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PATHWISE STOCHASTIC TAYLOR EXPANSIONS AND STOCHASTIC VISCOSITY SOLUTIONS FOR FULLY NONLINEAR STOCHASTIC PDEs

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In this paper we study a new type of "Taylor expansion" for Itô-type random fields, up to the second order. We show that an Itô-type random field with reasonably regular "integrands" can be expanded, up to the second order, to the linear combination of increments of temporal and spatial variables, as well as the driven Brownian motion, around even a random (t, x)-point. Also, the remainder can be estimated in a "pathwise" manner. We then show that such a Taylor expansion is valid for the solutions to a fairly large class of stochastic differential equations with parameters, or even fully-nonlinear stochastic partial differential equations, whenever they exist. Using such analysis we then propose a new definition of stochastic viscosity solution for fully nonlinear stochastic PDEs, in the spirit of its deterministic counterpart. We prove that this new definition is actually equivalent to the one proposed in our previous works [2] and [3], at least for a class of quasilinear SPDEs.

1. Introduction. In this paper we are interested in random fields of the following *Itô-form*:

$$u(t,x)=u(0,x)+\int_0^t u_1(s,x)\,ds+\int_0^t u_2(s,x)\,dB_s,\qquad (t,x)\in[0,T]\times\mathbb{R}^n, \tag{1.1}$$

where B is a standard Brownian motion. We would like to ask the following questions:

(I) Suppose that u_1 and u_2 are both smooth in the variables (t, x). Is it possible to obtain a *Taylor expansion* of u at any given point (τ, ξ)? For example, can we write

$$u(t,x)=u(\tau,\xi)+\sum_{\substack{i,j\ge 0\\ 1\le i+j\le N}} a_{ij}(x-\xi,\dots,x-\xi)(t-\tau)^j+R_N(t,x),$$

where $a_{ij}(\cdots)$'s are some i-linear forms over $\mathbb{R}^n$?

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(II) What will happen if (τ, ξ) is random (e.g., τ is a stopping time)?

(III) How to evaluate the remainder $R_N(t,x)$? For instance, can we estimate them ω-wisely?

By a simple observation on (1.1) we see that the Taylor expansion proposed in question (I) is not quite realistic, because of the presence of the stochastic integral, and the fact that a Brownian motion is nowhere differentiable. A more reasonable form of the Taylor expansion could then, for example, be the following:

$$\begin{aligned} u(t,x) = u(\tau,\xi) + &\sum_{\substack{i,j\ge 0\\ 1\le i+2j+k\le N}} a_{ijk}(x-\xi,\dots,x-\xi)^i (t-\tau)^j (B_t - B_\tau)^j \\ &+ R_N(t,x), \end{aligned} \tag{1.2}$$

where a_{ijk}'s are i-linear forms. It seems that to date the Taylor expansion of this kind has not been addressed, to our best knowledge, in any literature. In fact, some of the answers to these questions [especially question (II)] turn out to be surprisingly technical, as we shall see in this paper.

We should note that the *stochastic Taylor expansion* was discussed in, for example, the book of Kloeden and Platen [6]. But our expansion differs from theirs in several ways. First, the Taylor expansion there mainly deals with functions of an Itô process, while ours deals with Itô-type random fields directly; second, the error term of the Taylor expansion in [6] are evaluated in the mean-square sense, while we require that they be handled in an ω-wise (or *pathwise*) manner.

At this point we should point out that our study of such Taylor expansion has been strongly motivated by our previous works on the "*stochastic viscosity solutions*" for nonlinear stochastic PDEs ([2] and [3]). In these two papers we introduced a notion of stochastic viscosity solution, in the spirit of the "maximum principle"-type definition of a viscosity solution in deterministic PDE theory (see [4] or [5]), as well as the stochastic characteristic method proposed by Lions and Souganidis [7, 8] for stochastic PDEs. We proved in [2] and [3], among other things, the existence and uniqueness of such stochastic viscosity solutions for a class of quasilinear SPDEs, by relating it to the so-called Backward Doubly SDEs (BDSDE), initiated by Pardoux and Peng [10]. However, the notion of stochastic viscosity solution introduced in [2] and [3] depends rather heavily on a Doss–Sussmann-type transformation of the equation, which inevitably causes the limitation in its generality. This paper is in a sense an attempt to make the notion of stochastic viscosity solution more tractable.

