The Annals of Probability
1994, Vol. 22, No. 1, 264–283

ALGEBRAIC L^2 DECAY OF ATTRACTIVE CRITICAL PROCESSES ON THE LATTICE[1]

BY JEAN-DOMINIQUE DEUSCHEL

E T H Zürich

We consider a special class of attractive critical processes based on the transition function of a transient random walk on $\mathbb{Z}^d$. These processes have infinitely many invariant distributions and no spectral gap. The exponential L^2 decay is replaced by an algebraic L^2 decay. The paper shows the dependence of this algebraic rate in terms of the dimension of the lattice and the locality of the functions under consideration. The theory is illustrated by several examples dealing with locally interacting diffusion processes and independent random walks.

0. Introduction. Let E be a Polish space and consider a Markov semigroup $\{P_t: t > 0\}$ acting on $C(E;\mathbb{R})$, the space of bounded continuous functions on E. Suppose that μ is an extremal $\{P_t: t > 0\}$-invariant probability distribution on E. Then by Jensen's inequality, for each $f \in C(E, \mathbb{R})$,

$$\|P_t f - \langle f \rangle_\mu\|_{L^2(\mu)} \quad \text{is decreasing in } t \in (0, \infty),$$

where we have introduced $\langle f \rangle_\mu = \int_E f \, d\mu$. Under good ergodic properties of the process, the above convergence occurs exponentially fast: There is a $\gamma > 0$ such that

$$\|P_t f - \langle f \rangle_\mu\|_{L^2(\mu)} \le e^{-\gamma t}\| f - \langle f \rangle_\mu\|_{L^2(\mu)}, \qquad t > 0 \text{ and } f \in C(E;\mathbb{R});$$

cf. Holley and Stroock [9], Liggett [11] and Deuschel and Stroock [4]. In the case where μ is $\{P_t: t > 0\}$-reversing, the largest such γ is precisely the *spectral gap* between 0 and the first nonzero eigenvalue of the generator of the process.

The object of this paper is to describe critical situations where no such $\gamma > 0$ exists; in terms of spectral decomposition this corresponds to a continuous spectrum at 0. Following Liggett [12], the above exponential decay will be replaced by an algebraic decay. More precisely, we will find some $\alpha > 0$ and nonnegative functional V_α defined on a dense subset $\mathscr{L}(E)$ of $L^2(\mu)$ such that

$$\|P_t f - \langle f \rangle_\mu\|^2_{L^2(\mu)} \le t^{-\alpha} V_\alpha^2(f), \qquad t > 0 \text{ and } f \in \mathscr{L}(E).$$

We will be working in the setting of attractive, linear processes. These processes, based on the transition function of a transient random walk on $\mathbb{Z}^d$,

Received October 1990.
[1]Supported in part by NSF Grant DMS-88-02667 at Cornell University and the Swiss National Science Foundation Grant 21-298333.90
AMS 1991 *subject classification*. 60K35.
Key words and phrases. Interacting particle systems, algebraic rates of convergence, critical branching random walk, critical Ornstein–Uhlenbeck process.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability*. www.jstor.org

have the property that the semigroup P_t acts linearly on linear functions. They are critical in the sense of having infinitely many extremal invariant distributions, the covariances of which are given by the Green function of the random walk.

The paper is divided into three sections. In Section 1, we derive the abstract algebraic decay results. The argument, based on the linearity and attractiveness of the processes, illustrates the dependence of the algebraic rate α on terms of the dimension d of the lattice and the choice of the functional V_α. Roughly speaking, we show qualitatively how *local* functions f have a faster algebraic decay than *nonlocal* ones.

In Section 2 we present three examples of interacting diffusion processes where the theory of the first section applies. These processes are known in the literature as the *critical Ornstein–Uhlenbeck process* (cf. Deuschel [3]), the *measure-valued critical branching random walk* or *super random walk* (cf. Dawson [2] and Dynkin [5]) and the *stepping stone model* (cf. Shiga and Shimizu [14]). Although not carried out in this paper, the same type of technique would yield similar results for closely related processes such as *critical branching random walks*, the *voter model* and the *simple exclusion process*.

The setting of Section 3 is quite different, since there the extremal invariant measures are of product type. We show that this increases the algebraic decay rate α by 1, and we give an example dealing with the counting process of independent random walks.

Further results describing algebraic decays of infinite-dimensional critical processes can be found in Liggett [12], which motivated the present paper. However, they cannot be handled using our approach, since they lack the property of linearity.

1. General results. In this section we present the general frame of attractive critical processes on the lattice and derive the algebraic L^p decay results.

We start with some useful notation. Let $\mathscr{S}(\mathbb{Z}^d)$ be the set of rapidly decreasing configurations $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\sum_{\mathbf{k}\in\mathbb{Z}^d}(1+|\mathbf{k}|)^{2p}|x(\mathbf{k})|^2<\infty$, for all $p\ge 1$. The set $\mathscr{S}'(\mathbb{Z}^d)$, the dual of $\mathscr{S}(\mathbb{Z}^d)$, is the set of tempered configurations $\mathbf{x}\in\mathbb{R}^{\mathbb{Z}^d}$ with $\sum_{\mathbf{k}\in\mathbb{Z}^d}(1+|\mathbf{k}|)^{-2p}|x(\mathbf{k})|^2<\infty$, for some $p\ge 1$. We write $\|\mathbf{x}\|_p=(\sum_{\mathbf{k}}|x(\mathbf{k})|^p)^{1/p}$ for the usual $L^p(\mathbb{Z}^d)$ norm, $\mathbf{x}*\mathbf{y}(\mathbf{k})=\sum_{\mathbf{j}}x(\mathbf{j})y(\mathbf{k}-\mathbf{j})$ for the convolution operator and $\langle\mathbf{x},\mathbf{y}\rangle=\sum_{\mathbf{k}}x(\mathbf{k})y(\mathbf{k})$ for the $L^2(\mathbb{Z}^d)$ scalar product.

Next let I be a closed subset of $\mathbb{R}$ and let $E=I^{\mathbb{Z}^d}\cap\mathscr{S}'(\mathbb{Z}^d)$. Let $\mathscr{L}(E)$ be the set of Lipschitz continuous $f\colon E\to\mathbb{R}$ such that

$$|f(\mathbf{x})-f(\mathbf{y})|\le\sum_{\mathbf{k}\in\mathbb{Z}^d}\delta_{\mathbf{k}}(f)|x(\mathbf{k})-y(\mathbf{k})|,\quad\text{with }\|\delta(f)\|_1<\infty,$$

where $\delta(f)=\{\delta_{\mathbf{k}}(f)\colon\mathbf{k}\in\mathbb{Z}^d\}$ denotes the oscillation of f:

$$\delta_{\mathbf{k}}(f)\equiv\sup\left\{\frac{|f(\mathbf{x})-f(\mathbf{y})|}{x(\mathbf{k})-y(\mathbf{k})}: x(\mathbf{k})>y(\mathbf{k})\text{ and }x(\mathbf{j})=y(\mathbf{j})\text{ for }\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{k}\}\right\}.$$

Let $L(E)$ be the set of linear $f \in \mathscr{L}(E)$ of the form

$$(1.1) \qquad f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a(\mathbf{k})x(\mathbf{k}), \qquad \mathbf{x} \in E, \text{ for some } \mathbf{a} \in \mathscr{S}(\mathbb{Z}^d).$$

Finally, $\mathscr{L}_0(E)$ and $L_0(E)$ will denote the set of local Lipschitz and local linear, respectively, functions depending on finitely many coordinates.

Our process is based on a *stationary*, *irreducible*, *transition matrix* $\mathbf{Q}$ on $\mathbb{Z}^d$:

$$Q(\mathbf{k},\mathbf{j}) \geq 0, \qquad \mathbf{k} \neq \mathbf{j}, \qquad -Q(\mathbf{k},\mathbf{k}) = \sum_{\mathbf{j}\neq\mathbf{k}} Q(\mathbf{k},\mathbf{j}) = 1,$$

$$Q(\mathbf{k},\mathbf{j}) = Q(\mathbf{0},\mathbf{j}-\mathbf{k}),$$

its symmetrized matrix $\tilde{\mathbf{Q}}$

$$\tilde{Q}(\mathbf{k},\mathbf{j}) = \frac{Q(\mathbf{k},\mathbf{j}) + Q(\mathbf{j},\mathbf{k})}{2}, \qquad \mathbf{j},\mathbf{k} \in \mathbb{Z}^d,$$

and the corresponding time-continuous transition functions $\mathbf{A}_t$ and $\tilde{\mathbf{A}}_t$, $t \geq 0$:

$$\mathbf{A}_t = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{Q}^n, \qquad \tilde{\mathbf{A}}_t = \sum_{n=0}^{\infty} \frac{t^n}{n!}\tilde{\mathbf{Q}}^n.$$

Of course $\mathbf{A}_t$ and $\tilde{\mathbf{A}}_t$ are stationary (here and below, stationary always refers to shift invariance on $\mathbb{Z}^d$) and we write $A_t(\mathbf{j}-\mathbf{k}) = A_t(\mathbf{0},\mathbf{j}-\mathbf{k}) = A_t(\mathbf{k},\mathbf{j})$. We will assume that $\mathbf{Q}$ is rapidly decaying: $\{Q(\mathbf{0},\mathbf{k})\colon \mathbf{k} \in \mathbb{Z}^d\} \in \mathscr{S}(\mathbb{Z}^d)$. Using Fourier analysis, one verifies that, for each $t \geq 0$, $\mathbf{A}_t$ maps $\mathscr{S}'(\mathbb{Z}^d)$ into $\mathscr{S}'(\mathbb{Z}^d)$ (cf. [14]).

