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ON CONDITIONING A RANDOM WALK TO STAY NONNEGATIVE

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Let S be a real-valued random walk that does not drift to ∞, so $P(S_k \geq 0 \text{ for all } k) = 0$. We condition S to exceed n before hitting the negative half-line, respectively, to stay nonnegative up to time n. We study, under various hypotheses, the convergence of these conditional laws as $n \to \infty$. First, when S oscillates, the two approximations converge to the same probability law. This feature may be lost when S drifts to $-\infty$. Specifically, under suitable assumptions on the upper tail of the step distribution, the two approximations then converge to different probability laws.

1. Introduction. This paper is concerned with two different interpretations of conditioning a real-valued random walk $S = (S_k, k \geq 0)$ to stay nonnegative. Our motivation comes from a continuous time analogue. More precisely, it is known that the three-dimensional Bessel process can be viewed as a standard Brownian motion conditioned to stay nonnegative; see Doob (1957) and McKean (1963). Deep connections between the two processes were stressed first by Williams (1974) and Pitman (1975), and we refer to Rogers and Williams (1987) and Revuz and Yor (1991) for recent overviews. It is therefore interesting to search for a random walk analogue of the three-dimensional Bessel process.

When the random walk drifts to ∞, the event

$$\Lambda = \left\{S_k \geq 0 \text{ for all } k \geq 0\right\}$$

has positive probability, and in this case, there is just one sensible meaning for the conditioning. Henceforth, we will concentrate on the case when S does not drift to ∞, which is equivalent to $P(\Lambda) = 0$. The obvious thing to do is to define conditioning with respect to Λ as the limit of conditioning with respect to Λ_n, where Λ_n is an approximation to Λ. We shall focus on two natural choices for Λ_n. The first is

$$\Lambda_n^{(1)} = \left\{S \text{ hits } [n, \infty) \text{ before it hits } (-\infty, 0)\right\},$$

which was considered first by Pitman (1975) in the case of the simple symmetric random walk. Notice that this event is time homogeneous. That is, $\theta_k(\Lambda_n^{(1)}) = \Lambda_n^{(1)}$ whenever S does not exit from $[0, n)$ before time k, where θ stands for the shift operator. The second is

$$\Lambda_n^{(2)} = \{S_k \geq 0 \text{ for all } 0 \leq k \leq n\},$$

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which has been considered by Keener (1992) for certain integer-valued random walks with negative drift. In this case, time homogeneity is lost. To show that these methods can lead to different answers, we discuss the case of the simple random walk.

Assume here that $P(S_1 = 1) = p, P(S_1 = -1) = 1 - p$, with $0 < p \leq 1/2$. When $p = 1/2$, the two methods yield the same limit; that is, the law of the Markov chain on the nonnegative integers with transition probability q given for $x \geq 0$ by

$$q(x, x+1) = \frac{x+2}{2(x+1)},$$
$$q(x, x-1) = \frac{x}{2(x+1)}.$$

When $p < 1/2$, the two methods yield two different limits, which both are the laws of strict homogeneous Markov chains. The transition probabilities are given for $x \geq 0$ by

$$q^{(1)}(x, x+1) = p \, \frac{\exp\big(\omega(x+2)\big) - 1}{\exp\big(\omega(x+1)\big) - 1},$$
$$q^{(1)}(x, x-1) = (1-p) \frac{\exp(\omega x) - 1}{\exp\big(\omega(x+1)\big) - 1},$$

with $\omega = \log(1/p - 1)$, and

$$q^{(2)}(x, x+1) = \frac{x+2}{2(x+1)},$$
$$q^{(2)}(x, x-1) = \frac{x}{2(x+1)}.$$

Notice that $q^{(1)}$ is expressed in the form $(h(y)/h(x))\, p(x,y), x, y \geq 0$, where $p(x,y)$ is the transition function of S. This means that h is a harmonic function for the random walk killed as it enters $(-\infty, 0)$, and that $q^{(1)}$ corresponds to an h-transform of the killed random walk. Moreover, if S^* denotes the so-called associated random walk that has $P(S_1^* = 1) = 1 - p, P(S_1^* = -1) = p$ and drifts to ∞, then one can check that $q^{(1)}$ coincides with the transition function of S^* conditioned to stay nonnegative. On the other hand, $q^{(2)}$ corresponds to an h-transform not of S, but rather of the simple *symmetric* random walk. Specifically, $q^{(2)}$ is the transition function of the simple symmetric random walk conditioned to stay nonnegative, which was given previously. This was discovered by Keener (1992); see also Good (1968).

The main object of this paper is to extend the foregoing results to a broad class of random walks. Technically, the key step consists of showing that the ratio $P_x(\Lambda_n^{(i)})/P(\Lambda_n^{(i)})$ converges as $n \to \infty$ (where P_x denotes the law of the random walk started at x and $P = P_0$) and expressing the limit in terms of a certain renewal function. We find first that for any oscillating random walk, conditioning by $\Lambda_n^{(1)}$ or by $\Lambda_n^{(2)}$ always yields the same strict Markovian limit (in the sense of weak convergence of finite-dimensional distributions), and the limit corresponds to an h-transform of S killed as it enters $(-\infty, 0)$. When the random walk drifts to $-\infty$, the situation is more complex. If the upper tail of the step distribution is regularly varying, then conditioning by $\Lambda_n^{(1)}$ or by $\Lambda_n^{(2)}$ yields the same limit, which is now only sub-Markovian. On the other hand, if the upper tail of the step distribution satisfies a certain exponential moment condition of Cramér's type, then conditioning by $\Lambda_n^{(1)}$ or by $\Lambda_n^{(2)}$ yields two strictly Markovian limits that are distinct. More precisely, conditioning by $\Lambda_n^{(1)}$ leads to the law of S^* conditioned to stay nonnegative, where S^* is the associated random walk that drifts to ∞. Conditioning by $\Lambda_n^{(2)}$ leads to the law of $\widetilde{S}$ conditioned to stay nonnegative, where $\widetilde{S}$ is a certain oscillating random walk. The fact that two different approximations of conditioning with respect to the same event of zero probability can produce two distinct limits is already known. We refer to Knight (1969) for another striking example arising from Brownian motion.

