The Annals of Probability
1995, Vol. 23, No. 3, 1300–1314

ASYMPTOTIC LAWS FOR ONE-DIMENSIONAL DIFFUSIONS CONDITIONED TO NONABSORPTION

By Pierre Collet, Servet Martínez and Jaime San Martín

Ecole Polytechnique, Universidad de Chile and Universidad de Chile

If (X_t) is a one-dimensional diffusion corresponding to the operator $\mathscr{L} = \frac{1}{2}\partial_{xx} - \alpha\partial_x$ starting from $x > 0$ and T_a is the hitting time of a, we prove that under suitable conditions on the drift coefficient the following limit exists: $\forall\, s > 0$, $\forall\, A \in \mathscr{F}_s$, $\lim_{t\to\infty} \mathbb{P}_x(X \in A \mid T_0 > t)$. We characterize this limit as the distribution of an h-like process, h satisfying $\mathscr{L}h = -\eta h$, $h(0) = 0$, $h'(0) = 1$, where $\eta = -\lim_{t\to\infty}(1/t)\log \mathbb{P}_x(T_0 > t)$. Moreover, we show that this parameter η can only take two values: $\eta = 0$ or $\eta = \underline{\lambda}$, where $\underline{\lambda}$ is the smallest point of increase of the spectral distribution of the operator $\mathscr{L}^* = \frac{1}{2}\partial_{xx} + \partial_x(\alpha\,\cdot)$.

1. Introduction and statement of main results. We denote by $\mathbb{P}_x$ the probability law of a Brownian motion (B_t) starting at x. Consider the diffusion (X_t) given by

$$X_t = B_t - \int_0^t \alpha(X_s)\, ds,$$

where we assume α to be C^1.

Denote by T_a the hitting time of a, $T_a = \inf\{t > 0 \colon X_t = a\}$.

We consider the sub-Markovian semigroup given by

$$\overline{P}_t f(x) = \mathbb{E}_x(f(X_t), T_0 > t),$$

and we denote by $\bar{p}(t, x, y)$ its transition density.

Let $\gamma(x) = 2\int_0^x \alpha(z)\, dz$. We assume the following condition holds:

Hypothesis H.

$$\int_0^\infty e^{\gamma(x)}\left(\int_0^x e^{-\gamma(z)}\, dz\right) dz = \int_0^\infty e^{-\gamma(x)}\left(\int_0^x e^{\gamma(z)}\, dz\right) dx = \infty.$$

This means that ∞ is the natural boundary of the process [see Feller (1952), page 487].

Under Hypothesis H we have [see Azencott (1974)]

$$\lim_{x\to\infty} \mathbb{P}_x(T_0 > s) = 1 \quad \text{for any } s > 0. \tag{1}$$

A relevant function in our study is

$$\Lambda(x) = \int_0^x e^{\gamma(z)}\, dz.$$

Received December 1993; revised November 1994.
AMS 1991 *subject classifications.* Primary 60J60; secondary 60F99.
Key words and phrases. One-dimensional diffusions, h-processes, absorption.

It satisfies $\mathscr{L}\Lambda \equiv 0$, $\Lambda(0) = 0$, where $\mathscr{L} = \frac{1}{2}\partial_{xx} - \alpha\partial_x$. Hence we also have

$$\mathbb{E}_x(\Lambda(X_s), s < T_0) \leq \Lambda(x). \tag{2}$$

This follows immediately from Itô's formula. In fact,

$$\begin{aligned}\Lambda(x) &= \mathbb{E}_x\big(\Lambda(X_{T_0 \wedge T_M \wedge s})\big)\\ &= \mathbb{E}_x(\Lambda(M), T_M < T_0 \wedge s) + \mathbb{E}_x(\Lambda(X_s), s < T_M \wedge T_0)\\ &\geq \mathbb{E}_x(\Lambda(X_s), s < T_M \wedge T_0).\end{aligned}$$

Then (2) follows from the monotone convergence theorem. Also note that the equality $\Lambda(x) = \Lambda(M)\mathbb{P}_x(T_M < T_0)$ for $x \in (0, M)$ gives

$$\mathbb{P}_x(T_M < T_0) = \frac{\Lambda(x)}{\Lambda(M)}. \tag{3}$$

Let $\mathscr{L}^* = \frac{1}{2}\partial_{xx} + \partial_x(\alpha\,\cdot)$ be the formal adjoint of $\mathscr{L}$. Under Hypothesis H we are in the limit point case, which means there exists a solution of $\mathscr{L}^*\varphi = 0$, $\varphi(0) = 0$, $\varphi'(0) = 1$, which does not belong to $L^2(d\Lambda)$. In fact, it suffices to take $\varphi = e^{-\gamma}\Lambda$. Denote by φ_λ the solution of $\mathscr{L}^*\varphi_\lambda = -\lambda\varphi_\lambda$, $\varphi_\lambda(0) = 0$, $\varphi_\lambda'(0) = 1$, and by ψ_λ the solution of $\mathscr{L}\psi_\lambda = -\lambda\psi_\lambda$, $\psi_\lambda(0) = 0$, $\psi_\lambda'(0) = 1$. The spectral properties of $\mathscr{L}^*$ and $\mathscr{L}$ are related by the equality $\varphi_\lambda = e^{-\gamma}\psi_\lambda$. Let $\rho(\lambda)$ be the spectral distribution of the operator $\mathscr{L}^*$. We assume ρ is left continuous [see Coddington and Levinson (1955), Chapter 9]. Let $\underline{\lambda}$ be the smallest point of increase of $\rho(\lambda)$. In Mandl [(1961), Lemma 2] it was shown that

$$\underline{\lambda} = \sup\{\lambda\colon \varphi_\lambda(\cdot) \text{ does not change sign}\},$$

Since $\varphi_0 = e^{-\gamma}\Lambda$ does not change sign, we have $\underline{\lambda} \geq 0$. Observe that Hypothesis H implies $\int \varphi_0(x)\,dx = \int e^{-\gamma(x)}\Lambda(x)\,dx = \infty$. Therefore, if $\varphi_{\underline{\lambda}} \in \mathscr{L}^1(dx)$, then necessarily $\underline{\lambda} > 0$.

We recall that a main role in the proof of the results obtained by Mandl is played by the unitary transformation $\pi\colon L^2(d\Lambda) \to L^2(d\rho)$, defined by

$$(\pi h)(\lambda) = \lim_{A\to\infty}\,(L^2(d\rho))\int_0^A h(x)\varphi_\lambda(x)\,d\Lambda(x) \quad \text{for } h \in L^2(d\Lambda),$$

and by L_0, the subset of functions $h \in L^2(d\Lambda)$, which are nonnegative, not equivalent to zero and whose image πh is bounded from below in some right neighborhood of $\underline{\lambda}$.