Our idea is the following: since in deterministic case there are two equivalent definitions of a viscosity solution, we shall first explore the possibility of extending the other definition, that is, the one uses "*jets*" (or *subdifferentials*), to the stochastic case. This is exactly where the proposed stochastic Taylor expansion comes into play, because it is a building block of the so-called *stochastic* "*jets/subdifferentials*," as we have seen in the deterministic theory.

The main results of this paper can be briefly described as follows. Suppose that $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^n\}$ is an Itô-type random field of the form (1.1). Then, under reasonable regularity assumptions on the integrands u_1 and u_2, the following stochastic "Taylor expansion" holds: *for any stopping time τ and any $\mathcal{F}_\tau$-measurable, square-integrable random variable ξ, and for any sequence of random variables $\{(\tau_k, \xi_k)\}$ where τ_k's are stopping times such that either $\tau_k > \tau$, $\tau_k \downarrow \tau$; or $\tau_k < \tau$, $\tau_k \uparrow \tau$, and ξ_k's are all $\mathcal{F}_\tau$-measurable, square integrable random variables, it holds almost surely that*

$$\begin{aligned} u(\tau_k,\xi_k) &= u(\tau,\xi) + a(\tau_k-\tau) + b(\xi_k-\xi) + \frac{c}{2}(B_{\tau_k}-B_\tau)^2 \\ &\quad + \langle p, \xi_k-\xi \rangle + \langle q, \xi_k-\xi \rangle (B_{\tau_k}-B_\tau) + \tfrac{1}{2}\langle X(\xi_k-\xi), \xi_k-\xi \rangle \\ &\quad + o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2), \end{aligned} \tag{1.3}$$

where (a,b,c,p,q,X) are all $\mathcal{F}_\tau$-measurable random variables, and the remainder $o(\zeta_k)$ are such that $o(\zeta_k)/\zeta_k \to 0$ as $k \to \infty$, in probability. Furthermore, the six-tuple (a,b,c,p,q,X) can be determined explicitly in terms of u_1, u_2 and their derivatives.

By choosing u_1 and u_2 in different forms, we shall then extend the Taylor expansion to random fields that are the solutions of stochastic differential equations, including fully nonlinear stochastic PDEs. In the case when the coefficients are less regular, we then introduce the notion of the *stochastic super*(*sub*)*jets* (or stochastic subdifferentials), from which an alternative definition of stochastic viscosity solution will be produced. We should note here that the new definition of the stochastic viscosity solution given in this paper does not depend on any type of transformation of the equation, hence it should be applicable to general *fully nonlinear SPDEs*. We shall prove, however, in the special quasilinear situation our new definition indeed coincides with the one defined in [2] and [3]. This not only justfies, in a sense, both of our definitions, but we hope it paves the way for further extensions of the results regarding viscosity solutions, from the deterministic theory to a stochastic one.

As a final remark, we should point out that at this point we have not been able to use our new definition of stochastic viscosity solution to extend our uniqueness result in [3] to more general nonlinear SPDEs. The main obstacles still seem to be the comparison between the martingale integrands of SPDEs, and finding suitable stochastic version of the "sup convolutions" that possesses special measurabilities (e.g., adaptedness) on the temporal variables. We hope to be able to address these issues in our future publications.