A different problem about conditioning a random walk to stay nonnegative has been intensively treated in the literature. Specifically, motivated by Donsker's invariance principle, several authors have studied the convergence of the law of the broken line process $(S_{[tn]}, t \in [0,1])$ properly scaled and conditioned on $\Lambda_n^{(2)}$. We refer in particular to Iglehart (1974a) for the oscillating case, and for the negative drift case, to Kao (1978) when a Cramér type condition holds and to Durrett (1980) when the upper tail of the step distribution is regularly varying. Convergence in law of a rescaled broken line process conditioned on $\Lambda_n^{(1)}$ was investigated by Asmussen (1982); see also Anantharam (1989). Roughly speaking, these works are concerned with the long range effect of conditioning to stay nonnegative, whereas the present paper focusses on the consequences of such conditioning for the initial behaviour of the random walk.

2. Preliminaries. The purpose of this section is to introduce the notation and recall classical results in fluctuation theory for random walks. Spitzer (1964) and Feller (1971) are the standard references; see also Gut (1988) for a recent exposition.

For every real number x, we denote the law of the random walk S started at x by P_x. For simplicity, we put $P = P_0$. We will always assume that $0 < P(S_1 > 0) < 1$. The expectation under P_x of a random variable X is denoted by $E_x(X)$. More generally, if $A_1, \dots, A_k$ are measurable sets, then $E_x(X, A_1, \dots, A_k)$ is the P_x-expectation of $1_A X$, where $A = A_1 \cap \cdots \cap A_k$.

The first entrance times, respectively, in $(-\infty, 0)$ and in $[n, \infty)$ are denoted by

$$\tau = \min\{k \geq 1\colon\ S_k < 0\},$$
$$\sigma(n) = \min\{k \geq 1\colon\ S_k \geq n\}.$$

Here, we make the convention that $\min \emptyset = \infty$. In particular, $\Lambda_n^{(1)} = \{\sigma(n) < \tau\}$ and $\Lambda_n^{(2)} = \{\tau > n\}$. Let $(H, T) = ((H_k, T_k), k \geq 0)$ be the strict ascending ladder point process of the reflected random walk $-S$. That is, $T_0 = 0$ and

$$H_k = -S_{T_k}, \qquad T_{k+1} = \min\{j > T_k\colon\ -S_j > H_k\},$$

with the convention $H_k = \infty$ when $T_k = \infty$. The variable H_1 (respectively, $T_1 = \tau$) is known as the first strict ascending ladder height (respectively, epoch) of $-S$. The *renewal function* associated with H_1 is

$$V(x) = \sum_{k=0}^{\infty} P(H_k \le x), \qquad x \ge 0.$$

It is a nondecreasing right-continuous function. Its left limit at $x > 0$ is denoted by $V(x-)$.

Because this renewal function plays a crucial role in this work, we recall now various expressions for V. First, according to the duality lemma, we have

$$V(x) = E\left(\sum_{j=0}^{\sigma(0)-1} 1_{\{-x \le S_j\}} \right).$$

We say that the random walk S drifts to ∞ if $P(\tau = \infty) > 0$. In this case,

$$V(x) = \frac{P_x(\tau = \infty)}{P(\tau = \infty)}.$$

The random walk S drifts to $-\infty$ if $-S$ drifts to ∞. In this case, $E_x(\tau) < \infty$ for all $x \ge 0$ and

$$V(x) = \frac{E_x(\tau)}{E(\tau)}.$$

Moreover, when $E(S_1) < \infty$, Wald's identity yields also

$$V(x) = \frac{x - E_x(S_\tau)}{E(-S_\tau)}.$$

Finally, the random walk S oscillates if it does not drift to ∞ or to $-\infty$.

When S drifts to ∞ or oscillates, the renewal function V is invariant for the random walk killed as it enters the negative half-line. That is,

$$\left\{ 1_{\{k < \tau\}} V(S_k),\ k \ge 0 \right\}$$

is a P_x martingale for every $x \ge 0$. When S drifts to $-\infty$, V is just superharmonic and the preceding process is a P_x-supermartingale for all $x \ge 0$.

We denote by P_x^V the h-transform of P_x by the function V. That is, P_x^V is the law of the homogeneous Markov chain on the nonnegative real numbers, with transition function

$$p^V(x,y) = \frac{V(y)}{V(x)}\, p(x,y), \qquad x, y \ge 0.$$

(Observe that p^V is strictly Markovian iff V is invariant.) Alternatively, if $f(S) = f(S_0, S_1, \ldots, S_k)$ is a functional that depends only on the k first steps of the random walk (where k is a fixed integer), then

$$E_x^V\bigl(f(S)\bigr) = \frac{1}{V(x)} E_x\bigl(V(S_k) f(S),\ k < \tau\bigr).$$

When S drifts to ∞, P^V is the law of S conditioned to stay nonnegative (in the usual sense) and the associated chain is strict Markov. When S oscillates, the transition function p^V is still honest and the chain is strict Markov. On the other hand, when S drifts to $-\infty$, the transition function p^V is defective and the chain is sub-Markov. We refer to Tanaka (1989) and Bertoin (1993) for pathwise constructions of Markov chains with law P^V from the random walk S.