Denote by $\mathscr{D}_0^0(\mathbb{R}_+)$ the set of nonnegative continuous probability densities with compact support. Since $\varphi_\lambda(x)$ is continuous in (λ, x), the set $\mathscr{D}_0^0(\mathbb{R}_+)$ is included in L_0. Hence the conclusion of the following theorem shown in Mandl [(1961), Theorem 2] holds for $h \in \mathscr{D}_0^0(\mathbb{R}_+)$.

THEOREM 0. *Assume Hypothesis H holds. Then, for any $h \in L_0$,*

$$\frac{\mathbb{P}_h(X_t \le y \mid T_0 > t)}{\mathbb{P}_h(X_t \le z \mid T_0 > t)} \to \frac{\int_0^y \varphi_{\underline{\lambda}}(x)\,dx}{\int_0^z \varphi_{\underline{\lambda}}(x)\,dx} \quad \text{as } t \to \infty,$$

where

$$\mathbb{P}_h(X_t \in A \mid T_0 > t) = \frac{\int h(x)\mathbb{P}_x(X_t \in A, T_0 > t)\,dx}{\int h(x)\mathbb{P}_x(T_0 > t)\,dx}.$$

Our main results are the following ones for which we assume Hypothesis H holds.

THEOREM A. *The limit $\eta = -\lim_{t\to\infty}(1/t)\log \mathbb{P}_x(T_0 > t)$ exists and it is independent of $x > 0$. Moreover:*

(a) *For any $x > 0$ and $s > 0$,*

$$\frac{\mathbb{P}_x(T_0 > t + s)}{\mathbb{P}_x(T_0 > t)} \to e^{-\eta s} \quad \text{as } t \to \infty.$$

(b) *For any $x \ge 0$,*

$$\frac{\mathbb{P}_x(T_0 > t)}{\mathbb{P}_1(T_0 > t)} \to W(x) \quad \text{as } t \to \infty,$$

where $W \in C^2$ and satisfies $\frac{1}{2}W'' - \alpha W' = -\eta W$ with $W(0) = 0$, $W(1) = 1$.

THEOREM B. *For any $s > 0$ and for any $A \in \mathscr{F}_s$,*

$$\lim_{t\to\infty} \mathbb{P}_x(X \in A \mid T_0 > t) = Q_x(A),$$

where Q_x is the distribution of a diffusion with transition probability densities given by

$$q(s,x,y) = e^{\eta s}\frac{W(y)}{W(x)}\bar{p}(s,x,y),$$

with W and η given by Theorem A.

THEOREM C. (a) *If $\eta > 0$, then $\eta = \underline{\lambda}$, $c \equiv \int_{\mathbb{R}_+} \varphi_{\underline{\lambda}}(z)\,dz < \infty$ and, for any Borel set $E \subset \mathbb{R}_+$,*

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in E \mid T_0 > t) = c^{-1}\int_E \varphi_{\underline{\lambda}}(z)\,dz \quad \text{for every } x > 0.$$

(b) *If $\eta = 0$, then $\lim_{t\to\infty} \mathbb{P}_x(X_t \in E \mid T_0 > t) = 0$ for any bounded Borel set $E \subset \mathbb{R}_+$, for every $x > 0$.*

If $\mathbb{P}_x(T_0 = \infty) > 0$, we are in the case $\eta = 0$ and Theorems A, B and C can be proved directly by using the equalities

$$W(x) = \frac{\mathbb{P}_x(T_0 = \infty)}{\mathbb{P}_1(T_0 = \infty)} = \frac{\Lambda(x)}{\Lambda(1)}.$$

Therefore, in the proof of the theorems we restrict ourselves to the case $\mathbb{P}_x(T_0 = \infty) = 0$, that is, $\Lambda(\infty) = \infty$.

From Theorem C it is deduced that if $\int_{\mathbb{R}_+} \varphi_{\underline{\lambda}}(z)\,dz = \infty$, then $\eta = 0$, and the pointwise version of Mandl [(1961), Theorem 3] is satisfied. On the other hand, if $\int_{\mathbb{R}_+} \varphi_{\underline{\lambda}}(z)\,dz < \infty$ and $\int_{\mathbb{R}_+} e^{-\gamma(z)}\,dz < \infty$ hold, then by using Mandl [(1961), Theorem 5] it is deduced that $\eta = \underline{\lambda} > 0$.

For $\eta = 0$ we can give another characterization of the limit studied in Theorem B. For this purpose consider the process (Z_t), the unique solution of

$$dZ_t = dB_t + \left(\frac{\Lambda'(Z_t)}{\Lambda(Z_t)} - \alpha(Z_t)\right) dt, \qquad Z_0 = x. \tag{4}$$

PROPOSITION D. *If* $\eta = 0$, *then*:

(a) $\Lambda(X_s)1_{T_0 > s}$ *is a* $\mathbb{P}_x$*-martingale.*

(b) *For any* $x > 0$, $\mathbb{P}_x(T_0^Z < \infty$ *or* $\lim_{a \nearrow \infty} T_a^Z < \infty) = 0$.

(c) $\lim_{t\to\infty} \mathbb{P}_x(X \in A \mid T_0 > t) = \lim_{M\to\infty} \mathbb{P}_x(X \in A \mid T_M < T_0) = R_x(A)$ *for any* $A \in \mathscr{F}_s$, *where* R_x *is the distribution of the diffusion* (Z_t) *whose transition probability densities are given by*

$$r(s,x,y) = \frac{\Lambda(y)}{\Lambda(x)}\bar{p}(s,x,y).$$

The study of general one-dimensional diffusions can be reduced, under suitable conditions on the diffusion coefficient, to the previous setting. In fact, consider the solution of

$$dY_t = \sigma(Y_t)\,dB_t - h(Y_t)\,dt, \qquad Y_0 = y,$$

where we assume $\sigma > 0$, $\sigma \in C^1(\mathbb{R})$, $\int_0^\infty (1/\sigma(u))\,du = \infty$. Take $F(y) = \int_0^y 1/(\sigma(u))\,du$. Then $F \in C^2(\mathbb{R})$ and F^{-1} exists. By Itô's formula we obtain

$$F(Y_t) = F(y) + B_t - \int_0^t \alpha(F(Y_s))\,ds,$$

where $\alpha(x) = (h/\sigma + \frac{1}{2}\sigma') \circ F^{-1}(x)$.

If we consider the process $X_t = F(Y_t)$, $X_0 = x = F(y)$, then

$$dX_t = dB_t - \alpha(X_t)\,dt.$$

Since the hitting time of 0 is the same for both processes, $T_0^X = T_0^Y$, we get

$$\mathbb{P}(X \in A, T_0^X > t, X_0 = x) = \mathbb{P}(Y \in F^{-1}(A), T_0^Y > t, Y_0 = y).$$

Therefore, for our purposes it is enough to study the semigroup associated with the process (X_t).

