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A SKOROHOD-TYPE LEMMA AND A DECOMPOSITION OF REFLECTED BROWNIAN MOTION[1]

BY KRZYSZTOF BURDZY AND ELLEN TOBY

University of Washington and Texas A&M University

We consider two-dimensional reflected Brownian motions in sharp thorns pointed downward with horizontal vectors of reflection. We present a decomposition of the process into a Brownian motion and a process which has bounded variation away from the tip of the thorn. The construction is based on a new Skorohod-type lemma.

1. Introduction. We will present a construction of reflected Brownian motion (RBM) Y in a two-dimensional thorn pointing downward with horizontal vectors of reflection. In the special case when the thorn is a wedge, the results of Varadhan and Williams [10] show that the process exists, visits the vertex infinitely often and does not have a semimartingale representation. It is also known that if the vectors of reflection in a wedge do not point at each other, then Y has a semimartingale representation if and only if there is a convex combination of the directions of reflection that points up into the wedge away from the corner (see [11]). In a sense, we are dealing with the critical case in this paper. Our construction of RBM Y_t in a thorn shows that when the thorn is a wedge, Y_t can be decomposed into a sum of Brownian motion X_t and a continuous process $(K_t, 0)$ which is of bounded variation strictly inside the excursion intervals from the vertex. This type of decomposition may be of interest because the stochastic calculus has been generalized to some extent to processes which are sums of martingales and continuous processes with zero quadratic variation (see, e.g., [6] and [8]). We do not know whether our RBM belongs in this class of processes. We will examine the variation of K_t over a single Y-excursion interval from the vertex—it may be finite or infinite depending on the thickness of the thorn.

The main idea of the construction of RBM in a thorn is based on a new Skorohod-type lemma. We start with a review of the classical result.

Suppose that $z \geq 0$ and $x(\cdot) = \{x(t);\ 0 \leq t < \infty\}$ is a real-valued continuous function such that $x(0) = 0$. Skorohod's lemma states that there exists a unique continuous function $k(\cdot) = \{k(t);\ 0 \leq t < \infty\}$ such that:

1. $y(t) \equiv z + x(t) + k(t) \geq 0$.
2. $k(0) = 0$, $k(\cdot)$ is nondecreasing.
3. $k(\cdot)$ is flat off $\{t \geq 0;\ y(t) = 0\}$.

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This function is given by

$$k(t) = \max\left[0, \max_{0 \le s \le t}\{-(z + x(s))\}\right], \qquad 0 \le t < \infty.$$

In the case where $x(\cdot)$ is standard Brownian motion starting from $z \ge 0$, the process $y(\cdot)$ is equivalent in law to $|z + x(\cdot)|$. In particular, y is a semimartingale representation of RBM in $\mathbb{R}_+ \equiv [0, \infty)$.

Skorohod's lemma is an analytic result not relying on Brownian motion. Therefore it is reasonable to conjecture that a two-dimensional version of Skorohod's lemma might exist for thorns, even in cases where it would give an expression for RBM when no semimartingale representation exists. In these cases the function analogous to $k(\cdot)$ would not be locally of bounded variation.

We prove a "two-sided" version of Skorohod's lemma for functions $X(\cdot)$ taking values in $\mathbb{R}_+^2 \equiv \{(x, y)\colon\ y \ge 0\}$. Suppose that L and R are continuous real functions defined on $[0, \infty)$ and such that $L(0) = R(0) = 0$ and $L(y) < R(y)$ for all $y > 0$. Let D be given by

$$D \equiv \{(x, y) \in \mathbb{R}^2\colon\ y \ge 0,\ L(y) \le x \le R(y)\}. \tag{1.1}$$

Let $\partial D^1 = \{(x, y) \in \partial D\colon\ x = L(y)\}$, $\partial D^2 = \{(x, y) \in \partial D\colon\ x = R(y)\}$ and let D^0 be the interior of D.

We remark parenthetically that the assumption of continuity of L and R is not used in this paper in an essential way. We imposed it in order to avoid jumps in the reflected Brownian motion in D. Burdzy and Marshall [1] discuss some domains in which RBM may have jumps on the boundary.

THEOREM 1. *Suppose L, R and D satisfy the above conditions. Let $z \in D$ be a given point and suppose $X\colon \mathbb{R}_+ \to \mathbb{R}_+^2$ is a continuous (nonrandom) function with $X(0) = z$. Then there exists a unique continuous real-valued function $K(\cdot) = \{K(t);\ 0 \le t < \infty\}$, such that:*

(i) $Y(t) \equiv X(t) + (K(t), 0) \in D;\ 0 \le t < \infty$.

(ii) $K(0) = 0$, *$K(\cdot)$ is nondecreasing on $\{t \ge 0\colon\ Y(t) \notin \partial D^2\}$ and is nonincreasing on $\{t \ge 0\colon\ Y(t) \notin \partial D^1\}$.*

(iii) *$K(\cdot)$ is constant on each maximal time interval in $\{t \ge 0\colon\ Y(t) \in D^0\}$.*

Without loss of generality we will restrict to processes X that are reflecting Brownian motions in the upper half-space. Such a process can be obtained from an ordinary Brownian motion starting in the upper half-space by applying the Skorohod reflection mapping (discussed at the beginning of the Introduction) to the vertical component of the Brownian motion.

Theorem 1 is a purely function-theoretic result. Let $X = (X_1, X_2)$. Defining $K(t)$ up until the first time $X_2(t) = 0$ is straightforward. One can use a simple modification of Skorohod's lemma to define the push necessary to keep the function inside D if we have to deal with only one part of the boundary ∂D^1 or ∂D^2 at a time. Since X is continuous, the time that elapses between the consecutive visits of Y to the opposite sides of the boundary ∂D^1 and ∂D^2 is strictly positive as long as X_2 stays away from 0. The difficulty arises, for example, when we try to define K for functions X starting from $(0,0)$. We may need an infinite amount of push in both directions just after the clock starts. We will show in Theorem 3(i) that this possibility is indeed realized in some cases. It seems that one cannot define K in such cases using an elegant formula. Instead, we will define it as the limit of an approximating sequence.

Generally speaking, an RBM is thought of as a continuous process in a domain G which behaves like Brownian motion in the interior of G and reflects instantaneously at the boundary of G in some given direction. There are many precise mathematical definitions of an RBM which incorporate these properties. One frequently used definition is to describe RBM as a family $\{P_x\colon x \in D\}$ of solutions of a submartingale problem [see (3.1)–(3.3) below].

Suppose $X(\cdot)$ is RBM in $\mathbb{R}^2_+$, D is a closed set satisfying (1.1) and $Y(\cdot)$ is a pathwise transformation of X defined as in Theorem 1. Let Ω_D be the set of continuous functions ω from $[0,\infty)$ into D. Let $\mathscr{M}$ denote $\sigma\{Z_t\colon 0 \le t < \infty\}$, where $Z_t(\omega) \equiv \omega(t)$, $0 \le t$. Denote by Q_z the probability measure on $(\Omega_D, \mathscr{M})$ which is the law of $Y(\cdot)$ in D starting from z.

THEOREM 2. *Let Y and the family $\{Q_z\colon z \in D\}$ be as defined above. Then $\{Q_z\colon z \in D\}$ is a solution to the submartingale problem on D starting from $z \in D$. Moreover, Y is a continuous process that behaves in the interior of D like a Brownian motion and it is confined to D by reflection at the boundary in horizontal directions on $\partial D \setminus \{0\}$ and vertical reflection at the origin. Furthermore the origin is a point of positive recurrence and Y spends zero Lebesgue time there.*

Consider a wedge $D \subset \mathbb{R}^2_+$ with neither side horizontal. For any fixed $t > 0$, it follows from the results of Varadhan and Williams [10] that on $[0,t]$ the total variation of K is not locally of bounded variation if Y visits 0 at some time in this interval. It follows from Proposition 1 below that the same is true for every domain D_1 contained in such a wedge. However, this does not mean that K is of unbounded variation during any single excursion from 0 by Y.