Finally, for a proper application of renewal theory, we need to distinguish between the lattice and the nonlattice case. We say that the random walk has a *lattice distribution with span* $\lambda > 0$ if its step distribution is supported by the centered lattice $\lambda\mathbb{Z} = \{0, \pm\lambda, \pm 2\lambda, \dots\}$ and no centered sublattice thereof. We say that S has a *nonlattice distribution* if its step distribution is not supported by $\lambda\mathbb{Z}$ for any $\lambda > 0$. Plainly, for every $\lambda > 0$, the distribution of H_1 is supported by the lattice $\lambda\mathbb{Z}$ (i.e., H_1 has a lattice distribution with span that is an integer multiple of λ) iff the renewal function V is constant over the intervals $[n\lambda, (n+1)\lambda)$ for all integers $n \geq 0$. By the duality lemma, this implies that the step distribution of S is also supported by $\lambda\mathbb{Z}$. Indeed, $P(k\lambda < S_1 < (k+1)\lambda) = 0$ for all integers $k < 0$; otherwise, the distribution of H_1 would not be supported by $\lambda\mathbb{Z}$. Now fix an integer $k \geq 0$. The probability that the $k+1$ first steps of S are all in $\{-\lambda, -2\lambda, \dots\}$ is positive, and thus, there exists an integer $k' > k$ such that

$$P(S_{k+1} = -k'\lambda, k+1 < \sigma(0)) > 0.$$

Now, if the probability that $k\lambda < S_1 < (k+1)\lambda$ was positive, then we would have $P((k-k')\lambda < S_{k+2} < (k+1-k')\lambda, k+2 < \sigma(0)) > 0$, and according to the duality lemma, V would not be constant on $((k'-k-1)\lambda, (k'-k)\lambda)$. Replacing λ by an integer multiple of λ, we see that H_1 has a lattice distribution with span λ (resp. a nonlattice distribution) iff S has a lattice distribution with span λ (resp. a nonlattice distribution).

3. The oscillating case. We will assume throughout this section that the random walk oscillates. It should be clear that the asymptotic behaviour of the ratio $P_x(\Lambda_n^{(i)})/P(\Lambda_n^{(i)})$ will play a crucial role in the study of conditioning by $\Lambda_n^{(i)}$. We will see that for $i = 1$, one can calculate the limit without evaluating each term of the ratio.

LEMMA 1. *For every $x \geq 0$, we have*

$$\lim_{n\to\infty} \frac{P_x(\Lambda_n^{(1)})}{P(\Lambda_n^{(1)})} = V(x).$$

PROOF. To begin, we show that for every fixed $a > 0$, the probability that S hits $[n+a, \infty)$ before it hits $(-\infty, 0)$ is equivalent to the probability that S hits $[n, \infty)$ before it hits $(-\infty, 0)$ as n goes to ∞. That is,

$$P(\Lambda_{n+a}^{(1)})/P(\Lambda_n^{(1)}) \to 1 \quad \text{as } n \text{ goes to } \infty. \tag{1}$$

Indeed, by the Markov property, the left-hand side of (1) can be expressed as

$$P\big(S_{\sigma(n)} \geq n + a \mid \Lambda_n^{(1)}\big) + \int_{x \in [0,a)} P_{n+x}\big(\Lambda_{n+a}^{(1)}\big)\, P\big(S_{\sigma(n)} \in n + dx \mid \Lambda_n^{(1)}\big),$$

but

$$P_{n+x}\big(\Lambda_{n+a}^{(1)}\big) = P_x\big(S \text{ hits } [a,\infty) \text{ before it hits } (-\infty,-n)\big).$$

Because S oscillates, the preceding quantity goes to 1 uniformly on $x \in [0,a]$ as n goes to ∞. This implies (1).

By (1), we need only show that

$$P\big(S \text{ hits } [n,\infty) \text{ before it hits } (-\infty,-x)\big) \sim V(x)\, P\big(\Lambda_n^{(1)}\big). \tag{2}$$

Recall from Section 2 that (H_n, T_n) stands for the nth strict ascending ladder point of $-S$, and denote by $M_n = \max\{S_k + H_{n-1},\ T_{n-1} \leq k \leq T_n\}$, the maximum of the nth excursion of S above its past minimum. Note that the variables $(H_1, M_1), (H_2 - H_1, M_2), \dots$ are i.i.d. Now, we rewrite the left-hand side of (2) as

$$\begin{aligned} &P(M_1 \geq n) + \sum_{k=1}^{\infty} P(H_k \leq x,\ M_1 < n,\ M_2 < n + H_1, \dots, \\ &\qquad\qquad M_k < n + H_{k-1},\ M_{k+1} \geq n + H_k). \end{aligned} \tag{3}$$

