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SURVIVAL OF ONE-DIMENSIONAL CELLULAR AUTOMATA UNDER RANDOM PERTURBATIONS

BY MAURY BRAMSON[1] AND CLAUDIA NEUHAUSER[2]

University of Wisconsin and University of Southern California

Cellular automata have been the subject of considerable recent study in the statistical physics literature, where they provide examples of easily accessible nonlinear phenomena. We investigate a class of nearest neighbor cellular automata taking values $\{0, 1\}$ on $\mathbb{Z}$. In the deterministic setting, this class includes rules which yield fractal-like patterns when starting from a single occupied site. We are interested here in the asymptotic behavior of systems subjected to small random perturbations. In this context, one wishes to ascertain under which conditions such systems survive with positive probability. We show here that, except in trivial cases, these systems in fact always survive, and they possess densities which remain bounded away from 0.

1. Introduction. In this paper, we consider discrete time Markov processes for which the state at time n (ξ_n) is a subset of $\mathbb{Z}$, the set of integers. We write $\xi_n(x) = 1$ for $x \in \xi_n$, in which case we say that there is a particle at x or that x is occupied; otherwise we say the site is vacant and write $\xi_n(x) = 0$. The values of ξ_n are assumed to be updated simultaneously at each time step according to local rules which satisfy the following:

(i) $P(x \in \xi_{n+1} | \xi_n) = f(\xi_n(x-1), \xi_n(x), \xi_n(x+1))$, that is, the value of ξ_{n+1} at x depends on just ξ_n at $x+1$, x and $x-1$.

(ii) Given ξ_n, the events $\{x \in \xi_{n+1}\}$, $x \in \mathbb{Z}$, are independent.

We let $f(\cdot)$ denote a function from $\{0,1\}^3 \to \{0, 1-\varepsilon\}$ with $0 \le \varepsilon < 1$. The processes ξ_n are examples of discrete time interacting particle systems. In the statistical physics literature the designation *cellular automata* is used; we employ this latter terminology here.

For $\varepsilon = 0$, the cellular automaton is deterministic: Given the initial configuration ξ_0, ξ_n is determined for all n. It is easy to check that there are $2^8 = 256$ possible cellular automata whose local rules satisfy (i) and (ii). If $\varepsilon > 0$, we say that the cellular automaton is randomly perturbed. We will first look at the deterministic case and define the class of models we want to investigate. This class consists of the one-dimensional cellular automata which were systematically studied and classified in two important papers by Wolfram

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(1983, 1984). The first paper includes a short history of cellular automata and an extensive list of references. Wolfram (1983) imposed the following restrictions on the possible rules:

(a) States consisting solely of vacant sites remain unchanged. That is, the empty set is a trap.

(b) Rules are reflection symmetric. That is, interchanging the values at the right and left neighbor of a given site does not change the value of that site in the next time step.

These two restrictions leave 32 cellular automata satisfying (i) and (ii), which Wolfram (1983) called "legal rules."

In the paper cited above, Wolfram used a labelling scheme for these rules, which we now describe. Let $\alpha_i \in \{0,1\}$ for $i = 0, 1, \dots, 7$, and let $\beta_j \in \{0,1\}$ for $j = 0, 1, 2$. The value at site x at time $n+1$ depends only on the values at sites $x-1$, x and $x+1$ at time n. Set $\beta_0 = \xi_n(x+1)$, $\beta_1 = \xi_n(x)$ and $\beta_2 = \xi_n(x-1)$, and set $\alpha_i = f(\beta_2, \beta_1, \beta_0)$ for $i = \sum_{k=0}^{2} \beta_k 2^k$. The integer $\sum_{i=0}^{7} \alpha_i 2^i$ then defines the "number" of the rule. For instance, consider the following rule where the transitions are given by

$$\begin{matrix} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{matrix}. \tag{1.1}$$

Each triple in the first line is the binary expansion of some $i \in \{0, 1, \dots, 7\}$. The value on the second line is the corresponding α_i (with α_7 on the left and α_0 on the right). Since 01011010 is the binary expansion of 90, we call this transition *rule 90*. (Rule 90 is one of the most studied among the 32 rules mentioned above, and we will refer to it several times below.)

Having defined Wolfram's legal rules, we now specify which of these rules we wish to investigate. The only further restriction we impose on the deterministic rules is:

(c) α_1 (and, by symmetry, α_4) is equal to 1. That is, $f(0,0,1) = 1$ (and $f(1,0,0) = 1$).

This leaves us with 16 rules. (For the convenience of the reader, these 16 rules are listed in Table 1.) Condition (c) assures us that 1's can propagate when starting from a finite set. The other 16 rules which violate (c) are not very interesting since components of ξ_0 which are separated by at least two consecutive vacant sites cannot interact with each other. That is, 1's emanating from any given component cannot propagate into neighboring components since α_1 and α_4 are both 0.

It is easy to implement these deterministic rules on a computer. Wolfram (1983) investigated two different types of initial configurations: one being the state with only a single occupied site and every other site vacant, the other

TABLE 1

Rule	111	110	101	100	011	010	001	000
18	0	0	0	1	0	0	1	0
22	0	0	0	1	0	1	1	0
50	0	0	1	1	0	0	1	0
54	0	0	1	1	0	1	1	0
90	0	1	0	1	1	0	1	0
94	0	1	0	1	1	1	1	0
122	0	1	1	1	1	0	1	0
126	0	1	1	1	1	1	1	0
146	1	0	0	1	0	0	1	0
150	1	0	0	1	0	1	1	0
178	1	0	1	1	0	0	1	0
182	1	0	1	1	0	1	1	0
218	1	1	0	1	1	0	1	0
222	1	1	0	1	1	1	1	0
250	1	1	1	1	1	0	1	0
254	1	1	1	1	1	1	1	0

being product measure. A number of the rules yield a nontrivial pattern when starting from a configuration with only one occupied site (i.e., rules 18, 22, 90, 126, 146, 150 and 218). More precisely, the resulting space-time picture on the computer screen is self-similar and exhibits fractal behavior. That is, certain geometrical shapes can be found on all length scales and the configuration of 1's is of a lower dimension than the space it is embedded in. An introduction to fractals and self-similarity can be found in Mandelbrot (1977). Wolfram (1983) observed that, when started from product measure, the rules which exhibit this fractal-like behavior generate correlations between far apart sites. This phenomenon manifests itself in the appearance of open "triangles" on all length scales in the space-time evolution. By this we mean the following: The rules are such that certain configurations of 1's and 0's result in all 0's in the next time step. [For instance, a sequence of consecutive 1's in rule 90 exhibits this behavior since $f(1,1,1) = 0$.] Therefore, during the time evolution of these rules, we encounter stretches of 0's. These stretches appear on all length scales. Because of condition (c), the length of these stretches is reduced by 1 on either side in each time step. This creates the triangular structure, in which the initial stretch of 0's forms the base of the triangle.

The motivation for this paper comes from the rules which exhibit the fractal behavior mentioned above, although our analysis works for all 16 rules. For concreteness, we focus our attention for the moment on rule 90, whose dynamics were given in (1.1). It is easy to check that, when starting from only one occupied site, triangles of all sizes are generated. Furthermore, when starting from a single occupied site, the Cesaro average in time over the "region of occupied sites" tends to zero. This region spreads out linearly and covers an interval of length $2n + 1$ at time n. More precisely, if ξ_n^0 denotes the set of occupied sites at time n when starting from $\xi_0^0 = \{0\}$, then

$$\frac{1}{N}\sum_{n=1}^{N}\frac{|\xi_n^0|}{2n+1} \to 0 \tag{1.2}$$

as $N \to \infty$. ($|\xi_n^0|$ denotes the number of elements in ξ_n^0.) This observation prompted Kinzel (1985) to conjecture for rule 90 that "the ordered state... is not dense enough to support a transition away from the deterministic limit." Kinzel's conjecture was based on a crude extrapolation from finite size scaling. He conjectured that, for any $\varepsilon > 0$, rule 90 would die out locally starting from any initial configuration, and thus (1.2) would still hold for the randomly perturbed system. [For pictures of the space-time evolution of rule 90 in both the deterministic and randomly perturbed case, see Durrett (1988), page 116.]