The case α constant was studied in previous works. We note that in this case Hypothesis H always holds. For $\alpha = 0$, Theorem B follows, respectively, from Knight (1969) and Williams (1970). For $\alpha < 0$ the limit distribution of Theorem B corresponds to a three-dimensional Bessel process BES$(3, |\alpha|)$; see Rogers and Pitman (1981). For $\alpha > 0$, Theorem B was shown in Martínez and San Martín (1994). Moreover, Theorem C follows from the fact that the distribution of X_t on $\{T_0 > t\}$ is explicitly known. For $\alpha \leq 0$, $\lim_{t\to\infty} \mathbb{P}_x(X_t > b \mid T_0 > t) = 1$ for any b and for $\alpha > 0$, $\lim_{t\to\infty} \mathbb{P}_x(X_t \in E \mid T_0 > t) = \int_E \alpha^2 y e^{-\alpha y}\, dy$ for any $E \subset \mathbb{R}_+$.

The multidimensional analogues of Theorems B and C have been studied by Pinsky (1985) in bounded regions G. The methods he used are based upon the theory of Stroock and Varadhan about the martingale problem. We remark that for nonbounded regions, as occurs in our case where $G = (0, \infty)$, Hypotheses 2 and 3 of Pinsky do not hold in general. This is due to the fact that the spectrum of $\mathscr{L}$ is not necessarily discrete. For instance, if the drift α is a positive constant, Hypothesis 2 of Pinsky is not satisfied because $(\mathbb{P}_x(T_0 > t))/(W(x)\exp(-(\alpha^2/2)t)) = o(1/\sqrt{t})$, and similarly it can be proved that his Hypothesis 3 also fails.

2. Proof of the results. First, let us study the quantities

$$v_t(x) = \frac{\mathbb{P}_x(T_0 > t)}{\mathbb{P}_1(T_0 > t)}.$$

For any $t > 0$, $v_t(x)$ increases in x and satisfies $v_t(0) = 0$, $v_t(1) = 1$.

Take $x > y > 0$. We have

$$\begin{aligned}\mathbb{P}_y(T_0 > t) &\geq \mathbb{P}_y(T_0 > t + T_x, T_0 > T_x)\\ &= \mathbb{E}_y(T_0 > T_x, \mathbb{P}_x(T_0 > t))\\ &= \mathbb{P}_y(T_0 > T_x)\mathbb{P}_x(T_0 > t)\\ &= \frac{\Lambda(y)}{\Lambda(x)}\mathbb{P}_x(T_0 > t),\end{aligned}$$

where the last equality follows from (3). Then, dividing both sides by $\mathbb{P}_1(T_0 > t)$, we get

$$\forall\, x \geq y > 0, \qquad v_t(x) \leq \frac{\Lambda(x)}{\Lambda(y)} v_t(y). \tag{5}$$

Now $v_t(x) \leq 1$ for $x \leq 1$ and $v_t(x) \leq (\Lambda(x))/\Lambda(1))$ for $x \geq 1$, so

$$v_t(x) \leq \max\left(\frac{\Lambda(x)}{\Lambda(1)}, 1\right) \leq \left(\frac{\Lambda(x)}{\Lambda(1)} + 1\right) \quad \text{for } x \geq 0. \tag{6}$$

LEMMA 1. *There exists a sequence* $t_n \to \infty$ *and a function* $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ *which is continuous in* $\mathbb{R}_+ - \{0\}$ *such that,*

$$v_{t_n}(x) = \frac{\mathbb{P}_x(T_0 > t_n)}{\mathbb{P}_1(T_0 > t_n)} \to w(x) t_n \quad \text{as } t_n \to \infty \text{ for any } x \in \mathbb{R}_+.$$

Moreover, w *is increasing,* $w(0) = 0$, $w(1) = 1$ *and*

$$0 < w(x) \le \max\left(\frac{\Lambda(x)}{\Lambda(1)}, 1\right) \le \left(\frac{\Lambda(x)}{\Lambda(1)} + 1\right),$$

for $x > 0$.

PROOF. Since $v_t(x)$ is increasing in x and the family of functions $(v_t)_{t>0}$ is bounded by the continuous function $\max(\Lambda(x)/\Lambda(1), 1)$, we can apply the Helley theorem [see Coddington and Levinson (1955), page 233] to get the existence of a subsequence $t_n \to \infty$, such that $v_{t_n} \to w$ as $t_n \to \infty$. Obviously w is increasing, $w(0) = 0$, $w(1) = 1$ and $w(x) \le \max(\Lambda(x)/\Lambda(1), 1)$.

Let us show $w(y) > 0$ for $y \in (0, 1)$. From (5) we get $w(x) \le (\Lambda(x)/\Lambda(y))w(y)$ for any $x \ge y$. From $w(1) = 1$ we conclude $w(y) > 0$.

Let us show the continuity of w for $x > 0$. Since w is increasing, it is enough to prove that $w(x+) \le w(x-)$. We have that $w(x+h) \le (\Lambda(x+h)/\Lambda(x-h))w(x-h)$ for $h > 0$. Thus the continuity of Λ implies $w(x+) \le w(x-)$. □

We remark that w is also continuous at $x = 0$; that is, $\lim_{x \to 0+} w(x) = 0$. This will be proved as a consequence of Lemma 4.

Now fix a probability density h on $\mathbb{R}_+$. Consider the distribution function $G_t^{(h)}$ of X_t conditioned to nonabsorption up to time t, when the starting density function is h:

$$G_t^{(h)}(y) = \mathbb{P}_h(X_t \le y \mid T_0 > t).$$

In the sequel, $\mathscr{H} = \{h_n\}$ is a fixed countable family of probability densities on $\mathbb{R}_+$.

LEMMA 2. *Let* $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ *be the one point compactification of* $\mathbb{R}_+$. *Then there exists a sequence* $t_n \to \infty$ *and* $F^{(\mathscr{H})}\colon \mathbb{R}_+ \to [0,1]$ *a right-continuous increasing function with* $F^{(\mathscr{H})}(0) = 0$ *and* $F^{(\mathscr{H})}(\infty) = 1$, *such that for any continuous and bounded function* $\eta\colon \overline{\mathbb{R}}_+ \to \mathbb{R}$:

$$\int_{\mathbb{R}_+} \eta(y)\, dG_{t_n}^{(h)}(y) \to \int_{\overline{\mathbb{R}}_+} \eta(y)\, dF^{(\mathscr{H})}(y) \quad \text{as } t_n \to \infty, \text{ for } h \in \mathscr{H}.$$

PROOF. This follows from the compactness of the set of probability measures on $\overline{\mathbb{R}}_+$. □

Notice that we can take the same sequence $t_n \to \infty$ in Lemmas 1 and 2. In the sequel $F^{(\mathscr{H})}$ will denote the distribution as well as the measure defined by it.