Consider now a time-continuous Markov process on E with semigroup $\{P_t\colon t > 0\}$. We suppose the existence of a stationary, extremal $\{P_t\colon t > 0\}$-invariant probability distribution μ on E. For $p \geq 1$ and $f \in L^p(\mu)$, we write $\langle f \rangle_\mu = \int_E f\,d\mu$ and $\|f\|_{\mu,p} = \|f\|_{L^p(\mu)}$. Our results will be based on the following hypotheses.

(H-1) (Linearity). *For each $t > 0$, P_t maps $L(E)$ into $L(E)$ with*

$$P_t f(\mathbf{x}) = \sum_{\mathbf{k},\mathbf{j}\in\mathbb{Z}^d} a(\mathbf{k})A_t(\mathbf{k},\mathbf{j})x(\mathbf{j}) = \langle \mathbf{a} * \mathbf{A}_t, \mathbf{x}\rangle, \qquad \mathbf{x} \in E.$$

(H-2) (Attractivity). *For each $t > 0$, P_t maps $\mathscr{L}_0(E)$ into $\mathscr{L}(E)$ with*

$$\delta_{\mathbf{j}}(P_t f) \leq \sum_{\mathbf{k}\in\mathbb{Z}^d} \delta_{\mathbf{k}}(f)A_t(\mathbf{k},\mathbf{j}) = \delta(f) * \mathbf{A}_t(\mathbf{j}), \qquad \mathbf{j} \in \mathbb{Z}^d.$$

(H-3) (Transience). *The lattice dimension d satisfies $d \geq 3$ and therefore $\tilde{\mathbf{Q}}$ is* transient. *Moreover, there is a $\sigma_2(\mu) \in \mathbb{R}^+$ such that the covariances of μ are given by*

$$\mathrm{Cov}_\mu(x(\mathbf{k}), x(\mathbf{j})) = \sigma_2^2(\mu)G(\mathbf{k},\mathbf{j}), \qquad \mathbf{k},\mathbf{j} \in \mathbb{Z}^d,$$

where $\mathbf{G} = \int_0^\infty \tilde{\mathbf{A}}_t\,dt$ is the Green operator.

(H-4) [$L^p(\mu)$-contractivity]. *Let $p \geq 2$. Then $\mathscr{L}(E) \subseteq L^p(\mu)$ and there is a $\sigma_p(\mu) \in \mathbb{R}^+$ such that*

$$\|f - \langle f \rangle_\mu\|_{p,\mu}^2 \leq \sigma_p^2(\mu) \sum_{\mathbf{k},\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{k}}(f) G(\mathbf{k},\mathbf{j}) \delta_{\mathbf{j}}(f), \qquad f \in \mathscr{L}(E).$$

REMARK 1.2.

(i) (H-2) is called *attractivity* for the following reason: Assume that for two initial configurations $\mathbf{x}$ and $\mathbf{y} \in E$ we can construct two coupled processes $\{\mathbf{X}_t(\mathbf{x}): t \geq 0\}$ and $\{\mathbf{X}_t(\mathbf{y}): t \geq 0\}$ on the same probability space $(\Omega, \mathscr{A}, P)$ such that $P(\mathbf{X}_0(\mathbf{x}) = \mathbf{x}) = 1$, $P(\mathbf{X}_0(\mathbf{y}) = \mathbf{y}) = 1$ and $E_P[f(X_t(\mathbf{x}))] = P_t f(\mathbf{x})$, $E_P[f(X_t(\mathbf{y}))] = P_t f(\mathbf{y})$, $f \in \mathscr{L}(E)$, where E_P denotes the expectation with respect to P. If, coordinatewise,

$$\mathbf{x} \leq \mathbf{y} \quad \text{implies} \quad \mathbf{X}_t(\mathbf{x}) \leq \mathbf{X}_t(\mathbf{y}), \quad P\text{-a.s.,} \tag{1.3}$$

then we get (H-2) from (H-1). Namely, take $\mathbf{x}, \mathbf{y} \in E$ with $x(\mathbf{j}) = y(\mathbf{j})$, $\mathbf{j} \neq \mathbf{k}$ and $x(\mathbf{k}) < y(\mathbf{k})$. Then (H-1) and (1.3) imply

$$\begin{aligned} |P_t f(\mathbf{y}) - P_t f(\mathbf{x})| &= |E_P[f(\mathbf{X}_t(\mathbf{y})) - f(\mathbf{X}_t(\mathbf{x}))]| \\ &\leq \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j}}(f) E_P[|X_t(\mathbf{j},\mathbf{y}) - X_t(\mathbf{j},\mathbf{x})|] \\ &= \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j}}(f) E_P[X_t(\mathbf{j},\mathbf{y}) - X_t(\mathbf{j},\mathbf{x})] \\ &= \sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j}}(f) A_t(\mathbf{k}-\mathbf{j})(y(\mathbf{k}) - x(\mathbf{k})). \end{aligned}$$

Thus $\delta_{\mathbf{k}}(P_t f) \leq \delta(f) * \mathbf{A}_t(\mathbf{k})$.

(ii) For $p = 2$, (H-4) is of the form

$$\mathrm{Var}_\mu(f) \leq \sigma_2^2(\mu) \sum_{\mathbf{k},\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{k}}(f) G(\mathbf{k},\mathbf{j}) \delta_{\mathbf{j}}(f), \qquad f \in \mathscr{L}(E),$$

which is very similar to the estimate one obtains via Dobrushin's contraction technique in the *subcritical* regime when $\sum_{\mathbf{j} \neq \mathbf{k}} Q(\mathbf{k},\mathbf{j}) < 1$ (cf. Föllmer [7]). However, here we are at criticality and Dobrushin's technique is not applicable. Note also that (H-4) can be viewed as a tightening of (H-3), since (H-4) holds for linear $f \in L(E)$.

We can now state our first main result:

THEOREM 1.4. *Assume* (H-2) *and* (H-4) *with $p \geq 2$. Choose any $q \in [1, 2d/[d+2])$. Then there is a constant $C(q,d) < \infty$ depending on q, p and* $\mathbf{G}$ *only, such that*

$$\begin{aligned} &\sup\left\{\|P_t f - \langle f \rangle_\mu\|_{\mu,p}^2 : f \in \mathscr{L}(E), \|\delta(f)\|_q^2 \leq 1\right\} \\ &\qquad \leq t^{-\alpha(q,d)} \sigma_p^2(\mu) C(q,d), \qquad t > 0, \end{aligned} \tag{1.5}$$

where $\alpha(q,d) = [2d - q(d+2)]/(2q)$.

Proof. The basic tool will be harmonic analysis as explained in Spitzer [15, Chapter 2]. Let $\mathscr{C} = (-\pi, \pi]^d$ be the d-dimensional cube and λ the normalized Lebesgue measure on $(-\pi, \pi]^d$. For $\mathbf{b} \in L^2(\mathbb{Z}^d)$, let $\hat{b}$ be the Fourier transform of $\mathbf{b}$:

$$\hat{b}(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} b(\mathbf{k})e^{i\mathbf{k}\cdot\theta} \quad \text{where } \mathbf{k}\cdot\theta = k_1\theta^1 + \cdots + k_d\theta^d \text{ and } i = \sqrt{-1}.$$

Define

$$\phi(\theta) \equiv -\sum_{\mathbf{k}\in\mathbb{Z}^d} Q(\mathbf{0},\mathbf{k})e^{i\mathbf{k}\cdot\theta} \quad \text{and} \quad \tilde{\phi}(\theta) \equiv -\sum_{\mathbf{k}\in\mathbb{Z}^d} \tilde{Q}(\mathbf{0},\mathbf{k})e^{i\mathbf{k}\cdot\theta}, \qquad \theta \in \mathscr{C}.$$

Then the irreducibility of $\tilde{\mathbf{Q}}$ implies the existence of $\varepsilon_1 > 0$ such that

$$\tilde{\phi}(\theta) \geq \varepsilon_1|\theta|^2, \qquad \theta \in \mathscr{C}; \tag{1.6}$$

cf. [15, Section II.7]. Next let $\hat{A}_t$ and $\hat{G}$ be the Fourier transforms of $\mathbf{A}_t$ and $\mathbf{G}$:

$$\hat{A}_t(\theta) = \exp[-t\phi(\theta)] \quad \text{and} \quad \hat{G}(\theta) = \frac{1}{\tilde{\phi}(\theta)}, \qquad \theta \in \mathscr{C}.$$

Using (H-2), (H-4) and Parseval's equality, we then have

$$\begin{aligned}
\|P_t f - \langle f\rangle_\mu\|_{\mu,p}^2 &\leq \sigma_p^2(\mu)\sum_{\mathbf{k},\mathbf{j}} \delta_{\mathbf{k}}(P_t f)G(\mathbf{k},\mathbf{j})\delta_{\mathbf{j}}(P_t f)\\
&= \sigma_p^2(\mu)\langle\delta(P_t f), \mathbf{G} * \delta(P_t f)\rangle\\
&\leq \sigma_p^2(\mu)\langle \mathbf{A}_t * \delta(f), \mathbf{G} * \mathbf{A}_t * \delta(f)\rangle\\
&= \sigma_p^2(\mu)\| \,|\delta(f)|^2 \hat{G}|\hat{A}_t|^2\|_{\lambda,1} = \sigma_p^2(\mu)\left\| |\hat{\delta}(f)|^2 \frac{\exp(-2t\tilde{\phi})}{\tilde{\phi}}\right\|_{\lambda,1}.
\end{aligned}$$

Take $q \in [1, 2d/[d+2])$ as above and set $p' = q/(2-q)$, $q' = p'/(p'-1)$. Then, by Hölder's inequality and the Hausdorff–Young inequality, we obtain

$$\begin{aligned}
\left\| |\hat{\delta}(f)|^2 \frac{\exp(-2t\tilde{\phi})}{\tilde{\phi}}\right\|_{\lambda,1} &\leq \|\hat{\delta}(f)\|_{\lambda,2q'}^2 \left\|\frac{\exp(-t\tilde{\phi})}{\tilde{\phi}}\right\|_{\lambda,p'}\\
&\leq t^{-\alpha(q,d)}C(q,d)\|\delta(f)\|_q^2,
\end{aligned}$$

where we have used (1.6) to show

$$\left\|\frac{\exp(-2t\tilde{\phi})}{\tilde{\phi}}\right\|_{\lambda,p'} \leq t^{1-d/(2p')}C(q,d) = t^{-\alpha(q,d)}C(q,d),$$

for some $0 < C(q,d) < \infty$. □

Remark 1.7

(i) The usual application of Theorem 1.4 is for $q = 1$, which corresponds to *local* functions with $\alpha(1,d) = (d-2)/2$. We have a slower convergence rate for *nonlocal* functions with $\lim_{q\to 2d/(d+2)} \alpha(q,d) = 0$. The algebraic rate

$\alpha(q,d)$ does not depend on $p \geq 2$, but of course $\sigma_p(\mu)$ depends on p. In general we have $\lim_{p\to\infty} \sigma_p(\mu) = \infty$ and no decay occurs in the uniform norm.