THEOREM 3. *Suppose X is RBM in $\mathbb{R}^2_+$, L, R and D satisfy* (1.1) *and K and Y are defined as in Theorems* 1 *and* 2.

(i) *Suppose there exists $\varepsilon > 0$ such that*

$$R(y) - L(y) \le y^2$$

for $y \in (0,\varepsilon)$. Then the total variation of $K(\cdot)$ during any single excursion of Y from 0 *is infinite a.s.*

(ii) *Suppose there exist $a < 2$ and $\varepsilon > 0$ such that*

$$R(y) - L(y) \ge y^a$$

for $y \in (0,\varepsilon)$. Moreover, assume that both L and R are Lipschitz functions. Then the total variation of $K(\cdot)$ during any single excursion of Y from 0 *is finite a.s.*

Suppose that (a, b) is an interval of a Y-excursion from 0, that is, $Y(a) = Y(b) = 0$, but $Y(t)$ is not equal to 0 for $t \in (a, b)$. It is easy to see that the total variation of K is finite on every interval (a_1, b_1), where $a < a_1 < b_1 < b$.

It is not hard to construct an example showing that Theorem 3(ii) is false without the Lipschitz assumption on L and R. However, Proposition 1 below shows that the conclusion of Theorem 3(ii) holds under a much weaker assumption. We will show that the total variation of K is monotonic with respect to regions. For any $0 \le s < t < \infty$ the total variation of K over the time interval $[s, t]$ will be denoted by $\hat{K}[s, t]$. Suppose that (L_α, R_α) and (L_β, R_β) are two pairs of functions satisfying the same conditions as L and R and let D_α and D_β be the corresponding domains defined by (1.1).

PROPOSITION 1. *Suppose D_α and D_β are defined as above and $D_\alpha \subset D_\beta$. Assume that $z \in D_\alpha$ and X: $\mathbb{R}_+ \to \mathbb{R}_+^2$ is a continuous function with $X(0) = z$. Let $K_\alpha(\cdot)$ [resp. $K_\beta(\cdot)$] denote the function defined in Theorem 1 for X and the set D_α (resp. D_β). Suppose that (a, b) is a maximal open interval in the set $\{t > 0\colon X_2(t) > 0\}$, that is, (a, b) is an interval of a Y-excursion from 0. Then*

$$\hat{K}_\beta[a, b] \le \hat{K}_\alpha[a, b].$$

For $a > 1$ let

$$C_a = \{(x, y)\colon\ y \ge |x|^{1/a}\}. \tag{1.2}$$

DeBlassie and Toby [4, 5] proved that RBM in C_a starting from the origin with normal direction of reflection has a semimartingale representation if and only if $a < 2$. The RBM Y in C_a with normal direction of reflection and starting from 0 never returns to this point, so we can think of it as a single excursion from 0. The result of DeBlassie and Toby says that if $a < 2$, then Y can be represented as $Y = X + K$, where X is a Brownian motion and K is a process with a finite variation on any interval $(0, t)$, where $t < \infty$. If a similar decomposition of Y existed for $a \ge 2$, the process K would have to have infinite variation over any interval $(0, t)$, where $t > 0$. At an intuitive level, our Theorem 3 is very close to the result of DeBlassie and Toby because the normal vector in C_a is very close to horizontal in a small neighborhood of 0. In our case, C_2 is also the critical domain. The methods used in both papers are quite different, though. The approach of DeBlassie and Toby is based on Itô's formula and some very accurate estimates for analytic functions. As a result, it is limited to thorns C_a defined in (1.2).

We prove Theorems 1–3 and Proposition 1 in Sections 2–5.

The constants $c_1, c_2, \dots$ will be always strictly positive and finite. They may change values from one proof to another.

2. A Skorohod-type lemma. First will be given a construction of the function $K(t)$ satisfying (i)–(iii) in Theorem 1. Next, $K(t)$ will be shown to be the unique function with such properties. Assume L, R, $z = (z_1, z_2)$, D and

$X(t) = (X_1(t), X_2(t))$ satisfy the assumptions of Theorem 1. Let $B = \{t \geq 0\colon\ X_2(t) = 0\}$ and let A denote the set of left endpoints of the maximal open intervals in $(0,\infty)\backslash B$. The set A is countable. If it does not contain 0, we may represent it as $A = \{a_n\colon\ n > 0\}$ (a_n's are not ordered in any particular way). If 0 is an element of A, then we will let $A = \{a_n\colon\ n \geq 0\}$ with $a_0 = 0$. If the set A is finite, we set $a_n = \infty$ for any n greater than the number of elements in A. Let

$$g_t = \sup\{s \leq t\colon\ s \in A\}, \qquad \sup\varnothing = -\infty,$$

$$d_t = \inf\{s > t\colon\ s \in B\}, \qquad \inf\varnothing = \infty,$$

$$T_0 = \inf\{t \geq 0\colon\ X_2(t) = 0\}.$$

For each $t \notin B$, there corresponds exactly one n such that $a_n = g_t$. If n and t are related in this way, let $b_n = d_t$.

We will define K as the uniform limit of a sequence of continuous functions K_i. Further, each K_i will be defined as a countable sum of functions, K_i^n. Each $K_i^n(\cdot)$ will be identically zero outside (a_n, b_n). The function $K_i^n(\cdot)$ represents the amount of push in both directions which is necessary to add to the excursion of X over (a_n, b_n) in order to keep Y inside D. The idea of its definition is quite simple (it is essentially identical to that in the classical Skorohod lemma), but the formal definition is quite complicated. Neverthless we feel obliged to present it.

Fix some $n > 0$. Find ε_i such that $R(\varepsilon_i) - L(\varepsilon_i) = 2^{-i}$ and $R(y) - L(y) \leq 2^{-i}$ for all $y \in [0, \varepsilon_i]$. By the definition of D, if $i \to \infty$, then $\varepsilon_i \to 0$. The characteristic function of a set U will be denoted by I_U. Let

$$\tau_i^n = \inf\{r > a_n\colon\ X_2(r) \geq \varepsilon_i\},$$

$$l_i^n(t) = I_{B^c \cap (a_n, \tau_i^n \wedge b_n]}(t)\,(L(X_2(t)) - X_1(t)).$$

Below will be defined $\sigma_m^{i,n}$, $u_m^{i,n}$ and $K_l^{i,n}$. We shall write τ, σ_m, u_m and K_l for τ_i^n, $\sigma_m^{i,n}$, $u_m^{i,n}$ and $K_l^{i,n}$ in cases where there is no ambiguity. Let

$$\sigma_0^{i,n} = \tau_i^n,$$

$$u_1^{i,n}(t) = L(X_2(\tau)) - X_1(\tau) + \sup_{\tau \leq r \leq t}\{L(X_2(r)) - X_1(r) - L(X_2(\tau)) + X_1(\tau)\} \vee 0,$$

$$\sigma_1^{i,n} = \inf\{r > \tau\colon\ X(r) + (u_1(r), 0) \in \partial D^2\},$$

$$u_2^{i,n}(t) = \sup_{\sigma_1 \leq r \leq t}\{X_1(r) - R(X_2(r)) + u_1(\sigma_1)\} \vee 0,$$

$$K_l^{i,n}(t) = l_i^n(t) + \sum_{m=1}^{l}(-1)^{m+1} u_m(\sigma_m \wedge t) I_{(\sigma_{m-1}, b_n)}(t),$$

where if m is odd,

$$u_m(t) = \sup_{\sigma_{m-1}\le r\le t} \{L(X_2(r)) - X_1(r) - K_{m-1}(\sigma_{m-1})\} \vee 0,$$

$$\sigma_m = \inf\{r > \sigma_{m-1}\colon\ X(r) + (K_{m-1}(\sigma_{m-1}) + u_m(r), 0) \in \partial D^2\},$$

but if m is even,

$$u_m(t) = \sup_{\sigma_{m-1}\le r\le t} \{X_1(r) - R(X_2(r)) + K_{m-1}(\sigma_{m-1})\} \vee 0,$$

$$\sigma_m = \inf\{r > \sigma_{m-1}\colon\ X(r) + (K_{m-1}(\sigma_{m-1}) - u_m(r), 0) \in \partial D^1\}.$$