LEMMA 3. *Let $\mathscr{H} \subset \mathscr{D}_0^0(\mathbb{R}_+)$ (continuous probability densities with compact support). For any $h \in \mathscr{H}$ and for any $s > 0$,*

$$\frac{\int_{\mathbb{R}_+} h(y)\mathbb{E}_y(T_0 > s, w(X_s))\, dy}{\int_{\mathbb{R}_+} h(y)w(y)\, dy} = \int_{\overline{\mathbb{R}}_+} \mathbb{P}_y(T_0 > s)\, dF^{(\mathscr{H})}(y).$$

PROOF. For $s > 0$ fixed, consider the function $\eta^{(s)}(y) = \mathbb{P}_y(T_0 > s)$ for $y \in \overline{\mathbb{R}}_+$, where $\eta^{(s)}(\infty) = \mathbb{P}_\infty(T_0 > s) = 1$. Using (1) this function is continuous on $\overline{\mathbb{R}}_+$ and bounded. Now, take $\theta_t^{(h)}(s) = \mathbb{P}_h(T_0 > t + s \mid T_0 > t)$. We have

$$\theta_t^{(h)}(s) = \frac{\mathbb{E}_h\big(T_0 > t, \mathbb{P}_{X_t}(T_0 > s)\big)}{\mathbb{P}_h(T_0 > t)} = \mathbb{E}_h\big(\eta^{(s)}(X_t) \mid T_0 > t\big).$$

Then, by definition of $G_t^{(h)}$ we obtain $\theta_t^{(h)} = \int \eta^{(s)}\, dG_t^{(h)}$. From Lemma 2,

$$\begin{aligned} \theta_{t_n}^{(h)}(s) &\to \int_{\overline{\mathbb{R}}_+} \eta^{(s)}(y)\, dF^{(\mathscr{H})}(y) \quad (\text{as } t_n \to \infty) \\ &= \int_{\overline{\mathbb{R}}_+} \mathbb{P}_y(T_0 > s)\, dF^{(\mathscr{H})}(y). \end{aligned} \tag{7}$$

On the other hand,

$$\theta_t^{(h)}(s) = \frac{\int_{\mathbb{R}_+} h(y)\mathbb{E}_y\big(T_0 > s, \mathbb{P}_{X_s}(T_0 > t)\big)\, dy}{\int_{\mathbb{R}_+} h(y)\mathbb{P}_y(T_0 > t)\, dy}.$$

Dividing the numerator and denominator by $\mathbb{P}_1(T_0 > t)$ and considering $t = t_n$, we get

$$\theta_{t_n}^{(h)}(s) = \frac{\int h(y)\mathbb{E}_y\big(T_0 > s, \mathbb{P}_{X_s}(T_0 > t_n)/\mathbb{P}_1(T_0 > t_n)\big)\, dy}{\int h(y)\big(\mathbb{P}_y(T_0 > t_n)/\mathbb{P}_1(T_0 > t_n)\big)\, dy}. \tag{8}$$

From (6), (2) and Lemma 1, we can apply the dominated convergence theorem to get:

$$\text{for any } y \in \mathbb{R}_+, \quad \mathbb{E}_y\left(T_0 > s, \frac{\mathbb{P}_{X_s}(T_0 > t_n)}{\mathbb{P}_1(T_0 > t_n)}\right) \to \mathbb{E}_y(T_0 > s, w(X_s)) \text{ as } t_n \to \infty.$$

Let $[0, a]$ be an interval containing the support of h. From (6) and (2), for any $y \le a$ we have

$$\begin{aligned} h(y)\mathbb{E}_y\left(T_0 > s, \frac{\mathbb{P}_{X_s}(T_0 > t_n)}{\mathbb{P}_1(T_0 > t_n)}\right) &\le h(y)\mathbb{E}_y\left(T_0 > s, \frac{\Lambda(X_s)}{\Lambda(1)} + 1\right) \\ &\le h(y)\left(\frac{\Lambda(y)}{\Lambda(1)} + 1\right) \end{aligned} \tag{9}$$

and

$$h(y)\frac{\mathbb{P}_y(T_0 > t_n)}{\mathbb{P}_1(T_0 > t_n)} \le h(y)\left(\frac{\Lambda(y)}{\Lambda(1)} + 1\right).$$

The function on the right of (9) is integrable with respect to y. Thus by the dominated convergence theorem, we can pass to the limit in the numerator and denominator of $\theta_{t_n}^{(h)}(s)$ in expression (8). Since $w(y)$ is strictly positive for $y > 0$, we get

$$\theta_{t_n}^{(h)}(s) \to \frac{\int_{\mathbb{R}_+} h(y)\mathbb{E}_y(T_0 > s, w(X_s))\,dy}{\int_{\mathbb{R}_+} h(y)w(y)\,dy} \quad \text{as } t \to \infty. \tag{10}$$

Hence, from (7) and (10) we obtain the result. □

Let $M(s) = \int_{\mathbb{R}_+} \mathbb{P}_y(T_0 > s)\,dF^{(\mathscr{H})}(y)$. From the previous lemma we have for $s > 0$,

$$\int h(y)\mathbb{E}_y(T_0 > s, w(X_s))\,dy = M(s)\int h(y)w(y)\,dy \quad \text{for any } h \in \mathscr{H}. \tag{11}$$

We assume the family $\mathscr{H} = \{h_n\}$ is contained in $\mathscr{D}_0^0(\mathbb{R}_+)$ with extra property that for any bounded interval with rational extremes (c, d) there exists a function $h_n \in \mathscr{H}$ whose support is contained in (c, d).

LEMMA 4. *There exists $\beta \ge 0$ such that for any $x \ge 0$ and $s > 0$,*

$$\mathbb{E}_x(T_0 > s, w(X_s)) = e^{-\beta s}w(x). \tag{12}$$

PROOF. From the choice of the family $\mathscr{H}$ and the monotonicity properties of $\mathbb{E}_x(T_0 > s, w(X_s))$ and $w(x)$ we deduce from (11) that the following equality holds:

$$\mathbb{E}_x(T_0 > s, w(X_s)) = M(s)w(x), \tag{13}$$

for $s > 0$ and dx-a.e. Since both sides of (13) are increasing in x and w is continuous on $\mathbb{R}_+ - \{0\}$ the equality holds for every $x > 0$. For $x = 0$, (13) is obviously satisfied.

