions $F_n$ depending on $n \in \mathbb{N}$, the inequality $F_n(z)^{1/n} \geq F_{n+1}(z)^{1/(n+1)}$, $z \in \mathbb{R}$, will hold true. This type of inequality has been proved by other methods in Finner (1990) for the distribution function $F_n$ of the range of $n$ iid random variables with arbitrary distribution function $F$.

**2. A generalization of Hölder's inequality.** To establish notation, let $I_n = \{1, \dots, n\}$, $n \in \mathbb{N}$, and let $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i \in I_n$, be measure spaces. For $\varnothing \neq S \subseteq I_n$ let $(\Omega_S, \mathscr{A}_S, \mu_S)$ be the product space of $(\Omega_i, \mathscr{A}_i, \mu_i)$ for $i \in S$ with $\Omega_S = \times_{i \in S} \Omega_i$, $\mathscr{A}_S = \otimes_{i \in S} \mathscr{A}_i$ and $\mu_S = \otimes_{i \in S} \mu_i$.

THEOREM 2.1. *Let $m \in \mathbb{N}$, $M = \{1, \dots, m\}$, $\varnothing \neq S_j \subseteq I_n$, $p_j \geq 1$ for $j \in M$, $M_i = \{j \in M\colon S_j \ni i\}$ for $i \in I_n$ such that $\sum_{j \in M_i} p_j^{-1} = 1$ for all $i \in I_n$. Let $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i \in I_n$, be measure spaces and $f_j \in L_{p_j}(\Omega_{S_j}, \mathscr{A}_{S_j}, \mu_{S_j})$, $j \in M$. Then $\prod_{j \in M} |f_j| \in L_1(\Omega_{I_n}, \mathscr{A}_{I_n}, \mu_{I_n})$ and*

$$\int \prod_{j \in M} |f_j| \, d\mu_{I_n} \leq \prod_{j \in M} \left( \int |f_j|^{p_j} \, d\mu_{S_j} \right)^{1/p_j}. \tag{2.1}$$

*To characterize equality in (2.1), assume without loss of generality and for the sake of simplicity that $M_r \neq M_s$ for all $r, s \in I_n$ with $r \neq s$ and $\int |f_j|^{p_j} \, d\mu_{S_j} > 0$ for all $j \in M$. Then equality holds in (2.1) if and only if*

(2.2) *for all $j \in M$ and $i \in S_j$ there exist functions $f_{ji} \in L_{p_j}(\Omega_i, A_i, \mu_i)$ and constants $A_{ji} > 0$ such that:*

(a) $|f_j| = \prod_{i \in S_j} |f_{ji}| \, [\mu_{S_j}]$ *for all* $j \in M$,

(b) $A_{ri} |f_{ri}|^{p_r} = A_{si} |f_{si}|^{p_s} [\mu_i]$ *for all* $i \in I_n$, $r, s \in M_i$, $r \neq s$.

PROOF. The proof is by induction over $n$. For $n = 1$, (2.1) reduces to the classic generalized Hölder inequality (1.1). Now let (2.1) hold true for $n - 1$

with $n \geq 2$. Let $i \in I_n$ and set $J_i = I_n \setminus \{i\}$. Then it follows from Fubini's theorem and (1.1) that

$$\begin{aligned}
&\int_{\Omega_{I_n}} \prod_{j \in M} |f_j| \, d\mu_{I_n} \\
&= \int_{\Omega_{J_i}} \int_{\Omega_i} \prod_{j \in M} |f_j| \, d\mu_i \, d\mu_{J_i} \\
&= \int_{\Omega_{J_i}} \prod_{j \in M \setminus M_i} |f_j| \left( \int_{\Omega_i} \prod_{j \in M_i} |f_j| \, d\mu_i \right) d\mu_{J_i} \\
&\leq \int_{\Omega_{J_i}} \prod_{j \in M \setminus M_i} |f_j| \prod_{j \in M_i} \left( \int_{\Omega_i} |f_j|^{p_j} \, d\mu_i \right)^{1/p_j} d\mu_{J_i} \\
&= \prod_{j \in M_i:\, S_j = \{i\}} \left( \int_{\Omega_i} |f_j|^{p_j} \, d\mu_i \right)^{1/p_j} \\
&\quad \times \int_{\Omega_{J_i}} \prod_{j \in M \setminus M_i} |f_j| \prod_{j \in M_i:\, S_j \neq \{i\}} \left( \int_{\Omega_i} |f_j|^{p_j} \, d\mu_i \right)^{1/p_j} d\mu_{J_i} \\
&= A \quad (\text{say}).
\end{aligned}$$