This paper is organized as follows. In Section 2 we give all the notations, as well as the simplest cases of stochastic Taylor expansions. In Section 3 we extend the expansion to the solutions of stochastic differential equations, and in Section 4 we prove a technical lemma appeared in Section 3. We turn our attention to the solutions of fully nonlinear stochastic PDEs in Section 5. We will first study the stochastic jets and their properties, and then introduce the stochastic viscosity solution. In Section 6 we consider a special class of quasilinear SPDEs, and prove that the new definition is also "closed" under the so-called Doss–Sussmann transformation, as was seen in [2] and [3]. Finally, in Section 7 we prove that the stochastic viscosity solutions defined in this paper is indeed equivalent to the one proposed in [2] and [3], in the special case considered in Section 6.

2. Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which is defined a 1-dimensional standard Brownian motion $B = \{B_t : t \in [0, T]\}$, where $T > 0$ is some fixed time horizon. We denote by $\mathbf{F} \triangleq \{\mathcal{F}_t\}_{t\ge 0}$ the natural filtration generated by B, augmented by all the P-null sets in $\mathcal{F}$.

Throughout this paper we let $\mathbb{E}$ be a generic Euclidean space, with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and we shall denote:

- For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}_T$ and $p \ge 0$, $L^p(\mathcal{G}; \mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables such that $E|\xi|^p < \infty$. When there is no danger of confusion, we often write $\xi \in \mathcal{G}$ whenever $\xi \in L^0(\mathcal{G}; \mathbb{E})$ for simplicity.
- For any $q \ge 0$, $L^q(\mathbf{F}, [0, T]; \mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes ψ, such that $E \int_0^T |\psi_t|^q \, dt < \infty$. In particular, $L^0(\mathbf{F}, [0, T]; \mathbb{E})$ denotes all $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes; and $L^\infty(\mathbf{F}, [0, T]; \mathbb{E})$ denotes those processes in $L^0(\mathbf{F}, [0, T]; \mathbb{E})$ that are uniformly bounded.
- $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ to be the space of all $\mathbb{E}_1$-valued functions defined on $[0, T] \times \mathbb{E}$ which are k-times continuously differentiable in $t \in [0, T]$ and ℓ-times continuously differentiable in $x \in \mathbb{E}$; $C_b^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ to be the subspace of $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ in which all functions have uniformly bounded partial derivatives; and $C_p^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ to be the subspace of $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ in which all the partial derivatives are of at most polynomial growth.
- For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}_T$, $C^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$) to be the space of all $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$)-valued random variables that are $\mathcal{G} \otimes \mathcal{B}([0, T] \times \mathbb{E})$-measurable.
- $C^{k,\ell}(\mathbf{F}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ [resp. $C_b^{k,\ell}(\mathbf{F}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathbf{F}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$] to be the space of all random fields $\varphi \in C^{k,\ell}(\mathcal{F}_T, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathcal{F}_T, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathcal{F}_T, [0, T] \times \mathbb{E}; \mathbb{E}_1)$), such that for fixed $x \in \mathbb{E}$, the mapping $(t, \omega) \mapsto \varphi(t, x, \omega)$ is $\mathbf{F}$-progressively measurable.
- $\mathcal{M}_{0,T}$ to be all the $\mathbf{F}$-stopping times τ such that $0 \le \tau \le T$, P-a.s., and $\mathcal{M}_{0,\infty}$ to be all $\mathbf{F}$-stopping times that are almost surely finite.

When the context is clear, we shall use the simplified notations such as: $C^{k,\ell}([0, T] \times \mathbb{E}) = C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{R})$; $C([0, T] \times \mathbb{E}; \mathbb{E}_1) = C^{0,0}([0, T] \times \mathbb{E}; \mathbb{E}_1)$; and $C(\mathbf{F}, [0, T] \times \mathbb{E}) = C^{0,0}(\mathbf{F}, [0, T] \times \mathbb{E}), \dots$, etc.

Furthermore, for $(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$, we denote $D=D_x=(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$, $D^2=D_{xx}=(\partial^2_{x_ix_j})^n_{i,j=1}$, $\frac{\partial}{\partial y}=D_y$, and $\frac{\partial}{\partial t}=D_t$. The meaning of D_{xy}, D_{yy}, etc., should be clear.