Let us prove now that $M(s) = e^{-\beta s}$ for some $\beta \ge 0$. To this end it will be enough to prove $\forall\ s, t \ge 0$: $M(s + t) = M(s)M(t)$, because M is monotone and bounded by 1. Using the Markov property and (13), we obtain

$$\begin{aligned} M(s+t)w(x) &= \mathbb{E}_x(T_0 > s+t, w(X_{s+t})) = \mathbb{E}_x\big(T_0 > s, \mathbb{E}_{X_s}(T_0 > t, w(X_t))\big) \\ &= \mathbb{E}_x(T_0 > s, M(t)w(X_s)) = M(t)M(s)w(x). \end{aligned}$$

Since $w(x) > 0$ for any $x > 0$, the result follows. □

Now let us show w is continuous at $x = 0$. We have

$$\mathbb{E}_x(T_0 > s, w(X_s)) \le \mathbb{E}_x\left(T_0 > s, \frac{\Lambda(X_s)}{\Lambda(1)} + 1\right) \le \frac{\Lambda(x)}{\Lambda(1)} + \mathbb{P}_x(T_0 > s)$$
$$= \frac{\Lambda(x)}{\Lambda(1)} + \mathbb{P}_x(T_0 > s \ge T_1) + \mathbb{P}_x(T_0 > s, T_1 > s)$$
$$\le 2\frac{\Lambda(x)}{\Lambda(1)} + \mathbb{P}_x(T_0 > s, T_1 > s).$$

This last term converges to 0. In fact, $\Lambda(x) \to 0$ as $x \to 0$ and $P_x(T_0 > s, T_1 > s) \to 0$ as $x \to 0+$ because the drift is bounded on $[0, 1]$.

From (12), β depends only on w (not on the particular family $\mathscr{H}$).

LEMMA 5. *If* $\beta = 0$, *then* $F^{(\mathscr{H})}(\{+\infty\}) = 1$. *If* $\beta > 0$, *then* $F^{(\mathscr{H})}(\mathbb{R}_+) = 1$, $c \equiv \int_{\mathbb{R}_+} \varphi_{\underline{\lambda}}(x)\,dx < \infty$ *and* $dF^{(\mathscr{H})}(x) = c^{-1}\varphi_{\underline{\lambda}}(x)\,dx$.

PROOF. Since $\mathbb{P}_y(T_0 > s) \to 0$ as $s \to \infty$ for any $y \in \mathbb{R}_+$ and $P_\infty(T_0 > s) = 1$ for any $s > 0$, we get

$$M(s) = \int_{\overline{\mathbb{R}}_+} \mathbb{P}_y(T_0 > s)\,dF^{\mathscr{H}}(y) = e^{-\beta s} \to F^{\mathscr{H}}(\{+\infty\}) \quad \text{as } s \to \infty.$$

Therefore, $\beta = 0$ implies $F^{\mathscr{H}}(\{+\infty\}) = 1$ and if $\beta > 0$, then $F^{\mathscr{H}}(\{+\infty\}) = 0$.

For the rest of the proof we assume $\beta > 0$. Take A, B bounded $F^{\mathscr{H}}$-continuous sets in $\mathbb{R}_+$ of strictly positive $F^{\mathscr{H}}$ measure. By Theorem 0 due to Mandl and Lemma 2, we get

$$\frac{F^{(\mathscr{H})}(A)}{F^{(\mathscr{H})}(B)} = \lim_{t_n \to \infty} \frac{\mathbb{P}_h(X_{t_n} \in A \mid T_0 > t_n)}{\mathbb{P}_h(X_{t_n} \in B \mid T_0 > t_n)} = \frac{\int_A \varphi_{\underline{\lambda}}(x)\,dx}{\int_B \varphi_{\underline{\lambda}}(x)\,dx}.$$

Hence

$$\int_B \varphi_{\underline{\lambda}}(x)\,dx = F^{(\mathscr{H})}(B)\frac{\int_A \varphi_{\underline{\lambda}}(x)\,dx}{F^{(\mathscr{H})}(A)}.$$

Taking $B \nearrow \mathbb{R}_+$, we get $c = \int_{\mathbb{R}_+} \varphi_{\underline{\lambda}}(x)\,dx < \infty$, and moreover $F^{(\mathscr{H})}(A) = c^{-1}\int_A \varphi_{\underline{\lambda}}(x)\,dx$ for any set A as above. Since $dF^{(\mathscr{H})}(x)$ and $c^{-1}\varphi_{\underline{\lambda}}(x)\,dx$ induce probability measures on $\mathbb{R}_+$, the result follows. □

LEMMA 6. *The function* w *is* C^2 *and it is the unique solution of the equation*

$$\tfrac{1}{2}w'' - \alpha w' = -\beta w, \qquad w(0) = 0 \text{ and } w(1) = 1.$$

PROOF. Assume $\beta > 0$. Consider U, the unique solution of

$$\tfrac{1}{2}U'' - \alpha U' = -\beta w, \qquad U(0) = 0 \text{ and } U'(0) = m. \tag{14}$$

The solution of (14) is given by

$$U(x) = \int_0^x e^{\gamma(z)} \left\{ m - 2\beta \int_0^z e^{-\gamma(\xi)} w(\xi)\, d\xi \right\} dz.$$

Now, choose m such that $U(1) = 1$. To achieve the result it is enough to show $U = w$.

Recall that $T_m = \inf\{t > 0\colon X_t = M\}$. Then $S_M = T_0 \wedge T_M$ is the exit time of $(0, M)$ when starting from $x \in (0, M)$. Since α is bounded in any compact of $\mathbb{R}_+$, $\mathbb{E}_x(S_M) < \infty$ for any $0 < x \le M$. By Itô's formula,

$$\begin{aligned} \mathbb{E}_x\big(U(X_{S_M})\big) &= U(x) + \mathbb{E}_x\left(\int_0^{S_M} \mathscr{L}U(X_s)\, ds\right) \\ &= U(x) + (-\beta)\mathbb{E}_x\left(\int_0^{S_M} w(X_s)\, ds\right). \end{aligned}$$

Since $U(0) = 0$, it follows that $\mathbb{E}_x(U(X_{S_M})) = U(M)\mathbb{P}_x(T_M < T_0)$. From (3) we get

$$U(M)\frac{\Lambda(x)}{\Lambda(M)} = U(x) - \beta \mathbb{E}_x\left(\int_0^{S_M} w(X_s)\, ds\right). \tag{15}$$

Since w is a positive function, we deduce from the monotone convergence theorem that

$$\lim_{M \to \infty} \mathbb{E}_x\left(\int_0^{S_M} w(X_s)\, ds\right) = \mathbb{E}_x\left(\int_0^{T_0} w(X_s)\, ds\right).$$

From the equality $\mathbb{E}_x(\int_0^{T_0} w(X_s)\, ds) = \int_0^\infty \mathbb{E}_x(w(X_s), T_0 > s)\, ds$ and Lemma 4, we get $\mathbb{E}_x(\int_0^{T_0} w(X_s)\, ds) = w(x)\int_0^\infty e^{-\beta s}\, ds = w(x)\beta^{-1}$.

Therefore, if we pass to the limit $M \to \infty$ in (15) we obtain

$$\lim_{M \to \infty} U(M)\frac{\Lambda(x)}{\Lambda(M)} = U(x) - w(x) \quad \text{for any } x > 0.$$

Since $\Lambda(1) \ne 0$ and $U(1) = w(1)$, we get $\lim_{M \to \infty} U(M)/\Lambda(M) = 0$. Therefore, $U \equiv w$ as required.

