The Annals of Probability
2003, Vol. 31, No. 1, 486–527

PINCHING AND TWISTING MARKOV PROCESSES

BY STEVEN N. EVANS[1] AND RICHARD B. SOWERS[2]

University of California, Berkeley and University of Illinois

We develop a technique for "partially collapsing" one Markov process to produce another. The state space of the new Markov process is obtained by a pinching operation that identifies points of the original state space via an equivalence relationship. To ensure that the new process is Markovian we need to introduce a randomized twist according to an appropriate probability kernel. Informally, this twist randomizes over the uncollapsed region of the state space when the process leaves the collapsed region. The "Markovianity" of the new process is ensured by suitable intertwining relationships between the semigroup of the original process and the pinching and twisting operations. We construct the new Markov process, identify its resolvent and transition function and, under some natural assumptions, exhibit a core for its generator. We also investigate its excursion decomposition. We apply our theory to a number of examples, including Walsh's spider and a process similar to one introduced by Sowers in studying stochastic averaging.

1. Introduction. *Walsh's spider* ([18]; see also [2, 3, 17]) can be thought of informally as a Markov process, the state space of is which n copies of $\mathbb{R}_+$ for some positive integer n. The process evolves as reflecting Brownian motion on a given copy until it hits the zero point of that copy, at which time it moves to the zero point of some (possibly the same) copy of $\mathbb{R}_+$, with the ith copy being chosen with some probability p_i independently of the previous evolution. Of course, the way to begin making rigorous sense of this prescription is to identify the n zero points of the copies of $\mathbb{R}_+$ as a single point and get a state space that can be thought of as n semi-infinite rays issuing from the origin in $\mathbb{R}^2$ (cf. Example 1 below). In addition to its intrinsic interest, the spider plays an important role in the work of Tsirelson [17] on the structure of Brownian filtrations (see also [3]).

Also, spiderlike processes are the fundamental building blocks for constructing diffusions on graphs. Processes that take values on graphs appear in the work of Freidlin and Wentzell [10], extending various classical stochastic averaging results for partial differential equations (PDEs). A higher dimensional diffusion with a structure somewhat akin to that of the spider plays a similar role in the related work of Sowers [16] on Hamiltonian systems perturbed by noise.

Received January 2001; revised July 2002.

[1]Research supported in part by NSF Grants DMS-97-03845 and DMS-00-71468.

[2]Research supported in part by NSF Grants DMS-96-15877 and DMS-00-7148 and ONR Grant N00014-01-1-0647.

AMS 2000 subject classifications. Primary 60J35; secondary 60J40, 60J60.

Key words and phrases. Markov function, intertwining, right process, Feller process, excursion theory, Walsh's spider, stratified space.

Rather than introduce the particular process from [16] now, we can describe a slightly simplified version of it as follows. We begin with reflecting Brownian motion on the unit ball in $\mathbb{R}^d$. When the process hits the boundary of the unit ball (i.e, the unit sphere), it is instantaneously restarted at another uniformly chosen point on the unit sphere. Again, the first step in making sense of this description as a nice Markov process involves identifying the points on the unit sphere as a single point. This identification turns the unit ball in $\mathbb{R}^d$ into a new state space that is homeomorphic to the unit sphere in $\mathbb{R}^{d+1}$ (cf. Example 2 below). We discuss the resulting process on the d sphere in Example 2 below and consider the actual process from [16] in Example 3.

Some of the key reasons the process on the d sphere described above is Markovian are that the radial part of the original reflecting Brownian motion in the ball is Markovian, that the process is given a random twist at a hitting time for the radial part process and that the randomization is over a level set of the radial part map (i.e., a sphere). Similar features are behind the Markov property for the spider. It is our aim in this article to study a general construction that covers both of these examples.

The results herein are also used in [1] to develop a general technique for constructing new Markov processes from existing ones. The new process and its state space are both projective limits of sequences built by an iterative scheme. The space at each stage in the scheme is obtained by taking disjoint copies of the space at the previous stage and quotienting to identify certain distinguished points. Away from the distinguished points, the process at each stage evolves like the one constructed at the previous stage on some copy of the previous state space, but when the process hits a distinguished point it enters at random another of the copies "pinned" at that point. Special cases of this construction produce diffusions on fractallike objects with interesting analytic properties that have been studied recently.

Our effort is organized as follows. First, we shall define a general topological setup involving a "pinching" map π that collapses an initial space E into the state space $\tilde{E}$ of the process we desire to construct (e.g., the spider space or the d sphere). To construct a Markov process $\tilde{X}$ on $\tilde{E}$ we first introduce a Markov process X on E and assume that the pinching operation intertwines in a suitable way with the evolution dynamics of X (Hypothesis 2.6). More specifically, the space E is decomposed into two pieces: a closed set A and its complement $E \setminus A$. The pinching map π is injective on $E \setminus A$ (intuitively, no pinching occurs on $E \setminus A$), whereas it is generally not injective on A. The process $\tilde{X}$ evolves according to the dynamics of $\pi \circ X$ when X is in the interior of either A and $E \setminus A$, and our intertwining assumption on π ensures that these dynamics are Markovian.

To complete the description of $\tilde{X}$, we need to describe how $\tilde{X}$ passes between $\pi(A)$ and $\pi(E \setminus A)$ (which we can identify with $E \setminus A$). This is accomplished by a "twist" operator K that describes the random mechanism by which $\pi(E \setminus A)$ is entered from $\pi(A)$. To ensure that the resulting dynamics for $\tilde{X}$ are

Markovian, this operator must also intertwine appropriately with the dynamics of X (Hypothesis 2.8).

Our basic result, Theorem 2.13, avers the existence of an appropriate Markov process $\tilde{X}$ on the desired space $\tilde{E}$. After discussing the random twist mechanism and related intertwining assumptions, we study the generator of $\tilde{X}$ and, under some simplifying assumptions, its excursion decomposition. Throughout our development, we focus on a number of examples.

We should also mention that our construction is perhaps not as general as one might like. The various intertwining relationships mentioned above impose a certain "homogeneity" on our processes. For example, we cannot use it to produce a process like Walsh's spider that evolves as an arbitrary Markov process on each leg (see, however, the continuation of Example 1 in Section 3 concerning processes like Walsh's spider that evolve as a different one-dimensional diffusion on each leg).

2. General setup.

2.1. *Topological structures.* Consider the following setup.

ASSUMPTION 2.1 (Spaces). Let E and $\hat{E}$ be two Hausdorff, locally compact, second countable topological spaces. Thus E and $\hat{E}$ are, in particular, Polish (i.e., metrizable as complete, separable metric spaces). Let $\psi : E \to \hat{E}$ be a continuous surjection. Let $A \subseteq E$ be closed and define

$$\tilde{E} \stackrel{\text{def}}{=} (E \setminus A) \cup \psi(A),$$

this being a disjoint union. We further assume that $\psi^{-1}(\psi(A)) = A$ and that $\psi^{-1}(K)$ is compact for any compact subset K of $\hat{E}$.

Informally, we get $\tilde{E}$ by pinching A into $\psi(A)$; that is, if $z \in \psi(A)$, we pinch all elements of $\psi^{-1}(z)$ into z (note that we are most definitely *not* assuming that ψ is one to one). Suppose now that we have a "nice" Markov process X with state space E (we will be precise about this in Section 2.2). Our goal is to construct in certain situations a Markov process $\tilde{X}$ on $\tilde{E}$ by pinching X to $\psi \circ X$ when X is in A, but retaining the original dynamics of X when it is in $E \setminus A$ (the rigorous result is given in Section 2.4). The interesting part of such a construction is what happens when $\tilde{X}$ "leaves" $E \setminus A$ and enters $\psi(A)$ or vice versa; this will be a central issue in our study (see the discussion preceding Hypothesis 2.8).

Let us start by introducing a topology on $\tilde{E}$ that is compatible with the pinching procedure.

ASSUMPTION 2.2 (Topological pinching). Define the map $\pi : E \to \tilde{E}$ by

$$\pi(x) \stackrel{\text{def}}{=} \begin{cases} x, & \text{if } x \in E \setminus A, \\ \psi(x), & \text{if } x \in A, \end{cases} \tag{2.1}$$

and give $\tilde{E}$ the topology induced by π. That is, $\mathscr{N} \subset \tilde{E}$ is open in the topology of $\tilde{E}$ if and only if $\pi^{-1}(\mathscr{N})$ is open in the topology of E. [Equivalently, we can think of $\tilde{E}$ as the quotient topological space of the topological space E under the equivalence relationship that declares two points x' and x'' equivalent if and only if $\pi(x') = \pi(x'')$.] We assume that $\tilde{E}$ with this topology is Hausdorff, locally compact and second countable (and hence Polish).

A more explicit understanding of the topology on $\tilde{E}$ might be of help.

LEMMA 2.3. *Fix a sequence* (x_n) *in* $\tilde{E}$ *that converges* (*in the topology of* $\tilde{E}$) *to* $x \in \tilde{E}$. *Then the following hold*:

(i) *If* $x \in E \setminus A$, *then* $x_n \in E \setminus A$ *for* n *sufficiently large and* $\lim_{n : x_n \in E \setminus A} x_n = x$ *in the topology of* E.

(ii) *If* $x \in \psi(A)$ *and* $x_n \in E \setminus A$ *for all* n, *then* $\lim_n \psi(x_n) = x$ *in the topology of* $\hat{E}$.

(iii) *If* $x \in \psi(A)$ *and* $x_n \in \psi(A)$ *for all* n, *then* $\lim_n x_n = x$ *in the topology of* $\hat{E}$.

The proof of this lemma is given in Section 7. Essentially, this result says that the topology of $\tilde{E}$ is equivalent to that of E on $E \setminus A$ and equivalent to that of $\hat{E}$ on $\psi(A)$, but that points on the boundary of $E \setminus A$ (in the topology of E) are identified with points on the boundary of $\psi(A)$ (in the topology of $\hat{E}$) via the map ψ.

Let us now introduce some illustrative examples that we will develop in the course of this article.

EXAMPLE 1 (Spider). Define $\mathbb{R} \stackrel{\text{def}}{=} [0, \infty)$ as usual and set $\mathcal{I}_n \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ for a positive integer n. Put $E \stackrel{\text{def}}{=} \mathbb{R}_+ \times \mathcal{I}_n$, $\hat{E} \stackrel{\text{def}}{=} \mathbb{R}_+$, $\psi(x, i) \stackrel{\text{def}}{=} x$ for all $(x, i) \in E$ and $A \stackrel{\text{def}}{=} \bigcup_{i \in \mathcal{I}_n} \{(0, i)\}$. Then all points $(0, i)$ are collapsed into a single point and $\tilde{E}$ is homeomorphic to the state space of Walsh's spider, that is, to a collection of n rays emanating from the origin of $\mathbb{R}^2$ (equipped with the subspace topology inherited from $\mathbb{R}^2$). Consequently, $\tilde{E}$ is indeed Hausdorff, locally compact and second countable—as required by our standing assumptions. See Figure 1.

EXAMPLE 2 (Ball to sphere). For future reference, let $B_d(0, 1) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} < 1\}$ be the open unit ball in $\mathbb{R}^d$ and let $S^{d-1} \stackrel{\text{def}}{=} \partial B_d(0, 1) = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} = 1\}$ be the unit sphere in $\mathbb{R}^d$. Define $E \stackrel{\text{def}}{=} \bar{B}_d(0, 1) = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} \le 1\}$ and $\hat{E} \stackrel{\text{def}}{=} [0, 1]$. Set $\psi(x) \stackrel{\text{def}}{=} \|x\|_{\mathbb{R}^d}$ for all $x \in E$ and let $A \stackrel{\text{def}}{=} S^{d-1}$. Thus all points on the boundary of a d-dimensional unit ball are

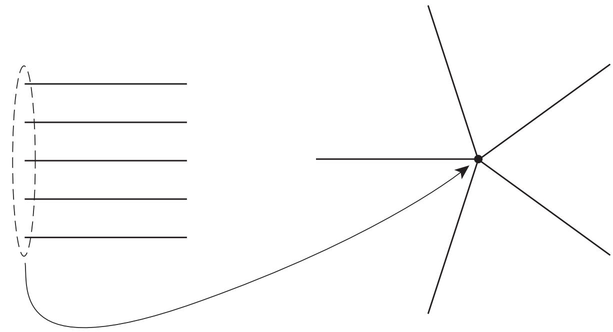

FIG. 1. *Walsh's spider. Left-hand endpoints are mapped into the vertex.*

collapsed into a point and $\tilde{E}$ can be identified with the d-dimensional unit sphere S^d by mapping points in $\bar{B}_d(0,1)$ that are at Euclidean distance r from the origin to points on S^d that are a spherical distance $\pi(1-r)$ from the north pole (so that the boundary of the ball is mapped into the north pole). In fact, this map gives a homeomorphism between $\tilde{E}$ and S^d (equipped with the subspace topology inherited from $\mathbb{R}^d$), and hence $\tilde{E}$ is certainly Hausdorff, locally compact and second countable. See Figure 2.

EXAMPLE 3 (Lollipop). This example is along the same lines as the previous example. Set $E \stackrel{\text{def}}{=} \mathbb{R}^d$ and $\hat{E} \stackrel{\text{def}}{=} \mathbb{R}_+$. Define $\psi(x) \stackrel{\text{def}}{=} \|x\|_{\mathbb{R}^d}$ for all $x \in \mathbb{R}^d$ and put $A = \mathbb{R}^d \setminus B_d(0,1)$. Each sphere (centered at the origin) of radius $r \geq 1$ is collapsed to the point r; thus the boundary of the unit ball is mapped into a point and the complement of the (closed) unit ball is mapped into the line segment $(1,\infty)$, which is attached to the point that represents the boundary of the unit ball. This amounts to mapping the unit ball in $\mathbb{R}^d$ into the unit sphere S^d, where the north pole of the unit sphere represents the boundary of the unit ball, and attaching a semi-infinite whisker to the north pole. We see that $\tilde{E}$ is homeomorphic to this subset of $\mathbb{R}^{d+1}$ (equipped with the subspace topology), and hence $\tilde{E}$ is Hausdorff, locally compact and second countable. See Figure 3.

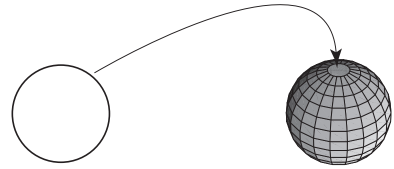

FIG. 2. *Ball to sphere. The boundary of the unit ball is mapped to the north pole of the sphere.*

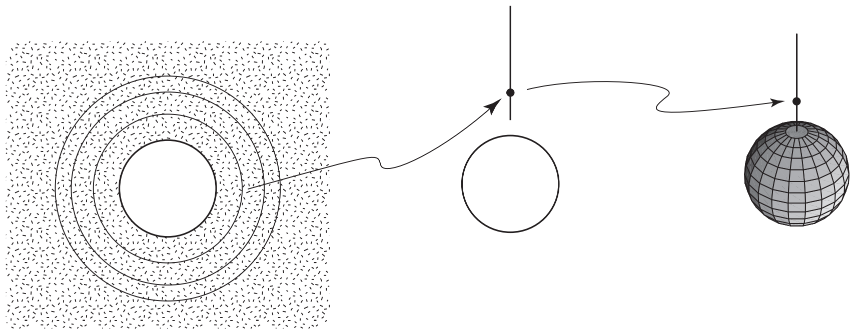

FIG. 3. *Lollipop. The unit ball is mapped into the sphere and the exterior of the unit ball is mapped into the line.*

EXAMPLE 4 (Skew product). This is something of a generalization of Example 1. Define $E \stackrel{\text{def}}{=} E' \times E''$ to be the Cartesian product of two second countable, Hausdorff spaces E' and E'', where E' is locally compact and E'' is compact. Let $A = A' \times E''$, where A' is a closed subset of E', and define $\psi(x', x'') \stackrel{\text{def}}{=} x'$ for all $(x', x'') \in E$. If we let d' be a metric on E' and let d'' be a metric on E'', we can define a metric $\tilde{d}$ that gives the topology of $\tilde{E}$ by $\tilde{d}(x', y') = d'(x', y')$ if x' and y' are in $\psi(A) = A'$, by $d(x', (y', y'')) = d'(x', y')$ if $x' \in A'$ and $(y', y'') \in E \setminus A$, and by

$$d\big((x', x''), (y', y'')\big) = [d'(x', y') + d''(x'', y'')] \wedge \inf_{z' \in A'} [d'(x', z') + d'(z', y')]$$

if (x', x'') and (y', y'') are both in $E \setminus A$. It is clear that $\tilde{E}$ is Hausdorff, locally compact and second countable.

Before introducing some probability, we need to fix a little more notation. First, observe that there is a natural map $\phi : \tilde{E} \to \hat{E}$ defined by

$$\phi(x) \stackrel{\text{def}}{=} \begin{cases} \psi(x), & \text{if } x \in E \setminus A, \\ x, & \text{if } x \in \psi(A). \end{cases}$$

We prove in Lemma 7.1 that ϕ is continuous. We have that $\psi = \phi \circ \pi$ or, equivalently, we have the commutative diagram

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[dr, "\psi"'] & \tilde{E} \arrow[d, "\phi"] \\ & \hat{E} \end{tikzcd} .$$

We will also use some standard notation from functional analysis. To fix this notation, let S denote a Hausdorff, locally compact, second countable topological space. We then let $B(S)$ be the Banach space of bounded real-valued functions on S and we let $B^+(S)$ be the collection of nonnegative elements of $B(S)$. Let $C_0(S)$ be the Banach space of real-valued continuous functions on S that vanish at infinity [if S is compact, then of course $C_0(S) = C(S)$, the Banach space of continuous functions on S].

For any subset R of S, define $B(S; R) \stackrel{\text{def}}{=} \{f \in B(S) : f|_R \equiv 0\}$ and $C_0(S; R) \stackrel{\text{def}}{=} C_0(S) \cap B(S; R)$.