On the one hand, we deduce from (3) that

$$\begin{aligned} &P\big(S \text{ hits } [n,\infty) \text{ before it hits } (-\infty,-x)\big) \\ &\quad \leq P(M_1 \geq n) + \sum_{k=1}^{\infty} P(H_k \leq x, M_{k+1} \geq n + H_k) \\ &\quad \leq P(M_1 \geq n) + \sum_{k=1}^{\infty} P(H_k \leq x) P(M_{k+1} \geq n) \\ &\quad \leq P(M_1 \geq n) V(x), \end{aligned}$$

where the last inequality comes from the definition of V. Thus,

$$\limsup_{n\to\infty} P\big(S \text{ hits } [n,\infty) \text{ before it hits } (-\infty,-x)\big) / P(M_1 \geq n) \leq V(x).$$

On the other hand, (3) is greater than or equal to

$$P(M_1 \geq n) + \sum_{k=1}^{\infty} P(H_k \leq x,\ M_1 < n,\ M_2 < n, \dots, M_k < n,\ M_{k+1} \geq n + x),$$

and because the $(k+1)$th excursion is independent of the k-preceding, the foregoing quantity equals

$$P(M_1 \geq n) + P(M_1 \geq n + x) \sum_{k=1}^{\infty} P(H_k \leq x,\ M_1 < n,\ M_2 < n, \dots, M_k < n).$$

Recall (1) and that $\Lambda_n^{(1)} = \{M_1 \geq n\}$. We deduce by monotone convergence that

$$\liminf_{n \to \infty} P\big(S \text{ hits } [n, \infty) \text{ before it hits } (-\infty, -x)\big)/P(M_1 \geq n) \geq V(x).$$

This proves (2). □

REMARK. The asymptotic behaviour of $P(\Lambda_n^{(1)})$ is characterized in Corollary 3 of Doney (1983) when the step distribution belongs to the domain of attraction of a stable law. In this case, Lemma 1 gives the asymptotic behaviour of $P_x(\Lambda_n^{(1)})$ for $x > 0$.

For $i = 2$, we have a weaker result (see also the remarks after the proofs of Lemma 2 and Theorem 1), which, however, will be sufficient for our purpose.

LEMMA 2. *For every $x \geq 0$, we have*

$$\liminf_{n \to \infty} \frac{P_x(\Lambda_n^{(2)})}{P(\Lambda_n^{(2)})} \geq V(x).$$

PROOF. Recall that (H_k, T_k) denotes the kth ascending ladder point of $-S$. By the Markov property for the following second inequality, we have

$$\begin{aligned} P_x(\tau > n) &\geq \sum_{k=0}^{\infty} P(H_k \leq x,\ T_k \leq n,\ T_{k+1} - T_k > n) \\ &\geq P(\tau > n) \sum_{k=0}^{\infty} P(H_k \leq x,\ T_k \leq n). \end{aligned}$$

By monotone convergence,

$$\begin{aligned} \lim_{n \to \infty} \sum_{k=0}^{\infty} P(H_k \leq x,\ T_k \leq n) &= \sum_{k=0}^{\infty} P(H_k \leq x) \\ &= V(x), \end{aligned}$$

which entails the lemma. □

REMARK. We also mention that in the lattice case with unit span and strong aperiodicity, Theorem 10 of Kesten (1963) gives that the ratio $P_x(\tau = n)/P(\tau = n)$ converges to $V(x)$ for every integer $x \geq 0$, which implies our Lemma 2.

Recall from Section 2 that P^V stands for the probability measure obtained from the law of S killed as it enters the negative half-line by h-transformation with $h = V$. The main result of this section is the following theorem.

THEOREM 1. *Consider a bounded functional $f(S) = f(S_0, S_1, \dots, S_k)$ that depends only on the k first steps of S. Then, for $i = 1$ or 2,*

$$\lim_{n \to \infty} E\big(f(S) \mid \Lambda_n^{(i)}\big) = E^V\big(f(S)\big).$$

PROOF. Assume first that $i = 1$. With no loss of generality, we may suppose that $0 \le f \le 1$. Applying the Markov property, we get

$$\begin{aligned} E\big(f(S), \Lambda_n^{(1)}\big) &\ge E\big(f(S), \Lambda_n^{(1)},\ k < \sigma(n) \wedge \tau\big) \\ &\ge E\Big(f(S)\, P_{S_k}\big(\Lambda_n^{(1)}\big),\ k < \sigma(n) \wedge \tau\Big). \end{aligned}$$

Recall from Lemma 1 that $P_{S_k}(\Lambda_n^{(1)})/P(\Lambda_n^{(1)})$ converges to $V(S_k)$, P-a.s. It follows from Fatou's lemma that

$$\liminf_{n\to\infty} E\big(f(S) \mid \Lambda_n^{(1)}\big) \ge E^V\big(f(S)\big).$$

Replacing f by $1 - f$, we get

$$\begin{aligned} \limsup_{n\to\infty} E\big(f(S) \mid \Lambda_n^{(1)}\big) &= 1 - \liminf_{n\to\infty} E\big((1-f)(S) \mid \Lambda_n^{(1)}\big) \\ &\le 1 - E^V\big((1-f)(S)\big) \\ &= E^V\big(f(S)\big). \end{aligned}$$

(*Nota bene*: The last equality comes from the fact that P^V is conservative and would be false otherwise.) This proves Theorem 1 for $i = 1$.