Applying the induction hypothesis to the functions $g_j = f_j$, $j \in M \setminus M_i$ and $g_j = (\int_{\Omega_i} |f_j|^{p_j} \, d\mu_i)^{1/p_j}$, $j \in M_i$ with $S_j \neq \{i\}$ now yields

$$\begin{aligned}
A &\leq \prod_{j \in M_i:\, S_j = \{i\}} \left( \int_{\Omega_i} |f_j|^{p_j} \, d\mu_i \right)^{1/p_j} \prod_{j \in M \setminus M_i} \left( \int_{\Omega_{S_j}} |f_j|^{p_j} \, d\mu_{S_j} \right)^{1/p_j} \\
&\quad \times \prod_{j \in M_i:\, S_j \neq \{i\}} \left( \int_{\Omega_{S_j \setminus \{i\}}} \int_{\Omega_t} |f_j|^{p_j} \, d\mu_i \, d\mu_{S_j \setminus \{i\}} \right)^{1/p_j} \\
&= \prod_{j \in M} \left( \int_{\Omega_{S_j}} |f_j|^{p_j} \, d\mu_{S_j} \right)^{1/p_j},
\end{aligned}$$

which completes the proof of (2.1).

The sufficiency of (2.2) for equality in (2.1) is obvious. Assume now equality in (2.1) and $\int |f_j|^{p_j} \, d\mu_{S_j} > 0$ for all $j \in M$. Then the necessity of (2.2) will be shown by induction. For $n = 1$, (2.2) reduces to the corresponding condition for equality in (1.1). For $n > 1$ it suffices to show part (a) of (2.2) since then the necessity of (2.2)(b) follows by Fubini's theorem and the equality condition for (1.1). The induction hypothesis implies that for every $i \in I_n$ the functions $g_j$ appearing in the proof of (2.1) must satisfy (2.2). Thus all functions $f_j$ with $j \in M \setminus M_i$ depending on at most $n - 1$ components $x_r$, $r \neq i$, satisfy (2.2). If $\bigcup_{i \in I_n} (M \setminus M_i) = M$, the proof is complete. Otherwise, there exists at least one $j \in M$ with $S_j = I_n$. In this case let $L_1 = \{j\colon\ S_j \neq I_n\}$ and $L_2 = \{j\colon\ S_j = I_n\}$. Note that the conditions $M_r \neq M_s$ for $r \neq s$ and $\sum_{j \in M_i} p_j^{-1} = 1$

imply $L_1 \neq \varnothing$ and $\bigcup_{j \in L_1} S_j = I_n$. Since (2.2)(a) is satisfied for all $f_j$ with $j \in L_1$, there exist $f_{ji} \in L_{p_j}(\Omega_i, A_i, \mu_i)$ such that $|f_j| = \prod_{i \in S_j} |f_{ji}|$ $[\mu_{S_j}]$ for all $j \in L_1$. Set $h_i = \prod_{j \in L_1 \cap M_i} |f_{ji}|$ and $q = (1 - \sum_{j \in L_2} p_j^{-1})^{-1}$. Noting that $\sum_{j \in L_1 \cap M_i} p_j^{-1} = q^{-1}$ for all $i \in I_n$, it may easily be seen that $h_i \in L_q(\Omega_i, A_i, \mu_i)$ for all $i \in I_n$. Furthermore, equality in (2.1) implies $(\int \prod_{j \in L_1} |f_j|^q \, d\mu_{I_n})^{1/q} = \prod_{j \in L_1} (\int |f_j|^{p_j} \, d\mu_{S_j})^{1/p_j}$. Since $\prod_{j \in L_1} |f_j| = \prod_{i \in I_n} |h_i|$ it follows by applying (1.1) that