We study the behavior of the above 16 rules under small random perturbations. In the randomly perturbed system, the function $f(\cdot)$ defined in (i) takes on the two values 0 and $1 - \varepsilon$, for some $0 < \varepsilon < 1$. That is, 0's are written down as in the deterministic system, but when a 1 appears in the deterministic system, a coin with probability $1 - \varepsilon$ of heads is tossed in the randomly perturbed system—if heads appears, we write down a 1, otherwise a 0. We will be interested in the survival properties of the randomly perturbed rules. It is easy to see that if ε is too large, then the system dies out locally with probability 1. This follows from comparison with a branching process. Each particle at time n can give rise to at most three particles at time $n + 1$. If $|\xi_n| = k$, then $E|\xi_{n+1}| \le 3k(1 - \varepsilon)$. So if $\varepsilon > \frac{2}{3}$, then $E(|\xi_{n+1}|\,|\xi_n)/|\xi_n| \le 3(1 - \varepsilon) < 1$ and the system will eventually die out.

The behavior of the rules is quite different for small positive ε. We need some notation before we can formulate the asymptotic behavior in this case. Let $\hat{\xi}_n^\zeta$ denote the deterministic process (i.e., when $\varepsilon = 0$) at time n with $\hat{\xi}_0^\zeta = \zeta$. To exclude initial configurations which result in the empty set at time 1, we define the following set of initial configurations: For I an interval in $\mathbb{Z}$, set

$$\mathscr{S}(I) = \left\{\zeta : \hat{\xi}_1^\zeta \cap I \ne \varnothing\right\}. \tag{1.3}$$

Note that on account of condition (c), $\zeta \in \mathscr{S}(\mathbb{Z})$ for $|\zeta| < \infty$. It is also easy to see that any product measure with density in $(0, 1)$ is almost surely in $\mathscr{S}(\mathbb{Z})$.

Observe that the processes ξ_n^ζ, with $\zeta = \{0\}$, exhibit distinct behavior depending on whether or not $f(0,1,0) = 0$. For all eight rules satisfying $f(0,1,0) = 0$, such as rule 90, it is easy to check that

$$\xi_{2n}^\zeta \subset 2\mathbb{Z}, \qquad \xi_{2n+1}^\zeta \subset 2\mathbb{Z} + 1; \tag{1.4}$$

we designate these rules as type (I). For the other eight rules, odd sites can be occupied at even times and vice versa; we designate them as type (II).

For a configuration $\xi_0 \notin \mathscr{S}(\mathbb{Z})$, one already has $\xi_1 = \varnothing$, and hence $\xi_n = \varnothing$, $n \ge 1$, for all $0 \le \varepsilon < 1$. On the other hand, for $\xi_0 \in \mathscr{S}(\mathbb{Z})$ and small ε, $\xi_1 \ne \varnothing$ with high probability. The following theorem, the main result of the paper, implies that the process ξ_n has positive probability of surviving for all time.

THEOREM. *For each type* (I) *rule and each* $\varepsilon > 0$ *sufficiently small, there exist* $\rho \equiv \rho(\varepsilon) > 0$ *and* $\gamma \equiv \gamma(\varepsilon) > 0$ *so that, for every initial configuration* $\xi_0 \in \mathscr{S}(\mathbb{Z})$,

$$\liminf_{n \to \infty} \min_{|x| < \gamma n} P(\xi_n(x) = 1 \text{ or } \xi_n(x+1) = 1) > \rho. \tag{1.5a}$$

For each type (II) *rule*,

$$\liminf_{n \to \infty} \min_{|x| < \gamma n} P(\xi_n(x) = 1) > \rho \tag{1.5b}$$

holds under the same conditions.

Note that, for any translation-invariant measure ξ_0 with $P(\xi_0 \in \mathscr{S}(\mathbb{Z})) > 0$,

$$\liminf_{n \to \infty} P(\xi_n(x) = 1) > \rho', \tag{1.6}$$

$\rho' > 0$, for rules of either type. It will follow from the proof of the Theorem that when starting from a finite set, the set of occupied sites will grow linearly in time whenever the process survives. One can also choose $\gamma(\varepsilon)$ so that $\gamma(\varepsilon) \to 1$ as $\varepsilon \to 0$, although the method employed here gives poor bounds for $\rho(\varepsilon)$. The statements in (1.5) imply that Kinzel's conjecture is wrong. (The space-time pictures of the randomly perturbed systems in fact look "thicker" than the ones of the deterministic systems exhibiting fractal behavior.)

By considering the Cesaro average of the distributions of ξ_m, $1 \le m \le n$, for any of the rules and taking the limit over an appropriate subsequence, one obtains an equilibrium ν. [This standard construction can be found in, e.g., Proposition 1.8 of Liggett (1985).] The Theorem implies that, for $\varepsilon > 0$ chosen small enough, ν has density bounded away from 0. An interesting question in this context is whether there is exactly one nonzero equilibrium for the process. A more difficult problem is to prove a convergence theorem for ξ_n. Some results in this spirit have been obtained for rule 90 when $\varepsilon = 0$. Rigorous treatment for this deterministic case is facilitated by the fact that the rule is additive in the sense that $\xi_{n+1}(x) = (\xi_n(x-1) + \xi_n(x+1)) \bmod 2$, which permits exact calculations with the aid of generating functions and ergodic theory. Miyamoto (1979) and, independently, Lind (1984) followed this approach to obtain rigorous results. It is easy to see that $\nu_{1/2}$, the product measure with density $1/2$, and $\delta_{\varnothing}$, the point mass at the all empty configuration, are equilibria. Miyamoto and Lind showed that starting from product measure with density θ, the distribution converges weakly only if $\theta \in \{0, \frac{1}{2}, 1\}$. The Cesaro averages over time, however, converge to $\nu_{1/2}$ when starting from product measure with density $\theta \in (0,1)$. Miyamoto showed that $\nu_{1/2}$ and $\delta_{\varnothing}$ are the only equilibria in the class of measures that are translation invariant and mixing. It is not known whether there are any other equilibria or what happens when starting from measures other than product measures. [See also Durrett (1988), Section 5d for a discussion of rule 90.] Similar questions can of course be studied for the other 15 rules considered here, in both the deterministic and the randomly perturbed settings.

The Theorem implies that each of the rules considered here has a nontrivial phase transition. That is, for some value of $\varepsilon > 0$, there is at least one nontrivial equilibrium. (Recall that, for $\varepsilon > \frac{2}{3}$, $\delta_\varnothing$ is the only equilibrium.) As is typically the case for nonattractive systems, we do not know whether there exists a unique critical value $\varepsilon_c > 0$ for a given rule, although this is presumably true. (That is, a nontrivial equilibrium exists for $0 < \varepsilon < \varepsilon_c$, but not for $\varepsilon_c < \varepsilon < 1$.) We also do not know whether the phase transition is continuous or discontinuous.

The paper is organized as follows. The proof of the theorem makes use of a rescaling argument which is laid out in Section 2. Two basic estimates involving a "tagged particle" are required to employ the argument. The first involves showing that this particle will move into an appropriately chosen interval within a fixed amount of time; this is done in Section 3. Once the particle has entered and thus "populated" such an interval, the interval will typically remain populated for a long period of time. To show this, one needs to employ explicitly the structure of the different rules; this is shown in Section 4. Both sections use random walk comparisons together with various bounds on the geometry of ξ_n.

2. The rescaling argument. The rescaling technique was developed by the first author and Durrett, and is reviewed in Durrett (1991). It is by now a standard technique and has been applied frequently [see, e.g., Bramson (1989), Bramson and Durrett (1988), Bramson, Ding and Durrett (1991) and Durrett and Neuhauser (1994)]. The basic idea is to show that for appropriate $\delta(\varepsilon) > 0$, the process under consideration (indexed by ε), when viewed on suitable length and time scales, dominates an oriented site percolation process in which sites are open with probability $1 - \delta(\varepsilon)$. (Sites may be j-dependent; in our case, $j = 2$.) One then shows that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since oriented site percolation percolates for δ close enough to 0, this will then imply that for small ε the process survives and has the desired properties.