In order to describe the *pathwise stochastic Taylor expansion*, we need the following definition.

DEFINITION 2.1. Let $\tau\in\mathcal{M}_{0,T}$, and $\xi\in\mathcal{F}_\tau$. We say that a sequence of random variables $\{(\tau_k,\xi_k)\}$ is a (τ,ξ)-*approximating sequence* if $(\tau_k,\xi_k)\in\mathcal{M}_{0,\infty}\times L^2(\mathcal{F}_\tau;\mathbb{R}^n)$, $\forall k$, such that:

(i) $\xi_k\to\xi$, in probability;

(ii) either $\tau_k\uparrow\tau$, a.s. and $\tau_k<\tau$ on the set $\{\tau>0\}$; or $\tau_k\downarrow\tau$, a.s., and $\tau_k>\tau$ on the set $\{\tau<T\}$.

If $\{\zeta_k\}$ is a sequence of random variables that converges to 0 in probability, then we shall denote $\eta_k=o(\zeta_k)$, $k=1,2,\dots$, to be any sequence of random variables such that $[o(\zeta_k)/\zeta_k]\mathbb{1}_{\{\zeta_k\neq0\}}\to0$, as $k\to\infty$, in probability.

We remark here that when $\tau_k\uparrow\tau$, then it is necessary that $\tau_k\in\mathcal{M}_{0,T}$, $\forall k$. However, we do not require that $\tau_k\in\mathcal{M}_{0,T}$ in the case when $\tau_k\downarrow\tau$. This is to avoid unnecessary technical subtlety in our future discussion.

To understand what the pathwise stochastic Taylor expansion is, let us first consider the following simple situation. Let $u\in C(\mathbf{F};[0,T]\times\mathbb{R}^n)$ be a random field. We say that u is of *Itô-type* if it can be written in the following form:

$$u(t,x)=u(0,x)+\int_0^t u_1(s,x)\,ds+\int_0^t u_2(s,x)\,dB_s,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\tag{2.1}$$

where $u_1,u_2\in C(\mathbf{F};[0,T]\times\mathbb{R}^n)$ such that

$$P\left\{\int_0^T|u_1(s,x)|\,ds+\int_0^T|u_2(s,x)|^2\,ds<\infty,\ \forall x\in\mathbb{R}^n\right\}=1.$$

Let us first assume that $u_2\equiv0$, and $u_1\in C^{0,2}(\mathbf{F};[0,T]\times\mathbb{R}^n)$. In this case, for any $(\tau,\xi)\in\mathcal{M}_{0,T}\times L^2(\mathcal{F}_\tau;\mathbb{R}^n)$, and $\{(\tau_k,\xi_k)\}$ be any (τ,ξ)-approximating sequence, it is readily seen that an application of usual Taylor expansion on u_1 in x would lead to that

$$\begin{aligned}u(\tau_k,\xi_k)=u(\tau,\xi)&+a(\tau_k-\tau)+\langle p,\xi_k-\xi\rangle+\tfrac12\langle X(\xi_k-\xi),\xi_k-\xi\rangle\\&+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2),\end{aligned}\tag{2.2}$$

where $a=u_1(\tau,\xi)$, $p=Du(\tau,\xi)$, $X=D^2u(\tau,\xi)$; and the remainder $o(\cdots)$ is in the sense of Definition 2.1. Since such an expansion holds ω-wisely, we call it a *pathwise stochastic Taylor expansion* of the random field $u(\cdot,\cdot)$.

Such a stochastic Taylor expansion becomes slightly more difficult when $u_2 \neq 0$, because of the presence of the stochastic integral, which cannot be analyzed in a *pathwise* (or "local") manner. We give a heuristic argument to illustrate this point.