(ii) Note that (H-1) and (H-3) imply

$$\sup\left\{\|P_t f - \langle f\rangle_\mu\|_{\mu,2}^2 : f \in L(E),\ \|\delta(f)\|_q^2 \leq 1\right\}$$
$$\leq t^{-\alpha(q,d)}\sigma_2^2(\mu)C(q,d), \qquad t \geq 0.$$

Namely, take $f \in L_0(E)$ of the form (1.1). Then, by (H-1) and (H-3), we have

$$(1.8) \qquad \begin{aligned}\|P_t f - \langle f\rangle_\mu\|_{\mu,2}^2 &= \sigma_2^2(\mu)\langle \mathbf{A}_t * \mathbf{a}, \mathbf{G} * \mathbf{A}_t * \mathbf{a}\rangle \\ &\leq \sigma_2^2(\mu)\langle \mathbf{A}_t * \delta(f), \mathbf{G} * \mathbf{A}_t * \delta(f)\rangle\end{aligned}$$

and proceed as above.

One may wonder whether the above theorem gives the correct algebraic rate. The answer is yes for $q = 1$ and almost for $q > 1$:

THEOREM 1.9. *Assume* (H-1) *and* (H-3). *Then, for each* $q \in [1, 2d/[d+2])$ *and* $0 < \varepsilon < d(q-1)/q$, *there is* $c(q,d,\varepsilon) > 0$ *depending on* q, d, ε *and* $\mathbf{G}$ *only, such that*

$$\sup\left\{\|P_t f - \langle f\rangle_\mu\|_{\mu,2}^2 : f \in L_0(E) \text{ with } \|\delta(f)\|_q = 1\right\}$$
$$\geq t^{-\alpha(q,d)-\varepsilon}\sigma_2^2(\mu)c(q,d,\varepsilon), \qquad t > 1.$$

Moreover, when $q = 1$, *we may take* $\varepsilon = 0$.

PROOF. Since $\mathbf{Q}$ decays rapidly, we have $\sum_{\mathbf{k}\in\mathbb{Z}^d}|\mathbf{k}|^2\tilde{Q}(\mathbf{0},\mathbf{k}) < \infty$, and therefore there is $\varepsilon_2 < \infty$ such that

$$(1.10) \qquad \phi(\theta) \leq \varepsilon_2|\theta|^2, \qquad \theta \in \mathscr{C}.$$

The lower bound is a simple application of (H-1) and (H-3): For given $q > 1$ and $0 < \varepsilon < d(q-1)/q$, let $\mathbf{a} \in L^q(\mathbb{Z}^d)$,

$$a(\mathbf{k}) = \begin{cases} 0, & \mathbf{k} = \mathbf{0}, \\ |\mathbf{k}|^{-\varepsilon-d/q}, & \mathbf{k} \neq \mathbf{0}, \end{cases}$$

and set $a_n(\mathbf{k}) = \chi_{\{|\mathbf{k}|\leq n\}}a(\mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^d$. Note that $\lim_{n\to\infty}\|\mathbf{a}_n\|_q = \|\mathbf{a}\|_q < \infty$ and $\lim_{n\to\infty}\hat{a}_n = \hat{a}$ in $L^2(\lambda)$, with

$$(1.11) \qquad \lim_{|\theta|\to 0}|\theta|^{d-\varepsilon-d/q}|\hat{a}(\theta)| = \gamma(d), \qquad \theta \in \mathscr{C},$$

for some $\gamma(d) > 0$. Next consider the sequence $\{f_n : n \in \mathbb{Z}^+\} \subseteq L_0(E)$,

$$f_n(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_n(\mathbf{k})x(\mathbf{k}), \qquad \mathbf{x} \in E.$$

Then (H-1) and (H-3) yield

$$\lim_{n\to\infty} \| P_t f_n - \langle f_n \rangle_\mu \|_{\mu,2}^2 = \sigma_2^2(\mu) \lim_{n\to\infty} \langle \mathbf{a}_n * \mathbf{A}_t, \mathbf{G} * \mathbf{A}_t * \mathbf{a}_n \rangle$$
$$= \sigma_2^2(\mu) \left\| |\hat{a}|^2 \frac{\exp(-t\tilde{\phi})}{\tilde{\phi}} \right\|_{\lambda,1}$$

and the lower bound follows from (1.10) and (1.11). Finally, for $q = 1$, we can simply take $f(\mathbf{x}) = x(\mathbf{0})$ and check the lower bound directly using (1.10). □

There are of course functions with a faster algebraic decay! For example, let $L_0^*(E)$ be the set of $f \in L_0(E)$ of the form (1.1) with $\sum_{\mathbf{k}\in\mathbb{Z}^d} a(\mathbf{k}) = 0$, and let us introduce the Sobolev-type norm

$$\|\mathbf{a}\|_{1,1} \equiv \sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathbf{k}|\, |a(\mathbf{k})|.$$

PROPOSITION 1.12. *Assume* (H-1) *and* (H-3). *Then there exist constants* $0 < k(d) < K(d) < \infty$ *such that*

$$t^{-d/2}\sigma_2^2(\mu) k(d) \le \sup\left\{ \| P_t f - \langle f \rangle_\mu \|_{\mu,2}^2 : f \in A_0^*(E) \text{ with } \|\delta(f)\|_{1,1} = 1 \right\}$$
$$\le t^{-d/2}\sigma_2^2(\mu) K(d), \qquad t \ge 1.$$

PROOF. Take $f \in L_0^*(E)$ as above. Note that $\hat{a}(0) = \sum_{\mathbf{k}} a(\mathbf{k}) = 0$ and

$$|\nabla \hat{a}(\eta) \cdot \theta| = \left| \sum_{\mathbf{k}\in\mathbb{Z}^d} a(\mathbf{k})(\mathbf{k}\cdot\theta) e^{i\mathbf{k}\cdot\eta} \right| \le |\mathbf{a}|_{1,1} |\theta|, \qquad \eta, \theta \in \mathscr{C},$$

and therefore, by the mean value theorem,

$$|\hat{a}|(\theta) < |\mathbf{a}|_{1,1} |\theta|, \qquad \theta \in \mathscr{C}.$$

From this and (1.6), we see that there is $K(d) < \infty$ such that

$$\left\| |\hat{a}|^2 \frac{\exp(-2t\tilde{\phi})}{\tilde{\phi}} \right\|_{\lambda,1} \le \left\| \frac{|\hat{a}|^2}{\tilde{\phi}} \right\|_{\lambda,\infty} \|\exp(-2t\tilde{\phi})\|_{\lambda,1} \le K(d) t^{-d/2} \|\mathbf{a}\|_{1,1}, \qquad t \ge 1.$$

This clearly implies the upper bound by (1.8). For the lower bound, choose

$$a(\mathbf{k}) = \begin{cases} \dfrac{\mathbf{k}\cdot\mathbf{1}}{2d}, & |\mathbf{k}| = 1, \\ 0, & |\mathbf{k}| \ne 1. \end{cases}$$

Then, as $\|\mathbf{a}\|_{1,1} = 1$ and

$$|\hat{a}(\theta)|^2 = \frac{|\sin(\theta^1) + \cdots + \sin(\theta^d)|}{d} \ge \frac{4}{(d\pi)^2} |\theta|^2 \quad \text{on } \mathscr{C}^+ \equiv [0, \pi/2]^d,$$

we get the lower bound in view of (1.8) and (1.10). □

The above proposition is not too surprising, considering the fact that the mean of each coordinate is a preserved quantity.

In the next section we will present a few interacting diffusion processes where the above theory applies.

2. Interacting diffusion processes. In this section we present three types of interacting diffusion processes which satisfy the above assumptions. For simplicity, we will suppose that the transition matrix $\mathbf{Q}$ is of finite range and that d, the dimension of the lattice, satisfies $d \geq 3$. Since $\tilde{\mathbf{Q}}$ is irreducible by assumption, this implies transience.