$$\begin{aligned} \int \prod_{j \in M} |f_j| \, d\mu_{I_n} &= \int \prod_{i \in I_n} |h_i| \prod_{j \in L_2} |f_j| \, d\mu_{I_n} \\ &\leq \left( \int \prod_{i \in I_n} |h_i|^q \, d\mu_{I_n} \right)^{1/q} \prod_{j \in L_2} \left( \int |f_j|^{p_j} \, d\mu_{I_n} \right)^{1/p_j}, \\ &= \prod_{j \in M} \left( \int |f_j|^{p_j} \, d\mu_{S_j} \right)^{1/p_j}, \end{aligned}$$

with equality iff there exist constants $B_j > 0$, $j \in L_2$, such that

$$B_j \prod_{i \in I_n} |h_i|^{q/p_j} = |f_j| \ [\mu_{I_n}] \quad \text{for all } j \in L_2.$$

Setting $f_{ji} = B_j |h_i|^{q/p_j}$ for $j \in L_2$ and $i \in I_n$, the proof is complete. □

REMARK 2.1. Note that $\sum_{j \in M} p_j^{-1} > 1$ iff $S_j \neq I_n$ for at least one $j \in M$. Furthermore, assuming $0 < \mu_i(\Omega_i) < \infty$, $i \in I_n$, (2.1) is an improvement of (1.1) in the following sense. Let $q_j \geq p_j$ with $\sum_{j \in M} q_j^{-1} = 1$, $\sum_{j \in M_i} p_j^{-1} = 1$, $i \in I_n$, and let $f_j \in L_{q_j}(\Omega_{S_j}, \mathscr{A}_{S_j}, \mu_{S_j})$ with $\int |f_j|^{q_j} \, d\mu_{S_j} > 0$, $j \in M$. Then the classic Hölder inequality (1.1) is given by

$$\int \prod_{j \in M} |f_j| \, d\mu_{I_n} \leq \prod_{j \in M} \left( \int |f_j|^{q_j} \, d\mu_{I_n} \right)^{1/q_j}.$$

On the other hand, (2.1) is given by

$$\int \prod_{j \in M} |f_j| \, d\mu_{I_n} \leq \prod_{j \in M} \left( \int |f_j|^{p_j} \, d\mu_{Sj} \right)^{1/p_j}.$$

Now (1.2) yields

$$\prod_{j \in M} \left( \int |f_j|^{p_j} \, d\mu_{S_j} \right)^{1/p_j} \leq \prod_{j \in M} \mu_{S_j}(\Omega_{S_j})^{1/p_j - 1/q_j} \prod_{j \in M} \left( \int |f_j|^{q_j} \, d\mu_{S_j} \right)^{1/q_j}.$$

Noting that

$$\prod_{j \in M} \mu_{S_j}(\Omega_{S_j})^{1/p_j - 1/q_j} = \prod_{j \in M} \mu_{I_n \setminus S_j}(\Omega_{I_n \setminus S_j})^{1/q_j},$$

it is easy to see that

$$\prod_{j\in M}\mu_{S_j}(\Omega_{S_j})^{1/p_j-1/q_j}\prod_{j\in M}\left(\int|f_j|^{q_j}\,d\mu_{S_j}\right)^{1/q_j}=\prod_{j\in M}\left(\int|f_j|^{q_j}\,d\mu_{I_n}\right)^{1/q_j}.$$

Thus

$$\prod_{j\in M}\left(\int|f_j|^{p_j}\,d\mu_{S_j}\right)^{1/p_j}\le\prod_{j\in M}\left(\int|f_j|^{q_j}\,d\mu_{I_n}\right)^{1/q_j},$$

with equality iff $|f_j| = \text{const}_j\ [\mu_{S_j}]$ for all $j \in M$ with $p_j < q_j$. This shows that (2.1) improves (1.1) in the case of finite measures.

We now consider some special versions of (2.1). It will be seen that these inequalities are again generalizations of known inequalities considered in the literature.