We proceed to review the above procedure. Define 2-dependent oriented site percolation with density $1 - \delta$ as follows. Let $\mathscr{L} = \{(z, k) \in \mathbb{Z}^2 : z + k \text{ is even}\}$. The oriented site percolation process is a collection of random variables $\{\omega(z, k) : (z, k) \in \mathscr{L}\}$, with values in $\{0, 1\}$, that indicate whether the sites in $\mathscr{L}$ are open (1) or closed (0). We say the process is *2-dependent with density* $1 - \delta$ if for any sequence (z_j, k_j), $1 \le j \le l$, satisfying $|z_i - z_j| > 4$ whenever both $i \ne j$ and $k_i = k_j$, then

$$P\big(\omega(z_j, k_j) = 0 \text{ for } 1 \le j \le l\big) = \delta^l.$$

By an open path from $(y, 0)$ to (z, k), we mean a sequence of points $z_0 = (y, 0)$, $z_1, z_2, \dots, z_k = (z, k)$ in $\mathscr{L}$ such that, for $0 \le j \le k - 1$, $z_{j+1} = z_j + (1, 1)$ or $z_{j+1} = z_j + (-1, 1)$, and all the z_j's with $j \ge 1$ are open.

Let

$$(2.1) \quad W_k^A = \{z : \text{there is an open path from } (y, 0) \text{ to } (z, k) \text{ for some } y \in A\}$$

and

$$\Omega_\infty = \bigcap_{k=0}^{\infty} \{W_k^{\{0\}} \neq \varnothing\}. \tag{2.2}$$

We think of W_k^A as the set of *wet sites* at time k connected to the *source* A at time 0. If Ω_∞ occurs, that is, if there is an infinite open path starting at (0, 0), we say *percolation occurs*. The following is, for 2-dependent percolation, the analog of results on 1-dependent percolation given in Durrett and Neuhauser [(1991), pages 204–205]; the proofs are the same. [A detailed exposition on independent oriented site percolation can be found in Durrett (1984).]

LEMMA 2.3. *Let* $\{\omega(z,k)\colon (z,k) \in \mathscr{L}\}$ *be a 2-dependent oriented site percolation process with density* $1-\delta$, $\delta \le 6^{-100}$. *Then* $P(\Omega_\infty) > 0$, *and there exist* $\rho > 0$ *and* $\gamma > 0$ *so that*

$$\liminf_{k\to\infty} \min_{\substack{|z|<\gamma k\\ (z,k)\in\mathscr{L}}} P\big(z \in W_k^{\{0\}}\big) > \rho. \tag{2.4}$$

Although we do not make use of it here, we note that the bound

$$\liminf_{k\to\infty} \big|W_k^{\{0\}}\big|\big/ k > 0 \quad \text{a.s. on } \Omega_\infty$$

also holds as in Durrett and Neuhauser (1991).

To compare the process ξ_n with oriented site percolation, we let $L = 10\lfloor\varepsilon^{-3/2}\rfloor$ and $T = 4L$. [One needs L to be sandwiched in between $\varepsilon^{-1}\log(1/\varepsilon)$ and ε^{-2}, with the upper bound needed for inequalities (3.7) and (3.11), and the lower bound for the inequalities starting with (4.15). Here $\lfloor y\rfloor$ denotes the integer part of y; for later purposes we want L to be divisible by 10.] Also, set

$$\begin{aligned}
\phi(z,k) &= (2zL, kT) \quad \text{for } (z,k)\in\mathscr{L},\\
B &= [-2L, 2L)\times(0,T],\\
B(z,k) &= \phi(z,k) + B \quad \text{for } (z,k)\in\mathscr{L},\\
I &= [-L, L), \qquad I_z = 2zL + I,\\
I^l &= [-L, 0), \qquad I_z^l = 2zL + I^l,\\
I^r &= [0, L), \qquad I_z^r = 2zL + I^r.
\end{aligned} \tag{2.5}$$

The sets $B(z,k)$, $(z,k)\in\mathscr{L}$, are $T\times T$ squares in space-time "centered" at $\phi(z,k)$ which partition $\mathbb{Z}\times\mathbb{Z}^+$, and I_z, I_z^l and I_z^r are the "full," "left" and "right" intervals "centered" at $2zL$. Note that $I_{z-1}^r \times \{(k+1)T\}$ and $I_{z+1}^l \times \{(k+1)T\}$ are contained in the left quarter and right quarter, respectively, of the upper boundary of $B(z,k)$.

Sections 3 and 4 of the paper are devoted to showing Proposition 2.6, which states that if ξ_{kT} is "admissible" in I_z, then with probability close to 1 the same is true for $\xi_{(k+1)T}$ in both I_{z-1}^r and I_{z+1}^l. For all rules except rule 22, the condition $\xi_{kT} \in \mathscr{S}(I_z)$ suffices. As will be explained shortly, for rule 22, it does not. In this case, we employ the condition $\xi_{kT} \in \mathscr{S}_\lambda^2(I_z)$, where

$$\mathscr{S}_\lambda^j(I) = \left\{\zeta\colon P\left(\xi_j^\zeta \cap I = \varnothing\right) \le \lambda\right\}.$$

The parameter $\lambda > 0$ will be a small number depending on ε and will be specified later. We say that ζ is *admissible* in I if $\zeta \in \mathscr{S}(I)$ for rules other than rule 22, and $\zeta \in \mathscr{S}_\lambda^2(I)$ for rule 22. Note that $\mathscr{S}(I) = \mathscr{S}_\varepsilon^1(I)$. [One could as well use $\mathscr{S}_\lambda^2(I)$ in defining admissible for all the rules. We prefer to employ the concrete $\mathscr{S}(I)$ when possible, i.e., except for rule 22, so as not to obscure the construction.]

PROPOSITION 2.6. *Let $\delta > 0$ and assume that $\varepsilon \equiv \varepsilon(\delta) > 0$ is sufficiently small. For $(z, k) \in \mathscr{L}$, if ξ_{kT} is admissible in I_z, then, with probability at least $1 - \delta$, $\xi_{(k+1)T}$ is admissible in both I_{z-1}^r and I_{z+1}^l.*

If $\xi_{(k+1)T}$ is admissible in both I_{z-1}^r and I_{z+1}^l for $(z, k) \in \mathscr{L}$, we say that (z, k) is *occupied*. Since $\xi_{(k+1)T}$ will be admissible with probability at least $1 - \delta$ if either $(z - 1, k - 1)$ or $(z + 1, k - 1)$ is occupied, repeated application of Proposition 2.6 will enable us to show that the set of occupied sites "tends to spread" as k increases. We require that ξ_{kT} be admissible in I_z in the hypothesis so that $\xi_{kT+1} \cap I_z \neq \varnothing$ ($\xi_{kT+2} \cap I_z \neq \varnothing$ for rule 22) with high probability; it turns out that there is afterwards enough randomness in the system to ensure that (z, k) will be occupied with high probability.