Finally, we set up some operators that map between various spaces of functions. If S' is a second locally compact space and ξ is a measurable map from S to S', we define $\xi^* : B(S') \to B(S)$ as $\xi^* f \stackrel{\text{def}}{=} f \circ \xi$. If ξ is continuous and $\xi^{-1}(K)$ is a compact subset of S for all compact subsets K of S', then $\xi^* : C_0(S') \to C_0(S)$. In terms of our commutative diagrams, we thus have

$$\begin{tikzcd} B(E) & B(\tilde{E}) \arrow[l, "\pi^*"'] \\ & B(\hat{E}) \arrow[ul, "\psi^*"] \arrow[u, "\phi^*"'] \end{tikzcd} \quad . \tag{2.2}$$

In fact, the results in Section 7 ensure that $\phi^* : C_0(\hat{E}) \to C_0(\tilde{E})$, $\pi^* : C_0(\tilde{E}) \to C_0(E)$ and $\psi^* : C_0(\hat{E}) \to C_0(E)$.

Define $\check{\pi}_* : B(E; A) \to B(\tilde{E}; \psi(A))$ by

$$(\check{\pi}_* f)(x) \stackrel{\text{def}}{=} \begin{cases} f(x), & \text{if } x \in E \setminus A, \\ 0, & \text{if } x \in \psi(A). \end{cases}$$

Let $I_{\tilde{E}}$ be the identity map on $B(\tilde{E})$.

LEMMA 2.4. *We have that*:

(a) $(I_{\tilde{E}} - \phi^* K \pi^*) B(\tilde{E}) \subseteq B(\tilde{E}; \psi(A))$;
(b) $\psi^* B(\hat{E}; \psi(A)) \subseteq B(E; A)$ *and* $\phi^* = \check{\pi}^* \psi^*$ *on* $B(\hat{E}; \psi(A))$;
(c) $\pi^* \check{\pi}_*$ *is the identity map on* $B(E; A)$.

PROOF. Claim (a) follows directly from claim (c) of Lemma 2.7. To see claim (b), fix $f \in B(\hat{E}; \psi(A))$. If $x \in A$, then $\psi^* f(x) = f(\psi(x)) = 0$. Second,

$$\check{\pi}^* \psi^* f(x) = \begin{cases} f(\psi(x)), & \text{if } x \in E \setminus A, \\ 0, & \text{if } x \in \psi(A), \end{cases}$$
$$= \begin{cases} f(\phi(x)), & \text{if } x \in E \setminus A, \\ 0, & \text{if } x \in \psi(A). \end{cases}$$

On the other hand, if $x \in \psi(A)$, then $\phi^* f(x) = f(x) = 0$. This gives us claim (b).

To see claim (c), fix $f \in B(E; A)$; then

$$\pi^* \check{\pi}_* f(x) = \begin{cases} \check{\pi}_* f(x), & \text{if } x \in E \setminus A, \\ \check{\pi}_* f(\psi(x)), & \text{if } x \in A, \end{cases}$$
$$= \begin{cases} f(x), & \text{if } x \in E \setminus A, \\ 0, & \text{if } x \in A, \end{cases}$$
$$= f(x). \qquad \square$$

2.2. *Intertwinings.* Let us next fix our basic stochastic process.

ASSUMPTION 2.5. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathbb{P}^x)$ be a conservative Borel right process with state space E and transition semigroup (P_t).

As we mentioned above, we want to pinch X when it is on A. We want this pinched process to be Markovian on $\psi(A)$, so we will impose:

HYPOTHESIS 2.6 (Dynkin intertwining relationship). *Suppose that there is a second Borel right process $\hat{X} = (\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\mathbb{P}}^x)$ with transition semigroup $(\hat{P}_t)$ such that*

$$P_t \psi^* = \psi^* \hat{P}_t. \tag{2.3}$$

This implies that the finite-dimensional distributions of $\psi \circ X$ under $\mathbb{P}^x$ are the same as those of $\hat{X}$ under $\hat{\mathbb{P}}^{\psi(x)}$ for any $x \in E$ (see [15], *Section* II.13).

Hypothesis 2.6 means that the evolution of $\hat{X}$ is Markovian as long as it stays in $\psi(A)$; we will also use in Section 5 the fact that $\hat{X}$ is Markovian even when X enters $E \setminus A$. Essentially, this will allow us to understand the excursions of X into $E \setminus A$ and $\psi(A)$ through the excursions of $\hat{X}$ into $\psi(E \setminus A)$ and $\psi(A)$. We also note that Hypothesis 2.6 enforces a certain invariance of the dynamics of X for starting points that have the same image under ψ. Note, however, the continuation of Example 1 in Section 3 to see that this invariance is not as restrictive as it might first seem.

Let us develop our examples.

EXAMPLE 1 (Spider, continued). Let (P'_t) be the semigroup of Brownian motion reflected at the origin; that is,

$$(P'_t f)(x) \stackrel{\text{def}}{=} \int_{y>0} f(y) \left\{ \frac{\exp[-(x-y)^2/(2t)]}{\sqrt{2\pi t}} + \frac{\exp[-(x+y)^2/(2t)]}{\sqrt{2\pi t}} \right\} dy.$$

Set $(P''_t f)(i) = f(i)$ for all $f : \mathcal{I}_n \to \mathbb{R}$ and all $i \in \mathcal{I}_n$. Define $P_t \stackrel{\text{def}}{=} P'_t \otimes P''_t$ for all $t \geq 0$. Then $\hat{P}_t = P'_t$.

EXAMPLE 2 (Ball to sphere, continued). Let X be Brownian motion on E reflected (with normal derivative) at S^d. Then $\hat{X}$ is a d-dimensional Bessel process reflected at 1.

EXAMPLE 3 (Lollipop, continued). Let X be a d-dimensional Brownian motion. Then $\hat{X}$ is a d-dimensional Bessel process.

EXAMPLE 4 (Skew product, continued). Let $X' = (\Omega', \mathscr{F}', \mathscr{F}'_t, X'_t, \theta'_t, \mathbb{P}^x_{\prime})$ be a Borel right process with state space E'. Let B be a perfect continuous additive functional of X'. Next, let $X'' = (\Omega'', \mathscr{F}'', \mathscr{F}''_t, X''_t, \theta''_t, \mathbb{P}^x_{\prime\prime})$ be a Borel right process with state space E'' and transition semigroup (P''_t). Define the semigroup

$$(P_t)f(x', x'') \stackrel{\text{def}}{=} \mathbb{P}^{x'}_{\prime} \otimes \mathbb{P}^{x''}_{\prime\prime}\big[f\big(X'(t), X''(B_t)\big)\big], \qquad f \in B(E' \times E'').$$

Equivalently, if $f \in B(E)$ is of the form $f(x', x'') = f'(x')f''(x'')$, where $f' \in B(E')$ and $f'' \in B(E'')$, then

$$(P_t f)(x', x'') = \mathbb{P}^{x'}_{\prime}\big[f'(X'_t)P''_{B_t}[f''](x'')\big].$$

It can be shown that (P_t) is the transition semigroup of a Borel right process called the *skew product of* X' *and* X'' *with clock* B (cf. [15], Section 16 for the special case of the *Cartesian product* for which $B_t = t$).

With Hypothesis 2.6 in hand, we can collapse X to $\hat{X}$ when X is in A. Our goal is to "splice" together X and $\hat{X}$ to give an $\tilde{E}$-valued process $\tilde{X}$ that behaves like X when it is on $E \setminus A$ and like $\hat{X}$ when it is on $\psi(A)$. A moment's thought shows that we need a further ingredient, however. Assume that we start at x on the boundary (in the sense of the topology on $\tilde{E}$) of $E \setminus A$ and $\psi(A)$. If $\tilde{X}$ decides to make an excursion into $\psi(A)$, it should do so using the dynamics of $\hat{X}$. But what happens if it decides to make an excursion into $E \setminus A$? Where should it "start" or, more precisely, what is its entrance law? Presumably, it should start the excursion at some point of $\psi^{-1}\{x\}$, but since $\psi^{-1}\{x\}$ will in general consist of more than one point, we should define a mechanism for selecting the particular element of $\psi^{-1}\{x\}$ from which the excursion into $E \setminus A$ starts. Let $k : \hat{E} \times \mathscr{B}(E) \to \mathbb{R}$ be a probability kernel; that is, for each $x \in \tilde{E}$, $k(x, \cdot)$ is a probability measure on $(E, \mathscr{B}(E))$, and the map $x \mapsto k(x, B)$ is Borel measurable for each $B \in \mathscr{B}(E)$. We define a linear operator $K : B(E) \to B(\hat{E})$ by $Kf(x) \stackrel{\text{def}}{=} \int_{y \in E} f(y)k(x, dy)$; the appropriate diagram is thus

$$\begin{tikzcd} B(E) \arrow[dr, "K"'] & \\ & B(\hat{E}) \end{tikzcd}.$$

We assume that

$$k(x, \psi^{-1}\{x\}) = 1 \tag{2.4}$$

for all $x \in \hat{E}$; that is, if $x \in \tilde{E}$, the probability measure $k(x, \cdot)$ gives a means to randomly select a point in $\psi^{-1}\{x\}$. Several relevant consequences of this assumption are listed in the following lemma.

LEMMA 2.7. *We have the following*:

(a) *The operator $K\psi^*$ on $B(\hat{E})$ is the identity.*

(b) *The operator $K\pi^*$ from $B(\tilde{E})$ into $B(\hat{E})$ satisfies $K\pi^* f(\psi(x)) = f(\psi(x))$ for all $f \in B(\tilde{E})$ and $x \in A$.*

(c) *The operator $\phi^* K\pi^*$ on $B(\tilde{E})$ satisfies $f(x) = \phi^* K\pi^* f(x)$ for all $f \in B(\tilde{E})$ and $x \in \psi(A)$.*

PROOF. Claim (a) is a simple consequence of (2.4). To see claim (b), we compute that for f and x as stated,

$$\begin{aligned}
(K\pi^* f)(\psi(x)) &= \int_{z\in E} f(\pi(z))\, k\big(\psi(x), dz\big) \\
&= \int_{z\in\psi^{-1}\psi\{x\}} f(\psi(z))\, k\big(\psi(x), dz\big) \qquad \text{[use (2.4) and (2.1)]} \\
&= f(\psi(x)) k\big(\psi(x), \psi^{-1}\psi\{x\}\big) = f(\psi(x)).
\end{aligned}$$

Claim (c) follows directly from claim (b). □

We now assume:

HYPOTHESIS 2.8 (Carmona–Petit–Yor intertwining relationship). *Assume that the semigroups (P_t) and $(\hat{P}_t)$ satisfy*

$$K P_t = \hat{P}_t K. \tag{2.5}$$

See [6, 7] *for other uses of this type of relationship. A special case of such an intertwining is discussed in* [19].

REMARK 2.9 (Rogers–Pitman intertwining relationship). By Lemma 2.7(a), we have that

$$K\psi^* = I,$$

and so Hypotheses 2.6 and 2.8 together imply that

$$\hat{P}_t = K P_t \psi^*$$

and

$$K P_t = \hat{P}_t K.$$

These last three relationships taken together are the intertwining introduced in [12]. From Theorem 2 of [12] we have

$$\mathbb{P}^{k(x,\cdot)}[f(X_t)|\psi \circ X_s,\, 0 \le s \le t] = \psi^* K f(X_t), \qquad x \in \hat{E},\ f \in B(E).$$

That is, if X has initial distribution $k(x,\cdot)$ for any $x \in \hat{E}$, then the conditional distribution of X_t given $\sigma\{\psi \circ X_s,\, 0 \le s \le t\}$ is $k(\psi \circ X_t, \cdot)$. See also Corollary 3.5 of [11]. This latter article has an extensive discussion of intertwinings for semigroups and their consequences, plus general results on establishing intertwinings for semigroups using the associated generator or martingale problem.

We discuss conditions that imply Hypothesis 2.8 in Section 3.

2.3. *Resolvents.* Our proofs will be based on various calculations using resolvents, so let us develop some appropriate notation.

NOTATION 2.10. Define the stopping times

$$\begin{aligned} T(\omega) &\stackrel{\text{def}}{=} \inf\{t \ge 0 : X_t(\omega) \in A\}, && \omega \in \Omega, \\ \hat{T}(\hat{\omega}) &\stackrel{\text{def}}{=} \inf\{t \ge 0 : \hat{X}_t(\hat{\omega}) \in \psi(A)\}, && \hat{\omega} \in \hat{\Omega}. \end{aligned}$$

Since $\psi^{-1}(\psi(A)) = A$, the $\mathbb{P}^x$ law of T is the same as the $\hat{\mathbb{P}}^{\psi(x)}$ law of $\hat{T}$ for any $x \in E$. Let (Q_t) and $(\hat{Q}_t)$ be, respectively, the semigroups for X stopped at T and $\hat{X}$ stopped at $\hat{T}$; that is,

$$\begin{aligned} (Q_t f)(x) &\stackrel{\text{def}}{=} \mathbb{P}^x[f(X_{t\wedge T})], && f \in B(E),\ x \in E, \\ (\hat{Q}_t f)(x) &\stackrel{\text{def}}{=} \hat{\mathbb{P}}^x[f(\hat{X}_{t\wedge \hat{T}})], && f \in B(\hat{E}),\ x \in \hat{E}. \end{aligned}$$

For $\alpha > 0$, define the operators

$$\begin{aligned} U^\alpha f(x) &\stackrel{\text{def}}{=} \int_0^\infty e^{-\alpha t} P_t f(x)\,dt = \mathbb{P}^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right], \\ P_T^\alpha f(x) &\stackrel{\text{def}}{=} \mathbb{P}^x[e^{-\alpha T} f(X_T)], \\ V^\alpha f(x) &\stackrel{\text{def}}{=} \int_0^\infty e^{-\alpha t} Q_t f(x)\,dt = \mathbb{P}^x\left[\int_0^\infty e^{-\alpha t} f(X_{t\wedge T})\,dt\right] \\ &= U^\alpha f(x) - P_T^\alpha U^\alpha f(x) + \alpha^{-1} P_T^\alpha f(x) \end{aligned} \tag{2.6}$$

for all $f \in B(E)$ and $x \in E$; U^α is the α-resolvent of the semigroup (P_t) and V^α is the α-resolvent of (Q_t). Similarly define

$$\tag{2.7}\begin{aligned}
\hat{U}^\alpha f(x) &\stackrel{\text{def}}{=} \int_0^\infty e^{-\alpha t} \hat{P}_t f(x)\, dt = \hat{\mathbb{P}}^x\left[\int_0^\infty e^{-\alpha t} f(\hat{X}_t)\, dt\right],\\
\hat{P}^\alpha_{\hat{T}} f(x) &\stackrel{\text{def}}{=} \hat{\mathbb{P}}^x\left[e^{-\alpha \hat{T}} f(\hat{X}_{\hat{T}})\right],\\
\hat{V}^\alpha f(x) &\stackrel{\text{def}}{=} \int_0^\infty e^{-\alpha t} \hat{Q}_t f(x)\, dt = \hat{\mathbb{P}}^x\left[\int_0^\infty e^{-\alpha t} f(\hat{X}_{t\wedge\hat{T}})\, dt\right]\\
&= \hat{U}^\alpha f(x) - \hat{P}^\alpha_{\hat{T}} \hat{U}^\alpha f(x) + \alpha^{-1} \hat{P}^\alpha_{\hat{T}} f(x)
\end{aligned}$$

for all $f \in B(\hat{E})$ and $x \in \hat{E}$.

Let us collect together several facts.

LEMMA 2.11. *We have that*:

(a) $\pi^* B(\tilde{E}; \psi(A)) \subseteq B(E; A)$;
(b) $Q_t B(E; A) \subseteq B(E; A)$;
(c) $V^\alpha B(E; A) \subseteq B(E; A)$;
(d) $(U^\alpha - P^\alpha_T U^\alpha) B(E) \subseteq B(E; A)$;
(e) $(\hat{U}^\alpha - \hat{P}^\alpha_{\hat{T}} \hat{U}^\alpha) B(\hat{E}) \subseteq B(\hat{E}; \psi(A))$;
(f) $P^\alpha_T B(E; A) = 0$.

PROOF. To prove (a), fix $f \in B(\tilde{E}; \psi(A))$ and $x \in A$. Then $\pi^* f(x) = f(\pi(x)) = f(\psi(x)) = 0$. To prove (b), fix $f \in B(E; A)$ and $x \in A$. Then $Q_t f(x) = \mathbb{P}^x[f(X_0)] = f(x) = 0$. Claim (b) directly implies claim (c). To see claim (e), fix $f \in B(E)$ and $x \in A$. Then $P^\alpha_T U^\alpha(x) = \mathbb{P}^x[(U^\alpha f)(X_0)] = U^\alpha f(x)$. Similarly, if $f \in B(\tilde{E})$ and $x \in \psi(A)$, then $\hat{P}^\alpha_{\hat{T}} \hat{U}^\alpha f(x) = \hat{U}^\alpha f(x)$. Finally, if $f \in B(E; A)$ and $x \in E$, then by right-continuity $f(X_T) = 0$ under any $\mathbb{P}^x$; this gives us claim (f). $\square$

2.4. *The basic theorem.* We are now ready to define what will turn out to be a semigroup on $\tilde{E}$.

DEFINITION 2.12. *For $t \geq 0$, $f \in B(\tilde{E})$ and $x \in \tilde{E}$, define*

$$\tilde{P}_t f(x) \stackrel{\text{def}}{=} \mathbb{P}^x\left[(\pi^* f)(X_t)\chi_{\{T>t\}}\right] + \hat{\mathbb{P}}^{\psi(x)}\left[\left(K P_{t-\hat{T}} \pi^* f\right)(\hat{X}_{\hat{T}})\chi_{\{\hat{T}\leq t\}}\right]$$

if $x \in E \setminus A$ and

$$(\tilde{P}_t f)(x) \stackrel{\text{def}}{=} (K P_t \pi^* f)(x)$$

if $x \in \psi(A)$.