Now assume $\beta = 0$. Since $\mathscr{L}\Lambda = 0$, we must show $w(x) = (\Lambda(x)/\Lambda(1))$ for any $x \ge 0$. Consider the function $G(x) = (\Lambda(x)/\Lambda(1)) - w(x)$. From Lemma 1, $G(x) \ge 0$ for any $x \ge 1$. Let us prove that in this case $G(x) \ge 0$ for every x. In fact, if there exists $x_0 < 1$ for which $G(x_0) < 0$, we obtain from (2) and (12),

$$-\max_{0 \le x \le 1} |G(x)| \mathbb{P}_{x_0}(X_s \le 1, T_0 > s) \le \mathbb{E}_{x_0}(G(X_s), T_0 > s) \le G(x_0) < 0.$$

Letting $s \to \infty$ and since $\mathbb{P}_{x_0}(T_0 = \infty) = 0$ we arrive at a contradiction. Therefore, $G \ge 0$.

From $G(1) = 0$, $G \geq 0$ and $\mathbb{E}_1(G(X_s), T_0 > s) \leq G(1) = 0$ we deduce $G = 0$, $\mathbb{P}_1(\cdot, T_0 > s)$-a.e. for any $s > 0$. Given that G is continuous, to achieve the result, that is, $G \equiv 0$, it is enough to prove that for any fixed interval $[a, b]$, $\mathbb{P}_1(X_s \in [a, b], T_0 > s)$ is strictly positive for some $s > 0$. Now $\mathbb{P}_1(X_s \in [a, b], T_0 > s) \geq \mathbb{P}_1(X_s \in [a, b], T_0 > s, T_M > s)$ and this quantity is strictly positive by the Cameron–Martín formula and the fact that it is so for the Brownian motion. □

LEMMA 7. *If $\beta = 0$, then $w(x) = (\Lambda(x)/\Lambda(1))$. If $\beta > 0$, then $\beta = \underline{\lambda}$, $w(x) = (e^{\gamma(x)}\varphi_{\underline{\lambda}}(x))/(e^{\gamma(1)}\varphi_{\underline{\lambda}}(1))$ for $x \geq 0$.*

PROOF. The case $\beta = 0$ has already been proved in the previous lemma. Now, if $\beta > 0$, it suffices to show $\beta = \underline{\lambda}$. In fact from Lemma 6, w will satisfy $\mathscr{L}w = -\underline{\lambda}w$, $w(0) = 0$, $w(1) = 1$, which implies $w(x) = (\psi_{\underline{\lambda}}(x)/\psi_{\underline{\lambda}}(1))$ for $x \geq 0$. The result then follows from the duality relation $\psi_{\underline{\lambda}} = e^{\gamma}\varphi_{\underline{\lambda}}$ between $\mathscr{L}$ and $\mathscr{L}^*$.

By Theorem 0 of Mandl we deduce that for f, g continuous on $\mathbb{R}_+$ with nonempty compact supports

$$\frac{\mathbb{E}_h(f(X_t) \mid T_0 > t)}{\mathbb{E}_h(g(X_t) \mid T_0 > t)} \to \frac{\int f(x)\varphi_{\underline{\lambda}}\,dx}{\int g(x)\varphi_{\underline{\lambda}}\,dx} \quad \text{as } t \to \infty.$$

Notice that $\eta^f(x) = \mathbb{E}_x(f(X_s), T_0 > s)$ is a continuous bounded function on $\mathbb{R}_+$ (the same for g). From Lemma 5, $F^{\mathscr{H}}(\mathbb{R}_+) = 1$ and $dF^{\mathscr{H}}(x) = c^{-1}\varphi_{\underline{\lambda}}(x)\,dx$. Therefore, from Lemma 2 we get

$$\frac{\mathbb{E}_h\big(f(X_{t_n+s}) \mid T_0 > t_n + s\big)}{\mathbb{E}_h\big(g(X_{t_n+s}) \mid T_0 > t_n + s\big)} = \frac{\mathbb{E}_h\big(\eta^f(X_{t_n}) \mid T_0 > t_n\big)}{\mathbb{E}_h\big(\eta^g(X_{t_n}) \mid T_0 > t_n\big)}$$

$$\to \frac{\int \eta^f(x)\varphi_{\underline{\lambda}}(x)\,dx}{\int \eta^g(x)\varphi_{\underline{\lambda}}(x)\,dx} \quad \text{as } t_n \to \infty.$$

Therefore, letting $g \nearrow 1$ and since $\int \varphi_{\underline{\lambda}}(x)\,dx < \infty$, we obtain

$$c^{-1}\int f(x)\varphi_{\underline{\lambda}}(x)\,dx = \frac{\int \mathbb{E}_x(f(X_s), T_0 > s)\varphi_{\underline{\lambda}}(x)\,dx}{\int \mathbb{P}_x(T_0 > s)\varphi_{\underline{\lambda}}(x)\,dx}.$$

Given that (see Lemma 5) $c^{-1}\int \mathbb{P}_x(T_0 > s)\varphi_{\underline{\lambda}}(x)\,dx = M(s) = e^{-\beta s}$, we get

$$e^{-\beta s}\int f(x)\varphi_{\underline{\lambda}}(x)\,dx = \int \mathbb{E}_x(f(X_s), T_0 > s)\varphi_{\underline{\lambda}}(x)\,dx, \tag{16}$$

which, by integrating s on $(0, t)$ yields

$$\frac{1 - e^{-\beta t}}{\beta}\int f(x)\varphi_{\underline{\lambda}}(x)\,dx = \int \mathbb{E}_x\left(\int_0^t f(X_s)1_{T_0 > s}\,ds\right)\varphi_{\underline{\lambda}}(x)\,dx.$$

Let g be a C^2 function with compact support contained in $(0, \infty)$ and consider $f = \mathscr{L}g$. Then the last equality and Itô's formula give

$$\frac{1 - e^{-\beta t}}{\beta}\int \mathscr{L}g(x)\varphi_{\underline{\lambda}}(x)\,dx = \int \big(\mathbb{E}_x(g(X_t), T_0 > t) - g(x)\big)\varphi_{\underline{\lambda}}(x)\,dx.$$

Let $t \to \infty$ to obtain

$$\int \mathscr{L} g(x) \varphi_{\underline{\lambda}}(x)\, dx = -\beta \int g(x) \varphi_{\underline{\lambda}}(x)\, dx.$$

Finally, integration by parts gives

$$\int \mathscr{L} g(x) \varphi_{\underline{\lambda}}(x)\, dx = \int g(x) \mathscr{L}^* \varphi_{\underline{\lambda}}(x)\, dx$$

and this last term is $-\underline{\lambda} \int g(x)\varphi_{\underline{\lambda}}(x)\, dx$. This shows that $\beta = \underline{\lambda}$. □

Proof of Theorem A. We have shown that for any sequence (t_n) there exists a subsequence (t'_n) such that $v_{t'_n} \to w$ as $t'_n \to \infty$. The function w satisfies $\mathscr{L} w = -\beta w$, $w(0) = 0$, $w(1) = 1$, where β depends on w and is either 0 or $\underline{\lambda}$. This means that the only possible limit points of (v_t) are of the form $w(x) = \Lambda(x)/\Lambda(1)$ for any $x \geq 0$ or $w(x) = \psi_{\underline{\lambda}}(x)/\psi_{\underline{\lambda}}(1)$ for any $x \geq 0$.