We now consider the case where $n = 0$. If $z = (0,0)$, then define $K_m^{i,0}(t)$ in the same way as for other n. Otherwise let

$$\tau_i^0 = \inf\{t \ge 0\colon\ X(t) \in \partial D\},$$

$$\sigma_0^{i,0} = \tau_i^0,$$

$$u_1^{i,0}(t) = I_{\{X(\sigma_0)\in\partial D^1\}} \sup_{0\le r\le t} \{L(X_2(r)) - X_1(r)\} \vee 0,$$

$$\sigma_1^{i,0} = \inf\{r \ge 0\colon\ X(r) + (u_1(r), 0) \in \partial D^2\},$$

$$K_l^{i,0}(t) = \sum_{m=1}^{l} (-1)^{m+1} u_m(\sigma_m \wedge t) 1_{(\sigma_{m-1}, b_n)}(t).$$

The rest of the definitions are the same as for when $n > 0$.

For $n \ge 0$ let

$$Y_m^{i,n}(t) = X(t) + (K_m^{i,n}(t), 0).$$

If $t \in (a_n, \tau_i^n]$, then $Y_m^{i,n}(t)$ is a continuous function which takes values in D (if $n > 0$, it takes values in ∂D^1). When $t \in (\tau_i^n, \sigma_m^{i,n} \wedge b_n)$, then $Y_m^{i,n}(t)$ is a continuous function taking values in D and $\lim_{t\to\tau^+} Y_m^{i,n}(t) = Y_m^{i,n}(\tau)$. We will show that

$$\lim_{m\to\infty} \sigma_m^{i,n} \ge b_n.$$

To see this, assume that $\lim_{m\to\infty} \sigma_m^{i,n} = c < b_n$. Let $X_2(c) = y_0$. Since X is continuous and $y_0 > 0$, we can find $c_1 < c$ and $\gamma, \gamma_1 > 0$ such that $|X_2(t) - y_0| < \gamma_1$ for $t \in [c_1, c]$ and

$$\sup_{y\in[y_0-\gamma_1, y_0+\gamma_1]} L(y) < \inf_{y\in[y_0-\gamma_1, y_0+\gamma_1]} R(y) - \gamma.$$

For m odd let

$$\tilde{\sigma}_m = \sup\{r < \sigma_m\colon\ X(r) + (K_{m-1}(\sigma_{m-1}) + u_m(r), 0) \in \partial D^1\},$$

but if m is even,

$$\tilde{\sigma}_m = \sup\{r < \sigma_m\colon\ X(r) + (K_{m-1}(\sigma_{m-1}) - u_m(r), 0) \in \partial D^2\}.$$

Note that the intervals $(\tilde{\sigma}_m, \sigma_m)$ are disjoint and u_m is constant on $(\tilde{\sigma}_m, \sigma_m)$. It follows that

$$|X_1(\tilde{\sigma}_m) - X_1(\sigma_m)| > \gamma$$

for large m. Hence, X is not continuous at c. This contradiction proves our claim.

For each n and i let

$$K^{i,n}(t) = \lim_{m\to\infty} K_m^{i,n}(t).$$

It is evident that for $t \in (a_n, b_n)$, $X(t) + (K^{i,n}(t), 0)$ is a continuous function taking values in D. Recall that for $t \notin (a_n, b_n)$, $K^{i,n}(t)$ is identically 0.

For all $t \geq 0$ let

$$K^i(t) = \sum_{n=1}^{\infty} K^{i,n}(t) - I_B(t) X_1(t),$$

$$Y^i(t) = (Y_1^i(t), Y_2^i(t)) = (X_1(t), X_2(t)) + (K^i(t), 0).$$

Since $\lim_{y\to 0^+}(R(y) - L(y)) = 0$, the functions $Y^i(t)$ are continuous for all $t \geq 0$.

LEMMA 1. *Let $K^{i,n}$, K^i and Y^i be defined as above. For any i and n:*

(i) *$K^i(t)$ is constant on each maximal time interval in $\{t\colon Y^i(t) \in D^0\}$.*

(ii) *$K^i(t)$ is nondecreasing for $\{t \in (\tau_i^n, b_n)\colon Y^i(t) \notin \partial D^2\}$ and nonincreasing for $\{t \in (\tau_i^n, b_n)\colon Y^i(t) \notin \partial D^1\}$.*

(iii) *If $j < i$ and there exists some $t_0 \in [\tau_j^n, b_n)$ such that $Y^i(t_0) = Y^j(t_0)$, then $Y^i(s) = Y^j(s)$ for all $s \in [t_0, b_n)$.*

PROOF. From the definition of $u_m^{i,n}(t)$ we see that $K^i(t)$ is nondecreasing for $\{t \in (\tau_i^n, b_n)\colon Y^i(t) \notin \partial D^2\}$ and nonincreasing for $\{t \in (\tau_i^n, b_n)\colon Y^i(t) \notin \partial D^1\}$, which proves part (ii).

For (i) choose an arbitrary $\delta > 0$ and set

$$D_\delta^0 = \{(x, y)\colon L(y) + \delta < x < R(y) - \delta\}.$$

It suffices to show K^i is flat whenever $Y^i \in D_\delta^0$. Let (s_1, s_2) be a subset of $\{t\colon Y^i(t) \in D_\delta^0\}$. Either there exists $m \geq 0$ such that $(s_1, s_2) \subset (\tau_i^m, b_m)$ or $z \in D \setminus \partial D$ and $s_2 < \tau_i^0$. However, in this latter case, $K^{i,0}(t) = 0$ for all $t \in (s_1, s_2)$. So we will only concern ourselves with the former case. Unless $n = m$, $K^{i,n}(t) \equiv 0$ for all $t \in (s_1, s_2)$. Therefore, for $t \in (s_1, s_2)$, $Y_1^i(t) = X_1(t) + K^{i,m}(t)$. Let $N = 1 + \sup\{n\colon \sigma_n^{i,m} < s_1\}$. Then $\sigma_{N-1}^{i,m} < s_1 < s_2 < \sigma_N^{i,m}$. Suppose first that N is odd. This implies that

$$K_N^{i,m}(s_2) - K_N^{i,m}(s_1) = u_N^{i,m}(s_2) - u_N^{i,m}(s_1) \geq 0.$$

In what follows we write $u_N(t)$ for $u_N^{i,m}(t)$. Note that

$$\begin{aligned} u_N(s_2) &= \max\Big\{u_N(s_1), \sup_{s_1\le r\le s_2}\{L(X_2(r)) - X_1(r) - K_N^{i,m}(\sigma_{N-1})\}\Big\} \\ &= \max\Big\{u_N(s_1), \sup_{s_1\le r\le s_2}\{L(X_2(r)) - Y_1^i(r) + u_N(r)\}\Big\} \\ &\le \max\Big\{u_N(s_1), \sup_{s_1\le r\le s_2}\{L(X_2(r)) - Y_1^i(r) + u_N(s_2)\}\Big\} \\ &\le \max\{u_N(s_1), u_N(s_2) - \delta\}. \end{aligned}$$

The last line follows from $Y_1^i(r) - L(X_2(r)) \ge \delta$ when $Y^i(t) \in D_\delta^0$. Hence, $u_N^{i,m}(s_2) = u_N^{i,m}(s_1)$ and, therefore, $K_N^{i,m}(s_2) = K_N^{i,m}(s_1)$. A similar argument holds when N is even. Therefore, $K^{i,m}(t)$, and hence $K^i(t)$, is flat on (s_1, s_2). This completes the proof of part (i).