COROLLARY 2.1. *Given the assumptions of Theorem* 2.1, *let* $S_j$ *denote all subsets of* $I_n$ *of the size* $k < n$ *and let* $p_j = t = \binom{n-1}{k-1}$, $j = 1,\dots, m = \binom{n}{k}$. *Furthermore, let* $(\Omega_i, \mathscr{A}_i, \mu_i) = (\Omega, \mathscr{A}, \mu)$, $i \in I_n$, $\Omega_{(n)} = X_{i=1}^n\Omega$, $\mathscr{A}_{(n)} = \otimes_{i=1}^n\mathscr{A}$, $\mu_{(n)} = \otimes_{i=1}^n\mu$ *and* $f_j \in L_t(\Omega_{(k)}, \mathscr{A}_{(k)}, \mu_{(k)})$, $j \in M$. *Then*

$$\int\prod_{j=1}^m|f_j|\,d\mu_{(n)}\le\prod_{j=1}^m\left(\int|f_j|^t\,d\mu_{(k)}\right)^{1/t}. \tag{2.3}$$

*In the particular case of* $k = n - 1$,

$$\int\prod_{j=1}^n\left|f_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\right|d\mu_{(n)}(x_1,\dots,x_n)$$
$$\le\prod_{j=1}^n\left(\int\left|f_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\right|^{n-1}\cdot d\mu_{(n-1)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\right)^{1/(n-1)}.$$

REMARK 2.2. If in Corollary 2.1 $\mu$ is the Lebesgue measure $\lambda_1$ over $(\mathbb{R}^1, \mathscr{B}^1)$, inequality (2.3) is known as Gagliardo's inequality [Gagliardo (1958); see, e.g., Adams (1975), pages 101–103]. This inequality is the basis for the proof of the "Sobolev imbedding theorem"; see Adams (1975), page 97, Theorem 5.4. It should be noted that the proof of Gagliardo's inequality given there is somewhat more complicated than the proof of the more general inequality (2.1). On the other hand, (2.1) may be considered as a generalization of Gagliardo's inequality.

COROLLARY 2.2. *Under the assumptions of Theorem* 2.1, *let* $A \in \mathscr{A}_{I_n}$, *such that the projections* $A_j$ *of* $A$ *onto the components with indices* $i \in S_j$ *satisfy*

$A_j \in \mathscr{A}_{S_j}$. *Then*

$$\mu_{I_n}(A) \le \int \prod_{j \in M} I_{A_j}\, d\mu_{I_n} \le \prod_{j \in M} \mu_{S_j}(A_j)^{1/p_j}. \tag{2.4}$$

*To characterize equality in* (2.4), *assume without loss of generality that* $M_r \ne M_s$ *for all* $r, s \in I_n$ *with* $r \ne s$ *and* $\mu_{S_j}(A_j) > 0$ *for all* $j \in M$. *Then equality holds in* (2.4) *if and only if there exist sets* $A_i \in \mathscr{A}_i$ *such that* $A = A_1 \times \cdots \times A_n$ $[\mu_{I_n}]$. *Furthermore, using the assumptions of Corollary* 2.1,

$$\mu_{(n)}(A) \le \int \prod_{j=1}^{m} I_{A_j}\, d\mu_{(n)} \le \prod_{j=1}^{m} \mu_{(k)}(A_j)^{1/t}, \tag{2.5}$$

*with* $t = \binom{n-1}{k-1}$, *and for* $k = n - 1$,

$$\mu_{(n)}(A) \le \int \prod_{j=1}^{n} I_{A_j}\, d\mu_{(n)} \le \prod_{j=1}^{n} \mu_{(n-1)}(A_j)^{1/(n-1)}. \tag{2.6}$$

*If A is such that the projections* $A_j$ *satisfy* $A_1 = \cdots = A_n = B$ (*say*), *which is the case, for example, if A is permutation invariant, or if* $A_i \subseteq B$ *for all* $i = 1, \dots, n$, *then*

$$\mu_{(n)}(A)^{1/n} \le \mu_{(n-1)}(B)^{1/(n-1)}. \tag{2.7}$$

REMARK 2.3. If $\mu = \lambda_1$ then the volume inequality (2.6) is known as the Loomis–Whitney inequality [Loomis and Whitney (1949)]. This inequality is closely related to an isoperimetric inequality; see, for example, Burago and Zalgaller (1980), pages 94 and 95, or Hadwiger (1957), pages 162–164. The proof of Gagliardo's inequality is similar to the original proof of the Loomis–Whitney inequality but is not cited in Gagliardo's article.