Assume now that $i = 2$. The Markov property entails that for $k \le n$,

$$\begin{aligned} E\big(f(S), \Lambda_n^{(2)}\big) &= E\Big(f(S) P_{S_k}\big(\Lambda_{n-k}^{(2)}\big), \tau > k\Big) \\ &\ge E\Big(f(S) P_{S_k}\big(\Lambda_n^{(2)}\big), \tau > k\Big). \end{aligned}$$

We deduce from Lemma 2 and Fatou's lemma that

$$\begin{aligned} \liminf_{n\to\infty} E\big(f(S) \mid \Lambda_n^{(2)}\big) &\ge E\big(f(S) V(S_k), \tau > k\big) \\ &\ge E^V\big(f(S)\big). \end{aligned}$$

The converse inequality for lim sup follows as in the case $i = 1$. □

REMARK. The following reinforcement of Lemma 2 derives easily from Theorem 1 and Theorem 26 on page 44 in Dellacherie and Meyer (1975). For any fixed integer $k \ge 0$, the ratio $P_{S_k}(\tau > n)/P(\tau > n)$ converges in L^1 to $V(S_k)$ as $n \to \infty$. In particular, in the 1-lattice case, $\lim P_x(\tau > n)/P(\tau > n) = V(x)$ for any integer $x \ge 0$. Finally, in the nonlattice case and when Spitzer's condition holds, the result on page 381 in Bingham, Goldie and Teugels (1987) is available. Then the argument of Lemma 3 can be adapted to show that $\lim P_x(\tau > n)/P(\tau > n) = V(x)$ for any real number $x \ge 0$.

4. The negative drift case. We will assume throughout this section that the random walk drifts to $-\infty$. The first result of this section (Theorem 2) claims that when the step distribution satisfies a Cramér type condition, then conditioning by $\Lambda_n^{(1)}$ or by $\Lambda_n^{(2)}$ yields the laws of two distinct strict Markov chains. To give a rigorous statement, we introduce the cumulant of the random walk

$$\kappa(\theta) = \log E\big(\exp(\theta S_1)\big) \in (-\infty, \infty] \quad \text{for } \theta \in \mathbb{R}.$$

Recall that the cumulant is convex and denote by κ' its derivative. We consider the following hypotheses.

HYPOTHESIS 1. *There exists $\omega > 0$ such that $\kappa(\omega) = 0$ and $E(S_1 e^{\omega S_1}) < \infty$.*

HYPOTHESIS 2. *There exists $\nu > 0$ such that $\kappa'(\nu) = 0$ and $\kappa < \infty$ on some open interval that contains ν.*

Hypothesis 1 is known as Cramér's condition. Hypothesis 2, together with the assumptions that $\kappa'(0) > -\infty$ and the step distribution of S is not supported by any noncentered lattice, corresponds to the Class II in Doney (1989a, b) and to the conditions of Theorem 1.2 in Keener (1992) in the integer-valued case. It also appears as a key hypothesis in Iglehart (1974b), Veraverbeke and Teugels (1975) and Deheuvels and Steinebach (1989); see also the references therein. Due to the convexity of the cumulant, Hypothesis 1 always implies Hypothesis 2 (and then $\nu < \omega$), but the converse is false.

When Hypothesis 1 is fulfilled, one introduces the so-called *associated* random walk S^*; see Feller (1971), Section XII.4(b). Specifically, the transition probability p^* of S^* is given by

$$p^*(x, dy) = \exp\big(\omega(y - x)\big)\, p(x, dy).$$

It is easy to see that S^* drifts to ∞. We observe the following relation of absolute continuity:

$$dP^*_{x\,|\,\mathcal{F}(k)} = \exp\big(\omega(S_k - x)\big)\, dP_{x\,|\,\mathcal{F}(k)}.$$

Here, P^*_x denotes the law of S^* started at x, and $\mathcal{F}(k)$ is the σ-field generated by the first k steps of the random walk. Plainly, this relation also holds if the fixed index k is replaced by a $\mathcal{F}(k)$-stopping time. Finally, we denote the renewal function of the first strict ascending ladder height of $-S^*$ by V^*, and P^{*V^*} will stand for the corresponding h-transform of S^* killed as it enters the negative half-line. Observe that P^* is an h-transform of P and, therefore, P^{*V^*} corresponds to an h-transform of S killed at time τ.

When Hypothesis 2 is fulfilled, one introduces similarly the transition function

$$\widetilde{p}(x, dy) = \exp\big(\nu(y - x) - \kappa(\nu)\big)\, p(x, dy).$$

It is the transition function of another random walk, $\widetilde{S}$. We denote its law by $\widetilde{P}$ and we notice the relation of absolute continuity

$$d\widetilde{P}_{x\,|\,\mathcal{F}(k)} = \exp\big(\nu(S_k - x) - k\kappa(\nu)\big)\, dP_{x\,|\,\mathcal{F}(k)}.$$

Plainly, $\widetilde{E}(\widetilde{S}_1) = 0$ and, therefore, $\widetilde{S}$ oscillates. We denote the renewal function of the first strict ascending ladder height of $-\widetilde{S}$ by $\widetilde{V}$. [*Warning*: the reader will check easily that the function $\widetilde{V}$ coincides with that denoted by $e^{-\nu x}\widetilde{V}(x)$ in Doney (1989b).] Finally, let $\widetilde{P}^{\tilde{V}}$ be the corresponding h-transform of $\widetilde{S}$ killed as it enters the negative half-line. Here, $\widetilde{P}$ is not an h-transform of P, and $\widetilde{P}^{\tilde{V}}$ does not correspond to any h-transform of S killed at time τ.