Let I be a closed interval of $\mathbb{R}$, and set $E = I^{\mathbb{Z}^d} \cap \mathscr{S}'(\mathbb{Z}^d)$. Let $\Omega = C([0,\infty); E)$ be the space of continuous paths, $\mathbf{X}_t$: $\Omega \to E$ the coordinate mapping and $\{\mathscr{F}_t: t > 0\}$ the canonical filtration. Let $a \in C^2(I; \mathbb{R}^+)$ be strictly positive in the interior of I and zero on the boundary. We also assume that a' and a'', the first and second derivatives of a, are bounded. For each $\mathbf{x} \in E$, let $P_\mathbf{x}$ be the law on Ω of the infinite system of Itô stochastic differential equations

$$(2.1)\quad \begin{aligned} X_t(\mathbf{k}) &= x(\mathbf{x}) + \int_0^t \sum_{\mathbf{j}\in\mathbb{Z}^d} Q(\mathbf{k},\mathbf{j}) X_s(\mathbf{j})\, ds \\ &\quad + \int_0^t \sqrt{a(X_s(\mathbf{k}))}\, dW_s(\mathbf{k}), \qquad \mathbf{k}\in\mathbb{Z}^d, \end{aligned}$$

where $\{W_t(\mathbf{k}): t > 0, \mathbf{k} \in \mathbb{Z}^d\}$ is a family of independent real-valued Wiener processes. Existence and uniqueness of (2.1) has been established by Shiga and Shimizu [14]. Let $C_0^2(E)$ be the set of bounded functions f: $E \to \mathbb{R}$ depending on finitely many coordinates with bounded partial derivatives of first and second order. We can also view the Markovian family $\{P_\mathbf{x}: \mathbf{x} \in E\}$ as the unique solution to the martingale problem for the linear operator L defined on $C_0^2(E)$ by

$$Lf(\mathbf{x}) \equiv \sum_{\mathbf{k}\in\mathbb{Z}^d} \left[\tfrac{1}{2} a(x(\mathbf{k})) D_\mathbf{k}^2 + \sum_{\mathbf{j}\in\mathbb{Z}^d} Q(\mathbf{k},\mathbf{j}) x(\mathbf{j}) D_\mathbf{k} \right] f(\mathbf{x}),$$

where $D_\mathbf{k} = \partial/\partial x(\mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^d$.

We will be interested in the following situations.

1. *The critical Ornstein–Uhlenbeck process*. This is a Gaussian process with

$$I = \mathbb{R} \quad \text{and} \quad a(x) \equiv 1,\ x \in \mathbb{R};$$

cf. [3]. For each $\tau \in \mathbb{R}$, let $\mu = \mu(\tau)$ be the Gaussian field on E with constant mean $\tau \in \mathbb{R}$ and covariances

$$\mathrm{Cov}_\mu(x(\mathbf{k}), x(\mathbf{j})) = G(\mathbf{k},\mathbf{j}), \qquad \mathbf{k},\mathbf{j} \in \mathbb{Z}^d.$$

Then μ is an extremal stationary $\{P_t: t > 0\}$-invariant distribution (cf. [3]).

2. *The measure-valued critical branching random walk or super random walk*. Set

$$I = \mathbb{R}^+ \quad \text{and} \quad a(x) = x, \qquad x \in \mathbb{R}^+.$$

This is a discrete version of the critical measure Brownian motion of Dawson and Watanabe where the underlying Brownian motion on $\mathbb{R}^d$ has been replaced by the random walk on $\mathbb{Z}^d$ with transition function $\mathbf{Q}$ (cf. [2] and [5]). For each $\tau \in (0, \infty)$, define $\boldsymbol{\tau} \in E$ $\boldsymbol{\tau}(\mathbf{k}) = \tau$, $\mathbf{k} \in \mathbb{Z}^d$. Then there is an extremal stationary $\{P_t: t > 0\}$-invariant measure $\mu = \mu(\tau)$ on E such that, for all $f \in \mathscr{L}_0(E)$,

$$\lim_{t \to \infty} P_t f(\boldsymbol{\tau}) = \langle f \rangle_\mu \quad \text{and} \quad \langle x(\mathbf{k}) \rangle_\mu = \tau, \qquad \mathbf{k} \in \mathbb{Z}^d. \tag{2.2}$$

3. *The stepping stone model*. Set

$$I = [0, 1] \quad \text{and} \quad a(x) = x(1 - x), \qquad x \in [0, 1].$$

This is a popular model in population genetics (cf. [13]). For each $\tau \in (0, 1)$, Shiga [13] has shown the existence an extremal stationary $\{P_t: t > 0\}$-invariant measure $\mu = \mu(\tau)$ on E satisfying (2.2).

Our objective is to show that the assumptions (H-1)–(H-4) hold for the above processes. The first step, the linearity assumption (H-1), follows immediately from (1.1) and Itô's formula (cf. [3], Proposition 2.13). Also, since $\sqrt{a}$ is Hölder-continuous with exponent $\frac{1}{2}$, the process is monotone in the sense of (1.3) (cf. [14]), and (H-2) follows from Remark 1.2. Our next step identifies the covariances of the invariant measure μ.

LEMMA 2.3. *Set* $\sigma_2^2(\mu) = \frac{1}{2}\|a\|_{\mu,1}$. *Then*

$$\operatorname{Cov}_\mu(x(\mathbf{k}), x(\mathbf{j})) = \sigma_2^2(\mu) G(\mathbf{k}, \mathbf{j}), \qquad \mathbf{k}, \mathbf{j} \in \mathbb{Z}^d. \tag{2.4}$$

PROOF. Set

$$R(\mathbf{k}, \mathbf{j}) \equiv \operatorname{Cov}_\mu(x(\mathbf{k}), x(\mathbf{j})), \qquad \mathbf{k}, \mathbf{j} \in \mathbb{Z}^d.$$

Then, using Itô's formula, one can check that $\mathbf{R}$ is characterized by the following linear equations:

$$0 = \sum_{i \in \mathbb{Z}^d} Q(\mathbf{j}, \mathbf{i}) R(\mathbf{i}, \mathbf{k}) + \sum_{\mathbf{i} \in \mathbb{Z}^d} Q(\mathbf{k}, \mathbf{i}) R(\mathbf{i}, \mathbf{j}) + \delta(\mathbf{j}, \mathbf{k}) \|a\|_{\mu,1}, \qquad \mathbf{j}, \mathbf{k} \in \mathbb{Z}^d,$$

where

$$\delta(\mathbf{j}, \mathbf{k}) = \begin{cases} 1, & \mathbf{j} = \mathbf{k}, \\ 0, & \mathbf{j} \neq \mathbf{k}; \end{cases}$$

cf. [3]. By stationarity of μ we have $R(\mathbf{k}, \mathbf{j}) = R(\mathbf{0}, \mathbf{j} - \mathbf{k})$, $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$, and therefore

$$0 = 2 \sum_{\mathbf{i} \in \mathbb{Z}^d} \tilde{Q}(\mathbf{0}, \mathbf{i}) R(\mathbf{0}, \mathbf{k} - \mathbf{j} - \mathbf{i}) + \delta(\mathbf{j}, \mathbf{k}) \|a\|_{\mu,1}.$$

Since $\tilde{\mathbf{Q}}$ is transient, $1/\tilde{\phi} \in L^1(\mathscr{C})$ (cf. [15]), and if $\hat{R}(\theta) \equiv \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{i\mathbf{k}\cdot\theta} R(\mathbf{0}, \mathbf{k})$, $\theta \in \mathscr{C}$, denotes the Fourier transform of $\mathbf{R}$, then the above equation yields

$$\hat{R}(\theta)\tilde{\phi}(\theta) = \sigma_2^2(\mu), \quad \text{that is,} \quad \hat{R}(\theta) = \sigma_2^2(\mu)/\tilde{\phi}(\theta),$$

which implies (2.4) □

LEMMA 2.5. *For each* $f \in C_0^2(E)$,

$$\lim_{t\to\infty} \| P_t f - \langle f \rangle_\mu \|_{\mu,2} = 0. \tag{2.6}$$

PROOF. Let us first look at the Ornstein–Uhlenbeck process. Let $\{P_t'\colon t > 0\}$ and $\{\tilde{P}_t\colon t > 0\}$ be the Markov semigroups associated with the Ornstein–Uhlenbeck processes with matrices $\mathbf{Q}'$ (the adjoint of $\mathbf{Q}$) and $\tilde{\mathbf{Q}}$, respectively. One can then check that $\{P_t'\colon t > 0\}$ and $\{\tilde{P}_t\colon t > 0\}$ are the semigroups corresponding to L^* (the μ-adjoint of L) and $\tilde{L} = (L + L^*)/2$. In particular, μ is $\{\tilde{P}_t\colon t > 0\}$-reversing and, since L and L^* commute,

$$\| P_t f - \langle f \rangle_\mu \|_{\mu,2} = \| \tilde{P}_t f - \langle f \rangle_\mu \|_{\mu,2}.$$

Also, μ is an extremal $\{\tilde{P}_t\colon t > 0\}$-invariant distribution, and this implies

$$\lim_{t\to\infty} \| \tilde{P}_t f - \langle f \rangle_\mu \|_{\mu,2} = 0,$$

by the spectral decomposition theorem (cf. [9]).

Consider next the super random walk and stepping stone model. Since f is bounded, (2.6) will follow from

$$\lim_{n\to\infty} P_{t_n} f = \langle f \rangle_\mu \quad \text{in probability with respect to } \mu, \tag{2.7}$$

for a sequence $\{t_n\colon n \in \mathbb{Z}^+\} \subseteq \mathbb{R}^+$ with $t_n \nearrow \infty$. In case of the stepping stone model, we know by Theorem 1.3 of [13] that, for each $\tau \in [0,1]$, $\mu = \mu(\tau)$, we have

$$\lim_{t\to\infty} P_t f(\mathbf{x}) = \langle f \rangle_\mu \tag{2.8}$$

if $\mathbf{x} \in E$ satisfies

$$\lim_{t\to\infty} \mathbf{A}_t * \mathbf{x}(\mathbf{k}) = \lim_{t\to\infty} P_t(f_{\mathbf{k}})(\mathbf{x}) = \langle f_{\mathbf{k}} \rangle_\mu = \tau, \quad \text{for all } \mathbf{k} \in \mathbb{Z}^d,$$

where $f_{\mathbf{k}}(\mathbf{x}) = x(\mathbf{k})$. However, we know that $\lim_{t\to\infty} \|P_t(f_{\mathbf{k}}) - \tau\|_{\mu,2} = 0$ [cf. Remark 1.7(ii)], and (2.7) follows.