**3. Some probability inequalities.** This section discusses some applications of the various inequalities of the last section in terms of probability measures and random variables. Using the assumptions of Theorem 2.1 let $\mu_{S_j} = P_{S_j} = \otimes_{i \in S_j} P_i$, $j \in M$, be probability measures and let $X_j \in L_{p_j}(\Omega_{S_j}, \mathscr{A}_{S_j}, P_{S_j})$ be real-valued random variables. Since $P_i(\Omega_i) = 1$, (2.1) may be written as

$$\int \prod_{j \in M} |X_j|\, dP_{I_n} \le \prod_{j \in M} \left( \int |X_j|^{p_j}\, dP_{S_j} \right)^{1/p_j} \tag{3.1}$$

or equivalently,

$$E \prod_{j \in M} |X_j| \le \prod_{j \in M} \left( E|X_j|^{p_j} \right)^{1/p_j}.$$

Remember that $\sum_{j \in M} p_j^{-1} > 1$ iff $S_j \ne I_n$ for at least one $j \in M$. Furthermore, (3.1) remains true if the conditions $\sum_{j \in M_i} p_j^{-1} = 1$, $i \in I_n$, are weakened to $\sum_{j \in M_i} p_j^{-1} \le 1$. The most interesting applications seem to appear if the $X_j$ are indicator variables. Then we have the same situation as in Corollary 2.2.

With (2.4)–(2.7) it is possible to calculate upper bounds for the probability of $n$-dimensional sets $A$ with the help of probabilities of lower-dimensional sets $A_j$. Here we consider an interesting application of (2.7). Let $(\Omega_i, \mathscr{A}_i, P_i) = (\Omega, \mathscr{A}, P)$, $i \in \mathbb{N}$, and let $A \in \mathscr{A}_{I_{n+1}} = \otimes_{i=1}^{n+1}\mathscr{A}$ such that the projections $A_j$ satisfy $A_j \in \mathscr{A}_{S_j} \equiv \otimes_{i=1}^{n}\mathscr{A}$. Assume $A_1 = \cdots = A_{n+1} = B$ (say) or let $A_i \subseteq B$ for all $i = 1, \dots, n+1$. Then (2.7) may be written as

$$P_{(n+1)}(A)^{1/(n+1)} \le P_{(n)}(B)^{1/n}. \tag{3.2}$$

This approach now yields a special type of inequality for various cumulative distribution functions. Let $X_i$, $i \in \mathbb{N}$, be iid random variables with values in $\Omega_i = \Omega$ and let $T_n$: $(\Omega^n, \otimes_{i=1}^{n}\mathscr{A}) \to (\mathbb{R}, \mathscr{B})$, $n \in \mathbb{N}$, be real-valued Borel-measurable statistics. Let $D \in \mathscr{B}$ and define $A = \{T_{n+1} \in D\}$. If the projections $A_j = \{T_{n+1} \in D\}_j$ satisfy

$$A_j \subseteq \{T_n \in D\} \quad \text{for all } j = 1, \dots, n+1, \tag{3.3}$$

then it follows that

$$P(T_{n+1} \in D)^{1/(n+1)} \le P(T_n \in D)^{1/n}. \tag{3.4}$$

If $D = (-\infty, z]$ and $F_n(z) = P(T_n \in D)$ for $z \in \mathbb{R}$, $n \in \mathbb{N}$, the last inequality is equivalent to

$$F_{n+1}(z)^{1/(n+1)} \le F_n(z)^{1/n}. \tag{3.5}$$

If $D = (z, \infty)$ and $\bar{F}_n(z) = 1 - F_n(z) = P(T_n \in D)$, then (3.4) is equivalent to

$$\bar{F}_{n+1}(z)^{1/(n+1)} \le \bar{F}_n(z)^{1/n}. \tag{3.6}$$

The condition $\{T_{n+1} \le z\}_j \subseteq \{T_n \le z\}$ is fulfilled if the statistics $T_n$ satisfy the monotonicity property