We now explain the first of these two statements. (The set of occupied sites "tends to spread.") Introduce the random variables $\omega(z, k)$, $(z, k) \in \mathscr{L}$, with $\omega(z, k) = 0$ or 1, as follows. If ξ_{kT} is admissible in I_z, choose

$$\{\omega(z, k) = 1\} \subset \{(z, k) \text{ is occupied}\}, \tag{2.7}$$

with $P(\omega(z, k) = 1) = 1 - \delta$, so that the occupied sites are "trimmed" independently of everything else. That is, an occupied site remains open [i.e., $\omega(z, k) = 1$] with probability $1 - \delta$; this is possible because of Proposition 2.6. If ξ_{kT} is not admissible in I_z, set $\omega(z, k) = 1$ with probability $1 - \delta$ independently of everything else. Clearly,

$$P(\omega(z, k) = 1 | \xi_{kT}) = P(\omega(z, k) = 1) = 1 - \delta \quad \text{for } (z, k) \in \mathscr{L}. \tag{2.8}$$

One can also check that $\{\omega(z, k)\colon (z, k) \in \mathscr{L}\}$ is 2-dependent. [Conditioned on ξ_{kT}, it is clear that $\{\omega(z, k)\colon (z, k) \in \mathscr{L}\}$ is 2-dependent, since the evolution of the rules given by (i) in Section 1 is determined by nearest-neighbor interactions, and so sites more than $8L = 2T$ apart cannot both depend on the same elementary event within the last T units of time. Since the conditional probabilities in (2.8) do not depend on ξ_{kT}, 2-dependence without conditioning follows.] So $\{\omega(z, k)\colon (z, k) \in \mathscr{L}\}$ is a 2-dependent oriented site percolation process.

Let $G = \{z\colon \xi_0 \in \mathscr{S}(I_z)\}$ for a given configuration ξ_0. Rather than use G directly in the comparison with oriented percolation, it is more natural to work with the random set

$$A = \{z: \xi_0 \in \mathscr{S}(I_z) \text{ and } \xi_T \text{ is admissible in } I^r_{z-1} \text{ and } I^l_{z+1}\}$$

as the source. One can show with a little work that $P(A \neq \varnothing) \geq \eta_0$, where $\eta_0 > 0$ does not depend on G for $G \neq \varnothing$; this follows from condition (c) of Section 1, with the reasoning being analogous to that leading up to (2.14). [The substitution of A for G is used to preserve the even–odd structure of the underlying lattice. Something like this is also necessary for the process corresponding to rule 22: The periodic configurations of the form $\dots 100100100 \dots$ are in $\mathscr{S}(\mathbb{Z})$, yet, starting from this state, $\hat{\xi}_2 = \varnothing$. The corresponding process ξ_n will survive because of the existence of "double errors" at neighboring sites at time 1. However, these will occur only with probability ε^2 and therefore typically not in $[-2L, 2L)$, which has only $40\varepsilon^{-3/2}$ sites. So in this case, η_0 will indeed be small. For all the other rules, one can employ a variation of Proposition 2.6 to conclude that $P(A \neq \varnothing) \geq 1 - \delta$ for $G \neq \varnothing$.]

Now let W_k^A be defined as in (2.1). By induction one can show that W_k^A only depends on the value of $\omega(z, k')$, $k' \leq k$, at sites (z, k') for which $\xi_{(k'+1)T}$ is admissible in I_z. [Whether or not (z, k') is open is not relevant if $(z-1, k'-1)$ and $(z+1, k'-1)$ are not occupied.] Because of the containment given by (2.7) when ξ_{kT} is admissible in I_z, it follows that, for $\delta > 0$ and for sufficiently small ε,

$$W_k^A \subset \{z: (z, k) \text{ is occupied}\}. \tag{2.9}$$

Employing (2.9) together with Lemma 2.3 (and the attractiveness and translation invariance of oriented site percolation), one obtains the following proposition.

PROPOSITION 2.10. *For sufficiently small ε, there exist $\rho_0 \equiv \rho_0(\varepsilon) > 0$ and $\gamma_0 \equiv \gamma_0(\varepsilon) > 0$ so that, for $G \neq \varnothing$,*

$$\liminf_{k \to \infty} \min_{\substack{|z| < \gamma_0 k \\ (z,k) \in \mathscr{L}}} P((z, k) \text{ is occupied}) > \rho_0. \tag{2.11}$$

By assuming Proposition 2.6, we have obtained (2.11), which is almost the statement in (1.5) of the Theorem. Having shown that any (z, k), $|z| < \gamma_0 k$, is with probability ρ_0 occupied, we need only extend the statement to all sites in $B(z, k)$. The Theorem then follows, with $\gamma = \gamma_0/2$. First of all, by (2.11), for $\gamma_0' < \gamma_0$, for large enough k and for $|z| < \gamma_0' k$, $\xi_{(k-1)T}$ is admissible in I^l_{z+1} $[= 2zL + [L, 2L)]$ with probability at least ρ_0. Therefore, for some site $x_1 \in I^l_{z+1}$,

$$P(\xi_N(x_1) = 1) > \rho_0',$$

for some $\rho_0' > 0$ not depending on (z, k), where $N \equiv (k-1)T + 1$. The probability that any site takes the value 0 at some given time is at least ε

(independently of everything else). So, in fact,

$$P(\xi_N(x_1) = 1, \xi_N(x_1+1) = \xi_N(x_1+2) = 0) > \rho_1, \tag{2.12}$$

for some $\rho_1 > 0$. The analog of (2.12) holds for x_1, $x_1 - 1$ and $x_1 - 2$.

By condition (c) of Section 1, $f(1,0,0) = f(0,0,1) = 1 - \varepsilon$. It therefore follows that, for some $\rho_2 \in (0, \rho_1]$,

$$P(\xi_{N+1}(x_1+1) = 1, \xi_{N+1}(x_1+2) = \xi_{N+1}(x_1+3) = 0) > \rho_2. \tag{2.13}$$

Iterating in this manner for up to $2T$ steps, shifting either to the left or right at each time step as desired, one can ensure that each x in $2zL + [-2L, 2L)$ will eventually "always" be occupied or adjacent to an occupied site. One obtains that

$$P(\xi_n(x) = 1 \text{ or } \xi_n(x+1) = 1) > \rho_{2T} \equiv \rho, \tag{2.14}$$

for any $(x, n) \in B(z, k)$, with (z, k) subject to $|z| < \gamma_0' k$. The bound ρ will be very small for ε small, but depends on nothing else. This implies (1.5a) of the Theorem. To obtain (1.5b) for type (II) rules, note that a configuration with

$$\xi_{n'}(x-1) = 0, \qquad \xi_{n'}(x) = 1, \qquad \xi_{n'}(x+1) = 0$$

has at least probability ε^2 of occurring at the time n' when x is first occupied. Since $f(0,1,0) = 1 - \varepsilon$ here, one can iterate as above, but this time specifying that the triple of sites remains centered at x up to time n. This implies (1.5b) of the Theorem.

We still need to demonstrate Proposition 2.6. To simplify notation, we set $z = k = 0$; because of the translation invariance of the dynamics of ξ_n, this suffices. Given that ξ_0 is admissible in I_0, we will show that, with probability close to 1, ξ_T will be admissible in $I_1^l = [L, 2L)$. Because of symmetry, the same estimate holds for I_{-1}^r. This implies Proposition 2.6. The estimate will be derived in two steps. In the first step we will show that if ξ_0 is admissible in I_0, then with probability close to 1 the interval $[7L/5, 8L/5)$ will not be empty for some $n \in [1, T+1]$ ($n \in [2, T+2]$ for rule 22). This will be done in Section 3. We will then show that $[7L/5, 8L/5)$ continues to contain particles for at least T more units of time with probability close to 1. This will be shown in Section 4.

3. Populating the target interval. In this section we define tagged particles and study their motion. This will allow us to conclude that, with probability close to 1, the *target interval* $[7L/5, 8L/5)$ is populated (i.e., contains particles) at some time $n \in [1, T+1]$ ($n \in [2, T+2]$ for rule 22), if ξ_0 is admissible in I_0. We first make the simplifying assumption that there are no particles to the right of $7L/5$ initially. To justify this, consider the case where $[7L/5, 8L/5)$ is empty at time 1 (2 for rule 22); otherwise we are done. Starting at time 1 (2), as long as $[7L/5, 8L/5)$ is empty, the evolution of the process to the left of $[7L/5, 8L/5)$ is independent of the evolution of the process to the right. All we need to show is that, with probability close to 1, particles will enter $[7L/5, 8L/5)$ from at least one side. If no particles enter from the right by time $T + 1$ ($T + 2$), the evolution of the process to the left of $[7L/5, 8L/5)$ will be the same as if there were at time 1 (2) no particles to the right of $7L/5$. Translating time by 1 (2) units, the problem is reduced to that with no particles to the right of $7L/5$ initially, and where one has T units of time to populate $[7L/5, 8L/5)$. Note that with probability ε (λ for rule 22), the initial state may not intersect with I_0.