Our basic theorem is that $(\tilde{P}_t)$ is a transition semigroup that satisfies certain desirable properties. We stress that Hypotheses 2.6 and 2.8 as well as the topological assumptions of Section 2.1 are in force.

THEOREM 2.13. *Suppose that* $U^\alpha C_0(E) \subseteq C_0(E)$ *and* $P_T^\alpha C_0(E) \subseteq C_0(E)$ *for each* $\alpha > 0$, *and that* $KC_0(E) \subseteq C_0(\hat{E})$. *Then the following hold*:

(a) *The collection* $(\tilde{P}_t)_{t\geq 0}$ *is the transition semigroup of a quasi-left-continuous Borel right process* $\tilde{X} = (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{\mathbb{P}}^x)$ *with* α*-resolvent* $\tilde{U}^\alpha$ *given by*

$$\tilde{U}^\alpha = \check{\pi}_* V^\alpha \pi^* \left(I_{\tilde{E}} - \phi^* K \pi^*\right) + \phi^* \hat{U}^\alpha K \pi^*, \tag{2.8}$$

this expression being well defined. An alternative representation of $(\tilde{P}_t)$ *is*

$$\tilde{P}_t = \check{\pi}_* Q_t \pi^* \left(I_{\tilde{E}} - \phi^* K \pi^*\right) + \phi^* \hat{P}_t K \pi^*, \tag{2.9}$$

this expression being well defined.

(b) *For each* $x \in \tilde{E}$, *the law of* $\phi \circ X$ *under* $\tilde{\mathbb{P}}^x$ *coincides with that of* $\hat{X}$ *under* $\hat{\mathbb{P}}^{\phi(x)}$; *in particular*, $\tilde{P}_t \phi^* = \phi^* \hat{P}_t$.

(c) *Define a stopping time by*

$$\tilde{T}(\tilde{\omega}) \stackrel{def}{=} \inf\{t \geq 0 : \tilde{X}_t(\tilde{\omega}) \in \psi(A)\}, \qquad \tilde{\omega} \in \tilde{\Omega},$$

and define a semigroup

$$\tilde{Q}_t f(x) \stackrel{def}{=} \tilde{\mathbb{P}}^x[f(\tilde{X}_{t\wedge\tilde{T}})], \qquad t \geq 0,\ f \in B(\tilde{E}),\ x \in \tilde{E}.$$

For each $x \in E$, *the* $\tilde{\mathbb{P}}^{\pi(x)}$ *law of* $\{\tilde{X}_t;\ 0 \leq t < \tilde{T}\}$ *is equal to the* $\mathbb{P}^x$ *law of* $\{\pi(X_t);\ 0 \leq t < T\}$; *in particular*, $\pi^* \tilde{Q}_t = Q_t \pi^*$.

(d) *The semigroup* $(\tilde{P}_t)$ *is Feller* [*i.e.*, $\tilde{P}_t C_0(\tilde{E}) \subseteq C_0(\tilde{E})$ *for each* $t \geq 0$ *and* $\lim_{t\downarrow 0} \sup_{x\in\tilde{E}} |\tilde{P}_t f(x) - f(x)| = 0$ *for all* $f \in C_0(\tilde{E})$].

PROOF. We begin with some consequences of our hypotheses that will be generally useful in what follows. It is immediate that

$$U^\alpha \psi^* = \psi^* \hat{U}^\alpha \tag{2.10}$$

and

$$K U^\alpha = \hat{U}^\alpha K \tag{2.11}$$

for all $\alpha > 0$. Approximate T from above by the sequence of discrete stopping times

$$T_n' \stackrel{\text{def}}{=} 2^{-n} \lceil 2^n T \rceil$$

for the filtration $(\sigma\{\psi \circ X_s : 0 \leq s \leq t\})_{t\geq 0}$ and use Hypotheses 2.6 and 2.8, Remark 2.9, the assumption that $KC_0(E) \subseteq C_0(\hat{E})$ and a monotone class argument to see that

$$P_T^\alpha \psi^* = \psi^* \hat{P}_{\hat{T}}^\alpha \tag{2.12}$$

and

$$KP_T^\alpha = \hat{P}_{\hat{T}}^\alpha K \tag{2.13}$$

for all $\alpha > 0$. Consequently,

$$V^\alpha \psi^* = \psi^* \hat{V}^\alpha \tag{2.14}$$

and

$$KV^\alpha = \hat{V}^\alpha K \tag{2.15}$$

for all $\alpha > 0$. Moreover, from the assumptions that $U^\alpha C_0(E) \subseteq C_0(E)$ and $P_T^\alpha C_0(E) \subseteq C_0(E)$, it is clear that $\hat{U}^\alpha C_0(\hat{E}) \subseteq C_0(\hat{E})$ $V^\alpha C_0(E) \subseteq C_0(E)$ and $\hat{V}^\alpha C_0(\hat{E}) \subseteq C_0(\hat{E})$ for all $\alpha > 0$.

To start the proof of (a), let us first see that each operator $\tilde{P}_t$ maps $B(\tilde{E})$ into itself and that $t \mapsto \tilde{P}_t f(x)$ is right-continuous for each $f \in C_0(\tilde{E})$. It is clear from the right assumption on X that $\alpha U^\alpha f \to f$ pointwise as $\alpha \to \infty$ for all $f \in C_0(E)$. Combining this with the assumption that U^α maps $C_0(E)$ into itself gives that X is quasi-left-continuous (see [15], Theorem 9.26). Thus if $G_1 \supseteq G_2 \supseteq \cdots$ are open subsets of E with $\bigcap_n G_n = A$ and $T_n'' \stackrel{\text{def}}{=} \inf\{t \geq 0 : X_t \in G_n\}$, then $\mathbb{P}^x\{T \neq \lim_n T_n''\} = 0$ for all $x \in E$. It then follows by standard arguments that $x \mapsto \tilde{P}_t f(x)$ is Borel measurable for each $f \in B(\tilde{E})$ and $t \geq 0$. Moreover, by the right assumption on X, the map $t \mapsto \tilde{P}_t f(x)$ is right-continuous for each $x \in \tilde{E}$ when $f \in C_0(\tilde{E})$.

Let us next prove the formulae for $\tilde{U}^\alpha$ and $\tilde{P}_t$. First of all, note that by claim (a) of Lemma 2.4 and claim (a) of Lemma 2.11, we have that

$$\pi^*(I_{\tilde{E}} - \phi^* K \pi^*) B(\tilde{E}) \subseteq B(E; A). \tag{2.16}$$

By claims (b) and (c) of Lemma 2.11, we know that

$$\begin{aligned} &V^\alpha \pi^*(I_{\tilde{E}} - \phi^* K \pi^*) B(\tilde{E}; A) \subseteq B(E; A), \\ &Q_t \pi^*(I_{\tilde{E}} - \phi^* K \pi^*) B(\tilde{E}; A) \subseteq B(E; A); \end{aligned} \tag{2.17}$$

hence the formulae for $\tilde{U}^\alpha$ and $\tilde{P}_t$ are well defined (i.e., $\breve{\pi}^*$ acts on its domain). Fix next $f \in B(\tilde{E})$. Note that if $x \in E \setminus A$, then

$$\tilde{P}_t f(x) = \mathbb{P}^x\big[(\pi^* f)(X_t)\chi_{\{T>t\}}\big] + \hat{\mathbb{P}}^{\psi(x)}\big[(KP_{t-\hat{T}}\pi^* f)(\hat{X}_{\hat{T}})\chi_{\{\hat{T} \leq t\}}\big].$$

Thus if $x \in E \setminus A$,

$$(\tilde{U}^\alpha f)(x) = (U^\alpha \pi^* f)(x) - (P_T^\alpha U^\alpha \pi^* f)(x) + \big(\hat{P}_{\hat{T}}^\alpha K U^\alpha \pi^* f\big)(\psi(x)).$$

On the other hand, if $x \in \psi(A)$, then

$$(\tilde{U}^\alpha f)(x) = (KU^\alpha \pi^* f)(x) = \big(\hat{P}_{\hat{T}}^\alpha K U^\alpha \pi^* f\big)(x).$$

Hence

$$\tilde{U}^\alpha = \check{\pi}_*(U^\alpha - P_T^\alpha U^\alpha)\pi^* + \phi^* \hat{P}_{\hat{T}}^\alpha K U^\alpha \pi^*.$$

Note that

$$\begin{aligned}
\phi^* \hat{P}_{\hat{T}}^\alpha K U^\alpha \pi^* &= \phi^* \hat{P}_{\hat{T}}^\alpha \hat{U}^\alpha K \pi^* \qquad \text{[use (2.5)]} \\
&= \phi^* \hat{U}^\alpha K \pi^* - \phi^* \big(\hat{U}^\alpha - \hat{P}_{\hat{T}}^\alpha \hat{U}^\alpha\big) K \pi^* \\
&= \phi^* \hat{U}^\alpha K \pi^* - \check{\pi}_* \psi^* \big(\hat{U}^\alpha - \hat{P}_{\hat{T}}^\alpha \hat{U}^\alpha\big) K \pi^* \\
&\qquad \text{[use claim (e) of Lemma 2.11 and claim (b) of Lemma 2.4]} \\
&= \phi^* \hat{U}^\alpha K \pi^* - \check{\pi}_* (U^\alpha - P_T^\alpha U^\alpha) \psi^* K \pi^* \\
&\qquad \text{[use (2.10) and note also claim (d) of Lemma 2.11]} \\
&= \phi^* \hat{U}^\alpha K \pi^* - \check{\pi}_* (U^\alpha - P_T^\alpha U^\alpha) \pi^* \phi^* K \pi^* \qquad \text{[use (2.2)].}
\end{aligned}$$

We therefore have that

$$\tilde{U}^\alpha = \check{\pi}_*(U^\alpha - P_T^\alpha U^\alpha)\pi^*(I_{\tilde{E}} - \phi^* K \pi^*) + \phi^* \hat{U}^\alpha K \pi^*.$$

Finally, we use (2.6), claim (f) of Lemma 2.11 and (2.16) to see that

$$\begin{aligned}
(U^\alpha - P_T^\alpha U^\alpha)\pi^*(I_{\tilde{E}} - \phi^* K \pi^*) &= (U^\alpha - P_T^\alpha U^\alpha - \alpha^{-1} P_T^\alpha)\pi^*(I_{\tilde{E}} - \phi^* K \pi^*) \\
&= V^\alpha \pi^* (I_{\tilde{E}} - \phi^* K \pi^*).
\end{aligned}$$

This implies (2.8). We get (2.9) by taking the inverse Laplace transform and using the right-continuity proved above.

To show that the collection of operators $(\tilde{P}_t)$ is a semigroup, it suffices, by uniqueness of Laplace transforms and the right-continuity of $t \mapsto \tilde{P}_t f(x)$ for $f \in C_0(\tilde{E})$, to show that the collection $(\tilde{U}^\alpha)$ satisfies the resolvent equation

$$\tilde{U}^\alpha = \tilde{U}^\beta + (\beta - \alpha)\tilde{U}^\alpha \tilde{U}^\beta. \tag{2.18}$$

Using (2.8), we have that

$$\begin{aligned}
\tilde{U}^\alpha \tilde{U}^\beta &= \check{\pi}_* V^\alpha \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \check{\pi}_* V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \\
&\quad + \phi^* \hat{U}^\alpha K \pi^* \phi^* \hat{U}^\beta K \pi^* \\
&\quad + \check{\pi}_* V^\alpha \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \phi^* \hat{U}^\beta K \pi^* \\
&\quad + \phi^* \hat{U}^\alpha K \pi^* \check{\pi}_* V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*).
\end{aligned}$$

First, note that

$$\begin{aligned}
&\phi^* \hat{U}^\alpha K \pi^* \phi^* \hat{U}^\beta K \pi^* \\
&\quad = \phi^* \hat{U}^\alpha K \psi^* \hat{U}^\beta K \pi^* \qquad \text{[use (2.2)]} \\
&\quad = \phi^* \hat{U}^\alpha \hat{U}^\beta K \pi^* \qquad \text{[use (a) of Lemma 2.7].}
\end{aligned}$$

Second, note that

$$\begin{aligned}
&\check{\pi}_* V^\alpha \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \phi^* \hat{U}^\beta K \pi^* \\
&\qquad = \check{\pi}_* V^\alpha (\pi^* \phi^* - \psi^* K \pi^* \phi^*) \hat{U}^\beta K \pi^* \\
&\qquad = \check{\pi}_* V^\alpha (\psi^* - \psi^* K \psi^*) \hat{U}^\beta K \pi^* \qquad \text{[use (2.2)]} \\
&\qquad = \check{\pi}_* V^\alpha (\psi^* - \psi^*) \hat{U}^\beta K \pi^* \qquad \text{[use (a) of Lemma 2.7]} \\
&\qquad = 0.
\end{aligned}$$

Third,

$$\begin{aligned}
&\phi^* \hat{U}^\alpha K \pi^* \check{\pi}_* V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \\
&\qquad = \phi^* \hat{U}^\alpha K V^\beta (\pi^* - \psi^* K \pi^*) \\
&\qquad\qquad\qquad\qquad \text{[use (2.17), claim (c) of Lemma 2.4 and (2.2)]} \\
&\qquad = \phi^* \hat{U}^\alpha V^\beta K (\pi^* - \psi^* K \pi^*) \qquad \text{[use (2.5)]} \\
&\qquad = \phi^* \hat{U}^\alpha V^\beta (K \pi^* - K \psi^* K \pi^*) \\
&\qquad = \phi^* \hat{U}^\alpha V^\beta (K \pi^* - K \pi^*) \qquad \text{[use (a) of Lemma 2.7]} \\
&\qquad = 0.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\check{\pi}_* V^\alpha \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \check{\pi}_* V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \\
&\quad = \check{\pi}_* V^\alpha \pi^* \check{\pi}_* V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) \\
&\qquad - \check{\pi}_* V^\alpha \pi^* \phi^* K \pi^* \check{\pi}_* V^\beta (\pi^* - \psi^* K \pi^*) \\
&\quad = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) - \check{\pi}_* V^\alpha \psi^* K V^\beta (\pi^* - \psi^* K \pi^*) \\
&\qquad\qquad\qquad\qquad \text{[use (2.2), claim (c) of Lemma 2.4 and (2.17)]} \\
&\quad = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) - \check{\pi}_* V^\alpha \psi^* \hat{V}^\beta K (\pi^* - \psi^* K \pi^*) \\
&\qquad\qquad\qquad\qquad \text{[use (2.5)]} \\
&\quad = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) - \check{\pi}_* V^\alpha \psi^* \hat{V}^\beta (K \pi^* - K \psi^* K \pi^*) \\
&\quad = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) - \check{\pi}_* V^\alpha \psi^* \hat{V}^\beta (K \pi^* - K \pi^*) \\
&\qquad\qquad\qquad\qquad \text{[use (a) of Lemma 2.7]} \\
&\quad = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*).
\end{aligned}$$

Thus,

$$\tilde{U}^\alpha \tilde{U}^\beta = \check{\pi}_* V^\alpha V^\beta \pi^* (I_{\tilde{E}} - \phi^* K \pi^*) + \phi^* \hat{U}^\alpha \hat{U}^\beta K \pi^*,$$

which, by the resolvent equations for (V^α) and $(\hat{U}^\alpha)$, immediately gives the resolvent equation (2.18) for $(\tilde{U}^\alpha)$.

To show that the semigroup $(\tilde{P}_t)$ is the transition semigroup of a quasi-left-continuous Borel right process, it suffices to show that

$$\tilde{U}^\alpha f \in C_0(\tilde{E}), \qquad f \in C_0(\tilde{E}), \tag{2.19}$$

and

$$\lim_{\alpha\to\infty} \alpha \tilde{U}^\alpha f = f \qquad \text{pointwise, } f \in C_0(\tilde{E}) \tag{2.20}$$

(see [15], Theorem 9.26 or [14], Section III.3). This will also establish part (c) (see [14], Lemma III.37.1). Property (2.19) follows fairly readily from our assumptions and Lemmas 7.3, 7.4 and 7.5. To prove property (2.20), we use the fact proved above that $t \mapsto \tilde{P}_t f(x)$ is right-continuous for each $x \in \tilde{E}$ to see that

$$\lim_{\alpha\to\infty} \alpha \tilde{U}^\alpha f = \tilde{P}_0 f \qquad \text{pointwise.}$$

We now note that

$$\begin{aligned} \tilde{P}_0 &= \check{\pi}_* \pi^*(I_{\tilde{E}} - \phi^* K \pi^* f) + \phi^* K \pi^* \\ &= \check{I}_{\tilde{E}} - \phi^* K \pi^* + \phi^* K \pi^* \qquad \text{[use claims (a) and (c) of Lemma 2.4]} \\ &= I_{\tilde{E}}. \end{aligned}$$

To see part (b), we note that

$$\begin{aligned} \tilde{U}^\alpha \phi^* &= \check{\pi}_* V^\alpha \pi^*(I_{\tilde{E}} - \phi^* K \pi^*)\phi^* + \phi^* \hat{U}^\alpha K \pi^* \phi^* \\ &= \check{\pi}_* V^\alpha (\pi^* \phi^* - \psi^* K \pi^* \phi^*) + \phi^* \hat{U}^\alpha K \pi^* \phi^* \\ &= \check{\pi}_* V^\alpha (\psi^* - \psi^* K \psi^*) + \phi^* \hat{U}^\alpha K \psi^* \qquad \text{[use (2.2)]} \\ &= \check{\pi}_* V^\alpha (\psi^* - \psi^*) + \phi^* \hat{U}^\alpha \qquad \text{[use claim (a) of Lemma 2.7]} \\ &= \phi^* \hat{U}^\alpha. \end{aligned}$$

The right-continuity of $(\hat{P}_t)$ and $(\tilde{P}_t)$ implies the Dynkin intertwining relationship $\tilde{P}_t \phi^* = \phi^* \hat{P}_t$; this is sufficient (see [15], Section II.13).