$$T_{n+1}(x) \ge T_n(x^j) \quad \text{for all } x \in \Omega^{n+1}, \tag{3.7}$$

where $x^j$ denotes the projection of $x$ onto the components $x_i$, $i \neq j$ of $x$. In this case (3.5) is an improvement of the elementary inequality $F_{n+1}(z) \le F_n(z)$. Similarly, $T_{n+1}(x) \le T_n(x^j)$ for all $x \in \Omega^{n+1}$ yields the condition $\{T_{n+1} > z\}_j \subseteq \{T_n > z\}$. In Finner (1990), inequality (3.5) is shown for the distribution function of the range of $n$ iid real-valued random variables with arbitrary distribution function $F$. The proof given there is based on a convex function inequality and a special representation for the distribution function of the range for discrete random variables. Finally, this article is born from the idea of generalizing the result for the range distribution to other distribution functions depending on a parameter $n \in \mathbb{N}$ and satisfying a condition like $\{T_{n+1} \le z\}_j \subseteq \{T_n \le z\}$ for $j = 1, \dots, n+1$. Since the range statistic $T_n = T_n(x_1, \dots, x_n) = \max_{1 \le i < j \le n}|x_i - x_j|$ satisfies (3.3) for $D = (-\infty, z]$, it follows directly that (3.5) holds true for its distribution function. But the concept given here yields inequality (3.5) [or (3.6)] for various other distribution functions. First consider the statistic

$$T_n = \sum_{i=1}^{n} |x_i|, \qquad n \in \mathbb{N},$$

for which the condition (3.7) obviously holds. This includes the special cases that $F_n$ is the distribution function of:

(a) the binomial distribution with parameter $p \in ]0, 1[$ and $n \in \mathbb{N}$,
(b) the Poisson distribution with parameter $\mu = \lambda n$ with $\lambda > 0$ and $n \in \mathbb{N}$,
(c) the chi-square distribution with degrees of freedom $n$,
(d) the gamma distribution with parameter $\alpha > 0$, $\gamma = n\beta$ with $\beta > 0$ and $n \in \mathbb{N}$ and Lebesgue density $p_{\alpha,\gamma}(x) = \alpha^{\gamma} e^{-\alpha x} x^{\alpha - 1}/\Gamma(\gamma)$, $x > 0$,
(e) the sum of $n$ uniformly distributed random variables on $[a, b]$, $a \geq 0$,
(f) the sum of $n$ independently identically distributed $p$-values which is sometimes used as a test statistic.

Furthermore, there are various statistics depending on the order statistics of the $X_i$ such that the distribution functions of these statistics satisfy (3.5). Let $X_{(1); n} \leq \cdots \leq X_{(n); n}$, $n \in \mathbb{N}$, be the order statistics of $X_1, \ldots, X_n$. Consider, for example, the statistics:

(a) $T_n = X_{(n-k); n}$, $k \in \{0, 1, \ldots, n-1\}$,
(b) $T_n = X_{(n-s); n} - X_{(r); n}$, $1 \leq r < n - s \leq n$,
(c) $T_n = \max_{1 \leq i < j \leq n}(X_i - X_j)$.

In each case (3.5) holds true. Setting $s = 0$ and $r = 1$ in (b) yields the range of the $X_i$, $i \in I_n$.

Some examples where the condition $\{T_{n+1} > z\}_j \subseteq \{T_n > z\}$ is satisfied and thus (3.5) holds are given by:

(d) $T_n = \min_{1 \leq i < j \leq n}|X_i - X_j|$,
(e) $T_n = \min_{1 \leq i < j \leq n}(X_i - X_j)$,
(f) $T_n = X_{(k); n}$, $k \in \{1, \ldots, n\}$.

A more general form of (3.2), (3.5) and (3.6) respectively may be derived from (2.4). Let $r \in \mathbb{N}$, $n_j \in I_n$, $j \in M$ such that $\Sigma_{j \in M} n_j = rn$. Using the corresponding assumptions then yields inequalities of the type

$$P(T_n \in A) \leq \prod_{j \in M} P\big(T_{n_j} \in A\big)^{1/r},$$

$$F_n(z) \leq \prod_{j \in M} F_{n_j}(z)^{1/r}$$

and

$$\overline{F}_n(z) \leq \prod_{j \in M} \overline{F}_{n_j}(z)^{1/r}.$$

A characterization of equality in the various inequalities of this section may easily be derived from the condition given in Corollary 2.2 and is omitted here.

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