For all $n \le T$, we let R_n denote the position of the rightmost particle at time n. (If $\xi_n = \varnothing$, set $R_n = -\infty$.) We call this the *tagged particle* and think of it as moving over time. On account of the nearest-neighbor property of ξ_n, $R_{n+1} - R_n \le 1$, although no corresponding lower bound holds. We wish to show in this section that the tagged particle will be, with probability close to 1, in the target interval $[7L/5, 8L/5)$ at some time $n \le T$. Since the farthest left it might start is at $-L$, this will occur if the tagged particle moves at least $12L/5$ units to the right. Proposition 3.1 shows that, for small ε, this occurs with probability close to 1. Because of translation invariance, we can assume that $R_0 = -L$.

PROPOSITION 3.1. *Let $\eta > 0$, and assume that $R_0 = -L$ and that there are no particles to the right of $-L$. For $\varepsilon > 0$ sufficiently small,*

$$P(R_n < 7L/5 \text{ for all } n \le T) \le \eta. \tag{3.2}$$

The proof of Proposition 3.1 requires several definitions and a better understanding of how the tagged particle moves. As we noted above, R_n may decrease by large amounts depending on the configuration to its left. [This may occur, for instance, in rule 90 if $\xi_n(x) = 1$ for $x \in [R_n - M, R_n]$, with M large, and $\xi_{n+1}(R_n) = \xi_{n+1}(R_n + 1) = 0$.] To control the amount R_n decreases, we define, for $n \ge 0$,

$$L_n = \max\{x \le R_n : \hat{\xi}_1^{\xi_n}(x) = 1\}. \quad (\text{If } R_n = -\infty, \text{ set } L_n = -\infty.) \tag{3.3}$$

The quantity L_n is obtained by running the process ξ_n up until time n and then setting $\varepsilon = 0$ at n for the additional unit of time. Observe that L_n is the first site to the left of $R_n + 1$ which might be occupied at time $n + 1$ given ξ_n. [Automatically, $\hat{\xi}_1^{\xi_n}(R_n + 1) = 1$.] Note that all the sites $L_n + 1, L_n + 2, \dots, R_n$ must be vacant at time $n + 1$. We call this stretch of vacant sites a *gap*. The size of the gap is given by

$$Y_n = R_n - L_n. \quad (\text{If } R_n = -\infty, \text{ set } Y_n = 0.) \tag{3.4}$$

To control the movement of R_n, there are two cases we need to consider: (1) The gap $[L_n + 1, R_n]$ is small, in which case we can rely on either side of the gap to give birth. (2) The gap is large at time n. Then, only if $R_n + 1$ is occupied at time $n + 1$ will we be able to control the movement of the rightmost tagged particle. We will see that (2) will not occur too often. Neglecting the exceptional behavior exhibited in (2), we will be able to compare R_n with a random walk with positive mean. Since the random walk will drift off to the right at a positive rate, the same will be true for R_n off a set of small probability. Proposition 3.1 will then follow.

We first show that in most realizations of the process, there will be a birth occurring at at least one side of the gap for all $n < T$. That is, either $\xi_{n+1}(R_n + 1) = 1$ or $\xi_{n+1}(L_n) = 1$ ("double errors" will not occur). Set

$$H_n = \{\xi_{n+1}(R_n + 1) = \xi_{n+1}(L_n) = 0\}, \tag{3.5}$$

for $n \geq 0$, and

$$H = \bigcup_{n=0}^{T-1} H_n; \tag{3.6}$$

H is the exceptional event we wish to avoid. [To avoid having to consider $L_n = -\infty$ at this point, we set $\xi_n(-\infty) = 1$.] Since $P(H_n) \leq \varepsilon^2$ for all n, it follows that

$$P(H) \leq \varepsilon^2 T = 40\varepsilon^2 \lfloor \varepsilon^{-3/2} \rfloor \leq 40\varepsilon^{1/2}; \tag{3.7}$$

this approaches 0 as $\varepsilon \to 0$.

The variable R_{n+1} is well behaved on H_n^c in the sense that

$$R_{n+1} \geq R_n - Y_n \quad \text{and} \quad P(R_{n+1} \neq R_n + 1; H_n^c) \leq \varepsilon. \tag{3.8}$$

We wish to show that Y_n is for most n not too large, that is, $Y_n \leq \varepsilon^{-3/4}$. It will follow that the probability that a *disaster* occurs for any $n < T$, that is, $Y_n > \varepsilon^{-3/4}$ and $\xi_{n+1}(R_n + 1) = 0$, is small. Together with (3.8), this will enable us to compare R_n with a random walk with positive drift as mentioned above.

We define a sequence of stopping times S_k, $k \geq 1$, that mark those times at which disasters may occur based on the configuration at time n:

$$\begin{aligned} S_1 &= \inf\{n \geq 0: Y_n > \varepsilon^{-3/4}\} \\ S_k &= \inf\{n > S_{k-1}: Y_n > \varepsilon^{-3/4}\} \quad \text{for } k \geq 2. \end{aligned} \tag{3.9}$$

The following lemma states that Y_n cannot be large very often.

LEMMA 3.10. *If* $\xi_{S_k+1}(R_{S_k} + 1) = 1$ *for given* $k \geq 1$, *then, on* H^c,

$$S_{k+1} - S_k > \tfrac{1}{2}\varepsilon^{-3/4}.$$

PROOF. We abbreviate S_k by N. By assumption, $R_N + 1$ is occupied by a particle at time $N + 1$. By definition, there are no particles at $L_N + 1, L_N + 2, \dots, R_N$ at time $N + 1$. Because of the nearest-neighbor property, the length of this gap can decrease by at most 1 on each side over each unit of time. Therefore, there will not be any particles at $L_N + m, L_N + m + 1, \dots, R_N - m + 1$ at time $N + m$, and it takes more than $\frac{1}{2}\varepsilon^{-3/4}$ units of time to close the gap. It follows that the "offspring" of the particle located at $R_N + 1$ at time $N + 1$, which lie within $J_m = [R_N - m + 2, R_N + m]$ at times $N + m$, are separated from all other particles for $m \leq \frac{1}{2}\varepsilon^{-3/4}$.

Assume now that J_m is not empty for $m \le \frac{1}{2}\varepsilon^{-3/4}$. Then $R_{N+m} \in J_m$, and so $L_{N+m} + 1 \in J_m$. Thus $Y_{N+m} < 2m \le \varepsilon^{-3/4}$, that is, $S_{k+1} > N + \frac{1}{2}\varepsilon^{-3/4}$, which is the desired result. To see that J_m is in fact not empty, assume by induction that J_{m-1} is not. Then $R_{N+m-1} + 1 \in J_m$ and $L_{N+m-1} \in J_m$. However, on H^c_{N+m-1}, one or the other must be occupied. □

It follows immediately from Lemma 3.10 that the probability of a disaster is small: The number of times n up until $T = 40\lfloor \varepsilon^{-3/2} \rfloor$ at which a disaster has a chance of occurring (given ξ_n) is bounded by $T/(\frac{1}{2}\varepsilon^{-3/4}) + 1$. At each such time, the probability of a disaster is ε since $\xi_{n+1}(R_n + 1) = 0$ must occur. Therefore,

$$(3.11) \quad P(\text{some disaster occurs on } [0, T)) \le \varepsilon(2T\varepsilon^{3/4} + 1) \le 90\varepsilon^{1/4};$$

this approaches 0 as $\varepsilon \to 0$. We let

$$(3.12) \qquad K = H \cup \{\text{some disaster occurs on } [0, T)\}.$$

Then K is the "bad set" on which we have little control over the behavior of R_n. On account of (3.7) and (3.11),

$$(3.13) \qquad P(K) \to 0 \quad \text{as } \varepsilon \to 0.$$

Here and in Section 4, we will need two elementary random walk estimates. Let $X_1, \dots, X_k, \dots$ be i.i.d. random variables with $P(X_k = 1) = 1 - \varepsilon$ and $P(X_k = -\varepsilon^{-3/4}) = \varepsilon$. Then $EX_k = 1 - \varepsilon^{1/4} - \varepsilon$, which is close to 1 for small ε. Set $V_n = \sum_{k=1}^n X_k$, with $V_0 = 0$.