Turning to part (c), define first $(\tilde{P}^\alpha_{\tilde{T}} f)(x) \stackrel{\text{def}}{=} \tilde{\mathbb{P}}^x[e^{-\alpha \tilde{T}} f(\tilde{X}_{\tilde{T}})]$ for all $x \in \tilde{E}$, $\alpha > 0$ and $f \in B(\tilde{E})$. Let $\tilde{V}^\alpha$ be the resolvent of $(\tilde{Q}_t)$; that is,

$$\tilde{V}^\alpha f(x) \stackrel{\text{def}}{=} \tilde{\mathbb{P}}^x\left[\int_0^\infty e^{-\alpha t} f(\tilde{X}_{t\wedge \tilde{T}})\, dt\right] = \tilde{U}^\alpha f(x) - \tilde{P}^\alpha_{\tilde{T}} \tilde{U}^\alpha f(x) + \alpha^{-1} \tilde{P}^\alpha_{\tilde{T}} f(x)$$

for $f \in B(\tilde{E})$ and $x \in \tilde{E}$. It suffices to show that $\pi^* \tilde{V}^\alpha f = V^\alpha \pi^* f$ for any $f \in B(\tilde{E})$. First, note that

$$\begin{aligned} \pi^* \tilde{U}^\alpha &= V^\alpha \pi^* - V^\alpha \pi^* \phi^* K \pi^* + \pi^* \phi^* \hat{U}^\alpha K \pi^* \\ &\qquad\qquad \text{[use (2.17) and claim (c) of Lemma 2.4]} \\ &= V^\alpha \pi^* - V^\alpha \psi^* K \pi^* + \psi^* \hat{U}^\alpha K \pi^* \qquad \text{[use (2.2)]} \\ &= V^\alpha \pi^* - V^\alpha \psi^* K \pi^* + U^\alpha \psi^* K \pi^* \qquad \text{[use (2.10)].} \end{aligned}$$

Second, note that $\tilde{P}^{\alpha}_{\tilde{T}} B(\tilde{E}; \psi(A)) = 0$, so

$$\begin{aligned}
\pi^* \tilde{P}^{\alpha}_{\tilde{T}} \tilde{U}^{\alpha} &= \pi^* \tilde{P}^{\alpha}_{\tilde{T}} \phi^* \hat{U}^{\alpha} K \pi^* \\
&= \pi^* \phi^* \hat{P}^{\alpha}_{\hat{T}} \hat{U}^{\alpha} K \pi^* \qquad \text{[use claim (b)]} \\
&= \psi^* \hat{P}^{\alpha}_{\hat{T}} \hat{U}^{\alpha} K \pi^* \qquad \text{[use (2.2)]} \\
&= P^{\alpha}_{T} U^{\alpha} \psi^* K \pi^* \qquad \text{[use (2.3)].}
\end{aligned}$$

Third, we have from claim (c) of Lemma 2.7,

$$\begin{aligned}
\pi^* \tilde{P}^{\alpha}_{\tilde{T}} f(x) &= \tilde{\mathbb{P}}^{\pi(x)}[e^{-\alpha \tilde{T}} f(\tilde{X}_{\tilde{T}})] \\
&= \tilde{\mathbb{P}}^{\pi(x)}[e^{-\alpha \tilde{T}} \phi^* K \pi^* f(\tilde{X}_{\tilde{T}})] \\
&= \pi^* \tilde{P}^{\alpha}_{\tilde{T}} \phi^* K \pi^* f(x) \\
&= \pi^* \phi^* \hat{P}^{\alpha}_{\hat{T}} K \pi^* f(x) \qquad \text{[use claim (b)]} \\
&= \psi^* \hat{P}^{\alpha}_{\hat{T}} K \pi^* f(x) \qquad \text{[use (2.2)]} \\
&= P^{\alpha}_{T} \psi^* K \pi^* f(x) \qquad \text{[use (2.12)].}
\end{aligned}$$

Combining everything together, we find that

$$\begin{aligned}
\pi^* \tilde{V}^{\alpha} &= V^{\alpha} \pi^* - V^{\alpha} \psi^* K \pi^* + U^{\alpha} \psi^* K \pi^* - P^{\alpha}_{T} U^{\alpha} \psi^* K \pi^* + \alpha^{-1} P^{\alpha}_{T} \psi^* K \pi^* \\
&= V^{\alpha} \pi^* - (V^{\alpha} - U^{\alpha} + P^{\alpha}_{T} U^{\alpha} - \alpha^{-1} P^{\alpha}_{T}) \psi^* K \pi^* \\
&= V^{\alpha} \pi^* \qquad \text{[use (2.6)]}
\end{aligned}$$

as required. □

REMARK 2.14. As noted in the proof of Theorem 2.13, the assumption that $U_{\alpha} C_0(E) \subseteq C_0(E)$ is equivalent to the semigroup (P_t) being Feller given that (P_t) is the semigroup of a Borel right process (see [14], Lemma III.37.1). Moreover, under the assumptions of Theorem 2.13 the semigroups (Q_t), $(\hat{P}_t)$ and $(\hat{Q}_t)$ are also Feller.

3. Sufficient conditions for Hypothesis 2.8. Suppose that all the assumptions of Section 2 on E, $\hat{E}$, ψ, (P_t) and $(\hat{P}_t)$ hold except for equation (2.4) and Hypothesis 2.8. In this section we discuss various conditions under which these extra conditions hold.

More precisely, suppose that μ is a Radon measure on E (in the sense of [8], Section III.46) and define $\hat{\mu} \stackrel{\text{def}}{=} \mu \circ \psi^{-1}$ (i.e., $\hat{\mu}$ is the push forward of μ by ψ).

By our assumption that the inverse image of compact sets by ψ also compact, the measure $\hat{\mu}$ is also Radon. Set

$$(f', g')_\mu \stackrel{\text{def}}{=} \int_E f'g'\,d\mu, \qquad f', g' \in B^+(E),$$

$$(f'', g'')_{\hat{\mu}} \stackrel{\text{def}}{=} \int_{\hat{E}} f''g''\,d\hat{\mu}, \qquad f'', g'' \in B^+(\hat{E}).$$

There is a disintegration $\mu(B) = \int_{\hat{E}} k(x, B)\,\hat{\mu}(dx)$, where k is a probability kernel such that (2.4) holds and $(Kf, g)_{\hat{\mu}} = (f, \psi^* g)_\mu$ for all $f \in B^+(E)$ and $g \in B^+(\hat{E})$. [Informally, if we think of the map ψ as a $\hat{E}$-valued random variable defined on $(E, \mathscr{B}(E))$ and equipped with the possibly infinite measure μ, then $k(x, B)$ is the "conditional probability" $\mu\{B|\psi = x\}$ and $(Kf)(x)$ is the "conditional expectation" $\mu[f|\psi = x]$.] Note also that $(f, g)_{\hat{\mu}} = (\psi^* f, \psi^* g)_\mu$ for all f and g in $B^+(\hat{E})$. We will investigate when this choice of K also satisfies Hypothesis 2.8.

EXAMPLE 4 (Skew product, continued). Suppose that μ' is a Radon measure and μ'' is a probability measure that is *invariant* for (P''_t); that is, $\mu'' P''_t = \mu''$. Set $\mu = \mu' \otimes \mu''$, so that $\hat{\mu} = \mu'$ and $Kf(x') = \int f(x', x'')\,\mu''(dx'')$.

Now if $f \in B(E)$ is of the form $f(x', x'') = f'(x')f''(x'')$, then

$$\begin{aligned}
KP_t f(x') &= \int_{E''} \mathbb{P}'^{x'} \otimes \mathbb{P}''^{x''}[f'(X'(t))f''(X''(B_t))]\,\mu''(dx'') \\
&= \int_{E''} \mathbb{P}'^{x'}\big[f'(X'(t))P''_{B_t} f''(x'')\big]\,\mu''(dx'') \\
&= \mathbb{P}'^{x'}\left[f'(X'(t)) \int_{E''} P''_{B_t} f''(x'')\,\mu''(dx'')\right] \\
&= \mathbb{P}'^{x'}\left[f'(X'(t)) \int_{E''} f''(x'')\,\mu''(dx'')\right] \\
&= \hat{P}_t K f(x')
\end{aligned}$$

and a monotone class argument gives Hypothesis 2.8.

EXAMPLE 3.1 (Symmetry). Suppose that the semigroup (P_t) is *symmetric* with respect to the measure μ [i.e., $(P_t f, g)_\mu = (f, P_t g)_\mu$ for all f and g in $B^+(E)$] and that μ has E as its support. Suppose also that $P_t C_0(E) \subseteq C_0(E)$ for all $t \geq 0$ [equivalently, $U^\alpha C_0(E) \subseteq C_0(E)$ for all $\alpha > 0$; see Remark 2.14] and that $KC_0(E) \subseteq C_0(\hat{E})$.

For $f \in B^+(E)$ and $g \in B^+(\hat{E})$, we have

$$\begin{aligned}
(KP_t f, g)_{\hat{\mu}} &= (P_t f, \psi^* g)_\mu = (f, P_t \psi^* g)_\mu \\
&\qquad = (f, \psi^* \hat{P}_t g)_\mu = (Kf, \hat{P}_t g)_{\hat{\mu}} = (\hat{P}_t K f, g)_{\hat{\mu}}.
\end{aligned}$$

Thus $KP_t f(x) = \hat{P}_t Kf(x)$ for $\hat{\mu}$-a.e. $x \in \hat{E}$. It is easy to see that the measure $\hat{\mu}$ has support of all $\hat{E}$ and so, by continuity, $KP_t f(x) = \hat{P}_t Kf(x)$ for all $x \in \hat{E}$ when $f \in C_0(E)$. A monotone class argument shows that Hypothesis 2.8 holds in this case.

EXAMPLE 1 (Spider, continued). Fix a measure μ'' on $\mathcal{I}_n$ and let μ' be Lebesgue measure on $\mathbb{R}_+$. Let $\mu = \mu' \otimes \mu''$. Then

$$Kf(x) = \int_{\mathcal{I}_n} f(x,i)\mu''(di)$$

for all $f \in B(E)$ and $x \in \mathbb{R}_+$. Recall that the process X for the spider is the Cartesian product of reflected Brownian motion on $\mathbb{R}_+$ and the trivial process on $\mathcal{I}_n$ that stays forever at its starting point. It is clear that μ'' is invariant for the semigroup of the latter process. Also, the semigroup (P_t) of X is certainly symmetric with respect to μ. It is, of course, also easy to see directly that Hypothesis 2.8 holds in this case.

Let us observe what happens in this spider example if we redefine $\psi(x,i) \stackrel{\text{def}}{=} s_i(x)$ for all $(x,i) \in E$, where s_i is the scale function of a regular diffusion on $\mathbb{R}_+$ (i.e., s_i is a continuous, strictly increasing function) with the added properties that $s_i(0) = 0$ and $\lim_{x\to\infty} s_i(x) = \infty$ for all i. We could then take $\hat{X}$ to be reflecting Brownian motion on $\mathbb{R}_+$ and X to be the process that evolves as $s_i^{-1} \circ \hat{X}$ on $\mathbb{R}_+ \times \{i\}$. It is not hard to see that the Hypotheses 2.6 and 2.8 hold in this case. By the classical scale and speed construction of one-dimensional regular diffusions from Brownian motion (see, e.g., Chapter V of [13]), we could, by the introduction of suitable time changes on each leg of the spider space, produce a process on the spider space that evolves as an arbitrary regular one-dimensional diffusion on each leg.

EXAMPLE 3.2 (Group equivariance). Assume that E is equipped with a compact, metrizable group G of homeomorphisms. Given $\tau \in G$, define $\Lambda_\tau : B(E) \to B(E)$ by $\Lambda_\tau f(x) \stackrel{\text{def}}{=} f(\tau x)$ for all $x \in E$ and $f \in B(E)$. Suppose that μ is invariant with respect to G (i.e., $\mu \circ \tau^{-1} = \mu$ for all $\tau \in G$) and that the support of μ is all of E. Suppose further that (P_t) is *equivariant* with respect to G; that is, $\Lambda_\tau P_t = P_t \Lambda_\tau$ for all $\tau \in G$. Third, suppose that ψ is G-invariant; that is, $\psi(x) = \psi(\tau x)$ for all $x \in E$ and $\tau \in G$. Equivalently, $\Lambda_\tau \psi^* = \psi^*$ for all $\tau \in G$. Last, suppose that $P_t C_0(E) \subseteq C_0(E)$ for all $t \geq 0$ [equivalently, $U^\alpha C_0(E) \subseteq C_0(E)$ for all $\alpha > 0$] and that $KC_0(E) \subseteq C_0(\hat{E})$.

Let ν denote normalized Haar measure on G and define an operator $L : B(E) \to B(E)$ by $Lf(x) \stackrel{\text{def}}{=} \int \Lambda_\tau f(x)\,\nu(d\tau) = \int f(\tau x)\,\nu(d\tau)$. We claim that for every $f \in B(E)$, $Lf(x) = \psi^* Kf(x)$ for μ-a.e. $x \in E$. First, note that for any $g \in B^+(\hat{E})$,

$$(\psi^* Kf, \psi^* g)_\mu = (Kf, g)_{\hat{\mu}} = (f, \psi^* g)_\mu.$$

On the other hand,

$$
\begin{aligned}
(Lf, \psi^* g)_\mu &= \int (\Lambda_\tau f, \psi^* g)_\mu \nu(d\tau) = \int (\Lambda_\tau f, \Lambda_\tau \psi^* g)_\mu \nu(d\tau) \\
&= (\Lambda_\tau f, \Lambda_\tau \psi^* g)_\mu.
\end{aligned}
$$

Thus $(Lf, \psi^* g)_\mu = (\psi^* Kf, \psi^* g)_\mu$ for any $g \in B^+(\hat{E})$. Note that $\psi^* Kf$ is ψ-measurable.

From the invariance of ψ with respect to G, we also have that Lf is ψ-measurable (cf. Remark (b) after [8], Theorem III.26). Thus indeed $Lf(x) = \psi^* Kf(x)$ for μ-a.e. $x \in E$.

For $f \in B^+(E)$ and $g \in B^+(\hat{E})$ we now find (noting that the equivariance assumption implies $P_t L = L P_t$) that

$$
\begin{aligned}
(K P_t f, g)_{\hat{\mu}} &= (\psi^* K P_t f, \psi^* g)_\mu = (L P_t f, \psi^* g)_\mu = (P_t L f, \psi^* g)_\mu \\
&= (P_t \psi^* K f, \psi^* g)_\mu = (\psi^* \hat{P}_t K f, \psi^* g)_\mu = (\hat{P}_t K f, g)_{\hat{\mu}}.
\end{aligned}
$$

Thus $K P_t f(x) = \hat{P}_t K f(x)$ for $\hat{\mu}$-a.e. $x \in \hat{E}$ and Hypothesis 2.8 follows as in the previous example.

EXAMPLE 2 (Ball to sphere, continued). Let $G = O(d)$, the group of orthogonal transformations of $\mathbb{R}^d$. Let μ be Lebesgue measure on E. It is easy to see that μ is invariant with respect to G and that (P_t) is equivariant with respect to G (since the Euclidean Laplacian is equivariant with respect to G, as is the normal derivative at ∂E). It is also easy to see that ψ is G-invariant. For $f \in B(E)$, we then have that

$$
\begin{aligned}
Kf(r) &= \frac{\int_{\|x\|_{\mathbb{R}^d} = r} f(x) \mathscr{H}^{d-1}(dx)}{\mathscr{H}^{d-1}\{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} = r\}} \\
&= d^{-1} \pi^{-d/2} \Gamma\left(\frac{d}{2} + 1\right) \int_{S^{d-1}} f(r\theta) \mathscr{H}^{d-1}(d\theta)
\end{aligned}
\tag{3.1}
$$

for $0 < r \leq 1$, where $\mathscr{H}^{d-1}$ is $(d-1)$-dimensional Hausdorff measure, Γ is the standard gamma function and

$$
Kf(0) = f(0).
$$

We could also establish Hypothesis 2.8 for this example (with the same operator K) by noting that (P_t) is symmetric with respect to a G-invariant measure on E.

EXAMPLE 3 (Lollipop, continued). We again let $G = O(d)$, the group of orthogonal transformations of $\mathbb{R}^d$ and let μ be Lebesgue measure on E. We get the same results as for the ball-to-sphere example, except that now (3.1) holds for all $r > 0$.

4. Generators and cores. We next study the generator of $(\tilde{P}_t)$. We will assume the conditions of Theorem 2.13. Recall from Remark 2.14 that the semigroups (P_t), (Q_t) and $(\hat{P}_t)$ are Feller. We will also make some further simplifying assumptions that are reasonably general and certainly apply to all of our examples.

PROPOSITION 4.1. *Let the Feller semigroups (P_t), (Q_t) and $(\hat{P}_t)$ have respective generators $\mathbf{G}$, $\mathbf{H}$ and $\hat{\mathbf{G}}$, that have domains, respectively, $\mathscr{D}(\mathbf{G})$, $\mathscr{D}(\mathbf{H})$ and $\mathscr{D}(\hat{\mathbf{G}})$. Let $\tilde{\mathbf{G}}$ denote the generator of the Feller semigroup $(\tilde{P}_t)$, with associated domain $\mathscr{D}(\tilde{\mathbf{G}})$. Consider a vector space of functions $D \subseteq C_0(\tilde{E})$ such that the following hold*:

(a) $K\pi^*(D) \subseteq \mathscr{D}(\hat{\mathbf{G}})$ [*hence* $\psi^* K\pi^*(D) \subseteq \mathscr{D}(\mathbf{G})$].
(b) $\phi^* K\pi^*(D) \subseteq D$.
(c) *There is an extension $\mathbf{G}^e$ of $\mathbf{G}|_{\psi^* K\pi^*(D)}$ with domain $\mathscr{D}(\mathbf{G}^e)$ such that*:

(c1) $\pi^*(D) \subseteq \mathscr{D}(\mathbf{G}^e)$;
(c2) $K\mathbf{G}^e = \hat{\mathbf{G}} K$ *on* $\pi^*(D)$;
(c3) $\{f \in \mathscr{D}(\mathbf{G}^e) : \mathbf{G}^e f(x) = 0$ *for all* $x \in A\} \subseteq \mathscr{D}(\mathbf{H})$ *and* $\mathbf{G}^e$ *agrees with* $\mathbf{H}$ *on left-hand set*;
(c4) $\mathbf{G}^e \pi^* f(x) = \mathbf{G}^e \pi^* f(y)$ *if* $f \in D$ *and* $\psi(x) = \psi(y) \in \psi(A)$.