For the super random walk, (2.8) is verified if, for all $0 \le \mathbf{c} \in L^1(\mathbb{Z}^d)$ with finite support,

$$\lim_{t\to\infty} \exp[-(V_t\mathbf{c}, \mathbf{x})] \equiv \lim_{t\to\infty} P_t(\exp[-h(\mathbf{c})])(\mathbf{x}) = \exp[-(\mathbf{c} - W\mathbf{c}, \boldsymbol{\tau})],$$

where $h(\mathbf{c}) \in L_0(E)$ is given by $h(\mathbf{c})(\mathbf{y}) \equiv \sum_{\mathbf{k} \in \mathbb{Z}^d} c(\mathbf{k}) y(\mathbf{k})$, $V_t\mathbf{c}\colon \mathbb{Z}^d \to \mathbb{R}$ is the solution to the nonlinear integral equation

$$V_t\mathbf{c}(\mathbf{k}) + \tfrac{1}{2}\int_0^t \mathbf{A}_{t-s} * [V_s\mathbf{c}]^2(\mathbf{k})\, ds = \mathbf{A}_t * \mathbf{c}(\mathbf{k}), \qquad \mathbf{k} \in \mathbb{Z}^d, t > 0,$$

and

$$W\mathbf{c}(\mathbf{k}) \equiv \tfrac{1}{2}\int_0^\infty [V_s\mathbf{c}]^2(\mathbf{k})\,ds, \qquad \mathbf{k} \in \mathbb{Z}^d;$$

cf. [5], Theorem 1.7. In order to show (2.7), it will be enough to verify

$$\lim_{t\to\infty} \|(V_t\mathbf{c}, \mathbf{x}) - (\mathbf{c} - W\mathbf{c}, \boldsymbol{\tau})\|_{\mu,2}^2 = 0. \tag{2.9}$$

For simplicity we assume that $\mathbf{Q} = \tilde{\mathbf{Q}}$; the nonsymmetric case works along the same lines. Note that

$$[V_s\mathbf{c}]^2(\mathbf{k}) \le [A_s * \mathbf{c}]^2(\mathbf{k}) \le A_s(\mathbf{0})\|\mathbf{c}\|_1 \cdot \mathbf{A}_s * \mathbf{c}(\mathbf{k}),$$

and since $\mathbf{Q}$ is transient, $\int_0^\infty A_s(\mathbf{0})\,ds = \int_{\mathscr{C}}(1/\phi)\,d\lambda = G(\mathbf{0},\mathbf{0}) < \infty$. We can replace $W\mathbf{c}$ in (2.9) by

$$W_t\mathbf{c}(\mathbf{k}) = \tfrac{1}{2}\int_0^t [V_s\mathbf{c}]^2(\mathbf{k})\,ds, \qquad \mathbf{k} \in \mathbb{Z}^d.$$

Using these two inequalities and the nonlinear integral equation, we get

$$\begin{aligned}
&\|(V_t\mathbf{c}, \mathbf{x}) - (\mathbf{c} - W_t\mathbf{c}, \boldsymbol{\rho})\|_{\mu,2}^2 \\
&\quad\le \left\|\frac{1}{2}\int_0^t \left(\mathbf{A}_{t-s} * [V_s\mathbf{c}]^2, \mathbf{x} - \boldsymbol{\rho}\right) ds - (\mathbf{A}_t * \mathbf{c}, \mathbf{x} - \boldsymbol{\rho})\right\|_{\mu,2}^2 \\
&\quad\le \frac{1}{2}\left\|\int_0^t \left(\mathbf{A}_{t-s} * [V_s\mathbf{c}]^2, \mathbf{x} - \boldsymbol{\tau}\right) ds\right\|_{\mu,2}^2 \\
&\qquad + 2\|(\mathbf{A}_t * \mathbf{c}, \mathbf{x} - \boldsymbol{\rho})\|_{\mu,2}^2 \\
&\quad\le \sum_{\mathbf{k},\mathbf{j}\in\mathbb{Z}^d} \mathbf{A}_t * \mathbf{c}(\mathbf{k}) R(\mathbf{k},\mathbf{j}) \mathbf{A}_t(\mathbf{j}) * \mathbf{c}(\mathbf{j}) \\
&\qquad \times \left[\frac{1}{2}\int_0^t\int_0^t A_s(\mathbf{0}) A_{s'}(\mathbf{0})\|\mathbf{c}\|_1^2\,ds\,ds' + 2\right] \\
&\quad\le \left[2 + G^2(\mathbf{0},\mathbf{0})\|\mathbf{c}\|_1^2\right] \left\| |\hat{a}|^2 \frac{\exp(-t\phi)}{\phi} \right\|_{\lambda,1}^2,
\end{aligned}$$

which converges to 0 as $t \to \infty$. □

LEMMA 2.10. *Let $C^2(E)$ be the set of bounded $f\colon E \to \mathbb{R}$ with bounded first and second derivatives such that*

$$\begin{aligned}
|||f|||_p \equiv &\sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + |\mathbf{k}|)^{2p}\|D_{\mathbf{k}}f\|_\infty \\
&+ \sum_{\mathbf{k},\mathbf{j}\in\mathbb{Z}^d} (1 + |\mathbf{k}|)^{2p}(1 + |\mathbf{j}|)^{2p}\|D_{\mathbf{k}}D_{\mathbf{j}}f\|_\infty < \infty,
\end{aligned}$$

for all $p \geq 1$. *Then, for each* $t > 0$, P_t *maps* $C_0^2(E)$ *into* $C^2(E)$ *and*

$$(2.11)\qquad \begin{aligned} P_t(f)(\mathbf{x}) &= f(\mathbf{x}) + \int_0^t \overline{L} P_s(f)(\mathbf{x})\, ds \\ &= f(\mathbf{x}) + \int_0^t P_s(Lf)(\mathbf{x})\, ds, \qquad \mathbf{x} \in E, \end{aligned}$$

for all $f \in C_0^2(E)$, *where* $\overline{L}$ *defined on* $C^2(E)$ *is given by*

$$\overline{L}f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \left[\tfrac{1}{2} a(x(\mathbf{k})) D_{\mathbf{k}}^2 + \sum_{\mathbf{j} \in \mathbb{Z}^d} Q(\mathbf{k},\mathbf{j}) x(\mathbf{j}) D_k \right] f(\mathbf{x}).$$

PROOF. Let $V_N \subseteq \mathbb{Z}^d$ be the box $[-N, N]^d$. We consider a finite-dimensional system defined on $E^N = I^{V_N}$ generated by L^N: $C^2(E^N) \to C^2(E^N)$:

$$L^N f(\mathbf{x}) = \tfrac{1}{2} \sum_{\mathbf{k} \in V_N} \left[a(x(\mathbf{k})) D_{\mathbf{k}}^2 + \sum_{\mathbf{j} \in V_N} Q(\mathbf{k},\mathbf{j}) x(\mathbf{j}) D_{\mathbf{k}} \right] f(\mathbf{x}).$$

The first step is to show that smoothness and boundedness of our coefficients imply that the statement holds for L^N and the corresponding semigroup $\{P_t^N: t > 0\}$. In case of the Ornstein–Uhlenbeck process, this is trivial since the diffusion coefficient a is constant. In case of the stepping stone model, Shiga shows that the Hille–Yoshida semigroup theory is applicable (cf. [13], 6°, page 241), and the result follows from an argument similar to that for the one-dimensional case treated by Ethier [6]. Finally, for critical branching random walks, one could either work with a dense class generated by exponential functions of the form $\exp[-h(\mathbf{c})]$ and the corresponding nonlinear integral equation in order to check the Hille–Yoshida theory as in [5], or use the fact that, due to the special nature of L^N, the semigroup P_t^N maps polynomials in $\mathbf{x}$ into polynomials.

Next let us prove that, for each $f \in C_0^2(E)$, $t > 0$ and $p \geq 1$, we can find $K_p < \infty$ such that

$$(2.12)\qquad \sup_{N \in \mathbb{Z}^+} |||P_t^N(f)|||_p \leq e^{K_p t} |||f|||_p.$$

This uniform estimate and the convergence of $P_t^N(f)$ to $P_t(f)$ on E will imply the result by the standard approximation procedure (cf. [14]).

Let us write

$$f_t^N(\mathbf{k}) = D_{\mathbf{k}} P_t^N(f), \qquad f_t^N(\mathbf{k},\mathbf{j}) = D_{\mathbf{k}} D_{\mathbf{j}} P_t^N(f)$$

and

$$u_t^N(\mathbf{k}) = \| D_{\mathbf{k}} P_t^N(f) \|_\infty, \qquad u_t^N(\mathbf{k},\mathbf{j}) = \| D_{\mathbf{k}} D_{\mathbf{j}} P_t^N(f) \|_\infty.$$