Informally, S^* can be viewed as S conditioned to drift to ∞ and $\widetilde{S}$ as S conditioned to oscillate. More precisely, using Cramér's estimate [cf. Feller (1971), Section XII.6(d)], one can check that if Hypothesis 1 holds, then

$$P\big(\cdot \mid S \text{ hits } [n,\infty)\big) \text{ converges to } P^*, \quad \text{as } n \to \infty$$

[see also Asmussen (1982)]. Here, the convergence is in the sense of the finite-dimensional distributions. Similarly, using the estimate of Theorem 3(ii) in Doney (1989a), one can check that if Hypothesis 2 holds and if the step distribution of S is not supported by any noncentered lattice (i.e., of the type $\{a + bk, k \in \mathbb{Z}\}$ with $0 < a < b$), then

$$P\big(\cdot \mid S_k \geq 0 \text{ for some } k \geq n\big) \text{ converges to } \widetilde{P}, \quad \text{as } n \to \infty.$$

Now, we make the following claim:

THEOREM 2. *Consider a bounded functional* $f(S) = f(S_0, \dots, S_k)$ *that depends only on the k first steps of S.*

(i) *If Hypothesis* 1 *holds, then*

$$\lim_{n\to\infty} E\big(f(S) \mid \Lambda_n^{(1)}\big) = E^{*V^*}\big(f(S^*)\big).$$

(ii) *If Hypothesis* 2 *holds and the step distribution of S is not supported by any noncentered lattice, then*

$$\lim_{n\to\infty} E\big(f(S) \mid \Lambda_n^{(2)}\big) = \widetilde{E}^{\tilde{V}}\big(f(\widetilde{S})\big).$$

The second part of Theorem 2 was obtained by Keener (1992) in the case of integer-valued random walks. The present approach, which is both simpler and more general, relies on the following lemma.

LEMMA 3. (i) *If Hypothesis* 1 *holds, then*

$$\lim_{n\to\infty} P_x\big(\Lambda_n^{(1)}\big)/P\big(\Lambda_n^{(1)}\big) = \exp(\omega x)V^*(x). \tag{4}$$

(ii) *If Hypothesis* 2 *holds and the step distribution of S is not supported by any noncentered lattice, then*

$$\lim_{n\to\infty} P_x\big(\Lambda_n^{(2)}\big)/P\big(\Lambda_n^{(2)}\big) = \exp(\nu x)\widetilde{V}(x). \tag{5}$$

Here, (4) *and* (5) *hold for every real number* $x \geq 0$ *if the step distribution is not supported by a lattice, and for every* $x = k\lambda$, *with* k *a nonnegative integer if the step distribution is supported by a lattice with maximal span* λ.

PROOF. The first part of the lemma follows readily in the nonlattice case from Lemma 1 of Iglehart (1972), which specifies the asymptotic behaviour of $P_x(\Lambda_n^{(1)})$ when Hypothesis 1 holds. The lattice case is similar; see for instance Lemma A of Karlin and Dembo (1992).

The second part is stated in Theorem II of Doney (1989b), except that (5) is claimed in the nonlattice case only at continuity points of $\widetilde{V}$ [Doney (1989b) makes the additional assumption that $E(S_1) > -\infty$, but actually this is not needed in his demonstration]. So, assume that $\widetilde{V}(x) > \widetilde{V}(x-)$ and introduce $\tau' = \min\{k \geq 0\colon S_k \leq 0\}$. Then, by the definition of $\widetilde{V}$ and with obvious notation,

$$\widetilde{V}(x) - \widetilde{V}(x-) = \widetilde{P}_x(\widetilde{S}_{\widetilde{\tau}'} = 0)$$
$$= \exp(-\nu x)E_x\Big(\exp\big\{-\kappa(\nu)\tau'\big\},\ S_{\tau'} = 0,\ \tau' < \infty\Big).$$

On the one hand, by the Markov property,

$$P_x(\tau > n) = P_x(\tau' > n) + \sum_{k=0}^{n} P_x(\tau' = k, S_k = 0)P(\tau > n - k)\,.$$

Moreover, $P(\tau > n) \sim r(n)\exp\{n\kappa(\nu)\}$, where $r(n)$ is regularly varying at ∞; see, for example, Doney (1989b). We deduce from Fatou's lemma that

$$\liminf_{n\to\infty} \sum_{k=0}^{n} P_x(\tau' = k,\ S_k = 0)P(\tau > n - k)/P(\tau > n)$$
$$\geq \sum_{k=0}^{\infty} P_x(\tau' = k,\ S_k = 0)\exp\big\{-\kappa(\nu)k\big\}$$
$$\geq \exp(\nu x)\big(\widetilde{V}(x) - \widetilde{V}(x-)\big).$$

Because $P_x(\tau' > n) \geq P_y(\tau > n)$ for all $y < x$ and because there exist continuity points $y < x$ of $\widetilde{V}$ that are arbitrarily close to x, we deduce that

$$\liminf_{n\to\infty} P_x(\tau > n)/P(\tau > n) \geq \exp(\nu x)\widetilde{V}(x).$$

On the other hand, $P_x(\tau' > n) \leq P_y(\tau > n)$ for all $y > x$ and because there exist continuity points $y > x$ of $\widetilde{V}$ that are arbitrarily close to x, we deduce that

$$\limsup_{n\to\infty} P_x(\tau > n)/P(\tau > n) \leq \exp(\nu x)\widetilde{V}(x).$$

This proves that (5) holds at x, and the proof of Lemma 3 is complete. □

REMARKS. 1. The asymptotic behaviour of $P(\Lambda_n^{(2)})$ when Hypothesis 2 holds has been specified originally by Iglehart (1974b), Theorem 2.1.

2. The feature that (5) holds for all x in the nonlattice case was observed independently by D. Korshunov (private communication).