Then $D \subseteq \mathscr{D}(\tilde{\mathbf{G}})$ and

$$\pi^* \tilde{\mathbf{G}} f = \mathbf{G}^e \pi^* f, \qquad f \in D. \tag{4.1}$$

Alternatively, we have

$$\tilde{\mathbf{G}} f(x) = \begin{cases} \mathbf{G}^e \pi^* f(x), & \text{if } x \in E \setminus A, \\ \hat{\mathbf{G}} K\pi^* f(x), & \text{if } x \in \psi(A). \end{cases} \tag{4.2}$$

If in addition:

(d) *D is dense in $C_0(\tilde{E})$*,
(e) *the range of $\tilde{\mathbf{G}}|_D - \lambda$ is dense in $C_0(\tilde{E})$ for each $\lambda > 0$*,

then D is a core for $\tilde{\mathbf{G}}$.

PROOF. Fix $f \in D$. First, we note that

$$\begin{aligned} &\lim_{t \searrow 0} \frac{1}{t}(P_t \psi^* K\pi^* f - \psi^* K\pi^* f) \\ &\quad = \lim_{t \searrow 0} \frac{1}{t}(\psi^* \hat{P}_t K\pi^* f - \psi^* K\pi^* f) \qquad \text{[use (2.3)]} \\ &\quad = \psi^* \hat{\mathbf{G}} K\pi^* f \qquad \text{[use assumption (a)]}. \end{aligned}$$

Thus $\psi^* K\pi^* f \in \mathscr{D}(\mathbf{G})$ and

$$\mathbf{G}\psi^* K\pi^* f = \psi^* \hat{\mathbf{G}} K\pi^* f. \tag{4.3}$$

Next, we note that for $x \in A$,

$$\begin{aligned} \mathbf{G}^e \pi^* f(x) &= \psi^* K \mathbf{G}^e \pi^* f(x) && \text{[by assumption (c4)]} \\ &= \psi^* \hat{\mathbf{G}} K\pi^* f && \text{[by assumption (c2)].} \end{aligned} \tag{4.4}$$

Further, note that the combination of (2.2) and assumptions (b) and (c1) ensures that $\pi^*\phi^* K\pi^*(D) \subseteq \mathscr{D}(\mathbf{G}^e)$. Thus we find that

$$\mathbf{G}\pi^*(I_{\tilde{E}} - \phi^* K\pi^*) f(x) = 0, \qquad x \in A.$$

Now using assumption (c3), we see that $\bar{f} \stackrel{\text{def}}{=} \pi^*(I_{\tilde{E}} - \phi^* K\pi^*) f$ is in the domain of $\mathbf{H}$ and furthermore $\mathbf{H}\bar{f} \in B(E; A)$. Hence

$$\begin{aligned} \pi^* \tilde{\mathbf{G}} f &= \pi^* \check{\pi}_* \mathbf{H}\bar{f} + \pi^*\phi^* \hat{\mathbf{G}} K\pi^* f \\ &= \mathbf{H}\bar{f} + \psi^* \hat{\mathbf{G}} K\pi^* f \qquad \text{[use claim (c) of Lemma 2.4 and (2.2)]} \\ &= \mathbf{G}^e \pi^* f - \mathbf{G}^e \psi^* K\pi^* f + \psi^* \hat{\mathbf{G}} K\pi^* f \\ &\qquad\qquad\qquad\qquad\qquad \text{[use assumption (c) and (2.2)]} \\ &= \mathbf{G}^e \pi^* f \qquad \text{[use (4.3)],} \end{aligned} \tag{4.5}$$

which is (4.1). To get (4.2), we simply use the definition of π to get the claimed formula for $x \in E \setminus A$. To get the formula when $x \in \psi(A)$, we use (4.4).

The result that D is a core is a direct consequence of Proposition 1.3.1 in [9]. $\square$

To better understand the generator, let us consider the case where X and $\hat{X}$ are Feller jump processes so that

$$\begin{aligned} (\mathbf{G}f)(x) &= \lambda(x) \int_{y \in E} \{f(y) - f(x)\}\, \nu(x, dy), && x \in E, \\ (\hat{\mathbf{G}}f)(x) &\stackrel{\text{def}}{=} \hat{\lambda}(x) \int_{y \in \hat{E}} \{f(y) - f(x)\} \hat{\nu}(x, dy), && x \in \hat{E}, \end{aligned} \tag{4.6}$$

where λ and $\hat{\lambda}$ are nonnegative and ν and $\hat{\nu}$ are probability kernels. By the Yosida approximation [9], any Feller generator can be approximated by ones of the form (4.6). Note that

$$(\mathbf{H}f)(x) = \lambda(x)\chi_{E \setminus A}(x) \int_{y \in E} \{f(y) - f(x)\}\, \nu(x, dy), \qquad x \in E.$$

Making all of the necessary assumptions, let us now write down the generator of $\tilde{X}$. Observe that Hypothesis 2.6 is equivalent to

$$\begin{aligned} \lambda &= \hat{\lambda} \circ \psi, \\ \nu(x, \cdot) \circ \psi^{-1} &= \hat{\nu}\big(\psi(x), \cdot\big), \qquad x \in E. \end{aligned}$$

Observe also that Hypothesis 2.8 is equivalent to

$$\int_{y\in E} \nu(y,\cdot)\,k(x,dy) = \int_{y\in\hat{E}} k(y,\cdot)\,\hat{\nu}(x,dy), \qquad x\in\hat{E},$$

when Hypothesis 2.6 and Equation (2.4) hold. If $x\in\psi(A)$, then define

$$\tilde{\nu}(x,S) \overset{\text{def}}{=} \hat{\nu}\big(x, S\cap\psi(A)\big) + \int_{y\in\hat{E}\setminus\psi(A)} k_y\big(S\cap(E\setminus A)\big)\,\hat{\nu}(x,dy)$$

for $S\in\mathscr{B}(\tilde{E})$, whereas if $x\in E\setminus A$, then define

$$\begin{aligned}\tilde{\nu}(x,S) &\overset{\text{def}}{=} \nu\big(x, S\cap(E\setminus A)\big) + \hat{\nu}\big(\psi(x), S\cap\psi(A)\big)\\ &= \nu\big(x, S\cap(E\setminus A)\big) + \nu\big(x, \psi^{-1}(S)\cap A\big)\end{aligned}$$

for $S\in\mathscr{B}(\tilde{E})$. Also set

$$\tilde{\lambda}(x) \overset{\text{def}}{=} \begin{cases}\lambda(x), & \text{if } x\in E\setminus A,\\ \hat{\lambda}(x), & \text{if } x\in\psi(A).\end{cases}$$

Then

$$\tilde{\mathbf{G}}f(x) = \tilde{\lambda}(x)\int_{y\in\tilde{E}}\{f(y)-f(x)\}\tilde{\nu}(x,dy). \tag{4.7}$$

It is important to note that if the hypotheses of Proposition 4.1 hold, then the dependence of $\tilde{\mathbf{G}}$ on K via the extension $\mathbf{G}^e$ is somewhat hidden [cf. (4.1)]. The effect of K is more directly visible in the *domain* of $\tilde{\mathbf{G}}$; that is, through assumption (b). We shall call assumption (b) the *glueing* condition. A treatment of our standard examples will clarify matters and justify this terminology. To apply Proposition 4.1 to our examples, let us first define

$$C_0^{\psi,A}(E) \overset{\text{def}}{=} \big\{f\in C_0(E): f(x)=f(y) \text{ if } x \text{ and } y \text{ are in } A \text{ and } \psi(x)=\psi(y)\big\}.$$

The proof of the following result is given in Section 7.

LEMMA 4.2. *The map π^* is a bijection from $C_0(\tilde{E})$ to $C_0^{\psi,A}(E)$.*

We will write $\pi_*: C_0^{\psi,A}(E)\to C_0(\tilde{E})$ for the inverse of π^*. Note that $C_0(E;A)\subseteq C_0^{\psi,A}(E)$ and the restriction of π_* to $C_0(E;A)$ is $\check{\pi}_*$. Most of the ensuing calculations are similar to some in [16].

EXAMPLE 1 (Spider, continued). Here

$$C_0^{\psi,A}(E) = \big\{f\in C_0(\mathbb{R}_+\times\mathcal{I}_n): f(0,i)=f(0,j) \text{ for all } i \text{ and } j \text{ in } \mathcal{I}_n\big\}.$$

Let

$$D' \stackrel{\text{def}}{=} \Big\{ f \in C_0^{\psi,A}(\mathbb{R}_+ \times \mathcal{I}_n) : f|_{\mathbb{R}_+\times\{i\}} \in C^2(\mathbb{R}_+) \text{ for all } i \in \mathcal{I}_n,$$
$$\frac{\partial^2 f}{\partial x^2}(0,i) = \frac{\partial^2 f}{\partial x^2}(0,j) \text{ for all } i \text{ and } j \text{ in } \mathcal{I}_n$$
$$\text{and} \int_{\mathcal{I}_n} \frac{\partial f}{\partial x}(0,i)\mu''(di) = 0 \Big\}.$$

We want to show that $\pi_* D'$ is a core for $\tilde{\mathbf{G}}$; that is, we want to verify the hypotheses of Proposition 4.1. Note that

$$\Big\{ f \in B(\mathbb{R}_+ \times \mathcal{I}_n) : f|_{\mathbb{R}_+\times\{i\}} \in C^2(\mathbb{R}_+)$$
$$\text{and } \frac{\partial^2 f}{\partial x^2}(0,i) = 0 \text{ for all } i \in \mathcal{I}_n \Big\} \subseteq \mathscr{D}(\mathbf{H}), \tag{4.8}$$
$$\Big\{ f \in C^2(\mathbb{R}_+) : \frac{\partial f}{\partial x}(0) = 0 \Big\} \subseteq \mathscr{D}(\hat{\mathbf{G}}).$$

Fix $f \in D'$. Then

$$\psi^* K f(x,i) = K f(x) = \int_{\mathcal{I}_n} f(x,i)\mu''(di) \qquad \text{for all } x \in \mathbb{R}_+ \text{ and } i \in \mathcal{I}_n.$$

Thus assumption (a) holds. Second, $\pi^* K f \in D'$, so assumption (b) holds. We now define

$$\mathbf{G}^e \bar{f}(x,i) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 \bar{f}}{\partial x^2}(x,i), \qquad x \in \mathbb{R}_+,\ i \in \mathcal{I}_n,$$

for all $\bar{f}$ in the set

$$\mathscr{D}(\mathbf{G}^e) \stackrel{\text{def}}{=} \Big\{ f \in C_0(\mathbb{R}_+ \times \mathcal{I}_n) : f|_{\mathbb{R}_+\times\{i\}} \in C^2(\mathbb{R}_+)$$
$$\text{and } \frac{\partial^2 f}{\partial x^2}(0,i) = \frac{\partial^2 f}{\partial x^2}(0,j) \text{ for all } i \text{ and } j \text{ in} \mathcal{I}_n \Big\}.$$

We easily see that $\mathbf{G}^e$ is indeed an extension of $\mathbf{G}|_{\psi^* K D'}$. Assumption (c1) is clearly true. Assumption (c2) is true since

$$K\mathbf{G}^e f(x) = \hat{\mathbf{G}} K f(x) = \frac{1}{2}\int_{\mathcal{I}_n} \frac{\partial^2 f}{\partial x^2}(x,i)\mu''(di)$$

for all $x \in \mathbb{R}_+$. Assumption (c3) holds by the first inclusion of (4.8). Assumption (c4) follows immediately from the definition of $\mathscr{D}(\mathbf{G}^e)$. Thus indeed $\pi_*(D') \subseteq \mathscr{D}(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{G}}\pi_* = \mathbf{G}^e f$.

We also clearly have that D' is dense in $C_0^{\psi,A}(E)$. Finally, fix $\varphi \in C_0^{\psi,A}(E)$ and $\lambda > 0$. For each $i \in \mathcal{I}_n$, solve

$$\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x,i) - \lambda g(x,i) = \varphi(x,i), \qquad x > 0,$$
$$g(0,i) = 0,$$
$$\frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x,i) - \lambda h(x,i) = 0, \qquad x > 0,$$
$$h(0,i) = 1.$$

By standard PDE results, g and h exist and have restrictions to $\mathbb{R}_+ \times \{i\}$ that are in $C^2(\mathbb{R}_+)$ for each $i \in \mathcal{I}_n$. Define now

$$f(x,i) \overset{\text{def}}{=} g(x,i) + Ch(x,i), \qquad (x,i) \in \mathbb{R}_+ \times \mathcal{I}_n,$$

where C is a constant to be determined. We want to show that $f \in D'$ and that

$$\pi^* \tilde{\mathbf{G}} \pi_* f - \lambda f = \varphi. \tag{4.9}$$

First, note that $f \in C_0^{\psi,A}(E)$ and f restricted to $\mathbb{R}_+ \times \{i\}$ is in $C^2(\mathbb{R}_+)$ for each $i \in \mathcal{I}_n$. Second,

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,i) - \lambda f(x,i) = \varphi(x,i), \qquad x > 0,\ i \in \mathcal{I}_n. \tag{4.10}$$

This implies that $\frac{\partial^2 f}{\partial x^2}(0,i)$ does not depend on i [since both f and φ are in $C_0^{\psi,A}(E)$]. To complete the proof that $f \in D'$, we need to check that

$$0 = \int_{\mathcal{I}_n} \frac{\partial f}{\partial x}(0,i)\mu''(di) = \int_{\mathcal{I}_n} \frac{\partial g}{\partial x}(0,i)\mu''(di) + C\int_{\mathcal{I}_n} \frac{\partial h}{\partial x}(0,i)\mu''(di).$$

Thus, we want to set

$$C \overset{\text{def}}{=} -\frac{\int_{\mathcal{I}_n} \frac{\partial g}{\partial x}(0,i)\mu''(di)}{\int_{\mathcal{I}_n} \frac{\partial h}{\partial x}(0,i)\mu''(di)}.$$

To do this we need to verify that

$$\int_{\mathcal{I}_n} \frac{\partial h}{\partial x}(0,i)\mu''(di) \neq 0. \tag{4.11}$$

Assume not; that is, assume that

$$\int_{\mathcal{I}_n} \frac{\partial h}{\partial x}(0,i)\mu''(di) = 0.$$

Then $h \in D'$ and $\tilde{\mathbf{G}}\pi_* h - \lambda h = 0$. Since $(\tilde{P}_t)$ is Feller, $\tilde{\mathbf{G}}$ satisfies the positive maximum principle (see [9], Theorem 2.1 of Chapter 4), so it is dissipative (see [9], Theorem 2.2 of Chapter 4). Since, as we already know, $\pi_*(D') \subset \mathscr{D}(\tilde{\mathbf{G}})$, we can conclude that $h = 0$, leading to a contradiction. Thus (4.11) must hold, and so $f \in D'$. Equation (4.10) implies (4.9), so indeed assumption (e) holds, completing the proof that $\pi_* D'$ is a core for $\tilde{\mathbf{G}}$.

EXAMPLE 2 (Ball to sphere, continued). Here

$$C_0^{\psi,A}(E) = \{f \in C_0(E) : f \text{ is constant on } S^{d-1}\}.$$

Let

$$D' \stackrel{\text{def}}{=} \biggl\{ f \in C_0^{\psi,A}(E) : f \in C^2(E), \Delta f \text{ is constant on } S^{d-1} \\ \text{and } \int_{S^{d-1}} \langle \nabla f(\theta), \theta \rangle_{\mathbb{R}^d} \mathscr{H}^{d-1}(d\theta) = 0 \biggr\}.$$

We again want to show that $\pi_* D'$ is a core for $\tilde{\mathbf{G}}$. Here we have that

$$\begin{aligned} \{f \in C^2(E) : \Delta f(\theta) = 0 \text{ for all } \theta \in S^{d-1}\} &\subseteq \mathscr{D}(\mathbf{H}), \\ \{f \in C^2([0,1]) : f'(1) = 0\} &\subseteq \mathscr{D}(\hat{\mathbf{G}}). \end{aligned} \tag{4.12}$$

Fix $f \in D'$. Then

$$\begin{aligned} \psi^* K f(x) &= K f(\|x\|) \\ &= d^{-1} \pi^{-d/2} \Gamma\left(\frac{d}{2} + 1\right) \int_{S^{d-1}} f(\|x\|_{\mathbb{R}^d} \theta) \mathscr{H}^{d-1}(d\theta) \end{aligned} \tag{4.13}$$

for all $x \in E$. Some straightforward calculations (that require some care at the origin) show that assumptions (a) and (b) indeed hold. We now define

$$\mathbf{G}^e \bar{f}(x) \stackrel{\text{def}}{=} \tfrac{1}{2} \Delta \bar{f}(x), \qquad x \in E, \tag{4.14}$$

for all $\bar{f}$ in

$$\mathscr{D}(\mathbf{G}^e) \stackrel{\text{def}}{=} \{f \in C^2(E) : \Delta f \text{ is constant on } S^{d-1}\}.$$

It is easy here too to see that $\mathbf{G}^e$ is indeed an extension of $\mathbf{G}|_{\psi^* K(D')}$. Assumption (c1) is clearly true. Assumption (c2) is true since

$$K \mathbf{G}^e f(r) = \hat{\mathbf{G}} K f(r) = \tfrac{1}{2} \int_{S^{d-1}} \Delta f(r\theta) \mathscr{H}^{d-1}(d\theta) \tag{4.15}$$

for all $0 < r < 1$ (use polar coordinates). Assumption (c3) holds by the first inclusion of (4.12). Assumption (c4) is true by definition of $\mathscr{D}(\mathbf{G}^e)$, so $\pi_*(D') \subseteq \mathscr{D}(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{G}} \pi_* f = \mathbf{G}^e f$.