From (2.11) we obtain the equations

$$\frac{d}{dt} f_t^N(\mathbf{k}) = L_{\mathbf{k}}^N f_t^N(\mathbf{k}) + \sum_{\mathbf{j} \in V_N} Q(\mathbf{k},\mathbf{j}) f_t^N(\mathbf{j}), \qquad \mathbf{k} \in V_N,$$

where $L^N_{\mathbf{k}}$ is the diffusion operator $L^N + \frac{1}{2}a'(x_{\mathbf{k}})D_{\mathbf{k}}$, and

$$\frac{d}{dt} f_t^N(\mathbf{k},\mathbf{j}) = L^N_{\mathbf{k},\mathbf{j}} f_t^N(\mathbf{k},\mathbf{j}) + \sum_{\mathbf{i}\in V_N} Q(\mathbf{k},\mathbf{i}) f_t^N(\mathbf{i},\mathbf{j}) + \sum_{\mathbf{i}\in V_N} Q(\mathbf{j},\mathbf{i}) f_t^N(\mathbf{i},\mathbf{k})$$
$$+ \frac{1}{2}\delta(\mathbf{k},\mathbf{j}) a''(x_{\mathbf{k}}) f_t^N(\mathbf{k},\mathbf{j}), \qquad \mathbf{k},\mathbf{j}\in V_N,$$

where $L^N_{\mathbf{k},\mathbf{j}} = L^N + \frac{1}{2}a'(x_{\mathbf{k}})D_{\mathbf{k}} + \frac{1}{2}a'(x_{\mathbf{j}})D_{\mathbf{j}}$. Integrating with respect to the semigroups generated by $L^N_{\mathbf{k}}$ and $L^N_{\mathbf{k},\mathbf{j}}$, respectively, and taking the supremum norm on both sides, yields the following integral inequalities:

$$u_t^N(\mathbf{k}) \le u_0^N(\mathbf{k}) + \sum_{\mathbf{j}\in V_N} |Q(\mathbf{k},\mathbf{j})| \int_0^t u_s^N(\mathbf{j})\, ds$$

and

$$u_t^N(\mathbf{k},\mathbf{j}) \le u_s^N(\mathbf{k},\mathbf{j}) + \sum_{\mathbf{i}\in V_N} |Q(\mathbf{k},\mathbf{i})| \int_0^t u_s^N(\mathbf{i},\mathbf{j})\, ds$$
$$+ \sum_{\mathbf{i}\in V_N} |Q(\mathbf{j},\mathbf{i})| \int_0^t u_s^N(\mathbf{i},\mathbf{k})\, ds + \tfrac{1}{2}\delta_{\mathbf{j},\mathbf{k}} \|a''\|_\infty \int_0^t u_s^N(\mathbf{j},\mathbf{k})\, ds.$$

Since $\mathbf{Q}$ is of finite range, (2.12) follows from Lemma 4.2 and Lemma 4.3 of [14]. □

We are now ready to prove the main step, namely, the L^p-contractivity:

THEOREM 2.13. *For any $p \ge 2$ and $f \in \mathscr{L}(E)$,*

$$\| f - \langle f\rangle_\mu \|^2_{\mu,p} \le (p-1)\frac{\|a\|_{\mu,p/2}}{2} \langle \delta(f), \mathbf{G} * \delta(f)\rangle. \tag{2.14}$$

PROOF. By approximation, it is enough to prove (2.14) for $f \in C_0^2(E)$ with $\langle f\rangle_\mu = 0$. For $t \ge 0$ write $f_t = P_t f \in C^2(E)$ (cf. Lemma 2.10). Since $C^2(E)$ is an algebra, $\Psi \circ f_t \in C^2(E)$ for any $\Psi \in C^2(\mathbb{R};\mathbb{R})$ and

$$\bar{L}(\Psi\circ f_t) = (\Psi'\circ f_t)\bar{L}f_t + \tfrac{1}{2}(\Psi''\circ f_t)\bar{\Gamma}(f_t, f_t),$$

where $\Psi'(z) = d\Psi(z)/dz$ and

$$\bar{\Gamma}(f_t, f_t) = \left(\bar{L}(f_t^2) - 2 f_t \bar{L} f_t\right) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a(x_{\mathbf{k}}) |D_{\mathbf{k}} f_t|^2;$$

cf. [4]. In particular if, for any $\varepsilon > 0$, we choose $\Psi_\varepsilon(z) = (z^2+\varepsilon)^{p/2}$ with

$$\Psi_\varepsilon''(z) = p(p-1)(z^2+\varepsilon)^{p/2-2}\left(z^2 + \varepsilon/(p-1)\right) \le p(p-1)\left(\Psi_\varepsilon(z)\right)^{1-2/p},$$

then we have

$$\bar{L}(\Psi_\varepsilon\circ f_t) \le (\Psi_\varepsilon'\circ f_t)\bar{L}f_t + \tfrac{1}{2}p(p-1)(\Psi_\varepsilon\circ f_t)^{1-2/p}\bar{\Gamma}(f_t, f_t).$$

Using $\langle \overline{L}(\Psi_\varepsilon \circ f_t)\rangle_\mu = 0$, we obtain

$$-\frac{d}{dt}\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{2/p} = -\frac{2}{p}\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{-1+2/p}\langle(\Psi_\varepsilon' \circ f_t)\overline{L}f_t\rangle_\mu$$
$$\le (p-1)\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{-1+2/p}\langle(\Psi_\varepsilon \circ f_t)^{1-2/p}\overline{\Gamma}(f_t, f_t)\rangle_\mu,$$

where

$$\overline{\Gamma}(f_t, f_t) \le \sum_{k\in\mathbb{Z}^d} a(x_{\mathbf{k}})\delta_{\mathbf{k}}(f_t)^2.$$

Hölder's inequality implies

$$\langle(\Psi_\varepsilon \circ f_t)^{1-2/p}\overline{\Gamma}(f_t, f_t)\rangle_\mu \le \sum_{\mathbf{k}\in\mathbb{Z}^d}\langle(\Psi_\varepsilon \circ f_t)^{1-2/p}a(x_{\mathbf{k}})\rangle_\mu\delta_{\mathbf{k}}(f_t)^2$$
$$\le \sum_{\mathbf{k}\in\mathbb{Z}^d}\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{1-2/p}\langle a(x_{\mathbf{k}})^{p/2}\rangle_\mu^{2/p}\delta_{\mathbf{k}}(f_t)^2$$
$$= \langle \Psi_\varepsilon \circ f_t\rangle_\mu^{1-2/p}\|a\|_{\mu,p/2}\sum_{\mathbf{k}\in\mathbb{Z}^d}\delta_{\mathbf{k}}(f_t)^2,$$

but, by (H-2)', this shows

$$-\frac{d}{dt}\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{2/p} \le (p-1)\|a\|_{\mu,p/2}\sum_{\mathbf{k}\in\mathbb{Z}^d}\delta_{\mathbf{k}}(f_t)^2$$
$$\le (p-1)\|a\|_{\mu,p/2}\|\mathbf{A}_t * \delta(f)\|_2^2.$$

On the other hand, since f is bounded, Lemma 2.5 implies $\lim_{t\to\infty}\langle \Psi_\varepsilon \circ f_t\rangle_\mu^{2/p} = \varepsilon$, and we obtain

$$\langle \Psi_\varepsilon \circ f\rangle_\mu^{2/p} - \varepsilon = \int_0^\infty -\frac{d}{ds}\langle \Psi_\varepsilon \circ f_s\rangle_\mu^{2/p}\,ds$$
$$\le (p-1)\|a\|_{\mu,p/2}\int_0^\infty \|\mathbf{A}_s * \delta(f)\|_2^2\,ds$$
$$= (p-1)\frac{\|a\|_{\mu,p/2}}{2}\langle \delta(f), \mathbf{G} * \delta(f)\rangle,$$

and the result follows by letting $\varepsilon \searrow 0$. □

REMARK 2.15.

(i) It should be noted that, apart from Lemma 2.5 and the first step of the proof of Lemma 2.10, we do not use the explicit form of the function a. In view of Ethier's one-dimensional result [6], one expects that Lemma 2.10 holds under quite general assumptions.

In particular, referring to the stepping stone model, we could look at a very general class of interacting diffusion processes in population genetics, where the diffusion coefficient $a \in C^2(I; \mathbb{R}^+)$ satisfies $a(x) > 0$, $x \in (0,1)$ and $a(0) = a(1) = 0$. Recently, Cox and Greven [1] have shown the existence of stationary extremal invariant distributions $\{\mu(\tau),\ \tau \in (0,1)\}$ satisfying (2.2). It is clear that (H-1)–(H-3) hold and therefore one has the lower bounds of Theorem 1.9 and Proposition 1.13. The only step missing in the proof of the L^p-contractivity is Lemma 2.5. Actually, it is very likely that a modification of the coupling technique of [1] would imply (2.6).

(ii) Another setting where the theory applies is the potlatch- and smoothing-type processes treated in the last chapter of Liggett's book (cf. [10], Examples 6.6 and 6.12). The linearity (H-1) is shown in Theorem 2.2 of [10]. The coupling techniques used for the proof of the attractivity condition (H-2) are explained in the first section of Chapter IX of [10], [cf. also [10], (2.16), (2.18) and (2.20)]. Also the covariances of the invariant distributions, condition (H-3), are expressed in terms of the Green function for the random walk [cf. [10], Theorem 3.17, equation (3.20)]. A verification of the L^p-contractivity (H-4) would then imply the full results of Section 1.

We conclude this section with the following observation: The idea in the proof of Theorem 1.4 is first to determine the correct algebraic rate for linear functions in $L(E)$ using the linearity of the semigroups $\{P_t\colon t > 0\}$ and the covariance structures [cf. (1.8)] and then to use a comparison argument based on the monotonicity or attractiveness of the processes. The functional $\|\delta(\cdot)\|_q$ is well adapted for comparison with linear functions, but would not work for polynomials. Actually, we expect a different decay for these functions. This fact can be explicitly verified for the Ornstein–Uhlenbeck process.