Let $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ be given, and let $\{(\tau_k, \xi_k)\}$ be any (τ, ξ)-approximating sequence. For simplicity let us assume that $\tau_k \uparrow \tau$. (In fact, the other case is similar, with less difficulty.) Let us denote

$$\Delta_1 = u(\tau_k, \xi_k) - u(\tau, \xi_k), \qquad \Delta_2 = u(\tau, \xi_k) - u(\tau, \xi), \tag{2.3}$$

then $u(\tau_k, \xi_k) - u(\tau, \xi) = \Delta_1 + \Delta_2$. Assume that $u \in C^{0,2}(\mathbf{F}; [0, T] \times \mathbb{R}^n)$. Then as before we can derive a pathwise expansion for Δ_2:

$$\Delta_2 = \langle Du(\tau, \xi), \xi_k - \xi \rangle + \tfrac{1}{2} \langle D^2 u(\tau, \xi)(\xi_k - \xi), \xi_k - \xi \rangle + o(|\xi_k - \xi|^2), \tag{2.4}$$

where $o(\cdots)$ is in the sense of Definition 2.1. Thus we turn our attention to Δ_1. From (2.1) we see that Δ_1 can be written as

$$\Delta_1 = -\int_{\tau_k}^{\tau} u_1(s, \xi_k)\, ds - \int_{\tau_k}^{\tau} u_2(s, x)\, dB_s\Big|_{x=\xi_k} \triangleq \Delta_1^1 + \Delta_1^2, \tag{2.5}$$

where Δ_1^1 and Δ_1^2 are the two integrals in (2.5). Again, by the same argument as before we can have a pathwise expansion for Δ_1^1:

$$\Delta_1^1 = -u_1(\tau, \xi_k)(\tau - \tau_k) + o(|\tau_k - \tau|) = u_1(\tau, \xi)(\tau_k - \tau) + o(|\tau_k - \tau|). \tag{2.6}$$

However, the pathwise expansion of Δ_1^2 is not so trivial. The main difficulty here is that all ξ_k's are only assumed to be $\mathcal{F}_\tau$-measurable, and $\tau \geq \tau_k$. Thus one cannot simply replace x by ξ_k in the stochastic integral and argue as before.

In this simple case, however, we can get around with this difficulty as follows. Assume that $u_2 \in C^{1,2}(\mathbf{F}; [0, T] \times \mathbb{R}^n)$. Then, for each $(s, x) \in [0, T] \times \mathbb{R}^n$ we have

$$u_2(s, x) = u_2(\tau_k, x) + \int_{\tau_k}^{s} \partial_t u_2(r, x)\, dr.$$

Denoting $U = \partial_t u_2$, we have

$$\int_{\tau_k}^{\tau} u_2(s, x)\, dB_s = -u_2(\tau_k, x)(B_{\tau_k} - B_\tau) + \int_{\tau_k}^{\tau} \int_{\tau_k}^{s} U(r, x)\, dr\, dB_s. \tag{2.7}$$

Using integration by parts (or Fubini's theorem), we have

$$\int_{\tau_k}^{\tau} \int_{\tau_k}^{s} U(r, x)\, dr\, dB_s = \int_{\tau_k}^{\tau} (B_\tau - B_r) U(r, x)\, dr.$$

Since the right-hand side above is no longer a stochastic integral, we obtain

$$\int_{\tau_k}^{\tau} \int_{\tau_k}^{s} U(r, x)\, dr\, dB_s\Big|_{x=\xi_k} = \int_{\tau_k}^{\tau} (B_\tau - B_r) U(r, \xi_k)\, dr = o(|\tau_k - \tau|), \tag{2.8}$$

thanks to the continuity of the Brownian path and that of $U(\cdot,\cdot)$. Furthermore, since $u_2 \in C^{1,2}(\mathbf{F}; [0,T]\times\mathbb{R}^n)$ we have