3. We also mention that Veraverbeke and Teugels (1975) have obtained an analogous result for the asymptotic behaviour of $P(n < \sigma(x) < \infty)$ when Hypothesis 2 holds.

PROOF OF THEOREM 2. Theorem 2 follows from Lemma 3, arguments based on Fatou's lemma and the Markov property analogous to those of Theorem 1. □

REMARK. When Hypothesis 2 holds, $\widetilde{S}_1$ is in the domain of attraction of the normal law. This hypothesis is required in Doney (1989b) solely in order to use classical large deviation results of Petrov to establish Lemma 4(ii) therein. We point out that whenever $\widetilde{S}_1$ is in the domain of attraction of a stable law of index $\alpha \in (1, 2]$, the result of Lemma 4(ii) is valid and can be established by using local limit theorems such as Theorems 8.4.1 and 8.4.2 of Bingham, Goldie and Teugels (1987). Thus in all such cases, Doney's Theorem II and hence our Theorem 2, part 2, holds. In particular, these results are valid in certain cases where $\kappa(\theta) = \infty$ for every $\theta > \nu$, a situation which is not covered by the results of Keener (1992).

Recall that Hypotheses 1 and 2 are only restrictions on the upper tail of the step distribution. We do not know whether they are necessary for the conditional probabilities $P(\cdot \mid \Lambda_n^{(1)})$ and $P(\cdot \mid \Lambda_n^{(2)})$ to converge toward the law of a strict Markov process. However, we show in the next theorem that some restrictions are indeed needed to get such a convergence.

THEOREM 3. *Assume that* $-\infty < E(S_1) < 0$ *and that*

$$P(S_1 \geq x) = x^{-\alpha} L(x),$$

where $1 < \alpha < \infty$ *and* L *is slowly varying at* ∞. *Then, for every bounded functional* $f(S) = f(S_0, S_1, \ldots, S_k)$, *which depends only on the* k *first steps of* S, *and for every* $0 < K < \infty$, *we have for* $i = 1$ *or* 2,

$$\lim_{n \to \infty} E\big(f(S), S_k < K \mid \Lambda_n^{(i)}\big) = E^V\big(f(S), S_k < K, k < \zeta\big).$$

Recall that P^V is the law of a sub-Markov chain with finite lifetime ζ. More precisely, for every $k > 0$, $P^V(k < \zeta) < 1$, Theorem 3 implies that if the random walk has a negative drift and the upper tail of the step distribution is regularly varying, then conditioning by $\Lambda_n^{(i)}$ ($i = 1$ or 2) induces an explosion as n goes to infinity. In the case $i = 2$, this explosion phenomenon is related to a result of Durrett (1980), who considers the rescaled broken line process $(S_{[tn]}/n, 0 \leq t \leq 1)$ conditioned on $\Lambda_n^{(2)}$.

PROOF. First, assume that $i = 2$. According to Theorem I in Doney (1989b),

$P(\tau > n)$ is regularly varying at ∞ with index $-\alpha$. Moreover,

$$\lim_{n \to \infty} P_x(\tau > n)/P(\tau > n) = V(x) \tag{6}$$

whenever V is continuous at x if the step distribution is not supported by a lattice, and for every $x = k\lambda$, where k is a nonnegative integer, if the step distribution is supported by a lattice with maximal span λ. Arguments analogous to those in Lemma 3(ii) show that (6) then holds for every x in the nonlattice case. It follows that for each fixed integer $k \geq 0$,

$$\lim_{n \to \infty} P_x(\tau > n - k)/P(\tau > n) = V(x).$$

Finally, it is easy to check that the convergence is dominated over compact intervals. The result then follows from the Markov property.

Now assume that $i = 1$. We shall treat only the lattice case, because the arguments can be easily adapted to the nonlattice case. With no loss of generality, we may now assume that S has a lattice distribution with unit span. Plainly, the proof amounts to showing that for every integer $x \geq 0$,

$$\lim_{n \to \infty} P_x\big(\Lambda_n^{(1)}\big)/P\big(\Lambda_n^{(1)}\big) = V(x).$$

Let

$$U(n) = \sum_{k=0}^{n} u(k), \qquad n \geq 0,$$

be the renewal function associated with the defective random variable $S_{\sigma(0)}$ (that is the first weak increasing ladder height of S). Because S drifts to $-\infty$, the process $1_{\{k < \sigma(0)\}} U(-S_k)$ is a P_{x-n}-martingale for all $n > 0$ and $x < n$. By translation, $1_{\{k < \sigma(n)\}} U(n - S_k)$ is a P_x-martingale. Applying Doob's optional sampling theorem at $\sigma(n) \wedge \tau$, we get

$$\begin{aligned} P_x\big(\Lambda_n^{(1)}\big) &= P_x\big(\sigma(n) < \tau\big) \\ &= \frac{1}{U(n-x)} E_x\big(U(n - S_\tau) - U(n - x),\ \tau < \sigma(n)\big). \end{aligned} \tag{7}$$

Because $S_{\sigma(0)}$ is defective, $U(n)$ increases to a finite limit as $n \to \infty$. We will prove below that

$$u(n) \sim c n^{-\alpha} L(n) \quad \text{as } n \to \infty. \tag{8}$$

Then, for every fixed k,

$$\lim_{n \to \infty} \big(U(n + k) - U(n)\big) n^{\alpha} L(n)^{-1} = ck$$

and, plainly, $(U(n + k) - U(n)) n^{\alpha} L(n)^{-1} \leq 2ck$ when n is large enough. Recall that $E(-S_\tau) < \infty$. By dominated convergence, we deduce from (7) that the ratio $P_x(\Lambda_n^{(1)})/P(\Lambda_n^{(1)})$ converges as $n \to \infty$ to

$$\frac{x - E_x(S_\tau)}{E(-S_\tau)} = V(x)$$

(this equality derives from Wald's identity, as we observed in Section 2). Thus, all that we need now is to prove (8).