We also clearly have that D' is dense in $C_0^{\psi,A}(E)$. Finally, fix $\varphi \in C_0^{\psi,A}(E)$ and $\lambda > 0$ and solve

$$\begin{aligned}
\tfrac{1}{2}\Delta g(x) - \lambda g(x) &= \varphi(x), \qquad x \in B_d(0,1),\\
g\big|_{S^{d-1}} &= 0,\\
\tfrac{1}{2}\Delta h(x) - \lambda h(x) &= 0, \qquad x \in B_d(0,1),\\
h\big|_{S^{d-1}} &= 1.
\end{aligned}$$

By standard PDE results, g and h exist and are in $C^2(E)$. Define now

$$f(x) \stackrel{\text{def}}{=} g(x) + Ch(x), \qquad x \in E,$$

where C is a constant to be determined. We want to show that $f \in D'$ and we again want to verify (4.9). Again, $f \in C_0^{\psi,A}(E)$ and $f \in C^2(E)$. We also have that

$$\tfrac{1}{2}\Delta f(x) - \lambda f(x) = \varphi(x), \qquad x \in B_d(0,1), \tag{4.16}$$

implying that Δf is constant on S^{d-1}. To complete the proof that $f \in D'$, we need to check that

$$\begin{aligned}
0 &= \int_{S^{d-1}} \langle \nabla f(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta)\\
&= \int_{S^{d-1}} \langle \nabla g(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta) + C\int_{S^{d-1}} \langle \nabla h(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta).
\end{aligned}$$

Thus, we want to set

$$C \stackrel{\text{def}}{=} -\frac{\int_{S^{d-1}} \langle \nabla g(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta)}{\int_{S^{d-1}} \langle \nabla h(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta)},$$

and to do this we need to verify that

$$\int_{S^{d-1}} \langle \nabla h(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta) \neq 0.$$

Assume not; that is, assume that

$$\int_{S^{d-1}} \langle \nabla h(\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta) = 0.$$

Then $h \in D'$ and $\tilde{\mathbf{G}}\pi_* h - \lambda h = 0$. As in Example 1, this implies that $h = 0$, leading to a contradiction and allowing us to complete the proof that $\pi_* D'$ is a core for $\tilde{\mathbf{G}}$.

EXAMPLE 3 (Lollipop, continued). Here

$$C_0^{\psi,A}(E) = \{f \in C_0(E) : f(x) = f(y) \text{ if } \|x\|_{\mathbb{R}^d} = \|y\|_{\mathbb{R}^d} \geq 1\}.$$

Let

$$D' \stackrel{\text{def}}{=} \Big\{ f \in C_0^{\psi,A}(E) : f\big|_{\bar{B}_d(0,1)} \in C^2(\bar{B}_d(0,1)),$$
$$f\big|_{\mathbb{R}^d \setminus B_d(0,1)} \in C^2(\mathbb{R}^d \setminus B_d(0,1)),$$
$$\lim_{x \to S^{d-1}, x \notin S^{d-1}} \Delta f(x) \text{ exists}$$
$$\text{and } \lim_{r \to 1, r \neq 1} \int_{S^{d-1}} \langle \nabla f(r\theta), \theta \rangle_{\mathbb{R}^d} \; \mathscr{H}^{d-1}(d\theta) \text{ exists} \Big\}.$$

Then

$$\pi_*(D') = \Big\{ f \in C_0(\tilde{E}) : f\big|_{\bar{B}_d(0,1)} \in C^2(\bar{B}_d(0,1)),\ f\big|_{[1,\infty)} \in C^2([1,\infty)),$$
$$\lim_{x \to S^{d-1}, x \in B_d(0,1)} \Delta f(x) = \lim_{r \searrow 1} \frac{1}{2} \ddot{f}(r) + \frac{1}{2r} \dot{f}(r)$$
$$\text{and } \lim_{r \nearrow 1} \int_{S^{d-1}} \langle \nabla f(r\theta), \theta \rangle_{\mathbb{R}^d} \; \mathscr{H}^{d-1}(d\theta)$$
$$= d\pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right) \lim_{r \searrow 1} \dot{f}(r) \Big\}.$$

Note that

$$\begin{aligned} \{ f \in C^2(E) : \Delta f(x) = 0 \text{ if } \|x\|_{\mathbb{R}^d} \geq 1 \} &\subseteq \mathscr{D}(\mathbf{H}), \\ C^2(\mathbb{R}_+) &\subseteq \mathscr{D}(\hat{\mathbf{G}}). \end{aligned} \tag{4.17}$$

Fix $f \in D'$. We again have (4.13), and assumptions (a) and (b) hold as in Example 2. We again use (4.14), where this time

$$\mathscr{D}(\mathbf{G}^e) = \{ f \in C^2(\mathbb{R}^2) : \Delta f(x) = \Delta f(y) \text{ if } \|x\|_{\mathbb{R}^d} = \|y\|_{\mathbb{R}^d} \geq 1 \}.$$

Thus $\mathbf{G}^e$ is an extension of $\mathbf{G}|_{\psi^* K(D')}$. Assumption (c1) is clearly true. Assumption (c2) is again true due to (4.15), which holds for all $r > 0$. Assumption (c3) holds by the first inclusion of (4.17), and assumption (c4) holds by definition of $\mathscr{D}(\mathbf{G}^e)$. Thus $\pi_*(D') \subseteq \mathscr{D}(\tilde{\mathbf{G}})$ and, for the third time, $\tilde{\mathbf{G}} \pi_* f = \mathbf{G}^e f$.

We also clearly have that D' is dense in $C_0^{\psi,A}(E)$. Fix $\varphi \in C_0^{\psi,A}(E)$ and $\lambda > 0$ and solve

$$\begin{aligned} \tfrac{1}{2} \Delta g_i(x) - \lambda g_i(x) &= \varphi(x), \qquad x \in B_d(0,1), \\ g_i\big|_{S^{d-1}} &= 0, \\ \tfrac{1}{2} \Delta h_i(x) - \lambda h_i(x), &= 0, \qquad x \in B_d(0,1), \\ h_i\big|_{S^{d-1}} &= 1. \end{aligned}$$

Now let $\varphi' \in C_0([1,\infty))$ be such that $\varphi(x) = \varphi'(\|x\|_{\mathbb{R}^d})$ when $\|x\|_{\mathbb{R}^d} \geq 1$. Then we also solve the PDEs

$$\begin{aligned}
\frac{1}{2}\ddot{g}_o(r) + \frac{1}{2r}\dot{g}_o(r) - \lambda g_o(r) &= \varphi'(r), \qquad r > 1,\\
g_o(1) &= 0,\\
\lim_{r\to\infty} g_o(r) &= 0,\\
\frac{1}{2}\ddot{h}_o(r) + \frac{1}{2r}\dot{h}_o(r) - \lambda h_o(r) &= 0, \qquad r > 1,\\
h_o(1) &= 0,\\
\lim_{r\to\infty} h_o(r) &= 0.
\end{aligned}$$

By standard PDE results, g_i and h_i exist and are in $C^2(\bar{B}_d(0,1))$ and g_o and h_o exist and are in $C^2([1,\infty))$. Define now

$$f(x) \stackrel{\text{def}}{=} \begin{cases} g_i(x) + Ch_i(x), & \text{if } x \in B_d(0,1),\\ g_o(\|x\|_{\mathbb{R}^d}) + Ch_o(\|x\|_{\mathbb{R}^d}), & \text{if } \|x\|_{\mathbb{R}^d} \geq 1,\end{cases}$$

where C is a constant to be determined. Clearly $f \in C_0^{\psi,A}(E)$ and $f \in C^2(B_d(0,1)) \cup C^2(\mathbb{R}^d \setminus \bar{B}_d(0,1))$. By standard calculations, we again have (4.16) on $\mathbb{R}^2 \setminus S^{d-1}$, so indeed $\lim_{x\in S^{d-1}, x\notin S^{d-1}} \Delta f(x)$ exists. Set now

$$\begin{aligned}
G_i &\stackrel{\text{def}}{=} \lim_{r\nearrow 1} \int_{S^{d-1}} \langle \nabla g_i(r\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta),\\
H_i &\stackrel{\text{def}}{=} \lim_{r\nearrow 1} \int_{S^{d-1}} \langle \nabla h_i(r\theta), \theta\rangle_{\mathbb{R}^d}\, \mathscr{H}^{d-1}(d\theta),\\
G_o &\stackrel{\text{def}}{=} d\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right) \lim_{r\searrow 1} \dot{g}_o(r),\\
H_o &\stackrel{\text{def}}{=} d\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right) \lim_{r\searrow 1} \dot{h}_o(r);
\end{aligned}$$

by some standard PDE calculations (see [16]), these limits exist. We now need to check that $G_i + CH_i = G_o + CH_o$. Thus we want to set

$$C \stackrel{\text{def}}{=} -\frac{G_i - G_o}{H_i - H_o}$$

and to do this we need to verify that $H_i \neq H_o$. Assume not; that is, that $H_i = H_o$. Then, as usual, we have found a nonzero element of D' such that $\tilde{\mathbf{G}}\pi_* h - \lambda h = 0$. This is impossible, so $H_i \neq H_o$, allowing us to complete the proof as we did in the previous examples.

5. Excursion theory. Suppose that the conditions of Theorem 2.13 hold. Suppose also that $\partial_{\tilde{E}}\psi(A) = \{x^*\} \subseteq \tilde{E}$, where $\partial_{\tilde{E}}\psi(A) = \partial_{\tilde{E}}(E \setminus A)$ is the boundary of both $\psi(A)$ and $E \setminus A$ in the topology of $\tilde{E}$; that is, $\partial_{\tilde{E}}\psi(A)$ is a single point x^*. We want to understand the structure of the excursions of $\tilde{X}$ from x^*. Note that since A is closed in the topology of E, $\psi(A)$ is closed in the topology of $\hat{E}$, so, by Lemma 2.3, $x^* \in \psi(A)$.

Suppose that x^* is a regular point for $\hat{X}$; then it is also a regular point for $\tilde{X}$. Let $\hat{l}$ be the local time of $\hat{X}$ at x^*, normalized so that $\hat{\mathbb{P}}^{x^*}[\int_0^\infty e^{-s}\,d\hat{l}_s] = 1$. Similarly, let $\tilde{l}$ be the local time of $\tilde{X}$ at x^*, normalized so that $\tilde{\mathbb{P}}^{x^*}[\int_0^\infty e^{-s}\,d\tilde{l}_s] = 1$. It is easy to see that the $\hat{\mathbb{P}}^{x^*}$ law of $\hat{l}$ is the same as the $\tilde{\mathbb{P}}^{x^*}$ law of $\tilde{l}$.

By standard results of excursion theory (see [5], Chapter IV or [13], Chapter VI) the paths of $\tilde{X}$ under $\tilde{\mathbb{P}}^{x^*}$ can be decomposed using the local time $\tilde{l}$ into a Poisson point process on $\mathbb{R}_+ \times \mathscr{E}$, where $\mathscr{E}$ is the space of excursion paths from x^*. That is, $\mathscr{E}$ is the space of cadlag paths $e : \mathbb{R}_+ \to \tilde{E}$ such that $e(t) = e(h(e)) = x^*$ for all $t \ge h(e) > 0$, where $h(e) \stackrel{\text{def}}{=} \inf\{t > 0 : e(t) = x^* \text{ or } e(t-) = x^*\}$. This Poisson process has intensity $\lambda \otimes \tilde{n}$, where λ is Lebesgue measure on $\mathbb{R}_+$ and $\tilde{n}$ is the σ-finite Itô excursion measure on $\mathscr{E}$. We can characterize $\tilde{n}$ as follows. Let $(\tilde{Q}_t)$ be the transition semigroup of $\tilde{X}$ stopped on first hitting x^*; that is,

$$\tilde{Q}_t(x, B) \stackrel{\text{def}}{=} \tilde{\mathbb{P}}^x\{\tilde{X}_{t\wedge\tilde{T}} \in B\}, \qquad B \in \mathscr{B}(\tilde{E}),\ x \in \tilde{E}.$$

Then $\tilde{n}$ is given by

$$\begin{aligned}&\tilde{n}\{e \in \mathscr{E} : e_{t_1} \in dx_1, \dots, e_{t_k} \in dx_k, h(e) > t_1\}\\ &\qquad = \tilde{n}_{t_1}(dx_1)\tilde{Q}_{t_2-t_1}(x_1, dx_2)\cdots\tilde{Q}_{t_k-t_{k-1}}(x_{k-1}, dx_k)\end{aligned}$$

for $0 < t_1 < \cdots < t_k < \infty$, where $(\tilde{n}_t)_{t>0}$ is a certain family of measures known as the *entrance law* of the excursion measure. By Theorem 2.13, $\tilde{Q}_t(x, B) = Q_t(\pi^{-1}(x), \pi^{-1}(B))$ for $x \in E \setminus A$ and $B \in \mathscr{B}(\tilde{E})$ such that $B \subset E \setminus A$; similarly, $\tilde{Q}_t(x, B) = \hat{Q}_t(\phi(x), \phi(B))$ for $x \in \psi(A)$ and $B \in \mathscr{B}(\hat{E})$ such that $B \subset \psi(A)$. To identify $(\tilde{n}_t)$, let $(\hat{n}_t)_{t>0}$ be the corresponding entrance law for the Itô excursion measure of $\hat{X}$.

From Equation VI.50.3 in [13], we have that

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\tilde{n}_t[f]\,dt &= \frac{\tilde{U}^\alpha f(x^*)}{\tilde{\mathbb{P}}^{x^*}[\int_0^\infty e^{-\alpha t}\,d\tilde{l}_t]} = \frac{\hat{U}^\alpha K\pi^* f(x^*)}{\hat{\mathbb{P}}^{x^*}[\int_0^\infty e^{-\alpha t}\,d\hat{l}_t]}\\ &= \int_0^\infty e^{-\alpha t}\hat{n}_t(K\pi^* f)\,dt.\end{aligned}$$

Thus

$$\tilde{n}_t(S) = \int_{y\in\hat{E}} k\big(y, \pi^{-1}(S)\big)\,\hat{n}_t(dy)$$

for all $S \in \mathscr{B}(\tilde{E})$. In particular, if $S \subset E \setminus A$, then

$$\tilde{n}_t(S) = \int_{y \in \hat{E}} k(y, S)\, \hat{n}_t(dy),$$

and if $S \subset \psi(A)$, then [by using (2.4)]

$$\tilde{n}_t(S) = \hat{n}_t(S).$$

Note the similarity between these calculations and those leading to (4.7). Hence, $\tilde{X}$ makes an excursion into $\psi(A) \setminus \{x^*\}$ with "probability" equal to that with which $\hat{X}$ makes an excursion into $\psi(A) \setminus \{x^*\}$, and $\tilde{X}$ makes an excursion into $E \setminus A$ with "probability" equal to that with which $\hat{X}$ makes an excursion into $\hat{E} \setminus \psi(A)$. If the excursion of $\tilde{X}$ is into $\psi(A)$, the entrance law and dynamics of the excursion are the same as the entrance law and dynamics of an excursion of $\hat{X}$ into $\psi(A)$. If the excursion of $\tilde{X}$ is into $E \setminus A$, the excursion enters $\psi(E \setminus A)$ with the entrance law of $\hat{X}$, and then randomizes over $E \setminus A$ according to the kernel k.

EXAMPLE 1 (Spider, continued). Let $\hat{n}_t$ be the entrance measure of reflected Brownian motion at 0 (i.e., the entrance law of $\hat{X}$). Then

$$\tilde{n}_t = \hat{n}_t \otimes \mu''.$$

In other words, excursions choose the ith leg of the spider with probability $\mu''(\{i\})$.

EXAMPLE 2 (Ball to sphere, continued). Here the measure $\tilde{n}_t$ is concentrated on the open ball $B_d(0, 1)$. Viewing the ball in polar coordinates, the measure $\tilde{n}_t$ is given by the product of $\hat{n}_t$ and normalized $(d-1)$-dimensional Hausdorff measure on the unit sphere S^{d-1}, where $\hat{n}_t$ is the entrance law of a d-dimensional Bessel process on $[0, 1]$ reflected at 1.

EXAMPLE 3 (Lollipop, continued). Here the measure $\tilde{n}_t$ is concentrated on the union of the open ball $B_d(0, 1)$ and the interval $(1, \infty)$. The component on $(1, \infty)$ is just the restriction to $(1, \infty)$ of the entrance law for excursions from 1 of a d-dimensional Bessel process, and the component on $B_d(0, 1)$ is, viewing the ball in polar coordinates, the product of the restriction to $(0, 1)$ of the entrance law for excursions from 1 of a d-dimensional Bessel process and normalized $(d-1)$-dimensional Hausdorff measure on the unit sphere S^{d-1}.