$$u_2(\tau_k,\xi_k) = u_2(\tau,\xi) + \partial_t u_2(\tau,\xi)(\tau_k-\tau) + \langle Du_2(\tau,\xi), \xi_k - \xi\rangle + \tfrac{1}{2}\langle D^2u_2(\tau,\xi)(\xi_k-\xi), \xi_k-\xi\rangle + o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2),$$

we derive from (2.7) and (2.8) that

$$\int_{\tau_k}^{\tau} u_2(s,x)\,dB_s\big|_{x=\xi_k} = -u_2(\tau,\xi)(B_{\tau_k}-B_\tau) - \langle Du_2(\tau,\xi),\xi_k-\xi\rangle(B_{\tau_k}-B_\tau) + o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2).$$

Plugging this into (2.5) and combining it with (2.6), (2.3) and (2.4) we have proved the following *pathwise Taylor expansion* for an Itô-type random field. As a matter of fact, such a result is new, to our best knowledge, even in such a simplest setting.

THEOREM 2.2. *Let $u \in C^{0,2}(\mathbf{F}, [0,T]\times\mathbb{R}^n)$ be a random field of Itô-type. Suppose that $u(0,\cdot)\in C(\mathbb{R}^n)$, $u_1 \in C^{0,2}(\mathbf{F}; [0,T]\times\mathbb{R}^n)$, and $u_2 \in C^{1,2}(\mathbf{F}; [0,T]\times\mathbb{R}^n)$. Then, for any $(\tau,\xi)\in\mathcal{M}_{0,T}\times L^2(\mathcal{F}_\tau;\mathbb{R}^n)$, and any (τ,ξ)-approximating sequence, $\{(\tau_k,\xi_k)\}$, the following pathwise Taylor expansion holds*:

$$\begin{aligned} u(\tau_k,\xi_k) = {} & u(\tau,\xi) + a(\tau_k-\tau) + b(B_{\tau_k}-B_\tau) \\ & + \langle p, \xi_k-\xi\rangle + \langle q,\xi_k-\xi\rangle(B_{\tau_k}-B_\tau) \\ & + \tfrac{1}{2}\langle X(\xi_k-\xi),\xi_k-\xi\rangle + o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2), \end{aligned} \tag{2.9}$$

where

$$\begin{cases} a = u_1(\tau,\xi), & p = Du(\tau,\xi); \qquad X = D^2u(\tau,\xi), \\ b = u_2(\tau,\xi), & q = Du_2(\tau,\xi). \end{cases} \tag{2.10}$$

3. Pathwise Taylor expansion for solutions of SDEs with parameters.

In this section we study the pathwise Taylor expansion formula for solutions of stochastic differential equations with parameters. To be more precise, we shall consider the following SDE:

$$u(t,x) = \varphi(x) + \int_0^t f\big(s,x,u(t,x)\big)\,ds + \int_0^t g\big(s,x,u(s,x)\big)\,dB_s, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \tag{3.1}$$

where $x\in\mathbb{R}^n$ is a parameter. Throughout this section we assume that

$$f, g \in C_b^{0,2,2}([0,T]\times\mathbb{R}^n\times\mathbb{R}), \qquad \varphi(\cdot)\in C_b^2(\mathbb{R}^n). \tag{3.2}$$

Then, by the standard theory of SDE we know that in such a case the solution $u(t,x)$ will be twice (stochastically) differentiable in the spatial variable x as well (the limit is understood as being taken in the sense of in probability). Since our main purpose of the paper is to establish the form of pathwise Taylor expansion of the solution, we will concentrate on such "smooth" solutions, and do not seek the mimimum conditions on the coefficients. To facilitate our discussion, let us give the following definition.

DEFINITION 3.1. A random field $u = \{u(t,x,\omega) : (t,x,\omega) \in [0,T] \times \mathbb{R}^n \times \Omega\}$ is called a "regular" solution to SDE (3.1) if:

(i) $u \in C^{0,2}(\mathbf{F}; [0,T] \times \mathbb{R}^n)$;
(ii) u is an Itô-type random field with the form

$$u(t,x) = \varphi(x) + \int_0^t u_1(s,x)\,ds + \int_0^t u_2(s,x)\,dB_s, \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where

$$u_1(t,x) = f\big(t,x,u(t,x)\big), \qquad u_2(t,x) = g\big(t,x,u(t,x)\big), \tag{3.3}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, P-a.s.