For part (iii), suppose $j < i$ and $Y^i(t_0) = Y^j(t_0)$ for some $t_0 \in [\tau_j^n, b_n)$. We may suppose that t_0 is the infimum of t with this property. Then, by (i), we know $Y^i(t_0) \in \partial D$. Assume $Y^i(t_0) \in \partial D^2$. Then there exist l, m even such that

$$\begin{aligned} K^i(t_0) &= K_l^{i,n}(t_0) = K_{l-1}^{i,n}(\sigma_{l-1}) - u_l^{i,n}(t_0), \\ K^j(t_0) &= K_m^{j,n}(t_0) = K_{m-1}^{j,n}(\sigma_{m-1}) - u_m^{j,n}(t_0). \end{aligned}$$

Below we will show for any $t \in (t_0, \min\{\sigma_l^{i,n}, \sigma_m^{j,n}\})$ that $K^i(t) = K^j(t)$. This implies that $\sigma_l^{i,n} = \sigma_m^{j,n}$, which in turn implies (iii). Suppose $t \in (t_0, \min\{\sigma_l^{i,n}, \sigma_m^{j,n}\})$ and let $K_{m-1}^{j,n}(\sigma_{m-1})$ be abbreviated as S_{m-1}^j:

$$\begin{aligned} K_m^{j,n}(t) &= S_{m-1}^j - \sup_{\sigma_{m-1}\le r\le t}\{X_1(r) + S_{m-1}^j - R(X_2(r))\} \vee 0 \\ &= S_{m-1}^j - \max\Big\{u_m(t_0), \sup_{t_0\le r\le t}\{X_1(r) + S_{m-1}^j - R(X_2(r))\}\Big\} \\ &= S_{m-1}^j + \min\Big\{-u_m(t_0), -\sup_{t_0\le r\le t}\{X_1(r) + S_{m-1}^j - R(X_2(r))\}\Big\} \\ &= \min\Big\{S_{m-1}^j - u_m(t_0), S_{m-1}^j + \inf_{t_0\le r\le t}\{R(X_2(r)) - X_1(r) - S_{m-1}^j\}\Big\} \\ &= \min\Big\{K_m^{j,n}(t_0), \inf_{t_0\le r\le t}\{R(X_2(r)) - X_1(r)\}\Big\}. \end{aligned}$$

Similarly,

$$K_l^{i,n}(t) = \min\Big\{K_l^{i,n}(t_0), \inf_{t_0\le r\le t}\{R(X_2(r)) - X_1(r)\}\Big\}.$$

However, $K_l^{i,n}(t_0) = K_m^{j,n}(t_0)$, implying $K^i(t) = K^j(t)$ for all $t \in (t_0, \min\{\sigma_l^{i,n}, \sigma_m^{j,n}\})$ and so by the above remarks, this shows (iii). □

PROOF OF THEOREM 1. Let

$$K(t) \equiv \lim_{i\to\infty} K^i(t), \qquad t \geq 0.$$

First we will prove that $K(t)$, $t \geq 0$, is well defined and continuous by showing that $\{K^i\}$ is a uniformly converging Cauchy sequence. Given $\varepsilon > 0$, choose M such that $2^{-M} < \varepsilon$. Suppose $i > j > M$, $t \geq 0$ and consider

$$|K^i(t) - K^j(t)|. \tag{2.1}$$

If $t \notin (\tau_i^n, b_n)$ for all $n \geq 0$, then (2.1) is equal to zero. Suppose this is not the case and $t \in (\tau_i^n, b_n)$ for some n. If $t \in (\tau_i^n, \tau_j^n]$, then because $i > j > M$,

$$|K^i(t) - K^j(t)| \leq R(X_2(t)) - L(X_2(t)) \leq 2^{-j} < \varepsilon.$$

Now consider $t \in [\tau_j^n, b_n)$. Since $Y^j(\tau_j^n) \in \partial D^1$, $K^j(\tau_j^n) \leq K^i(\tau_j^n)$. Let $\tau_0 = \inf\{t \geq \tau_j^n\colon Y^j(t) \in \partial D^2$ or $Y^i(t) \in \partial D^1\}$. Then $K^j(\tau_0) - K^i(\tau_0) = Y_1^j(\tau_0) - Y_1^i(\tau_0) \geq 0$ on $\{\tau_0 < b_n\}$. By Lemma 1(iii), $K^j \equiv K^i$ on $[\tau_0, b_n)$ (since $Y^j \equiv Y^i$ somewhere on $[\tau_0, b_n)$ if $\tau_0 < b_n$). On $[\tau_j^n, \tau_0)$ the only times t at which $K^i(t) - K^j(t)$ can change are when $Y^j(t) \in \partial D^1$ or $Y^i(t) \in \partial D^2$. However, during these times the difference can only decrease. Because $K^j(\tau_j^n) - K^i(\tau_j^n) \leq 2^{-j} < \varepsilon$, the preceding remarks imply $|K^i(t) - K^j(t)| < \varepsilon$ for all $t \in [\tau_j^n, b_n)$.

For (iii) choose an arbitrary $\delta > 0$ and recall the definition of D_δ^0 from Section 2. Let (r_1, r_2) be any subset of $\{t\colon Y(t) \in D_\delta^0\}$. Because $\{K^i\}$ is a uniformly converging sequence, there exists N so that for all $i > N$, $|K^i(t) - K(t)| < \delta/2$. For such i and for all $t \in (r_1, r_2)$, $Y^i(t) \in D_{\delta/2}^0$. By Lemma 1(i), K^i is constant in (r_1, r_2) for all $i > N$. This together with uniform convergence implies K is constant in (r_1, r_2). Because δ is arbitrary, (iii) follows.

To show $K(\cdot)$ is nonincreasing on every maximal open interval in $\{t \geq 0\colon Y(t) \notin \partial D^1\}$ it suffices to show for any $\eta > 0$ that $K(\cdot)$ is nonincreasing on every maximal open interval in $\{t \geq 0\colon Y \in E\}$, where $E = \{(x, y) \in \mathbb{R}\colon y \geq 0, L(y) + \eta < x \leq R(y)\}$. Let s_1, s_2 be arbitrarily chosen so that $[s_1, s_2] \subset \{t \geq 0\colon Y(t) \in E\}$. Then for some n, $[s_1, s_2] \subset (a_n, b_n)$. Moreover, because $L(y)$ and $R(y)$ are continuous functions with $L(0) = R(0)$, there exists an i such that $\tau_i^n \leq s_1$. Pick $j > i$ so that $|K(t) - K^j(t)| < \eta/2$ for all t. Then $Y^j(t) \notin \partial D^1$ for $t \in [s_1, s_2]$. By Lemma 1(ii), $K^j(\cdot)$ is nonincreasing on $[s_1, s_2]$. By the uniform convergence of the K^j to K, the function $K(\cdot)$ is nonincreasing on $[s_1, s_2]$. Since s_1, s_2 were arbitrarily chosen, this proves $K(\cdot)$ is nonincreasing on every maximal open interval in $\{t \geq 0\colon Y \in E\}$ and thus nonincreasing on maximal open intervals in $\{t \geq 0\colon Y(t) \notin \partial D^1\}$. The proof that $K(\cdot)$ is nondecreasing on $\{t \geq 0\colon Y(t) \notin \partial D^2\}$ follows from similar reasoning.