EXAMPLE 4 (Skew product, continued). Suppose that the additive functional B is given by $B_t = \int_0^t b(X'_s)\, ds$ for some nonnegative Borel function b that is bounded in a neighborhood of x^*. Assume that the conditions of Theorem 2.13 hold. If e is in the excursion space $\mathscr{E}$ for $\tilde{X}$, then $e(t) = (e'(t), e''(t)) \in E' \times E''$ for $0 < t < h(e)$. It follows from the above discussion that the excursion law $\tilde{n}$ can be described as follows. The E'-valued component of the excursion is chosen according to $\hat{n}$. Conditional on this component being e', the E''-valued component has the law of the process $(X''(\int_0^t b(e'(s))\,ds) : 0 < t < h(e'))$ under $\mathbb{P}''^{\mu''}$. That is, the E''-valued component evolves as an instance of X'' that is started at a random starting place chosen according to μ'' and time-changed according to the clock B driven by the E'-valued component.

6. Another example. Our aim in this section is to make some remarks about a process that arises in [16]. This process was the original motivation for our work. Our goal is to generalize in a sense the process of Example 3 by using the construction of Example 4.

We first construct a $(0,\infty) \times S^1$ process that can be thought of as an $\mathbb{R}^2 \setminus \{0\}$-valued process viewed in polar coordinates. We will use the skew product construction of Example 4 to carry out the details.

We begin with the radial part. Let X' be a process with state space $E' \stackrel{\text{def}}{=} (0,\infty)$ that, intuitively speaking, evolves as a two-dimensional Bessel process on $(0,1)$ and as the stochastic differential equation

$$dZ_t = \sigma(Z_t)\,dW_t + b(Z_t)\,dt \tag{6.1}$$

on $(1,\infty)$, where W is a Wiener process. Formally, define two generators

$$\mathscr{L}_1 f(x) \stackrel{\text{def}}{=} \frac{1}{2}\ddot{f}(x) + \frac{1}{2x}\dot{f}(x)$$

for $0 < x < 1$ if $f \in C^\infty(0,1)$, and define

$$\mathscr{L}_2 f(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\sigma^2(x)\ddot{f}(x) + b(x)\dot{f}(x)$$

for $x > 1$ if $f \in C^\infty(1,\infty)$, where $\sigma^2 > 0$ and b are both C^∞ and bounded. Define a scale function

$$s(x) \stackrel{\text{def}}{=} \begin{cases} \log x, & \text{if } 0 < x \le 1, \\ \displaystyle\int_1^x \exp\left[-2\int_1^z \frac{b(z')}{\sigma^2(z')}\,dz'\right] dz, & \text{if } 1 < x < \infty, \end{cases}$$

and a speed measure

$$m(B) = 2\int_{z\in B} \frac{1}{s'(z)}\left\{\chi_{(0,1)}(z) + \frac{1}{\sigma^2(z)}\chi_{(1,\infty)}(z)\right\} dz$$

for all $B \in \mathscr{B}((0,\infty))$. (See [13], Sections 46–48 for a discussion of the scale and speed description of one-dimensional diffusions in general and Bessel processes in particular.) We assume that s is finite on $[1,\infty)$. Then we can construct a Feller process X' with this scale and speed measure. Let us identify a core for the generator of X'. Define

$$\mathscr{D}(\mathbf{G}') \stackrel{\text{def}}{=} \Big\{ f \in C_0((0,\infty)) : \begin{array}{l} f\big|_{(0,1]} \in C^2((0,1]), \\ f\big|_{[1,\infty)} \in C^2([1,\infty)), \\ \lim_{r \to 1, r \neq 1} f'(r) \text{ exists} \\ \text{and } \lim_{r \nearrow 1} \mathscr{L}_1 f(r) = \lim_{r \searrow 1} \mathscr{L}_2 f(r) \Big\}. \end{array} \tag{6.2}$$

For $f \in \mathscr{D}(\mathbf{G}')$, define

$$\mathbf{G}' f(r) \stackrel{\text{def}}{=} \begin{cases} \mathscr{L}_1 f(r), & \text{if } 0 < r < 1, \\ \mathscr{L}_2 f(r), & \text{if } r > 1, \\ \lim_{r \to 1, r \neq 1} \mathbf{G}' f(r), & \text{if } r = 1. \end{cases}$$

Then $\mathbf{G}'$ with domain $\mathscr{D}(\mathbf{G}')$ is a core for X'.

Now let us define the angular part. Let X'' be Brownian motion on the unit circle $E'' \stackrel{\text{def}}{=} S^1$ and define X to be the skew product of X' and X'' with clock $B_t := \int_0^t (X_s')^{-2} \chi\{X_s' \leq 1\}\, ds$. The skew product lives on the cylinder $E \stackrel{\text{def}}{=} E' \times E'' = (0,\infty) \times S^1$. If we think of the $E' \times E''$-valued process X as being a process on $\mathbb{R}^2 \setminus \{0\}$ represented in polar coordinates, then we see from the skew product representation of two-dimensional Brownian motion (see, e.g., Section IV.35 of [13]) that this process evolves as two-dimensional Brownian motion on the (punctured) unit disk but each time the process leaves the unit disk it executes an excursion on the ray issuing from the origin and passing through the point at which it left the disk. This excursion is according to the dynamics of (6.1). Picturesquely, this latter process views $\mathbb{R}^2 \setminus \{0\}$ in the same way that an ant sees a daisy: the (punctured) unit disk is like the face of the daisy and the rays outside the unit disk are like petals along which the ant is constrained to move, being only able get from one petal to another by passing through the face. See Figure 4. We note in passing that the process X has a similar flavor to the *fiber Brownian motion* of Bass and Burdzy [4].

To continue with the constructive steps of Example 4, we now define $\psi(x', x'') \stackrel{\text{def}}{=} x'$ for all $(x', x'') \in E$ and we define $A \stackrel{\text{def}}{=} [1,\infty) \times S^1$. Thus $\tilde{E} = ((0,1) \times S^1) \cup [1,\infty)$. This collapses all rays in our daisy into a single ray, giving us essentially the lollipop of Figure 3. More specifically, it is easy to see that the map $\zeta : \tilde{E} \to \mathbb{R}^3$ defined by

$$\zeta(r,x,y) \stackrel{\text{def}}{=} \Big(x\sqrt{1-(2r-1)^2},\, y\sqrt{1-(2r-1)^2},\, 2r-1 \Big)$$

for $(r,x,y) \in E \setminus A = (0,1) \times S^1$ and

$$\zeta(r) \stackrel{\text{def}}{=} (r,0,0)$$

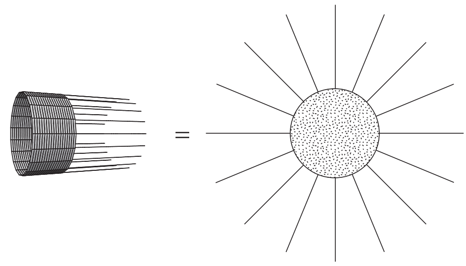

FIG. 4. *The daisy and cylinder.*

for $r \in [1, \infty) = \psi(A)$ defines a homeomorphism between $\tilde{E}$ and $(S^2 \setminus \{(0, 0, -1)\}) \cup ([1, \infty) \times \{0\} \times \{0\})$, that is, between $\tilde{E}$ and the space obtained by removing the south pole of the spherical portion of the lollipop.

Noting that normalized one-dimensional Hausdorff measure on S^1 is invariant for X'', it follows from our remarks about Example 4 in Section 3 that the kernel $k((x', x''), \cdot)$ given by the product of the delta measure at $x' \in (0, 1)$ with normalized one-dimensional Hausdorff measure on S^1 satisfies the relevant hypotheses. Identifying $\tilde{E}$ with the (punctured) lollipop, the corresponding operator K is just the one given by (3.1). Our ant wanders on the spherical surface of the lollipop, and each time it encounters the base of the stick it can either execute a one-dimensional excursion on the stick or execute an excursion on the sphere with a uniformly chosen "initial direction."

We finish by defining a core for the generator of $\tilde{X}$. The development is similar to Example 3 if we were to write those calculations in polar coordinates. Here

$$C_0^{\psi,A}(E) = \{f \in C_0(E) : f(r, \theta) = f(r, \theta') \text{ for all } \theta \text{ and } \theta' \text{ in } S^1 \text{ if } r \geq 1\}.$$

Let

$$\begin{aligned} D' \stackrel{\text{def}}{=} \Bigg\{ f \in C_0^{\psi,A}(E) : f|_{(0,1]\times S^1} \in C^2\big((0, 1] \times S^1\big), \\ f|_{[1,\infty)\times S^1} \in C^2\big([1, \infty) \times S^1\big), \\ \lim_{r \nearrow 1} \mathscr{L}_1 f(r, \theta) + \frac{1}{2r^2} \Delta_\theta f(r, \theta) \\ = \lim_{r \searrow 1} \mathscr{L}_2 f(r, \theta') \text{ for all } \theta, \theta' \in S^1 \\ \text{and } \lim_{r \nearrow 1} \frac{1}{2\pi} \int_{S^1} \frac{\partial f}{\partial r}(r, \theta)\, \mathscr{H}^1(d\theta) \\ = \lim_{r \searrow 1} \frac{\partial f}{\partial r}(r, \theta') \text{ for all } \theta' \in S^1 \Bigg\}. \end{aligned}$$

Then

$$\pi_*(D') = \Big\{ f \in C_0(\tilde{E}) : f|_{(0,1)\times S^1} \text{is the restriction of a function in } C^2((0,1]\times S^1),\ f|_{[1,\infty)} \in C^2([1,\infty)),\ \lim_{r\nearrow 1} \mathscr{L}_1 f(r,\theta) + \frac{1}{2r^2}\Delta_\theta f(r,\theta) = \lim_{r\searrow 1} \mathscr{L}_2 f(r) \text{ for all } \theta \in S^1 \text{ and } \lim_{r\nearrow 1} \frac{1}{2\pi}\int_{S^1} \frac{\partial f}{\partial r}(r,\theta)\,\mathscr{H}^1(d\theta) = \lim_{r\searrow 1} f'(r) \Big\}.$$

Note that

$$\begin{aligned} \Big\{ f \in C^2(E) : {} & f|_{(0,1]\times S^1} \in C^2((0,1]\times S^1),\\ & f|_{[1,\infty)\times S^1} \in C^2\big([1,\infty)\times S^1\big),\\ & \lim_{r\nearrow 1} \mathscr{L}_1 f(r,\theta) + \frac{1}{2r}\Delta_\theta f(r,\theta) = 0 \text{ for all } \theta \in S^1\\ & \text{and } \mathscr{L}_2 f(r,\theta) = 0 \text{ if } r \geq 1 \Big\} \subseteq \mathscr{D}(\mathbf{H}). \end{aligned} \tag{6.3}$$

Fix $f \in D'$. It is clear that

$$\psi^* K f(r,\theta) = K f(r) = \frac{1}{2\pi}\int_{S^1} f(r,\theta)\mathscr{H}^1(d\theta) \tag{6.4}$$

for all $r \in (0,\infty)$ and $\theta \in S^1$. It is clear that assumption (a) of Proposition 4.1 holds. By the nature of the skew product process and recalling (6.2), we have that $\mathscr{D}(\mathbf{G}') \subset \mathscr{D}(\hat{\mathbf{G}})$, so it is also clear that assumption (b) of Proposition 4.1 holds. We now define

$$\mathbf{G}^e f(r,\theta) \stackrel{\text{def}}{=} \begin{cases} \mathscr{L}_1 f(r,\theta) + \dfrac{1}{2r^2}\Delta_\theta f(r,\theta), & \text{if } (r,\theta) \in (0,1)\times S^1,\\ \mathscr{L}_2 f(r,\theta), & \text{if } (r,\theta) \in (1,\infty)\times S^1,\\ \displaystyle\lim_{r\to 1, r\neq 1} \mathbf{G}^e f(r,\theta), & \text{if } (r,\theta) \in \{1\}\times S^1, \end{cases} \tag{6.5}$$

for all $\bar{f}$ in

$$\begin{aligned}\mathscr{D}(\mathbf{G}^e) \stackrel{\text{def}}{=} \Big\{ f \in C_0(E) : f\big|_{(0,1]\times S^1} &\in C^2\big((0,1]\times S^1\big),\\ f\big|_{[1,\infty)\times S^1} &\in C^2\big([1,\infty)\times S^1\big),\\ \lim_{r\nearrow 1} \mathscr{L}_1 f(r,\theta) &+ \frac{1}{2r^2}\Delta_\theta f(r,\theta)\\ &= \lim_{r\searrow 1} \mathscr{L}_2 f(r,\theta) \text{ for all } \theta\in S^1\\ \text{and } \mathscr{L}_2 f(r,\cdot) &\text{ is constant on } S^1 \text{ for each } r\ge 1\Big\}.\end{aligned}$$

Assumptions (c1), (c3) and (c4) of Proposition 4.1 hold, and an easy calculation (involving an integration by parts on S^1) shows that assumption (c2) also holds. Thus indeed $\pi_*(D') \subseteq \mathscr{D}(\tilde{\mathbf{G}})$ and, as usual, $\tilde{\mathbf{G}}\pi_* f = \mathbf{G}^e f$.

We also clearly have that D' is dense in $C_0^{\psi,A}(E)$. Finally, fix $\varphi \in C_0(\tilde{E})$ and $\lambda > 0$. Solve

$$\begin{aligned}
\mathscr{L}_1 g_i(r,\theta) + \frac{1}{2r^2}\Delta_\theta f(r,\theta) - \lambda g_i(r,\theta) &= \varphi(r,\theta), \qquad (r,\theta)\in(0,1)\times S^1,\\
g_i(1,\cdot) &= 0,\\
\mathscr{L}_1 h_i(r,\theta) + \frac{1}{2r^2}\Delta_\theta h_i(r,\theta) - \lambda h_i(r,\theta) &= 0, \qquad (r,\theta)\in(0,1)\times S^1,\\
h_i(1,\cdot) &= 1,\\
\mathscr{L}_2 g_o(r) - \lambda g_o(r) &= \varphi(r), \qquad r>1,\\
g_o(1) &= 0,\\
\lim_{r\to\infty} g_o(r) &= 0,\\
\mathscr{L}_2 h_o(r) - \lambda h_o(r) &= 0, \qquad r>1,\\
h_o(1) &= 0,\\
\lim_{r\to\infty} h_o(r) &= 0.
\end{aligned}$$

By standard PDE results, g_i and h_i exist and are in $C^2((0,1]\times S^1)$ and g_o and h_o exist and are in $C^2([1,\infty))$. Define now

$$f(x) \stackrel{\text{def}}{=} \begin{cases} g_i(r,\theta) + Ch_i(r,\theta), & \text{if } (r,\theta)\in(0,1)\times S^1,\\ g_o(r) + Ch_o(r), & \text{if } r\ge 1.\end{cases}$$

We then proceed as before, showing that

$$\lim_{r\nearrow 1}\frac{1}{2\pi}\int_{S^1}\frac{\partial h_i}{\partial r}(r,\theta)\mathscr{H}^1(d\theta) \ne \lim_{r\searrow 1}\dot{h}_o(r).$$

7. Topological lemmas. Here we prove several topological results that we have used. To clarify our arguments, we will let $\mathscr{T}_E$, $\mathscr{T}_{\tilde{E}}$, and $\mathscr{T}_{\hat{E}}$ be, respectively, the topologies on E, $\tilde{E}$ and $\hat{E}$.

PROOF OF LEMMA 2.3. Assume first that $x \in E \setminus A$. Fix $\mathscr{N} \in \mathscr{T}_E$ such that $x \in \mathscr{N}$. Then $\mathscr{N}' \stackrel{\text{def}}{=} \mathscr{N} \cap (E \setminus A) \in \mathscr{T}_E$ and $\pi^{-1}(\mathscr{N}') = \mathscr{N}' \in \mathscr{T}_E$, so $\mathscr{N}' \in \mathscr{T}_{\tilde{E}}$. Thus $x_n \in \mathscr{N}'$ for n large, so $x_n \in E \setminus A$ for n large and $x_n \in \mathscr{N}$ for n large. Therefore $\lim_{n : x_n \in E\setminus A} x_n = x$ in $\mathscr{T}_E$, this limit existing.

Assume next that $x \in \psi(A)$. Fix $\mathscr{N} \in \mathscr{T}_{\hat{E}}$ such that $x \in \mathscr{N}$. Set $\mathscr{N}' \stackrel{\text{def}}{=} ((E \setminus A) \cap \psi^{-1}(\mathscr{N})) \cup (\psi(A) \cap \mathscr{N}) \subseteq \tilde{E}$. Then $\pi^{-1}(\mathscr{N}') = \psi^{-1}(\mathscr{N}) \in \mathscr{T}_E$ (since ψ is continuous). Thus $\mathscr{N}' \in \mathscr{T}_{\tilde{E}}$ and hence $x_n \in \mathscr{N}'$ for n large. If (x_n) is in $E \setminus A$, then $x_n \in \psi^{-1}(\mathscr{N})$ for n large, that is, $\psi(x_n) \in \mathscr{N}$ for n large. Thus $\lim_n \psi(x_n) = x$ in $\mathscr{T}_{\hat{E}}$. If (x_n) is in $\psi(A)$, then $x_n \in \mathscr{N}$ for n large, so $\lim_n x_n = x$ in $\mathscr{T}_{\hat{E}}$. $\square$

LEMMA 7.1. *The map ϕ is continuous.*

PROOF. The proof uses Lemma 2.3. Fix a sequence (x_n) in $\tilde{E}$ that converges in $\mathscr{T}_{\tilde{E}}$ to $x \in \tilde{E}$. We will extract a subsequence (x_{n_k}) such that $\lim_k \phi(x_{n_k}) = \phi(x)$ in $\mathscr{T}_{\hat{E}}$.

Assume first that $x \in E \setminus A$. Then by Lemma 2.3, $\lim_{n : x_n \in E\setminus A} x_n = x$ in $\mathscr{T}_E$. Thus there is a subsequence (x_{n_k}) contained in $E \setminus A$ with $\lim_k x_{n_k} = x$ (in $\mathscr{T}_E$), so since ψ is continuous, $\lim_k \phi(x_{n_k}) = \lim_k \psi(x_{n_k}) = \psi(x) = \phi(x)$, this limit existing in $\mathscr{T}_{\hat{E}}$.