LEMMA 3.14. *For small* $\varepsilon > 0$ *and* $\gamma_i, C_i > 0$, $i = 1, 2$,

$$(3.15) \qquad P(V_n < 12L/5 \text{ for all } n \le T) \le C_1 \exp\{-\gamma_1 \varepsilon^{-3/2}\}$$

and

$$(3.16) \qquad P(V_n \le -L/5 \text{ for some } n \le T) \le C_2 \exp\{-\gamma_2 \varepsilon^{-3/4}\}.$$

PROOF. One has

$$(3.17) \qquad P(V_n < 12L/5 \text{ for all } n \le T) \le P(V_{4L} < 12L/5).$$

For small ε,

$$E(X_1) \ge 4/5,$$

so the large deviation bound

$$P(V_{4L} < 12L/5) \le C_1 e^{-\gamma_1 L}$$

holds for appropriate $\gamma_1, C_1 > 0$, which gives (3.15). To estimate (3.16), we do not have to worry about V_n for $n < N \equiv L\varepsilon^{3/4}/5$ since V_n cannot reach $-L/5$ before N, even if it always jumps $\varepsilon^{-3/4}$ to the left. So another large deviation estimate gives

$$P(V_n \le -L/5 \text{ for some } n \le T) \le \sum_{n \ge N} C_1' e^{-\gamma_2 n} \le C_2 e^{-\gamma_2 N},$$

for appropriate $\gamma_2, C_1', C_2 > 0$. Since $N \ge \varepsilon^{-3/4}$ for small ε, (3.16) follows. □

Expressions (3.13) and (3.15) give us enough control over R_n to demonstrate Proposition 3.1.

Proof of Proposition 3.1. We want to show that R_n will typically move to the right of $7L/5$ by time T. For this, compare R_n with the random walk $\tilde{V}_n = -L + \sum_{k=1}^{n} X_k$, $n \ge 0$, where $X_k = 1$ if $\xi_{k+1}(R_k + 1) = 1$, and $X_k = -\varepsilon^{-3/4}$ otherwise. One has $P(X_k = 1) = 1 - \varepsilon$ and $P(X_k = -\varepsilon^{-3/4}) = \varepsilon$. One can check that since $\tilde{V}_0 = R_0 = -L$,

$$\tilde{V}_n \le R_n \quad \text{on } K^c, \tag{3.18}$$

for $n \le T$, where K is given in (3.12). By (3.13), $P(K) \to 0$ as $\varepsilon \to 0$. So (3.2) follows from (3.15). □

4. The repositioning algorithm. In Section 3 we showed how to populate the target interval $[7L/5, 8L/5)$ with high probability at some random time in $[1, T+1]$ ($[2, T+2]$ for rule 22). In this section, we will perform the second step of our construction, that is, we will show that if at time at least 1 (2), $[7L/5, 8L/5)$ is populated. Then with probability close to 1, ξ_T will be admissible in $[L, 2L)$. We will again define tagged particles, which we now wish to remain inside $[L, 2L)$. The setup is given by a repositioning algorithm whose basic outline is related to the one defined in Bramson, Ding and Durrett (1991). Roughly speaking, the repositioning algorithm will attempt to keep $[7L/5, 8L/5)$ populated; if the interval becomes empty at some time, one can then choose a tagged particle in $[L, 2L)$ which, with probability close to 1, reaches $[7L/5, 8L/5)$ without leaving $[L, 2L)$. This procedure will be repeated until time $T+1$ ($T+2$) is reached or until it fails. We will find it convenient to subdivide $[L, 2L)$ into five intervals of equal length, which we label from left to right by $D_1, D_2, \dots, D_5$. That is,

$$D_k = \left[L + 2(k-1)\lfloor \varepsilon^{-3/2} \rfloor, L + 2k\lfloor \varepsilon^{-3/2} \rfloor\right) \quad \text{for } k = 1, 2, \dots, 5. \tag{4.1}$$

Note that D_3 is our target interval; it has midpoint $3L/2$.

We now specify the repositioning algorithm. Let τ be the first time greater than or equal to 1 (≥ 2 for rule 22) for which D_3 is populated. Let σ_1 be the next time at which D_3 is empty, with R_0^1 denoting the position of the rightmost particle in D_2 if $\xi_{\sigma_1} \cap D_2 \ne \varnothing$, and the position of the leftmost particle in D_4 if D_2 is empty but $\xi_{\sigma_1} \cap D_4 \ne \varnothing$. If both D_2 and D_4 are empty, we say that the algorithm has failed. Tag the particle at R_0^1. If $R_0^1 \in D_2$, let R_n^1 denote the position of the rightmost particle at time $n + \sigma_1$ in $[L, 3L/2 - 1)$ until time τ_1, when the set $J = \{3L/2 - 1, 3L/2\}$ is next populated. [For $R_0^1 \in D_4$, define R_n^1 symmetrically, using $[3L/2 + 1, 2L)$ and J.] We also say the algorithm has failed if $[L, 3L/2 - 1)$ is ever empty over (σ_1, τ_1) (and similarly, for $R_0^1 \in D_4$). Repeat the procedure inductively up to time $T + 1$ $(T + 2)$, defining R_0^i as the position of the tagged particle in D_2 (or D_4) closest to J at the next time σ_i at which D_3 is empty, R_n^i as the position of the corresponding tagged particle and τ_i as the succeeding time at which J is populated. Note that the tagged particle R_n^i, $n \in [0, \tau_i - \sigma_i)$, evolves according to the same law as R_n of Section 3. We denote by E_i the event that the algorithm fails over (σ_i, τ_i). Let F denote the event that the repositioning step of the algorithm ever fails (i.e., $D_2 \cup D_4$ is empty at some time σ_i). It is easy to check that this repositioning need only be employed at most 40 times. If $R_0^i \in D_2$ (or D_4), then D_3 is empty, and R_0^i is separated from any particles on its right by at least $L/5 + 1$. Since this distance decreases by at most 2 over each time step, it will take at least time $L/10 = T/40$ for J to be populated again. We set $E = \bigcup_{i=1}^{40} E_i$.

On the event $(E \cup F)^c$, $[L, 2L)$ remains populated. Proposition 4.2 states that $E \cup F$ is unlikely to occur.

PROPOSITION 4.2. *For any $\eta > 0$, $\varepsilon > 0$ can be chosen sufficiently small so that, with probability at least $1 - \eta$, the algorithm will not fail for $n \le T + 1$ $(n \le T + 2$ for rule 22).*

Proposition 2.6 (with $z = k = 0$) follows from Propositions 3.1 and 4.2. As explained at the beginning of Section 3, if ξ_0 is admissible in I_0, then off a set Λ of probability at most ε (λ for rule 22), ξ_1 (ξ_2) can be compared with a configuration which is restricted to the left of D_3. Employing Proposition 3.1, one obtains that, except for further probability η, D_3 will be populated at some time τ in $[1, T + 1]$ ($[2, T + 2]$). This starts the above algorithm, which by Proposition 4.2 continues through $T + 1$ $(T + 2)$ except on another set of probability η. Off these exceptional sets, I_1^l is populated at time $T + 1$ $(T + 2)$. For all rules except 22, this implies that $\xi_T \in \mathscr{S}(I_1^l)$. Consequently, for ξ_0 admissible in I_0,

$$P\big(\xi_T \text{ is admissible in } I_1^l\big) \ge 1 - \varepsilon - 2\eta. \tag{4.3}$$

Since $\eta \to 0$ as $\varepsilon \to 0$, for fixed $\delta > 0$ and sufficiently small $\varepsilon > 0$, $\varepsilon + 2\eta \le \delta/2$. Together with the analog for I_{-1}^r, (4.3) implies Proposition 2.6 for all rules except 22.