Our main result of this section is the following theorem.

THEOREM 3.2. *Assume* (3.2), *and further that* $g \in C_b^{1,2,3}([0,T] \times \mathbb{R}^n \times \mathbb{R})$. *Let u be a regular solution of SDE* (3.1). *Then*, *for any* $(\tau,\xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ *and any* (τ,ξ)-*approximating sequence* $\{(\tau_k,\xi_k)\}$, *the following pathwise Taylor expansion holds*:

$$\begin{aligned} u(\tau_k,\xi_k) &= u(\tau,\xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 + \langle p, \xi_k - \xi \rangle \\ &\quad + \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \frac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle \\ &\quad + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2), \end{aligned} \tag{3.4}$$

where

$$\begin{cases} a = f\big(\tau,\xi,u(\tau,\xi)\big) - \frac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big), \\ b = g\big(\tau,\xi,u(\tau,\xi)\big), \qquad c = (g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big); \\ p = Du(\tau,\xi), \qquad q = \partial_x g\big(\tau,\xi,u(\tau,\xi)\big) + \partial_u g\big(\tau,\xi,u(\tau,\xi)\big) Du(\tau,\xi), \\ X = D^2 u(\tau,\xi). \end{cases} \tag{3.5}$$

In particular, *if τ and τ_k's are all deterministic*, *the extra assumption on g can be dropped*.

REMARK 3.3. (i) In the case when $\tau = 0$ or $\tau = T$, only $\tau_k \downarrow \tau$ or $\tau_k \uparrow \tau$ should be considered, respectively.

(ii) It is easy to see that, if f and g are independent of u, then (3.4) and (3.5) coincide with (2.9) and (2.10). Therefore Theorem 3.2 contains Theorem 2.2 as a special case. Furthermore, we see that it is the dependence of g on u that produces the "unusual" term involving $(B_{\tau_k} - B_\tau)^2$.

Before we prove the theorem, let us first introduce the so-called *Wick-square* of the Brownian motion, which is originated in the Wiener homogeneous chaos expansion. For any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and any (τ, ξ)-approximating sequence $\{(\tau_k, \xi_k)\}$, we define the "Wick-square" of $B_{\tau_k} - B_\tau$ to be

$$(B_{\tau_k} - B_\tau)^{\diamond 2} \triangleq (B_{\tau_k} - B_\tau)^2 - |\tau_k - \tau|. \tag{3.6}$$

The following explicit form, which can be fairly easily verified, will be useful in our future discussion:

$$(B_{\tau_k} - B_\tau)^{\diamond 2} = \begin{cases} 2\displaystyle\int_{\tau_k}^{\tau}\int_{\tau_k}^{s} dB_r\, dB_s = 2\int_{\tau_k}^{\tau}(B_s - B_{\tau_k})\, dB_s, & \text{if } \tau_k \le \tau, \\ 2\displaystyle\int_{\tau}^{\tau_k}\int_{\tau}^{s} dB_r\, dB_s = 2\int_{\tau}^{\tau_k}(B_s - B_\tau)\, dB_s, & \text{if } \tau_k \ge \tau. \end{cases} \tag{3.7}$$

We now give two lemmas that are interesting in their own rights. The first one is more or less conceivable, but the second one is more involved. In fact, since the proof of the second lemma is quite technical and lengthy, we shall postpone it to the next section in order not to disturb our discussion. Those readers who are not particularly interested in such level of technical details can even skip that section.