Assume next that $x \in \psi(A)$. Then there are two subcases. In the first subcase, there is a subsequence (x_{n_k}) contained in $\psi(A)$. By Lemma 2.3, we then have that $\lim_k x_{n_k} = x$ in $\mathscr{T}_{\hat{E}}$; thus $\lim_k \phi(x_{n_k}) = \lim_k x_{n_k} = x = \phi(x)$ in $\mathscr{T}_{\hat{E}}$. In the second subcase, there is a subsequence (x_{n_k}) contained in $E \setminus A$. By Lemma 2.3, we then have that $\lim_k \psi(x_{n_k}) = x$ in $\mathscr{T}_{\hat{E}}$; thus $\lim_k \phi(x_{n_k}) = \lim_k \psi(x_{n_k}) = x = \phi(x)$ in $\mathscr{T}_{\hat{E}}$. $\square$

LEMMA 7.2. *If $f \in C_0(\hat{E})$, then $\psi^* f \in C_0(E)$.*

PROOF. Because ψ is continuous, $\psi * f = f \circ \psi$ is also continuous. Fix $L > 0$. To complete the proof we need to show that $H \stackrel{\text{def}}{=} \{x \in E : |\psi^* f(x)| \geq L\}$ is compact (in $\mathscr{T}_E$). Defining $\hat{H} \stackrel{\text{def}}{=} \{x \in \hat{E} : |f(x)| \geq L\}$ [this set is compact because $f \in C_0(\hat{E})$], we have that $H = \{x \in E : \psi(x) \in \hat{H}\} = \psi^{-1}(\hat{H})$. Our assumption that pre-images of compact sets through ψ are compact ensures that H is indeed compact. $\square$

LEMMA 7.3. *If $f \in C_0(\tilde{E})$, then $\pi^* f \in C_0(E)$.*

PROOF. Since π is continuous by definition of $\mathscr{T}_{\tilde{E}}$, we know that $\pi^* f = f \circ \pi$ is certainly continuous. To see that in fact $\pi^* f \in C_0(E)$, fix $L > 0$ and define $H \stackrel{\text{def}}{=} \{x \in E : |\pi^* f(x)| \geq L\}$; we need to show that H is compact (in $\mathscr{T}_E$). Since $\pi^* f$ is continuous, we at least know that H is closed (in $\mathscr{T}_E$). Define now $\tilde{H} \stackrel{\text{def}}{=} \{x \in \tilde{E} : |f(x)| \geq L\}$. Since $f \in C_0(\tilde{E})$, we know that $\tilde{H}$ is compact in $\mathscr{T}_{\tilde{E}}$. Our immediate goal is to show that $H \subseteq \psi^{-1}(\phi(\tilde{H}))$.

First, fix $x \in H$ such that $x \in E \setminus A$. Then $\pi(x) = x$ and thus $|f(x)| = |f(\pi(x))| = |\pi^* f(x)| \geq L$. Consequently, $x \in \tilde{H}$. Also, $\phi(x) = \psi(x)$, so $\psi(x) = \phi(x) \subseteq \phi(\tilde{H})$, ensuring that $x \in \psi^{-1}(\phi(\tilde{H}))$.

Next, fix $x \in H$ such that $x \in A$. Then $\pi(x) = \psi(x)$ and thus $|f(\psi(x))| = |f(\pi(x))| = |\pi^* f(x)| \geq L$, so $z \stackrel{\text{def}}{=} \psi(x)$ is in $\tilde{H}$. Since $z \in \psi(A)$, we have that $\phi(z) = z$. Thus $\psi(x) = z = \phi(z) \subseteq \phi(\tilde{H})$, ensuring that $\psi(x) \in \phi(\tilde{H})$.

Because ϕ is continuous (Lemma 7.1) and $\tilde{H}$ is compact in $\mathscr{T}_{\tilde{E}}$ [because $f \in C_0(\tilde{E})$], $\phi(\tilde{H})$ is compact in $\mathscr{T}_{\hat{E}}$. Our assumption that pre-images of compact sets through ψ are compact then ensures that $\psi^{-1}(\phi(\tilde{H}))$ is compact. Hence H is a closed subset of a compact set and is therefore compact. $\square$

LEMMA 7.4. *If $f \in C_0(\hat{E})$, then $\phi^* f \in C_0(\tilde{E})$.*

PROOF. By Lemma 7.1, we know that $\phi^* f = f \circ \phi$ is continuous. Fix $L > 0$ and define $\tilde{H} \stackrel{\text{def}}{=} \{x \in \tilde{E} : |\phi^* f(x)| \geq L\}$; we need to show that $\tilde{H}$ is compact. Since $\phi^* f$ is continuous, we at least know that $\tilde{H}$ is closed. Next, define $\hat{H} \stackrel{\text{def}}{=} \{x \in \hat{E} : |f(x)| \geq L\}$. This set is compact because $f \in C_0(\hat{E})$. We will show that $\tilde{H} \subseteq \pi(\psi^{-1}(\hat{H}))$. Indeed, fix $x \in \tilde{H}$. If $x \in E \setminus A$, then $x = \pi(x)$ and $|f(\psi(x))| = |\phi^* f(x)| \geq L$, so $x \in \psi^{-1}(\hat{H})$. Thus $x = \pi(x) \subseteq \pi(\psi^{-1}(\hat{H}))$. On the other hand, if $x \in \psi(A)$, then $x = \psi(z) = \pi(z)$ for some $z \in A$ and $|f(x)| = |\phi^* f(x)| \geq L$. Thus $\psi(z) = x \in \hat{H}$, so $z \in \psi^{-1}(\hat{H})$ and $x = \pi(z) \subseteq \pi(\psi^{-1}(\hat{H}))$. Thus indeed $\tilde{H} \subseteq \pi(\psi^{-1}(\hat{H}))$. By the assumption that pre-images of compact sets through ψ are compact, $\psi^{-1}(\hat{H})$ is compact since $\hat{H}$ is compact. Hence $\pi(\psi^{-1}(\hat{H}))$ is the image of a compact set through the continuous mapping π and therefore it, and consequently $\tilde{H}$, is compact. $\square$

LEMMA 7.5. *If $f \in C_0(E; A)$, then $\check{\pi}_* f \in C_0(\tilde{E})$.*

PROOF. First, let us prove that $\check{\pi}_* f$ is continuous. Fix a sequence (x_n) in $\tilde{E}$ that converges in $\mathscr{T}_{\tilde{E}}$ to $x \in \tilde{E}$. We will show that there is a subsequence (x_{n_k}) such that $\lim_k (\check{\pi}_* f)(x_{n_k}) = (\check{\pi}^* f)(x)$. There are two cases to check: when $x \in E \setminus A$ and when $x \in \psi(A)$.

Assume first that $x \in E \setminus A$. Then by Lemma 2.3 there is a subsequence (x_{n_k}) contained in $E \setminus A$ such that $\lim_k x_{n_k} = x$ in $\mathscr{T}_E$. Since f is continuous, $\lim_k \check{\pi}_* f(x_{n_k}) = \lim_k f(x_{n_k}) = f(x) = \check{\pi}^* f(x)$.

Assume next that $x \in \psi(A)$. Then $\check{\pi}_* f(x) = 0$. By Lemma 2.3, there is a subsequence (x_{n_k}) contained in either $\psi(A)$ or $E \setminus A$. In the first case, where (x_{n_k}) is a subsequence contained in $\psi(A)$, we immediately have that $\lim_k (\check{\pi}_* f)(x_{n_k}) = \lim_k 0 = 0 = (\check{\pi}_* f)(x)$. Consider next the second case, where (x_{n_k}) is a subsequence contained in $E \setminus A$. Then, by Lemma 2.3, $\lim_k \psi(x_{n_k}) = x$. Now let us take a further subsequence $(x_{n_{k_j}})$ such that $(x_{n_{k_j}})$ converges in the topology of the one-point compactification of E. If the limit point is in E, then $x^* \stackrel{\text{def}}{=} \lim_j x_{n_{k_j}}$ exists (this limit being in $\mathscr{T}_E$) and, by continuity of ψ, $\psi(x^*) = \lim_j \psi(x_{n_{k_j}}) = x \in \psi(A)$. Thus $x^* \in \psi^{-1}(\psi(A)) = A$. Since $f \in C_0(E; A)$, $\lim_j \check{\pi}_* f(x_{n_{k_j}}) = \lim_j f(x_{n_{k_j}}) = f(x^*) = 0$. If the limit point of $(x_{n_{k_j}})$ is at infinity, then we use the fact that $f \in C_0(E; A) \subseteq C_0(E)$ to see that $\lim_j \check{\pi}_* f(x_{n_{k_j}}) = \lim_j f(x_{n_{k_j}}) = 0$.

Next, we need to prove that in fact $\check{\pi}_* f \in C_0(\tilde{E})$. Fix $L > 0$ and consider the set $\tilde{H} \stackrel{\text{def}}{=} \{x \in \tilde{E} : |\check{\pi}_* f(x)| \ge L\}$. We need to show that $\tilde{H}$ is compact (in $\mathscr{T}_{\tilde{E}}$). Since $\check{\pi}_* f$ is continuous, we at least know that $\tilde{H}$ is closed. Define now $H \stackrel{\text{def}}{=} \{x \in E : |f(x)| \ge L\}$. Then H is compact (in $\mathscr{T}_E$) since $f \in C_0(E; A)$. Fix $x \in \tilde{H}$. Then $x \in E \setminus A$, so $x = \pi(x)$ and $|f(x)| = |\check{\pi}_* f(x)| \ge L$. Thus $x \in H$, so $x = \pi(x) \subseteq \pi(H)$. Hence $\tilde{H} \subseteq \pi(H)$ and $\pi(H)$ is compact since it is the image of the compact set H through the continuous mapping π (recall that π is continuous by definition of $\mathscr{T}_{\tilde{E}}$). Thus $\tilde{H}$ is compact. $\square$

Finally, we have:

PROOF OF LEMMA 4.2. First, it is clear that $\pi^* C_0(\tilde{E}) \subseteq C_0^{\psi,A}(E)$.

To proceed, fix $f \in C_0^{\psi,A}(E)$. Since $\pi E = \tilde{E}$ (i.e., π is surjective), we define $f' : \tilde{E} \to \mathbb{R}$ by

$$f'(\pi(x)) \stackrel{\text{def}}{=} f(x), \qquad x \in E.$$

It is easy to see that, by definition of $C_0^{\psi,A}(E)$, f' is well defined. We want to show that $f' \in C_0(\tilde{E})$. To do so, fix a sequence (x_n) in $\tilde{E}$ that converges in $\mathscr{T}_{\tilde{E}}$ to $x \in \tilde{E}$. We will extract a subsequence (x_{n_k}) such that $\lim_k f'(x_{n_k}) = f(x)$. This will show that f' is at least continuous.

Assume first that $x \in E \setminus A$. Then, by Lemma 2.3, $\lim_{n : x_n \in E \setminus A} x_n = x$ in $\mathscr{T}_{\tilde{E}}$. Since $f \in C_0(E)$, we have that $\lim_{n : x_n \in E \setminus A} f'(x_{n_k}) = \lim_{n : x_n \in E \setminus A} f(x_{n_k}) = f(x) = f'(x)$.

Assume next that $x \in \psi(A)$. Then there are two subcases. In the first subcase, there is a subsequence (x_{n_k}) contained in $\psi(A)$. By Lemma 2.3, we then have that $\lim_k x_{n_k} = x$ in $\mathscr{T}_{\hat{E}}$. Since the x_{n_k}'s and x are all in $\psi(A)$, we can find (x'_{nk}) in A and $x' \in A$ such that $\psi(x'_{n_k}) = x_{n_k}$ and $\psi(x') = x$. Since the pre-images of compact sets through ψ are compact, we can then find a further subsequence $(x'_{n_{k_j}})$ such that $x'' \stackrel{\text{def}}{=} \lim_j x'_{n_{k_j}}$ exists in $\mathscr{T}_E$. Since A is closed, $x'' \in A$. Because ψ is continuous, $\psi(x'') = \lim_j \psi(x'_{n_{k_j}}) = \lim_j x_{n_{k_j}} = x$, this limit being in $\mathscr{T}_{\hat{E}}$. Thus $\lim_j f'(x_{n_{k_j}}) = \lim_j f(x'_{n_{k_j}}) = f(x'') = f'(\pi(x'')) = f'(\psi(x'')) = f'(x)$. In the second subcase, there is a subsequence (x_{n_k}) contained in $E \setminus A$. By Lemma 2.3, $\lim_k \psi(x_{n_k}) = x$ in $\mathscr{T}_{\hat{E}}$. Because pre-images of compact sets through ψ are compact, we can then find a further subsequence $(x_{n_{k_j}})$ such that $x'' \stackrel{\text{def}}{=} \lim_j x_{n_{k_j}}$ exists in $\mathscr{T}_E$. Because ψ is continuous, we thus have that $\psi(x'') = \lim_j \psi(x_{n_{k_j}}) = x$. Hence $\lim_j f'(x_{n_{k_j}}) = \lim_j f(x_{n_{k_j}}) = f(x'') = f'(\pi(x'')) = f'(\psi(x'')) = f'(x)$.

We next need to verify that $f' \in C_0(\tilde{E})$. Fix $L > 0$ and define $\tilde{H} \stackrel{\text{def}}{=} \{x \in \tilde{E} : |f'(x)| \geq L\}$ and $H \stackrel{\text{def}}{=} \{x \in E : |f(x)| \geq L\}$. Since $f \in C_0(E)$ by assumption, H is compact (in $\mathscr{T}_E$). We note that $\pi H = \tilde{H}$ and so, since π is continuous, $\tilde{H}$ is compact.

We now know that $f' \in C_0(\tilde{E}) \subset B(\tilde{E})$. Clearly $\pi^* f' = f$, so we now know that $C_0^{\psi,A}(\tilde{E}) \subseteq \pi^* C_0(\tilde{E})$. Thus in fact $C_0^{\psi,A}(\tilde{E}) = \pi^* C_0(\tilde{E})$. Hence π^* is a surjection of $C_0(\tilde{E})$ onto $C_0^{\psi,A}(E)$.

To finish, we need to check that π^* is also injective. Assume that $\pi^* f_1 = \pi^* f_2$ for some f_1 and f_2 in $C_0(\tilde{E})$. Since π is a surjection, we know that in fact $f_1 = f_2$. This completes the proof. $\square$

Acknowledgments. We thank Marc Yor for a number of useful comments on an early draft and the referee for a number of extremely helpful observations.

REFERENCES

[1] BARLOW, M. T. and EVANS, S. N. (2002). Markov processes on vermiculated state spaces. *Random Walks and Geometry* (V. Kaimanovich, ed.). de Gruyter, Berlin.

[2] BARLOW, M. T., PITMAN, J. and YOR, M. (1989). On Walsh's Brownian motions. *Séminaire de Probabilités XXXII. Lecture Notes in Math.* **1613** 30–36. Springer, Berlin.

[3] BARLOW, M. T., ÉMERY, M., KNIGHT, F. B., SONG, S. and YOR, M. (1998). Author d'un théorème de Tsirelson sur les filtrations Browniennes et non Browniennes. *Séminaire de Probabilités XXXII. Lecture Notes in Math.* **1686** 264–305. Springer, Berlin.

[4] BASS, R. F. and BURDZY, K. (2000). Fiber Brownian motion and the "hot spots" problem. *Duke Math. J.* **105** 25–58.

[5] BERTOIN, J. (1996). *Lévy processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press.

[6] BIANE, P. (1995). Interwining of Markov semi-groups, some examples. *Séminaire de Probabilités XXIX. Lecture Notes in Math.* **1613** 30–36. Springer, Berlin.

[7] CARMONA, P., PETIT, F. and YOR, M. (1998). Beta–gamma random variables and intertwining relations between certain Markov processes. *Rev. Mat. Iberoamericana* **14** 311–367.

[8] DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential.* North-Holland, Amsterdam.

[9] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence.* Wiley, New York.

[10] FREIDLIN, M. and WENTZELL, A. D. (1994). Random perturbations of Hamiltonian systems. *Mem. Amer. Math. Soc.* **109** viii+82.

[11] KURTZ, T. G. (1998). Martingale problems for conditional distributions of Markov processes. *Electronic J. Probab.* **3**(9).

[12] ROGERS, L. C. G. and PITMAN, J. W. (1981). Markov functions. *Ann. Probab.* **9** 573–582.

[13] ROGERS, L. C. G. and WILLIAMS, D. (1978). *Diffusions, Markov Processes, and Martingales*: *Itô Calculus* **2**. Wiley, Chichester.

[14] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales* **1**: *Foundations*, 2nd ed. Cambridge Univ. Press.

[15] SHARPE, M. (1988). *General Theory of Markov Processes.* Academic Press, San Diego.

[16] SOWERS, R. B. (2002). Stochastic averaging with a flattened Hamiltonian: A Markov process on a stratified space (a whiskered sphere). *Trans. Amer. Soc.* **354** 853–900.

[17] TSIRELSON, B. (1997). Triple points: From non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.* **7** 1096–1142.

[18] WALSH, J. B. (1978). A diffusion with discontinuous local time. *Temps Locaux. Astérisque* **52–53** 37–45.

[19] YOR, M. (1989). Une extension markovienne de l'algébre des lois béta–gamma. *C. R. Acad. Sci. Paris Sér. I Math.* **308** 257–260.

DEPARTMENT OF STATISTICS 3860
UNIVERSITY OF CALIFORNIA, BERKELEY
367 EVANS HALL
BERKELEY, CALIFORNIA 94720-3860
E-MAIL: evans@stat.berkeley.edu
WEB: http://www.stat.berkeley.edu/users/evans

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 60201
E-MAIL: r-sowers@uiuc.edu
WEB: http://www.math.uiuc.edu/˜r-sowers/