We need to introduce some additional notation: Let $\mathscr{P}_n(E)$ be the set of polynomials on E of degree less than or equal to n, $n \in \mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$ define $\delta^{(n)}\colon C_0^{\infty,n}(E) \equiv C_0^\infty(E) \cup \mathscr{P}_n(E) \to L^1(\otimes_n \mathbb{Z}^d)$

$$\delta^{(n)}_{(\mathbf{k}_1,\dots,\mathbf{k}_n)}(f) \equiv \sup_{\mathbf{x}\in E} \left| D_{\mathbf{k}_1} \dots D_{\mathbf{k}_n} f(\mathbf{x}) \right|, \qquad (\mathbf{k}_1,\dots,\mathbf{k}_n) \in \bigotimes_n \mathbb{Z}^d,$$

and set

$$\|\delta^{(n)}(f)\|_1 = \sum_{(\mathbf{k}_1,\dots,\mathbf{k}_n)\in \otimes_n \mathbb{Z}^d} \delta_{(\mathbf{k}_1,\dots,\mathbf{k}_n)}(f).$$

THEOREM 2.16. *Consider the critical Ornstein–Uhlenbeck process and define iteratively $D^{(1)}(\mu) = C_0^\infty(E)$,*

$$D^{(n)}(\mu) = \Big\{ f \in C_0^{\infty,n}(E) \colon \langle D_{\mathbf{k}_1} \dots D_{\mathbf{k}_m} f \rangle_\mu = 0, \text{ for all } (\mathbf{k}_1,\dots,\mathbf{k}_m) \in \bigotimes_m \mathbb{Z}^d \text{ and } 1 \le m \le n-1 \Big\}.$$

Then, for each $n \in \mathbb{Z}^+$, there exists $0 < c^{(n)}(d) < C^{(n)}(d) < \infty$ such that

$$t^{-(d/2-1)n} c^{(n)}(d) \le \sup\Big\{ \|P_t f - \langle f \rangle_\mu\|^2_{\mu,2} \colon f \in D^{(n)}(\mu) \text{ with } \|\delta^{(n)}(f)\|_1 = 1 \Big\} \le t^{-(d/2-1)n} C^{(n)}(d), \qquad t > 1.$$

PROOF. We proceed by induction. The case $n = 1$ is covered by Theorems 1.4 and 1.9. Assume that the upper bound holds for $n-1$, $n \geq 2$, and take $f \in D^{(n)}(\mu)$ with $\langle f \rangle_\mu = 0$. Then, using the ideas of the proof of Theorem 2.13 with $\Psi_0(z) = z^2$, we get

$$(2.17)\quad \|P_t f\|_{\mu,2}^2 = \int_t^\infty \langle \bar{\Gamma}(P_s f, P_s f)\rangle_\mu \, ds = \int_t^\infty \sum_{\mathbf{k}\in\mathbb{Z}^d} \left\langle |D_{\mathbf{k}}(P_s f)|^2 \right\rangle_\mu ds.$$

However, for the Ornstein–Uhlenbeck process we simply have

$$D_{\mathbf{k}}(P_s f) = P_s\Big(\sum_{\mathbf{j}\in\mathbb{Z}^d} D_{\mathbf{j}} f \cdot A_s(\mathbf{k}-\mathbf{j})\Big) = P_s(\mathbf{D}(f) * \mathbf{A}_s(\mathbf{k})), \qquad \mathbf{k}\in\mathbb{Z}^d,\ s>0$$

(cf. [3]), where $\mathbf{D}(f)*\mathbf{A}_s(\mathbf{k}) \equiv \sum_{\mathbf{j}\in\mathbb{Z}^d} D_{\mathbf{j}} f\cdot A_s(\mathbf{k}-\mathbf{j}) \in D^{(n-1)}(\mu)$ by assumption. Therefore,

$$(2.18)\quad \begin{aligned} &\|P_s(\mathbf{D}(f)*\mathbf{A}_s(\mathbf{k}))\|_{\mu,2}^2 \\ &\quad \leq s^{-(n-1)(1-d/2)} C^{(n-1)}(d) \|\delta^{(n-1)}(\mathbf{D}(f)*\mathbf{A}_s(k))\|_1^2, \end{aligned}$$

and the upper bound follows from (2.17) and (2.18) if we can find $\tilde{C}^{(n)}(d) \in (0,\infty)$ such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \|\delta^{(n-1)}(\mathbf{D}(f)*\mathbf{A}_s(\mathbf{k}))\|_1^2 \leq s^{-d/2}\tilde{C}^{(n)}(d)\|\delta^{(n)}(f)\|_1^2.$$

This a simple consequence of (1.6) and the following inequalities:

$$\begin{aligned} &\sum_{\mathbf{k}\in\mathbb{Z}^d} \|\delta^{(n-1)}(\mathbf{D}(f)*\mathbf{A}_s(\mathbf{k}))\|_1^2 \\ &\quad \leq \sum_{\mathbf{k}\in\mathbb{Z}^d}\Bigg(\sum_{(\mathbf{k}_1,\dots,\mathbf{k}_n)\in\otimes_n\mathbb{Z}^d} \delta_{(\mathbf{k}_1,\dots,\mathbf{k}_n)}(f)\, A_s(\mathbf{k}_n-\mathbf{k})\Bigg)^2 \\ &\quad \leq \Bigg(\sum_{(\mathbf{k}_1,\dots,\mathbf{k}_n)\in\otimes_n\mathbb{Z}^d} \delta_{(k_1,\dots,k_n)}(f)\Bigg)^2 \sum_{\mathbf{k}\in\mathbb{Z}^d}|A_s(\mathbf{k})|^2 \\ &\quad = \|\delta^{(n)}(f)\|_1^2 \, \|\exp(-2s\tilde{\phi})\|_{\lambda,1} \leq s^{-d/2}\tilde{C}^{(n)}(d)\|\delta^{(n)}(f)\|_1^2. \end{aligned}$$

As for the lower bound, take f_n of the form

$$f_n(\mathbf{x}) = \sum_{m=0}^{n} b_m x(\mathbf{0})^m,$$

where $b_n = 1/n!$ and b_m, $m = 0,\dots,n-1$, are chosen such that

$$\langle D_{\mathbf{0}}^m f\rangle_\mu = 0, \qquad m = 0,\dots,n-1.$$

In other words, f_n is the nth Hermite polynomial associated with the normal law of mean τ and variance $\langle |x(\mathbf{0})-\tau|^2\rangle_\mu = G(\mathbf{0},\mathbf{0})$. Note that $f_n \in D^{(n)}(E)$ with $\|\boldsymbol{\delta}^{(n)}(f_n)\|_1 = 1$. Using this f_n and the same inductive procedure as above, the verification of the lower bound is left as an exercise. □

3. Independent random walks. In the previous section we looked at attractive linear processes, the covariances of which were given by the Green function of a transient transition function. In this section we will assume that (H-1) and (H-2) holds, but replace the transience assumption (H-3) by the following:

(H-5) (Independence). *The* $\{P_t\colon t > 0\}$*-invariant measure* μ *is of the form* $\mu = \prod_{\mathbf{k}} \rho$, *where* $\rho \in \mathbf{M}_1(\mathbb{R})$ *satisfies*

$$\sigma_p(\rho) = \left(\iint |x-y|^p \rho(dx)\rho(dy)\right)^{1/p} < \infty \quad \textit{for some } p \geq 2.$$

We derive L^p algebraic decay in this situation and show that the counting process of independent random walks satisfies this hypothesis.

Note first that (H-5) implies $L^p(\mu)$-contractivity; more precisely, we have the following lemma.

LEMMA 3.1. *Assume* (H-5) *for some* $p \geq 2$. *Then there exists a constant* $c_p \in \mathbb{R}^+$ *depending on p only, such that*

$$\|f - \langle f\rangle_\mu\|_{p,\mu}^2 \leq c_p \sigma_p^2(\rho) \sum_{\mathbf{k}\in\mathbb{Z}^d} \delta_{\mathbf{k}}(f)^2, \qquad f \in \mathscr{L}(E). \tag{3.2}$$

PROOF. The argument, based on Rosenthal's inequality, is quite standard. We may assume that $f \in \mathscr{L}_0(E)$ depends on $x(\mathbf{k}_1), \dots, x(\mathbf{k}_n)$ only. Set $\mathscr{A}_0 = \{\varnothing, E\}$ and $\mathscr{A}_j = \sigma(x(\mathbf{k}_1), \dots, x(\mathbf{k}_j))$, and write

$$f - \langle f\rangle_\mu = \sum_{j=1}^n \Delta_j, \quad \text{where } \Delta_j = E_\mu[f|\mathscr{A}_j] - E_\mu[f|\mathscr{A}_{j-1}].$$

By Rosenthal's inequality (cf. Theorem 2.11 of [8]), there is a $C_p < \infty$ such that

$$\|f - \langle f\rangle_\mu\|_{p,\mu}^p \leq C_p\left(E_\mu\left[\left(\sum_{j=1}^n E_\mu[\Delta_j^2|\mathscr{A}_{j-1}]\right)^{p/2}\right] + \sum_{j=1}^n E_\mu[\Delta_j^p]\right).$$

Also, we have

$$\begin{aligned}
&E_\mu[\Delta_j^2|\mathscr{A}_{j-1}](x(\mathbf{k}_1), \dots, x(\mathbf{k}_{j-1}))\\
&\quad = \tfrac{1}{2}\iint \{E_\mu[f|\mathscr{A}_j](x(\mathbf{k}_1), \dots, x(\mathbf{k}_{j-1}), z)\\
&\qquad\qquad - E_\mu[f|\mathscr{A}_j](x(\mathbf{k}_1), \dots, x(\mathbf{k}_{j-1}), y)\}^2 \rho(dz)\rho(dy)\\
&\quad \leq \tfrac{1}{2}\iint (z-y)^2 \rho(dz)\rho(dy)\delta_{\mathbf{k}_j}(f)^2 \leq \sigma_2^2(\rho)\delta_{\mathbf{k}_j}(f)^2.
\end{aligned}$$

Similarly one shows

$$E_\mu\left[\Delta_j^p\right] \le \delta_{\mathbf{k}_j}(f)^p \int\left\{\int |z-y|\rho(dy)\right\}^p \rho(dz)$$
$$\le \delta_{\mathbf{k}_j}(f)^p \iint |z-y|^p \rho(dz)\rho(dy) = \sigma_p^p(\rho)\delta_{\mathbf{k}_j}(f)^p.$$

Putting things together, we obtain

$$\|f - \langle f\rangle_\mu\|_{p,\mu}^p \le C_p\left(\sigma_2^p(\rho)\left(\sum_{j=1}^n \delta_{\mathbf{k}_j}(f)^2\right)^{p/2} + \sigma_p^p(\rho)\sum_{j=1}^n \delta_{\mathbf{k}_j}(f)^p\right)$$
$$\le 2C_p\sigma_p^p(\rho)\left(\sum_{j=1}^n \delta_{\mathbf{k}_j}(f)^2\right)^{p/2}. \qquad \square$$

In the independent case, the formulation of the upper bound is the following:

THEOREM 3.3. *Assume* (H-2) *and* (H-5) *for some* $p \ge 2$, *and choose* $q \in [1,2)$. *Then there is* $C(q,d) < \infty$ *depending on* q, p *and* $\tilde{\mathbf{Q}}$ *only, such that*

$$\sup\left\{\|P_t f - \langle f\rangle_\mu\|_{\mu,p}^2 : f \in \mathscr{L}(E), \|\delta(f)\|_q^2 \le 1\right\}$$
$$\le t^{-\beta(q,d)} C(q,d)\sigma_p^2(\rho), \qquad t > 0, \tag{3.4}$$

where $\beta(q,d) = d(2-q)/2q = \alpha(q,d) + 1$.