The derivation for rule 22 requires a little more work. For ζ any configuration, set

$$g(\zeta) = P\big(\xi_2^\zeta \cap I_1^l = \varnothing\big). \tag{4.4}$$

Note that $\zeta \in \mathscr{S}_\lambda^2(I_1^l)$ if and only if $g(\zeta) \le \lambda$. By the above bounds,

$$E\big[g(\xi_T); \Lambda^c\big] \le 2\eta. \tag{4.5}$$

So far in the paper, λ has been left unspecified; we now set $\lambda = \sqrt{\eta}$. Applying

Markov's inequality to (4.5), we obtain that

$$P\left(\{\xi_T \notin \mathscr{S}^2_\Lambda(I_1^l)\} \cap \Lambda^c\right) \le 2\sqrt{\eta}\,. \tag{4.6}$$

Since $P(\Lambda) \le \sqrt{\eta}$, it follows that for ξ_0 admissible in I_0,

$$P\left(\xi_T \text{ is admissible in } I_1^l\right) \ge 1 - 3\sqrt{\eta}\,. \tag{4.7}$$

For fixed $\delta > 0$ and small $\varepsilon > 0$, $3\sqrt{\eta} \le \delta/2$. Together with its analog for I^r_{-1}, (4.7) implies Proposition 2.6 for rule 22.

To demonstrate Proposition 4.2, we first show that $P(E)$ is small. Referring back to Section 3, E_i is easy to investigate. Apply the strong Markov property to compare R^i_n with the random walk $\tilde{V}_n = v_0 + \Sigma^n_{k=1} X_k$ as in (3.18); here $v_0 = R^i_0$. One has $R^i_0 \in D_2 \cup D_4$, with the tagged particle able to "fall back" at least distance $L/5$ and still remain in $[L, 2L)$. By (3.16),

$$P(E_i \cap K_i^c) \le C_2 \exp\{-\gamma_2 \varepsilon^{-3/4}\} \quad \text{for all } i, \tag{4.8}$$

where K_i is the analog of K in (3.12). On account of (3.13), $P(K_i) \to 0$ as $\varepsilon \to 0$, and so $P(E_i) \to 0$. Summing over $i \in [1, 40]$, one obtains

$$P(E) \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

In order for F to occur, $D_2 \cup D_3 \cup D_4$ must be empty at some time $n + 1 \in [2, T + 1]$ ($[3, T + 2]$ for rule 22), even though D_3 is populated at time n. Denote these events by F_n. Proposition 4.10 provides a bound for $P(F)$ which, together with the previous paragraph, implies Proposition 4.2.

PROPOSITION 4.10. *For any $\eta > 0$, $\varepsilon > 0$ can be chosen sufficiently small so that, for $n \ge 1$ ($n \ge 2$ for rule 22), $P(F_n) \le \eta/T$.*

We note that as explained in Section 2, the restriction $n \ge 2$ is in fact necessary for rule 22.

The remainder of the paper is devoted to demonstrating Proposition 4.10. The following elementary lemma will be used. Here Q denotes the set of configurations for which the proportion of vacant sites over the interval $[0, L/5)$ is less than $\delta_1 > 0$, where δ_1 will be specified later, and Q^2 denotes those configurations with no pairs of adjacent vacant sites over $[0, L/5)$ (i.e., all vacant sites are isolated).

LEMMA 4.11. *For small $\varepsilon > 0$ and any ζ,*

$$P\left(\xi^\zeta_n \in Q^c \cap Q^2\right) \le \exp\{-\delta_1 \varepsilon^{-1/2}\}$$

for any $n \ge 1$.

PROOF. Label the vacant sites of ξ^ζ_n in $[0, L/5)$ from left to right by $1, 2, \dots, K$, and let $N_n(k)$ denote the position of the kth vacant site at time n. Then

$$P\left(N_n(k+1) = N_n(k) + 1 | \xi_{n-1}, N_n(l) \text{ for } l \le k\right) \ge \varepsilon. \tag{4.12}$$

Let $M = \lfloor \delta_1 L/5 \rfloor$, let G_k denote the set on which $K \geq k$, and let

$$A_k = \{N_n(k+1) \neq N_n(k) + 1\}.$$

Elementary decomposition implies that

$$\begin{aligned} P(A_1 \cap \cdots \cap A_{M-1} \cap G_M | \xi_{n-1}) &= \prod_{k=1}^{M-1} P(A_k \cap G_M | A_1 \cap \cdots \cap A_{k-1}, \xi_{n-1}) \\ &\leq \prod_{k=1}^{M-1} P(A_k | A_1 \cap \cdots \cap A_{k-1} \cap G_k, \xi_{n-1}) \\ &\leq (1-\varepsilon)^{M-1}. \end{aligned}$$

Since $Q^c \subset G_M$, it follows that, for small ε,

$$\begin{aligned} P(\xi_n^\zeta \in Q^c \cap Q^2) &\leq (1-\varepsilon)^{\delta_1 L/5 - 2} \\ &\leq (1-\varepsilon)^{\delta_1 \varepsilon^{-3/2}}. \end{aligned}$$

Since $\log(1-\varepsilon) \leq -\varepsilon$, this is

$$\leq \exp\{-\delta_1 \varepsilon^{-1/2}\}. \qquad \square$$

PROOF OF PROPOSITION 4.10. The proof varies, depending on the rule under consideration. We break the argument into four cases, which are presented in order of increasing difficulty. Case (i) consists of rule 18, (ii) of those rules with $\alpha_7 = 1$, (iii) of all the remaining rules except for rule 22 and (iv) of this last pesky rule. For notational simplicity, we set $n = 1$ for all cases except (iv), where we set $n = 2$. We show that for any choice of $\eta > 0$ and any initial configuration ζ,

$$P(\hat{F}_1) \leq \eta \varepsilon^{3/2} \tag{4.13}$$

in cases (i)–(iii), and

$$P(\hat{F}_2) \leq \eta \varepsilon^{3/2} \tag{4.14}$$

in case (iv), where $\hat{F}_j$ is the event that $\xi_j^\zeta \cap D_3 \neq \varnothing$ and $\xi_{j+1}^\zeta \cap (D_2 \cup D_3 \cup D_4) = \varnothing$. The proposition then follows immediately from (4.13) and (4.14).

(i) *Rule 18*. This rule is characterized by $\alpha_1 = \alpha_4 = 1$ with all other α_i's equal to 0. The key ingredient for rule 18 is to observe that it is impossible to get three occupied sites in a row at time 1. To see this, note that in order to get two occupied sites in a row, say at $x = 1$ and $x = 2$ at time 1, it is necessary for ζ to have the configuration 1001 on [0, 3]. Since only $\alpha_1 = \alpha_4 = 1$ and all other α_i's are equal to 0, it follows immediately that $\xi_1(0) = \xi_1(3) = 0$. Therefore, ξ_1 is always empty at $1/3 - \delta_2$ or more of the sites in both D_2 and D_4, where δ_2 is small since L is large. For $\delta_1 \leq 1/3 - \delta_2$, Lemma 4.11 implies that with probability at least $1 - 2\exp\{-\delta_1 \varepsilon^{-1/2}\}$, ξ_1 contains pairs of adjacent vacant sites in both D_2 and D_4. Let p_L (p_R) denote the position of the leftmost (rightmost) particle in $\xi_1^\zeta \cap D_3$. (Set $p_L = \infty$ and $p_R = -\infty$ if $\xi_1^\zeta \cap D_3 = \varnothing$.) It follows that under $\xi_1^\zeta \cap D_3 \neq \varnothing$, $\tilde{D}_2 = [6L/5, p_L]$ and $\tilde{D}_4 = [p_R, 9L/5)$ typically each possess at least one pair of such sites adjacent to an occupied site. Since $\alpha_1 = \alpha_4 = 1$, with probability at least $1 - \varepsilon^2$, one of these triples will produce a particle at time 2. Therefore,

$$P(\hat{F}_1) \leq 2\exp\{-\delta_1\varepsilon^{-1/2}\} + \varepsilon^2 \leq \eta\varepsilon^{3/2}, \tag{4.15}$$

for small enough $\varepsilon > 0$. This gives (4.13) for rule 18.