Assume that (v_t) does not converge. Therefore, there exists $x_0 \in \mathbb{R}_+$ (which is necessarily different from 0 and 1) and two subsequences $(t'_n), (t''_n)$ such that $v_{t'_n}(x_0) \to a'$ as $t'_n \to \infty$, $v_{t''_n}(x_0) \to a''$ as $t''_n \to \infty$ and $a' \neq a''$. Passing if necessary to subsequences, we can assume that $v_{t'_n}$ and $v_{t''_n}$ converge pointwise. Without loss of generality, we assume $v_{t'_n}(\cdot) \to \Lambda(\cdot)/\Lambda(1)$ as $t'_n \to \infty$ and $v_{t''_n}(\cdot) \to \psi_{\underline{\lambda}}(\cdot)/\psi_{\underline{\lambda}}(1)$ as $t''_n \to \infty$. Since $a' \neq a''$ and $v_t(x_0)$ is continuous in t, we can construct a sequence (t''_n) such that $v_{t'''_n}(x_0) \to (a' + a'')/2$ as $t'''_n \to \infty$, which is different from $a' = \Lambda(x_0)/\Lambda(1)$ and $a'' = \psi_{\underline{\lambda}}(x_0)/\psi_{\underline{\lambda}}(1)$. Now, by passing to a subsequence of (t'''_n) we can assume that $v_{t'''_n}$ converges pointwise either to $\Lambda(\cdot)/\Lambda(1)$ or to $\psi_{\underline{\lambda}}(\cdot)/\psi_{\underline{\lambda}}(1)$, which is a contradiction. Therefore, (v_t) converges and we denote its limit by W. Moreover, we denote by η the parameter β associated to $w = W$ in (12).

Let us show

$$\frac{\mathbb{P}_x(T_0 > t + s)}{\mathbb{P}_x(T_0 > t)} \to e^{-\eta s} \quad \text{as } t \to \infty, \text{for any } s > 0. \tag{17}$$

Since $\mathbb{P}_x(T_0 > t)/\mathbb{P}_1(T_0 > t) \to W(x)$ as $t \to \infty$ it suffices to show the result for $x = 1$. By the Markov property,

$$\frac{\mathbb{P}_1(T_0 > t + s)}{\mathbb{P}_1(T_0 > t)} = \mathbb{E}_1\left(T_0 > s, \frac{\mathbb{P}_{X_s}(T_0 > t)}{\mathbb{P}_1(T_0 > t)}\right) = \mathbb{E}_1(T_0 > s, v_t(X_s)).$$

We have $v_t(X_s) \to W(X_s)$ as $t \to \infty$. Using (6), (2), the dominated convergence theorem and (12) we obtain

$$\lim_{t\to\infty} \frac{\mathbb{P}_1(T_0 > t + s)}{\mathbb{P}_1(T_0 > t)} = \mathbb{E}_1(T_0 > s, W(X_s)) = e^{-\eta s} W(1) = e^{-\eta s}.$$

To finish the proof it suffices to show that $\eta = -\lim_{t\to\infty}(1/t)\log \mathbb{P}_x(T_0 > t)$. This is a direct consequence of (17), being easily deduced from the equality $(1/n)\log \mathbb{P}_x(T_0 > n) = (1/n)\sum_{k=0}^{n-1}\log(\mathbb{P}_x(T_0 > k + 1)/\mathbb{P}_x(T_0 > k))$. □

Since W and η are uniquely determined, from Lemmas 5 and 2 we deduce that for any $h \in \mathscr{D}_0^0(\mathbb{R}_+)$, $\mathbb{E}_h(X_t \in \cdot \mid T_0 > t)$ converges weakly to a distribution F independent of h, which is either δ_∞ if $\eta = 0$ or $dF(x) = c^{-1}\varphi_{\underline{\lambda}}(x)\,dx$ if $\eta > 0$. In the case $\eta > 0$, equation (16) implies that $\varphi_{\underline{\lambda}}\overline{P}_s = e^{-\lambda s}\varphi_{\underline{\lambda}}$. Hence, if the starting distribution is $dF(x) = c^{-1}\varphi_{\underline{\lambda}}(x)\,dx$, the hitting time T_0 is exponentially distributed with parameter $\underline{\lambda}$.

Proof of Theorem B. Let $A \in \mathscr{F}_s$. Then by the Markov property,

$$\begin{aligned}\mathbb{P}_x(X \in A \mid T_0 > t) &= \frac{\mathbb{P}_x(X \in A, T_0 > t)}{\mathbb{P}_x(T_0 > t)}\\ &= \frac{\mathbb{E}_x\big(X \in A, T_0 > s, \mathbb{P}_{X_s}(T_0 > t - s)\big)}{\mathbb{P}_x(T_0 > t)}\\ &= \frac{\mathbb{E}_x(X \in A, v_{t-s}(X_s))}{v_t(x)}\,\frac{\mathbb{P}_1(T_0 > t-s)}{\mathbb{P}_1(T_0 > t)}.\end{aligned}$$

From the dominated convergence theorem and Theorem A, we get

$$\lim_{t\to\infty}\mathbb{P}_x(X \in A \mid T_0 > t) = \mathbb{E}_x\left(X \in A, T_0 > s, \frac{W(X_s)}{W(x)}e^{\eta s}\right),$$

from which the result follows. □

Proof of Theorem C. Take $h \in \mathscr{D}_0^0(\mathbb{R}_+)$ with support $h \subset (x - \varepsilon, x)$. By monotonicity,

$$\frac{\mathbb{P}_h(X_t > a, T_0 > t)}{\mathbb{P}_h(T_0 > t)} \le \frac{\mathbb{P}_x(X_t > a, T_0 > t)}{\int h(y)\mathbb{P}_y(T_0 > t)\,dy}.$$

The term on the left-hand side converges to $F(a, +\infty]$. Now, dividing by $\mathbb{P}_x(T_0 > t)$, letting $t \to \infty$ and using Theorem A, we deduce

$$F(a, +\infty] \le \frac{1}{\int h(y)(W(y)/W(x))\,dy}\liminf_{t\to\infty}\mathbb{P}_x(X_t > a \mid T_0 > t).$$

Finally, let ε converge to 0 to deduce

$$F(a, +\infty] \le \liminf_{t\to\infty}\mathbb{P}_x(X_t > a \mid T_0 > t).$$

By a similar argument, considering support $h \subset (x, x + \varepsilon)$, we obtain $F(a, +\infty] \ge \limsup_{t+\infty}\mathbb{P}_x(X_t > a \mid T_0 > t)$. Then the result follows from Lemma 5. □

Proof of Proposition D. (a) When $\eta = 0$, we have $W(x) = \Lambda(x)/\Lambda(1)$. Therefore, from (12) we get $\mathbb{E}_x(\Lambda(X_s), T_0 > s) = \Lambda(x)$, which by the Markov property implies that $(\Lambda(X_s)1_{T_0 > s})_{s \ge 0}$ is a $\mathbb{P}_x$-martingale.