PROOF. The proof is almost the same as the proof of Theorem 1.4. Here the Green operator $\mathbf{G}$ is replaced by the identity $\mathbf{I}$, and $\hat{G}$ by the constant $\hat{I} = 1$. The absence of the pole $|\theta|^{-2}$ in $\hat{G}$ is responsible for the faster decay $\beta(q,d) = 1 + \alpha(q,d)$. Using (H-2) and (3.2), we have

$$\|P_t f - \langle f\rangle_\mu\|_{\mu,p}^2 \le c_p\sigma_p^2(\rho)\sum_{\mathbf{j}\in\mathbb{Z}^d} \delta_{\mathbf{j}}(P_t f)^2$$
$$= c_p\sigma_p^p(\rho)\langle\delta(P_t f), \delta(P_t f)\rangle$$
$$\le c_p\sigma_p^2(\rho)\langle\delta(f) * \mathbf{A}_t, \delta(f) * \mathbf{A}_t\rangle$$
$$= c_p\sigma^2(\rho)\left\| |\hat{\delta}(f)|^2 \exp(-2t\tilde{\phi})\right\|_{1,\lambda}.$$

Now take $q \in [1,2)$. Let $p' = q/(2-q)$ and $q' = p'/(p'-1)$. Then by Hölder's inequality and the Hausdorff–Young inequality we get (3.4) from the above inequality as

$$c_p\left\| |\hat{\delta}(f)|^2 \exp(-2t\tilde{\phi})\right\|_{\lambda,1} \le c_p\|\hat{\delta}(f)\|_{\lambda,2q'}^2 \|\exp(-2t\tilde{\phi})\|_{\lambda,p'}$$
$$\le t^{-\beta(q,d)} C(q,d)\|\boldsymbol{\delta}(f)\|_q^2,$$

where we have used (1.6) in

$$c_p\|\exp(-2t\tilde{\phi})\|_{\lambda,p'} \le t^{-d/(2p')} C(q,d) = t^{-\beta(q,d)} C(q,d),$$

for some $0 < C(q,d) < \infty$. $\square$

We state, without proofs, the corresponding lower bound and faster decay on $L_0^*(E)$. The argument follows the same ideas as the proofs of Theorems 1.9 and 3.3:

THEOREM 3.5. *Assume* (H-1) *and* (H-5). *Then, for each* $q \in [1,2)$ *and* $0 < \varepsilon < d(q-1)/q$, *there is a* $c(q,d,\varepsilon) > 0$ *such that*

$$\sup\left\{\|P_t f - \langle f\rangle_\mu\|_{\mu,2}^2 \colon f \in L_0(E) \text{ with } \|\delta(f)\|_q = 1\right\} \geq t^{-\beta(q,d)-\varepsilon} c(q,d,\varepsilon), \qquad t > 1.$$

In the case $q = 1$, *we may take* $\varepsilon = 0$.

PROPOSITION 3.6. *Assume* (H-1) *and* (H-5). *Then there exists* $0 < k(d) < K(d) < \infty$ *such that*

$$t^{-(d/2)-1}k(d) \leq \sup\left\{\|P_t f - \langle f\rangle_\mu\|_{\mu,2}^2 \colon f \in L_0^*(E) \text{ with } \|\delta(f)\|_{1,1} = 1\right\} \leq t^{-(d/2)-1}K(d), \qquad t > 1.$$

REMARK 3.7. Note that independence implies a faster algebraic decay $\beta(q,d) = \alpha(q,d) + 1$ than in the slowly decaying covariances of Section 1.

We conclude this section with an example showing that the above theory applies to the counting process of independent random walks.

Let $E = \mathbb{N}^{\mathbb{Z}^d} \cap \mathscr{S}'(\mathbb{Z}^d)$ and set $\Omega = D([0,\infty); E)$, the space of right-continuous paths. Let $\mathbf{X}_t \colon \Omega \to E$ be the coordinate mapping and let $\{\mathscr{F}_t \colon t > 0\}$ be the canonical filtration. Consider the linear operator L on $\mathscr{L}_0(E)$:

$$Lf(\mathbf{x}) \equiv \sum_{\mathbf{k},\mathbf{j} \in \mathbb{Z}^d} Q(\mathbf{k},\mathbf{j}) x(\mathbf{k}) \left[f(\mathbf{x}^{\mathbf{k},\mathbf{j}}) - f(\mathbf{x}) \right],$$

where

$$x^{\mathbf{k},\mathbf{j}}(\mathbf{i}) = \begin{cases} x(\mathbf{i}), & \text{if } \mathbf{i} \neq \mathbf{k},\mathbf{j}, \\ x(\mathbf{k}) - 1, & \text{if } \mathbf{i} = \mathbf{k}, \\ x(\mathbf{j}) + 1, & \text{if } \mathbf{i} = \mathbf{j}. \end{cases}$$

Then $\{P_{\mathbf{x}} \colon \mathbf{x} \in E\}$ is the Markov family associated with the unique solution to the martingale problem for L at $\mathbf{x} \in E$, and $\{P_t \colon t > 0\}$ is the corresponding Markov semigroup.

For each $\mathbf{x} \in E$, the process can be constructed as the counting process associated with a collection

$$\left\{\xi_t^{\mathbf{k}_n} \colon 1 \leq n \leq x(\mathbf{k}), \mathbf{k} \in \mathbb{Z}^d, t \geq 0 \text{ with } \xi_0^{\mathbf{k}_n} = \mathbf{k}\right\}$$

of independent random walks on $\mathbb{Z}^d$ with transition function $\mathbf{Q}$:

$$X_t(\mathbf{j}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{n=1}^{x(\mathbf{k})} \chi_{\{\mathbf{j}\}}\left(\xi_t^{\mathbf{k}_n}\right), \qquad \mathbf{j} \in \mathbb{Z}^d.$$

Fix $\tau > 0$ and let $\rho = \rho(\tau)$ be the Poisson distribution on $\mathbb{N}$ with intensity τ. It is a well-known fact that $\mu = \prod_{\mathbf{k} \in \mathbb{Z}^d} \rho$ is an extremal stationary $\{P_t: t > 0\}$-invariant probability measure.

A simple application of Kolmogorov's forward equation shows that the process is linear and (H-1) is satisfied. By construction, the process is monotone in the sense of (1.3); thus, (H-2) holds. Finally, the Poisson distribution has moments of all orders, that is, $\sigma_p(\rho) < \infty$ for any $p \geq 2$.

Acknowledgments. The idea of this paper was initiated at the Ninth Seminar on Stochastic Processes held in San Diego in 1989. The author would like to express his gratitude to T. Liggett for suggesting this interesting problem and to the organizers of the meeting for inviting him. The author would also like to thank the referee and an Associate Editor for suggesting some improvements of the original manuscript.

REFERENCES

[1] Cox, J. T. and Greven, A. (1992). Ergodic theorems for infinite systems of locally interacting diffusions. Unpublished manuscript.
[2] Dawson, D. A. (1977). The critical measure diffusion process. *Z. Wahrsch. Verw. Gebiete* **40** 125–145.
[3] Deuschel, J. D. (1989). Invariance principle and empirical mean large deviations of the critical Ornstein–Uhlenbeck process. *Ann. Probab.* **17** 74–90.
[4] Deuschel, J. D. and Stroock, D. W. (1990). Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising model. *J. Funct. Anal.* **89** 30–48.
[5] Dynkin, E. B. (1989). Three classes of infinite-dimensional diffusions. *J. Funct. Anal.* **86** 75–110.
[6] Ethier, S. N. (1978). Differentiability preserving properties of Markov semigroups associated with one-dimensional diffusions. *Z. Wahrsch. Verw. Gebiete* **45** 225–238.
[7] Föllmer, H. (1982). A covariance estimate for Gibbs measures. *J. Funct. Anal.* **46** 387–395.
[8] Hall, P. and Heyde, C. C.. *Martingale Limit Theory and Its Application*. Academic, New York.
[9] Holley, R. and Stroock, D. W. (1986). L_2 theory of the stochastic Ising model. *Z. Wahrsch. Verw. Gebiete* **35** 87–101.
[10] Liggett, T. (1985). *Interacting Particle Systems*. Springer, Berlin.
[11] Liggett, T. (1989). Exponential L_2 convergence of attractive reversible nearest particle systems. *Ann. Probab.* **17** 403–432.
[12] Liggett, T. (1991). L_2 rates of convergence for attractive reversible nearest particle systems: The critical case. *Ann. Probab.* **19** 935–959.
[13] Shiga, T. (1980). An interacting system in population genetics, I and II. *J. Math. Kyoto Univ.* **20** 213–242; 723–733.
[14] Shiga, T. and Shimizu, A. (1980). Infinite dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.* **20** 395–416.
[15] Spitzer, F. (1976). *Principle of Random Walk*, 2d ed. Springer, Berlin.

Department of Mathematics
ETH-Zentrum
CH-8092 Zürich
Switzerland