Last we show uniqueness. Suppose that $Y(t) \equiv X(t) + (K(t), 0)$ and $Z(t) \equiv X(t) + (H(t), 0)$ both satisfy conditions (i), (ii) and (iii) of the theorem. First note that by the geometry of D, if $t \in B$, then $Y(t) = Z(t)$. Suppose now that there exists n and $t_0 \in (a_n, b_n)$ such that $Y(t_0) \neq Z(t_0)$. Let

$$S = \sup\{t < t_0\colon Y(t) = Z(t)\}.$$

Because both K and H are flat off ∂D, $Y(S) = Z(S) \in \partial D$. Without loss of generality, suppose that $Y_1(t) < Z_1(t)$ on $(S, t_0]$. Therefore, $Z(t) \notin \partial D^1 \cup \{0\}$ for all $t \in (S, t_0]$ and so H is nonincreasing on $[S, t_0]$. Similar reasoning will show that K is nondecreasing on $[S, t_0]$. However, $K(S) = H(S)$ and the above implies $Y_1(t) > Z_1(t)$ on $(S, t_0]$. This contradicts our original premise that $Y_1(t) < Z_1(t)$ on $(S, t_0]$. □

3. Reflected Brownian motion in D. Let Ω_D be the set of continuous functions ω from $[0, \infty)$ into D. For $t \geq 0$ let $\mathscr{M}_t$ be the σ-algebra of subsets of Ω_D generated by the coordinate maps $Z_s(\omega) \equiv \omega(s)$, $0 \leq s \leq t$. Let $\mathscr{M}$ denote $\sigma\{Z_t\colon 0 \leq t < \infty\}$. Let $C_b^2(D)$ be the set of real-valued continuous functions that are defined and twice continuously differentiable on some domain containing D and that together with their first and second partial derivatives are bounded on D. Denote by Q_z the probability measure on $(\Omega_D, \mathscr{M})$ which is the law of $Y(\cdot)$ in D starting from z when X is a reflecting Brownian motion in $\mathbb{R}_+^2$.

Proof of Theorem 2. We will follow the approach of Stroock and Varadhan [9] and show that the family $\{Q_z\colon z \in D\}$ is a solution to the submartingale problem on D starting from $z \in D$. To accomplish this we must show that the following hold for each $z \in D$:

$$Q_z(\omega(0) = z) = 1; \tag{3.1}$$

for each $f \in C_b^2(D)$,

$$f(\omega(t)) - 1/2 \int_0^t \Delta f(\omega(s))\, ds \tag{3.2}$$

is a Q_z-submartingale on $(\Omega_D, \mathscr{M}, \{\mathscr{M}_t\})$ whenever f is constant in a neighborhood of 0 and $v_i \cdot \nabla f \geq 0$ on ∂D^i, where $v_i = (-1^{(i+1)}, 0)$ and $i = 1, 2$;

$$E^{Q_z}\left[\int_0^\infty I_{\{0\}}(\omega(s))\, ds\right] = 0. \tag{3.3}$$

Equations (3.1) and (3.3) follow immediately from the definition of Y because $K(0) = 0$ and $X_2(\cdot)$ spends 0 Lebesgue time at zero.

For (3.2), let $f \in C_b^2(D)$ be given, where f satisfies $v_i \cdot \nabla f \geq 0$ on ∂D^i and f is constant in a neighborhood of 0. Then there is some $\varepsilon > 0$ such that $f(x, y) = C$ for all $y \leq \varepsilon$. We want to apply Itô's formula to $f(Y(t))$. In order to do so, there needs to be a common filtration to which X, K and Y are adapted and relative to which X is a martingale. Consider the filtration generated by X. Denote it by $\{\mathscr{H}_t\}$. It is easily seen for each i that $K^i(t)$ is adapted to $\{\mathscr{H}_t\}$. Because K is the pointwise limit of the K^i, K and Y are also adapted to $\{\mathscr{H}_t\}$. The other consideration before using Itô's formula is the total variation of K. However, on the time intervals where $X_2(t) > \varepsilon$, $K(t)$ is locally of bounded variation. Therefore, we may apply Itô's formula to $f(Y(t))$. Recall that $v_i \cdot \nabla f \geq 0$ on ∂D^i, where $v_i = (-1^{(i+1)}, 0)$ and $i = 1, 2$ so that $\nabla f(Y_s)\, dK_s = |f_x(Y_s)|\, d|K_s|$. We may assume that $X_t = B_t + L_t$, where B is a standard Brownian motion and L is the local time of the vertical component of X at 0. Note that L is constant on the maximal open intervals in $\{t: X_2(t) > 0\}$ and ∇f is equal to 0 in a neighborhood of 0. Therefore, the term corresponding to L does not appear in the Itô formula given below. If $t \in (a_n, b_n)$ for some n, then we let $g_t = a_n$. Otherwise let $g_t = t$. We have

$$f(Y(t)) = f(Y(0)) + \int_0^t \nabla f(Y_s)\, dB_s + \sum_{b_n < g_t} \int_{a_n}^{b_n} |f_x(Y_s)|\, d|K_s|$$

$$+ \int_{g_t}^t |f_x(Y_s)|\, d|K_s| + 1/2 \int_0^t \Delta f(Y(s))\, ds.$$

Hence we may conclude that

$$f(Y(t)) - 1/2 \int_0^t \Delta f(Y(s))\, ds$$

is a submartingale.

The remaining claims of the theorem follow directly from the pathwise construction of Y from a reflected Brownian motion X in the upper half-plane. □

4. Monotonicity of $\hat{K}$ as a function of the domain.

PROOF OF PROPOSITION 1. Suppose that L_α, R_α, L_β, R_β, D_α and D_β satisfy the assumptions of the proposition. The definitions associated with D_α (resp. D_β) will be distinguished by an α (resp. β) subscript, for example, $Y_\alpha(t) \equiv X(t) + (K_\alpha(t), 0)$.

Consider a maximal open interval (a, b) in $\{t > 0: X_2(t) > 0\}$. Let $\{[u_k, w_k]\}$ be a family of intervals indexed by k in a finite or infinite subset of the integers with the following properties. For $t \in (u_k, w_k)$, $Y_\beta(t) \notin \partial D_\beta^1 \cup \partial D_\beta^2$; for even k, $Y_\beta(u_k) \in \partial D_\beta^1$, $Y_\beta(w_k) \in \partial D_\beta^2$ and $Y_\beta(t) \notin \partial D_\beta^1$ for $t \in [w_k, u_{k+1}]$; for odd k, $Y_\beta(u_k) \in \partial D_\beta^2$, $Y_\beta(w_k) \in \partial D_\beta^1$ and $Y_\beta(t) \notin \partial D_\beta^2$ for $t \in [w_k, u_{k+1}]$. We will assume that k may take any integer value, that $\lim_{k\to\infty} u_k = b$ and that $\lim_{k\to-\infty} u_k = a$. The other cases require only minor modifications. The second coordinates of Y_β and Y_α are identical, $D_\alpha \subset D_\beta$ and the process Y_β has to go from one side of the boundary of D_β to the other side on the interval $[u_k, w_k]$. Hence there must exist $v_k \in [u_k, w_k]$ such that $Y_\alpha(v_k) = Y_\beta(v_k)$. It follows that $K_\alpha(v_k) - K_\alpha(v_{k-1}) = K_\beta(v_k) - K_\beta(v_{k-1})$ for every k. By Theorem 1(ii), K_β is monotone on $[v_{k-1}, v_k]$ and so

$$\hat{K}_\beta[v_{k-1}, v_k] = |K_\beta(v_k) - K_\beta(v_{k-1})| = |K_\alpha(v_k) - K_\alpha(v_{k-1})| \le \hat{K}_\alpha[v_{k-1}, v_k].$$

This, the fact that $\hat{K}_\alpha[a, b] = \sum_k \hat{K}_\alpha[v_{k-1}, v_k]$ and a similar formula for K_β imply the proposition. □

5. Variation of K during one excursion. We start with the proof of Theorem 3(ii) because it is simpler than that of Theorem 3(i). In broad general terms, the main ideas used in this section are estimations of the amount of pushing that is done on the boundary, starting from some point away from the tip of the thorn until one reaches that tip and a time reversal argument. This time reversal argument only works because the directions of reflection point at each other.