(b) Consider the process (Z_t) introduced in (4). Let $T_a^Z = \inf\{t \ge 0: Z_t = a\}$ and $S^Z = \lim_{a \nearrow \infty} T_a^Z$.

Let $G(y) = 1/\Lambda(y)$ for $y > 0$. A direct computation shows that $\frac{1}{2}G'' + (\Lambda'/\Lambda - \alpha)G' = 0$. Therefore, by Itô's formula, if $0 < \varepsilon < x < a$,

$$G(x) = \mathbb{E}_x\big(G(Z_{T_a^Z \wedge T_\varepsilon^Z})\big) = G(\varepsilon)\mathbb{P}_x\big(T_\varepsilon^Z < T_a^Z\big) + G(a)\mathbb{P}_x\big(T_a^Z < T_\varepsilon^Z\big),$$

from which we deduce that $\mathbb{P}_x(T_a^Z < T_\varepsilon^Z) = (G(x) - G(\varepsilon))/(G(a) - G(\varepsilon))$. Since $G(0+) = \infty$, we obtain $\forall\, a > x$, $\mathbb{P}_x(T_a^Z < T_0^Z) = 1$. Hence

$$\mathbb{P}_x\big(S^Z \le T_0^Z\big) = 1. \tag{18}$$

An application of Girsanov's theorem [see Karatzas and Shreve (1988), page 191] leads to

$$\mathbb{E}_x\left(X \in A, \frac{\Lambda(X_s)}{\Lambda(x)}, T_a > s, T_\varepsilon > s\right) = \mathbb{P}_x\big(Z \in A, T_a^Z > s, T_\varepsilon^Z > s\big)$$

for $A \in \mathscr{F}_s$, $0 < \varepsilon < x < a$. Now, let $\varepsilon \to 0+$, $a \to \infty$ and use (18) to obtain

$$\forall\, s > 0, \forall\, A \in \mathscr{F}_s,$$

$$\mathbb{E}_x\left(X \in A, \frac{\Lambda(X_s)}{\Lambda(x)}, T_0 > s\right) = \mathbb{P}_x(Z \in A, S^Z > s). \tag{19}$$

From $\Lambda(x) = \Lambda(M)\mathbb{P}_x(T_M \le s,\ T_M < T_0) + \mathbb{E}_x(\Lambda(X_s),\ s < T_M \wedge T_0)$ and $\mathbb{P}_x(T_M < T_0) = \Lambda(x)/\Lambda(M)$, we get

$$\mathbb{P}_x(T_M > s \mid T_M < T_0) = \mathbb{E}_x\left(\frac{\Lambda(X_s)}{\Lambda(x)}, s < T_M \wedge T_0\right).$$

From the monotone convergence theorem and (19) we obtain

$$\lim_{M \to \infty} \mathbb{P}_x(T_M > s \mid T_M < T_0) = \mathbb{E}_x\left(\frac{\Lambda(X_s)}{\Lambda(x)}, s < T_0\right) = \mathbb{P}_x(S^Z > s).$$

The middle term is equal to 1 by (a). From (18) we get $\mathbb{P}_x(T_0^Z \ge S^Z > s) = 1$ for any $s > 0$ and (b) follows.

(c) We have

$$\begin{aligned}\mathbb{P}_x(X \in A, T_M > s \mid T_M < T_0) &= \frac{\mathbb{E}_x\big(X \in A, T_M > s, T_0 > s, \mathbb{P}_{X_s}(T_M < T_0)\big)}{\mathbb{P}_x(T_M < T_0)}\\ &= \mathbb{E}_x\left(X \in A, T_M > s, T_0 > s, \frac{\Lambda(X_s)}{\Lambda(x)}\right)\\ &= \mathbb{P}_x\big(Z \in A, T_M^Z > s\big).\end{aligned}$$

The last equality follows from (19). Therefore,

$$\lim_{M \to \infty} \mathbb{P}_x(X \in A, T_M > s \mid T_M < T_0) = \mathbb{P}_x(Z \in A, S^Z > s) = \mathbb{P}_x(Z \in A).$$

On the other hand, $\lim_{M \to \infty} \mathbb{P}_x(X \in A, T_M < s \mid T_M < T_0) \le \lim_{M \to \infty} \mathbb{P}_x(T_M < s \mid T_M < T_0) = 0$. Then the result follows. □

Acknowledgments. The authors are thankful for discussions with P. A. Ferrari (University of São Paulo) and K. Burdzy (University of Seattle). They are indebted to support from Fundacação Vitae, program Evaluation Cooperation Scientifique–Comisión Nacional Científica y Tecnológica and Comunidad Económica Europea CI1*-CT92 0046. S. M. and J. S. M. were partially supported by grants Fondo Nacional de Ciencia y Tecnología and Departamento Técnico de Investigación. P. C. is glad to thank the Departamento de Ingeniería Matemática of the Universidad de Chile for its kind hospitality. The authors thank the referee for calling their attention to an error in the previous version of Lemma 2 and for comments which allowed the results of this work to be improved.

REFERENCES

AZENCOTT, R. (1974). Behavior of diffusion semigroups at infinity. *Bull. Soc. Math. France* **102** 193–240.

CODDINGTON, E. A. and LEVINSON, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill, New York.

FELLER, W. (1952). The parabolic differential equation and the associated semi-groups of transformations. *Ann. Math.* **55** 468–519.

KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus*. Springer, New York.

KNIGHT, F. B. (1969). Brownian local times and taboo processes. *Trans. Amer. Math. Soc.* **143** 173–185.

MANDL, P. (1961). Spectral theory of semi-groups connected with diffusion processes and its applications. *Czech. Math. J.* **11** 558–569.

MARTÍNEZ, S. and SAN MARTÍN, J. (1994). Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.* **31** 911–920.

PINSKY, R. G. (1985). On the convergence of diffusion processes conditioned to remain in bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.* **13** 363–378.

ROGERS, L. and PITMAN, J. (1981). Markov functions. *Ann. Probab.* **9** 573–582.

WILLIAMS, D. (1970). Decomposing the Brownian path. *Bull. Amer. Math. Soc.* **76** 871–873.

PIERRE COLLET
C.N.R.S.
PHYSIQUE THÉORIQUE
ECOLE POLYTECHNIQUE
91128 PALAISEAU CEDEX
FRANCE

SERVET MARTÍNEZ
JAIME SAN MARTÍN
UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS
Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA
CASILLA 170-3, CORREO 3
SANTIAGO
CHILE