(ii) *Rules with* $\alpha_7 = 1$. As in rule 18, we consider the density of vacant sites in D_2 and D_4 at time 1. Here, however, the density of vacant sites may be small. If the density of vacant sites in both intervals is at least $1/7$ at time 1, we can then apply Lemma 4.11 and (4.15) as above. Otherwise, we use the following fact: In an interval of length $L/5$ in which the density of vacant sites is less than $1/7$, the number of triples for which all three sites are occupied is at least $L/10$. (A margin of error is included for sites on the boundary.) Since $\alpha_7 = 1$, each of these triples may produce a particle at time 2. The probability that all of them fail is at most $\varepsilon^{L/10}$. Combining the two cases, one obtains that

$$P(\hat{F}_1) \leq 2\exp\{-\varepsilon^{-1/2}/7\} + \varepsilon^2 + \varepsilon^{L/10} \leq \eta\varepsilon^{3/2}. \tag{4.16}$$

(iii) *Rules with* $\alpha_5 = 1$ *or* $\alpha_3 = \alpha_6 = 1$, *but* $\alpha_7 = 0$. As in cases (i) and (ii), we consider the density of empty sites in D_2 and D_4 at time 1; here, we do the computations for the two parts separately. If the density of empty sites in D_2 is at least $\frac{1}{3}$ at time 1, we can then employ Lemma 4.11 as before to deduce the presence of a pair of adjacent vacant sites next to an occupied site except for probability $\exp\{-\varepsilon^{-1/2}/3\}$. If, on the other hand, the density of empty sites in D_2 is less than $\frac{1}{3}$, then pairs of adjacent occupied sites will be present. Also, except on a set of probability at most $(1-\varepsilon)^{L/5-1} \leq \exp\{-\varepsilon^{-1/2}\}$, $\xi_1 \cap (6L/5, 7L/5)$ has at least one vacant site. One can therefore check that under $\xi_1 \cap D_3 \neq \varnothing$, $\tilde{D}_2$ must contain a triple of the form (1) 001 or 101 and (2) 001, 011, or 110. We apply (1) for rules with $\alpha_5 = 1$ and (2) for rules with $\alpha_3 = \alpha_6 = 1$. So we see that without any assumptions on the density of empty sites at time 1, the probability that no site in $\tilde{D}_2$ has a chance of producing a 1 at time 2 is small:

$$P\left(\xi_1 \cap D_3 \neq \varnothing, \hat{\xi}_1^{\xi_1} \cap \tilde{D}_2 = \varnothing\right) \leq \exp\{-\varepsilon^{-1/2}/3\} + \exp\{-\varepsilon^{-1/2}\} \leq 2\exp\{-\varepsilon^{-1/2}/3\}. \tag{4.17}$$

The analog of (4.17) with $\tilde{D}_4$ substituted for $\tilde{D}_2$ also holds. On $\{\hat{\xi}_1^{\xi_1} \cap \tilde{D}_k \neq \varnothing, k = 2, 4\}$, the probability that a 1 is produced on neither side is at most ε^2. It follows that

$$P(\hat{F}_1) \leq 4\exp\{\varepsilon^{-1/2}/3\} + \varepsilon^2 \tag{4.18}$$

$$\leq \eta \varepsilon^{3/2}.$$

(iv) *Rule* 22. The rule is characterized by $\alpha_1 = \alpha_2 = \alpha_4 = 1$ with all other α_i's equal to 0. We will show that ξ_2 typically contains pairs of adjacent vacant sites in both D_2 and D_4; as in (i), this implies that $\tilde{D}_2$ and $\tilde{D}_4$ each possess at least one pair of such sites adjacent to an occupied site, which will enable us to demonstrate (4.14). As in (iii), we do the computations for the two parts separately.

Subdivide D_2 into $\lfloor L/35 \rfloor$ disjoint intervals $J(k)$, $k = 1, 2, \dots,$ of length 7. A configuration ζ' is said to be *periodic* over $J(k)$ if every third site in $J(k)$ is occupied with the others being vacant, one being able to start at any point in the cycle. For example, 0100100 is included. It is easy to check that if ζ' is not periodic over $J(k)$, then $\xi_1^{\zeta'}$ restricted to $J(k)$ must include at least one vacant site. The set U will consist of those configurations which are periodic over at least $L/75$ of the intervals $J(k)$. We divide the initial configurations ζ into two cases, depending on whether or not $\hat{\xi}_1 \equiv \hat{\xi}_1^{\zeta} \in U$. (Recall that $\hat{\xi}_1^{\zeta}$ is not random.)

First note that if $\hat{\xi}_1$ is periodic over $J(k)$, then there is one occupied site in $J(k)$ which must have a pair of vacant sites as immediate neighbors on each side. That is, one has the five-tuple 00100. Call this site the *interior* 1 of $J(k)$; for $\hat{\xi}_1 \in U$, the configuration includes at least $L/75$ interior 1's. Let G_1 be the event that $\xi_1(x) = 1$ at all interior 1's. For $\hat{\xi}_1 \in U$ and small $\varepsilon > 0$,

$$P(G_1) \leq (1 - \varepsilon)^{L/75} \leq \exp\{-\varepsilon^{-1/2}/8\}. \tag{4.19}$$

But on G_1^c, $\xi_1|_{D_2}$ (ξ_1 restricted to D_2) includes five vacant adjacent sites somewhere, which implies that $\xi_2|_{D_2}$ contains a pair of (actually three) vacant adjacent sites.

Suppose now that $\hat{\xi}_1 \notin U$. Then $\hat{\xi}_1$ is not periodic over more than $L/75$ intervals $J(k)$. Each such interval will become periodic for ξ_1 with probability at most 7ε, which is at most $\frac{1}{8}$ for $\varepsilon \leq \frac{1}{56}$. (One can check that the first probability is in fact at most ε for $\varepsilon \leq \frac{1}{2}$.) Let G_2 denote the event that at least $L/150$ such changes occur over D_2. It is a simple large deviation estimate that

$$P(G_2) \leq \exp\{-\gamma_3 L\} \leq \exp\{-\gamma_3 \varepsilon^{-3/2}\}, \tag{4.20}$$

for some $\gamma_3 > 0$. On G_2^c, ξ_1 is not periodic over more than $L/150$ intervals $J(k)$; recall that ξ_2 restricted to such a $J(k)$ must include at least one vacant site. So on G_2^c, the proportion of vacant sites in D_2 for ξ_2 is at least $\frac{1}{30}$. Set

$$G_3 = G_2^c \cap \{\xi_2 \in Q^2\}.$$

From Lemma 4.11,

$$P(G_3) \leq \exp\{-\varepsilon^{-1/2}/30\}. \tag{4.21}$$

On $(G_2 \cup G_3)^c$, $\xi_2|_{D_2}$ contains a pair of vacant adjacent sites.

On account of (4.19)–(4.21), the probability that $\xi_2|_{D_2}$ does not contain a pair of vacant adjacent sites is, for all ζ, at most $2\exp\{-\varepsilon^{-1/2}/30\}$. The same statement holds as well for $\xi_2|_{D_4}$. If $\xi_2 \cap D_3 \neq \varnothing$ and both $\tilde{D}_2$ and $\tilde{D}_4$ contain pairs of vacant adjacent sites, the probability that a 1 is produced on neither side is at most ε^2. It follows that

$$P(\hat{F}_2) \le 4\exp\{-\varepsilon^{-1/2}/30\} + \varepsilon^2 \le \eta\varepsilon^{3/2}. \tag{4.22}$$ □

We have demonstrated Proposition 4.10, which implies Proposition 4.2, and hence Proposition 2.6. This completes the proof of the theorem.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
VAN VLECK HALL
MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
1042 W. 36TH PLACE, DRB 155
LOS ANGELES, CALIFORNIA 90089