PROOF OF THEOREM 3(ii). We will only discuss the case when L and R are Lipschitz with the constant equal to 1/8. The general case requires only some routine modifications. Recall the notation from Section 2 and that we are assuming $R(y) - L(y) \geq y^a$.

Fix arbitrary integers $n_0, n_1 > 0$. Let $n = n(n_0, n_1)$ be such that (a_n, b_n) corresponds to the n_0th excursion of Y from 0 which hits $\{(x, y) \in D\colon\ y = 1/n_1\}$. Let T be the first time Y_2 hits $1/n_1$ after a_n. Let $\tilde{K}[c, d]$ be the variation of K accumulated over the part of $[c, d]$ where X_2 is less than 1. We will show that $E\tilde{K}[T, b_n] < \infty$. One can show in a similar way, using time reversal, that $E\tilde{K}[a_n, T] < \infty$ and, therefore, $E\tilde{K}[a_n, b_n] < \infty$. Hence, $\tilde{K}[a_n, b_n] < \infty$ a.s. Since there is only a countable number of pairs (n_0, n_1) and they correspond to all excursions of Y from 0, the same is true w.p.1 for all excursions simultaneously. Because the contribution to the total variation $\hat{K}[a_n, b_n]$ when $X_2(\cdot) \geq 1$ is finite a.s., this will suffice to show that $\hat{K}[a_n, b_n] < \infty$ a.s.

It remains to show that $E\tilde{K}[T, b_n] < \infty$. We may assume without loss of generality that $n_1 = 1$. Note that T is a stopping time for X_2 and b_n is the first hitting time of 0 by X_2 after T. Hence, the post-T X_2 process is a reflected Brownian motion, by the strong Markov property. Let

$$C_{k,m} = \{(x, y) \in D\colon\ 2^{-k} + m2^{-ak} \leq y \leq 2^{-k} + (m+1)2^{-ak}\},$$

where m is an integer such that $0 \leq m \leq [2^{k(a-1)}]$. Let $L_{k,m}[s, t]$ represent the variation accumulated by the boundary process K during the times r such that $Y(r) \in C_{k,m}$ and $r \in [s, t]$. Then

$$E\,\tilde{K}[T, b_n] \leq \sum_{k=1}^{\infty} \sum_{m=0}^{[2^{k(a-1)}]} E\,L_{k,m}[T, b_n]. \tag{5.1}$$

Fix $m = 0, 1, \dots, [2^{k(a-1)}]$ and set

$$\begin{aligned}
U_1 &= \inf\{t > T\colon\ X_2(t) = 2^{-k} + m2^{-ak} \quad \text{or} \\
&\qquad X_2(t) = 2^{-k} + (m+1)2^{-ak}\},\\
V_1 &= \inf\{t > U_1\colon\ X_2(t) = 2^{-k} + (m-1)2^{-ak} \quad \text{or} \\
&\qquad X_2(t) = 2^{-k} + (m+2)2^{-ak}\},\\
U_i &= \inf\{t > V_{i-1}\colon\ X_2(t) = 2^{-k} + m2^{-ak} \quad \text{or} \\
&\qquad X_2(t) = 2^{-k} + (m+1)2^{-ak}\}, \qquad i \geq 2,\\
V_i &= \inf\{t > U_i\colon\ X_2(t) = 2^{-k} + (m-1)2^{-ak} \quad \text{or} \\
&\qquad X_2(t) = 2^{-k} + (m+2)2^{-ak}\}, \qquad i \geq 2.
\end{aligned}$$

The probability that the reflected Brownian motion X_2 starting from $2^{-k} + (m-1)2^{-ak}$ will hit 0 before hitting $2^{-k}+m2^{-ak}$ is equal to $2^{-ak}/(2^{-k}+m2^{-ak})$ and so it is bounded below by $c_1 2^{k(1-a)}$. The probability that the reflected Brownian motion X_2 starting from $2^{-k} + m2^{-ak}$ or $2^{-k} + (m+1)2^{-ak}$ will hit $2^{-k} + (m-1)2^{-ak}$ before hitting $2^{-k} + (m+2)2^{-ak}$ is not less than 1/3. It follows that the number of U_i's less than b_n is stochastically bounded above by a random variable having geometric distribution with mean less than or equal to $c_2 2^{k(a-1)}$ for some constant c_2 independent of k and m.

Next we will estimate $EL_{k,m}[U_i, V_i]$. Let us start with $k = 1$ and fix i. Since $R(y) - L(y) \geq y^a$ and L and R are Lipschitz with constant 1/8, Y can cross D from ∂D^1 to ∂D^2 (or vice versa) within $C_{1,m}$ only if X_1 changes by at least $2^{-a}/2$. Let $S_1 = U_i$ and

$$S_j = \inf\{t > S_{j-1}\colon\ |X_1(t) - X_1(S_{j-1})| \geq 2^{-a}/2\}$$

for $j \geq 2$. Suppose that the event $\{S_{j-1} < V_i\}$ has occurred. Since $X_1(t)$ and $X_2(t)$ are independent, the probability that the process $X_2(S_{j-1}+\cdot)$ will leave the interval $[X_2(S_{j-1}) - 3\cdot 2^{-a}, X_2(S_{j-1}) + 3\cdot 2^{-a}]$ before $X_1(S_{j-1}+\cdot)$ leaves $[X_1(S_{j-1}) - 2^{-a}/2, X_1(S_{j-1}) + 2^{-a}/2]$ is strictly positive. Hence, for every j, the probability of $\{S_j < V_i\}$ given $\{S_{j-1} < V_i\}$ is less than $c_3 < 1$. Thus, the expected number of S_j less than V_i is less than $c_4 < \infty$. The total variation of K cannot increase between S_{j-1} and S_j by more than

$$\sup_{t\in[S_{j-1},S_j]} |X_1(t) - X(S_{j-1})| + \sup_{y_1,y_2\in[2^{-1}+(m-1)2^{-a},\ 2^{-1}+(m+2)2^{-a}]} |L(y_1) - L(y_2)|$$
$$+ \sup_{y_1,y_2\in[2^{-1}+(m-1)2^{-a},\ 2^{-1}+(m+2)2^{-a}]} |R(y_1) - R(y_2)| \leq 2^{1-a}.$$

It follows that

$$EL_{1,m}[U_i, V_i] \leq c_4 2^{1-a}.$$

An analogous argument shows that

$$EL_{k,m}[U_i, V_i] \leq c_5 2^{-ak}.$$

This and the previous remarks on the expected number of U_i's imply that

$$EL_{k,m}[T, b_n] \leq c_5 2^{-ak} c_2 2^{k(a-1)} \leq c_6 2^{-k}.$$

Therefore, by (5.1),

$$E\,\tilde{K}[T, b_n] \leq \sum_{k=1}^{\infty} (1 + [2^{k(a-1)}]) c_6 2^{-k}.$$

When $a < 2$, this is finite, completing the proof of Theorem 3(ii). □

PROOF OF THEOREM 3(i). We will show that $\hat{K}[a_n, b_n] = \infty$ a.s., where (a_n, b_n) corresponds to the first excursion of Y which hits $\{(x, y) \in D\colon\ y = 1\}$. The result may be extended to all excursions using the same argument as in the proof of Theorem 3(ii).