First, define $v(n)$ for all $n \geq 0$ by the relation $V(n) = \sum_{p=0}^{n} v(p)$. According to the duality lemma,

$$v(n) = \sum_{p=0}^{\infty} P\big(S_p = -n,\ p < \sigma(0)\big).$$

We have

$$\begin{aligned} P\big(S_{\sigma(0)} = n\big) &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} P\big(S_p = -k,\ p < \sigma(0),\ S_{p+1} = n\big) \\ &= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} P(S_1 = n+k) P\big(S_p = -k,\ p < \sigma(0)\big) \\ &= \sum_{k=0}^{\infty} v(k) P(S_1 = n+k). \end{aligned}$$

Put $\mu = E(H_1)$. According to the renewal theorem, $v(k)$ converges to $1/\mu$ as $k \to \infty$. Fix $\varepsilon > 0$ and let K be such that $|v(k) - 1/\mu| < \varepsilon$ for all $k \geq K$. We have

$$\left(\frac{1}{\mu} - \varepsilon\right) P(S_1 > n+K) \leq \sum_{K+1}^{\infty} v(k) P(S_1 = n+k) \leq \left(\frac{1}{\mu} + \varepsilon\right) P(S_1 > n+K)$$

and

$$\sum_{0}^{K} v(k) P(S_1 = n+k) \leq c P(n \leq S_1 \leq n+K) = o(1) P(S_1 > n).$$

Thus,

$$P\big(S_{\sigma(0)} = n\big) \sim \frac{1}{\mu} n^{-\sigma} L(n) \quad \text{as } n \to \infty. \tag{9}$$

Now we show that (8) follows from (9). Denote by $(Z_k, k \geq 0)$ the weak ascending ladder height process of S (so $Z_1 = S_{\sigma(0)}$) and put $\lambda = P(\sigma(0) < \infty)$. We have

$$u(n) = \sum_{k=0}^{\infty} P(Z_k = n) = \sum_{k=0}^{\infty} \lambda^k P(Z'_k = n),$$

where $(Z'_k,\ k \geq 0)$ is a nonnegative random walk whose step distribution coincides with that of $S_{\sigma(0)}$ conditioned on $\sigma(0) < \infty$.

Assume now that $\alpha > 2$. Because by (9), $P(Z'_1 = n) \sim (\lambda\mu)^{-1} n^{-\alpha} L(n)$, $E(Z'_1) = \mu'$ is finite. According to Doney (1989c), $P(Z'_k = n) \sim k P(Z'_1 = n)$ as n goes to ∞, uniformly in $k \leq C \log n$, where $C > 0$ can be chosen arbitrarily large. Therefore,

$$\sum_{k \leq C \log n} \lambda^k P(Z'_k = n) \sim P(Z'_1 = n) \sum_{k \leq C \log n} k\lambda^k \quad \text{as } n \to \infty. \tag{10}$$

On the other hand, when C is chosen large enough,

$$\sum_{k > C \log n} \lambda^k P(Z'_k = n) \le (1-\lambda)^{-1} \lambda^{C \log n} = o\big(n^{-\alpha} L(n)\big). \tag{11}$$

Now (8) follows from (9), (10) and (11).

Similar arguments apply to the case $1 < \alpha \le 2$. More precisely, one has just to show that (10) holds when $E(Z'_1) = \infty$. The proof follows essentially from Corollary 1.5 of Nagaev (1979) and the arguments of Doney (1989c). □

REMARKS. 1. The hypothesis of Theorem 3 can be weakened. We believe that the conclusions hold provided that $\lim_{x \to \infty} P(S_1 > x + 1)/P(S_1 > x) = 1$, but we were not able to prove this in its full generality.

2. We observe that a proof of (4) of Lemma 3 can also be based on equation (7). For example, in the case that S_1 is lattice with span 1, $u_n^* \to 1/m^*$, where $m^* = E^*(S^*_{\sigma^*(0)}) < \infty$ and u_n^* stands for u_n evaluated for the associated random walk S^*. Because $u_n = e^{-n\omega} u_n^*$, we have $u_{n+k}/u_n \to e^{-k\omega}$ for each fixed k, and using this in (7) gives

$$P\big(\Lambda^{(1)}_{n+k}\big)/P\big(\Lambda^{(1)}_n\big) \to \exp(-k\omega). \tag{12}$$

We can then adapt the argument leading from (1) to (2) to show that (12) implies (4). Using this approach also allows us to deal with certain cases where Hypothesis 1 holds except that $E(S_1 e^{\omega S_1}) = \infty$. Specifically, suppose again we have the lattice case and $P^*(S_1^* > n) = n^{-\alpha} L(n)$, where $\alpha \in (\frac{1}{2}, 1)$. Then, by Theorem 1(b) of Veraverbeke (1977), $P^*(S^*_{\sigma^*(0)} > n) \sim c n^{-\alpha} L(n)$ and hence $u_n^* \sim c n^{-\alpha - 1} L(n)$; see Garsia and Lamperti (1963). Thus again $u_{n+k}/u_n \to e^{-k\omega}$, (12) and hence (4) holds and the conclusion of Theorem 2, part 1, is valid.

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