Let T be the first time Y hits $\{(x,y)\in D\colon\ y=1\}$. Recall that $Y=X+K$, where X is a normally reflecting Brownian motion in the upper half-plane. Without loss of generality suppose that $X(t)=(\hat{B}_1(t),|\hat{B}_2(t)|)$, where $\hat{B}(t)$ is a standard two-dimensional Brownian motion with $\hat{B}(0)=z$. Let $B(t)=(B_1(t),B_2(t))=(\hat{B}_1(T+t)-\hat{B}_1(T),\hat{B}_2(T+t))$. The process B is a standard planar Brownian motion starting from $(0,1)$. Let T_0 be the first time B_2 hits 0 and let L_t^y be one-half of the usual local time for B_2, jointly continuous in y and t. Let $u^+(y,\varepsilon,t)$ denote the number of upcrossings from height y to height $y+\varepsilon$ made by B_2 before time t. Our argument will be based on the well known fact that (see, e.g., [7])

$$L^y(T_0)=\lim_{\varepsilon\to 0}\varepsilon\, u^+(y,\varepsilon,T_0).$$

We will not make the last statement any more precise because we will not use it in this form.

Let

$$A_k=\Big\{\inf_{y\in[2^{-k-1},2^{-k})}L^y(T_0)\ge 2^{1-k}\Big\},$$

$$N_k=\{y\colon\ y=2^{-k-1}+n2^{3-2k},\ n=0,1,\dots,2^{k-4}-1\}.$$

Let N_k^0 be the set obtained from N_k by deleting its smallest and largest elements.

We will need the following two results about $L^y(T_0)$ and $u^+(y,\varepsilon,T_0)$.

LEMMA 2.

$$P(A_k\ \textit{occurs infinitely often})=1.$$

LEMMA 3. *There exists a constant c_1, independent of k, such that*

$$P\Big(\max_{y\in N_k}|L^y(T_0)-2^{3-2k}u^+(y,2^{3-2k},T_0)|>2^{-k}\Big)\le c_1 2^{-k}.$$

Proofs of Lemma 2 and 3 are deferred to later in this section.

Let

$$C_k=\{u^+(y,2^{3-2k},T_0)\ge 2^{k-3}\ \forall\ y\in N_k\}.$$

Lemmas 2 and 3 and the Borel–Cantelli lemma imply that

$$P(C_k\ \text{occurs infinitely often})=1.$$

Let $N_\infty=\bigcup_{k\ge 1}N_k$. Let S_k denote the consecutive times when B_2 hits new points in N_∞. More precisely, let $S_1=\inf\{t>0\colon\ B_2(t)\in N_\infty,\ B_2(t)\ne B_2(0)\}$

and

$$S_k = \inf\{t > S_{k-1}\colon\ B_2(t) \in N_\infty,\ B_2(t) \neq B_2(S_{k-1})\}.$$

Fix some sequence $\{y_i\}_{i\geq 1}$ of elements of N_∞ whose consecutive elements are neighbors in N_∞ (the same number may appear in the sequence more than once). Let

$$H \equiv \{B_2(S_1) = y_1,\ B_2(S_2) = y_2, \dots\}.$$

The strong Markov property implies that $\{B_2(t),\ 0 \leq t \leq S_j\}$ is independent of $\{B_2(t),\ t \geq S_j\}$ given the value of $B_2(S_j)$. An application of the strong Markov property to the process $\{B_2(t),\ t \geq S_j\}$ implies similarly that $\{B_2(t),\ S_j \leq t \leq S_{j+1}\}$ is independent of $\{B_2(t),\ t \geq S_{j+1}\}$ given the value of $B_2(S_{j+1})$. We conclude that the distribution of $\{B_2(t),\ S_j \leq t \leq S_{j+1}\}$ given H is that of Brownian motion starting from y_j and conditioned to hit y_{j+1} before hitting the other point in N_∞ closest to y_j. Suppose that $y_j \in N_i^0$. Then both neighbors of y_j in N_∞ are at the same distance 2^{3-2i} from y_j; in particular, $|y_j - y_{j+1}| = 2^{3-2i}$. By symmetry, the distribution of $S_{j+1} - S_j$ given H and $y_j \in N_i^0$ is the same as that of (unconditioned) Brownian motion stopped after hitting a point 2^{3-2i} units away from its starting point. With probability $c_2 > 0$, the supremum of the absolute value of Brownian motion starting from 0 taken over the time interval of length 2^{-4i} is less than 2^{3-2i}. Note that c_2 is independent of i by Brownian scaling. It follows that the conditional probability of $\{S_{j+1} - S_j > 2^{-4i}\}$ given H is greater than $c_2 > 0$. Recall that $Y(t) = X(t) + (K(t), 0)$ and that the clock for B_2 is shifted by T. The process X_1 is independent of B_2 given H since this event is defined only in terms of B_2. If the event $H \cap \{S_{j+1} - S_j > 2^{-4i}\}$ holds, then the increment of X_1 over the interval $(T + S_j, T + S_{j+1})$ will be greater than 2^{-2i+3} with probability greater than $c_3 > 0$ and with the same probability it will be less than -2^{-2i+3}. We have assumed that $R(y) - L(y) \leq y^2$. Hence, $R(y_{j+1}) - L(y_{j+1}) \leq 2^{-2i}$ because $y_j \in N_i^0$ and so $y_{j+1} \leq 2^{-i}$. If the increment of X_1 over the interval $(T + S_j, T + S_{j+1})$ is greater than 2^{-2i+3}, then the variation of K over $(T + S_j, T + S_{j+1})$ must be greater than 2^{-2i+1} in order to keep Y inside the domain. Suppose that $H \subset C_k$ and H occurred. There are more than 2^{k-5} points in N_k^0 and so there will be at least $2^{k-5}2^{k-3}$ different j such that S_j, S_{j+1} correspond to an upcrossing of B_2 between points in N_k^0. Let $\{y_{i_n}\}$ for $n = 1, 2, \dots, 2^{2k-8}$ correspond to the first 2^{2k-8} of the $y_j \in H$ that are also in N_k^0. Let $W_{i_n} = 1$ if $\sup_{S_{i_n} \leq t < S_{i_n+1}} |B_1(t) - B_1(S_{i_n})| > 2^{3-2k}$ and equal to zero otherwise. Then (given H) the W_{i_n} are independent random variables which equal 1 with probability not less than $c_2 \cdot c_3 = c_4$. Since c_4 is independent of k, by the central limit theorem there is some $c_5 > 0$ such that (given H) $P(\sum_{n=1}^{2^{2k-8}} W_{i_n} \geq c_4 2^{2k-9}) > c_5$ for each k. Thus under the assumption that $H \subset C_k$, the total variation of $K(t)$ accumulated when $Y_2(t) \in (2^{-k-1}, 2^{-k})$ is greater than $c_4 2^{2k-9} 2^{-2k+1}$ with probability at least c_5 (given H). Suppose that $j_1 < j_2 < \cdots < \infty$ and $H \subset \bigcap_{i\geq 1} C_{j_i}$. Let E_k denote the event that the total variation of $K(t)$ accumulated when $Y_2(t) \in (2^{-k-1}, 2^{-k})$ is greater than or equal to $c_4 2^{-8}$. Since the E_k are independent and each has the probability not less than c_5, an application of the Borel–Cantelli lemma shows that given H, $P(E_k \text{ i.o.}) = 1$. Thus the total variation of K accumulated when $Y_2(t) \in \bigcup_{i\geq 1}(2^{-j_i-1}, 2^{-j_i})$ is infinite with probability 1. The result follows when we integrate over all H because we know that infinitely many C_k's occur with probability 1. □

It now remains to prove Lemmas 2 and 3.

PROOF OF LEMMA 2. Let W_t be a two-dimensional Bessel process starting from 0, that is, the distribution of W is that of the norm of two-dimensional Brownian motion. Let

$$\tilde{A}_k = \Big\{ \inf_{t\in[2^{-k-1},2^{-k}]} W_t^2 \geq 2^{1-k} \Big\}.$$

It is clear that

$$P\Big\{ \inf_{t\in[2^{-2},2^{-1}]} W_t \geq 1 \Big\} \geq c_1 > 0.$$

Brownian scaling implies that for all $k \geq 1$,

$$P\Big\{ \inf_{t\in[2^{-k-1},2^{-k}]} W_t \geq 2^{(1-k)/2} \Big\} \geq c_1 > 0.$$

Hence

$$P(\tilde{A}_k) = P\Big(\inf_{t\in[2^{-k-1},2^{-k}]} W_t^2 \geq 2^{1-k} \Big) \geq c_1.$$

It follows that

$$P\Big(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \tilde{A}_k \Big) \geq c_1.$$

The event in the last formula belongs to the germ σ-field and so it has probability 1. This implies that $P(\tilde{A}_k \text{ infinitely often}) = 1$.

Since $\{L^x(T_0),\ 0 \leq x \leq 1\}$ has the same distribution as $\{W_x^2,\ 0 \leq x \leq 1\}$ (see [12], Section 3.1), the lemma follows. □

PROOF OF LEMMA 3. The following proof corresponds closely to an argument given by Chacon, Le Jan, Perkins and Taylor in [3] starting on page 206. Recall $P(B_2(0) = 1) = 1$. Let $\{\theta_t\}_{t\geq 0}$ denote the usual family of shift operators. Define the following stopping times for B_2:

$$\begin{aligned} U_0(x,\varepsilon) &= T_x, \\ V_i(x,\varepsilon) &= U_i(x,\varepsilon) + T(x+\varepsilon) \circ \theta_{U_i(x,\varepsilon)}, \\ U_{i+1}(x,\varepsilon) &= V_i(x,\varepsilon) + T(x) \circ \theta_{V_i(x,\varepsilon)}. \end{aligned}$$

We shall write U_i and V_i for $U_i(x, \varepsilon)$ and $V_i(x, \varepsilon)$ in cases where there is no ambiguity. Note that

$$(5.2)\quad \left(\sum_{i=0}^{\infty} I(U_i(x,\varepsilon) < T_0)\right) - 1 \le u^+(x,\varepsilon,T_0) \le \left(\sum_{i=0}^{\infty} I(U_i(x,\varepsilon) < T_0)\right).$$

We have

$$(5.3)\quad \begin{aligned} L^x(T_0) &= \sum_{i=0}^{\infty} I(U_i(x,\varepsilon) < T_0)\big(L^x_{V_i(x,\varepsilon)} - L^x_{U_i(x,\varepsilon)}\big) \\ &\quad - \sum_{i=0}^{\infty} I(U_i(x,\varepsilon) < T_0 < V_i(x,\varepsilon))\big(L^x_{V_i(x,\varepsilon)} - L^x_{T_0}\big) \\ &= \sum_{i=0}^{\infty} I(U_i < T_0) L^x_{T(x+\varepsilon)} \circ \theta_{U_i} \\ &\quad - \sum_{i=0}^{\infty} I(U_i < T_0 < V_i) L^x_{T(x+\varepsilon)} \circ \theta_{T_0}. \end{aligned}$$

It is well known that $L^x_{T(x+\varepsilon)} \circ \theta_{T_x}$ has an exponential distribution with mean 2ε. For $j = 0, 1, \dots, \infty$, let

$$M_j \equiv \sum_{i=0}^{j} I(U_i < T_0)(L^x_{T(x+\varepsilon)} \circ \theta_{U_i} - 2\varepsilon).$$

The strong Markov property shows that $\{(M_j, \mathscr{F}_{U_{j+1}});\ j = 0, 1, \dots\}$ is a martingale. It follows from (5.2) and (5.3) that for $x \in N_k$ and for some constant c_2 independent of k,

$$E[(L^x(T_0) - \varepsilon\, u^+(x,\varepsilon,T_0))^4] \le c_2(E[M_\infty^4] + \varepsilon^4).$$

If $m_j \equiv M_j - M_{j-1}$ ($M_{-1} = 0$), then a square function inequality for martingales (see [2], Theorem 21.1) implies, for some constant c_3 independent of k,

$$\begin{aligned} E[M_\infty^4] &\le c_3 E\left[\left(\sum_{i=0}^{\infty} E(m_i^2 \mid \mathscr{F}_{U_i})\right)^2 + \sum_{i=0}^{\infty} m_i^4\right] \\ &\le c_4 E\left[\left(\sum_{i=0}^{\infty} \varepsilon^2 I(U_i < T_0)\right)^2 + \sum_{i=0}^{\infty} \varepsilon^4 I(U_i < T_0)\right], \end{aligned}$$

where we have used the strong Markov property and the fact that $L^x_{T(x+\varepsilon)} \circ \theta_{T(x)}$ has variance $2\varepsilon^2$ and fourth moment bounded by $c_5\varepsilon^4$. Therefore,

$$\begin{aligned} E[M_\infty^4] &\le c_6\, \varepsilon^4 E\left[\sum_{i \le j}\sum I(U_i < T_0, U_j < T_0)\right] \\ &\le c_6\, \varepsilon^4 \left[1 + \sum_{j=1}^{\infty} j P(U_j < T_0)\right]. \end{aligned}$$

Substitute 2^{3-2k} for ε and suppose $x \in N_k$. Then for some constant c_7,

$$E[(L^x(T_0) - 2^{3-2k}u^+(x, 2^{3-2k}, T_0))^4] \le c_7 2^{12-8k}\left[2 + \sum_{j=1}^{\infty} jP(u^+(x, 2^{3-2k}, T_0) \ge j)\right].$$

The probability that Brownian motion starting from $x \in N_k$ will hit $x + 2^{3-2k}$ is bounded by $(2^{-k})/(2^{-k} + 2^{3-2k})$ and so

$$P(u^+(x, 2^{3-2k}, T_0) \ge j) \le \left(\frac{2^{-k}}{2^{-k} + 2^{3-2k}}\right)^j.$$

It is elementary to check that

$$\sum_{j=1}^{\infty} j\left(\frac{2^{-k}}{2^{-k} + 2^{3-2k}}\right)^j \le c_8 2^{2k}.$$

Therefore, for some constant c_9 independent of k,

$$E[(L^x(T_0) - 2^{3-2k}u^+(x, 2^{1-2k}, T_0))^4] \le c_9 2^{-6k}$$

for all k sufficiently large. By the Chebyshev inequality,

$$P(|L^x(T_0) - 2^{3-2k}u^+(x, 2^{3-2k}, T_0)| > 2^{-k}) \le c_9\, 2^{4k}2^{-6k}.$$

The number of elements in N_k is bounded by $c_{10}2^k$. Using this we find

$$P\Big(\max_{x \in N_k} |L^x(T_0) - 2^{3-2k}u^+(x, 2^{3-2k}, T_0)| > 2^{-k}\Big) \le c_{11}\, 2^{4k}2^{-6k}2^k \le c_{11}\, 2^{-k}.$$

This completes the proof of Lemma 3. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TEXAS 77843