LEMMA 3.4. *Let* $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, *and* $\{(\tau_k, \xi_k)\}$ *be any* (τ, ξ) *approximating sequence*. *Then*:

(i) $(B_{\tau_k} - B_\tau)^3 = o(|\tau_k - \tau|)$.
(ii) $(B_{\tau_k} - B_\tau)^{\diamond 2}(B_{\tau_k} - B_\tau) = o(|\tau_k - \tau|)$.

PROOF. Since assertion (ii) follows immediately from (i) and the definition (3.6), we only prove (i). Further, we shall assume without loss of generality that $\tau_k \uparrow \tau$, since the other case is similar.

First let us consider the following random variable:

$$\eta \triangleq \sup_{0<t\le 1} \frac{|B_t|}{\sqrt{t \log\log(1/t)}}.$$

By the law of the iterated logarithm and the continuity of the Brownian paths, one has $\eta < +\infty$, P-a.s. Thus, for any $0 < \delta \le 1$, it holds that

$$\begin{aligned}\sup_{0<t\le\delta}\left\{\frac{|B_t|^3}{t}\right\} &= \sup_{0<t\le\delta}\left\{\frac{|B_t|}{\sqrt{t\log\log(1/t)}}\right\}^3\sqrt{t\log^3\left(\log\frac{1}{t}\right)}\\ &\le \eta^3\sup_{0<t\le\delta}\left\{\sqrt{t\log^3\left(\log\frac{1}{t}\right)}\right\}\to 0, \qquad P\text{-a.s., as } \delta\to 0.\end{aligned}$$

Therefore, $\sup_{0<t\le\delta}\{|B_t|^3/t\}\to 0$, as $\delta\to 0$, in probability. Consequently, for any $\varepsilon, \varepsilon' > 0$, we can choose $\delta > 0$ so that $P\{\sup_{0<t\le\delta}\{|B_t|^3/t\} > \varepsilon\} \le \frac{\varepsilon'}{2}$. Let $K(\varepsilon,\varepsilon') > 0$ be such that $P\{\tau-\tau_k > \delta\} \le \frac{\varepsilon'}{2}$, $\forall k \ge K(\varepsilon,\varepsilon')$. Then

$$\begin{aligned}P\left\{\left|\frac{(B_\tau - B_{\tau_k})^3}{\tau-\tau_k}\right| > \varepsilon\right\} &\le P\left\{\left|\frac{(B_\tau - B_{\tau_k})^3}{\tau-\tau_k}\right| > \varepsilon, |\tau-\tau_k|\le\delta\right\} + P\{|\tau-\tau_k|>\delta\}\\ &\le P\left\{\sup_{0<t\le\delta}\frac{|B_t|^3}{t} > \varepsilon\right\} + \frac{\varepsilon'}{2} \le \varepsilon', \qquad k \ge K(\varepsilon,\varepsilon').\end{aligned}$$

To wit, $\frac{(B_\tau - B_{\tau_k})^3}{|\tau-\tau_k|}\to 0$, as $k\to\infty$, in probability, proving the lemma. □

LEMMA 3.5. *Assume* (3.2), *and denote* $G(t,x) \triangleq (g\partial_u g)(t,x,u(t,x))$. *Let* $(\tau,\xi)\in\mathcal{M}_{0,T}\times L^2(\mathcal{F}_\tau;\mathbb{R}^n)$ *be given and* $\{(\tau_k,\xi_k)\}$ *be any* (τ,ξ)-*approximating sequence*. *Assume further that* $g\in C_b^{1,2,3}([0,T]\times\mathbb{R}^n)$. *Then the following estimates hold*:

$$(3.8)\quad \begin{cases}\displaystyle\int_{\tau_k}^{\tau}\int_{\tau_k}^{s} G(r,x)\,dB_r\,dB_s\Big|_{x=\xi_k} = \tfrac{1}{2}G(\tau_k,\xi_k)(B_\tau - B_{\tau_k})^{\diamond 2} + o(|\tau_k-\tau|), & \tau > \tau_k,\\ \displaystyle\int_{\tau}^{\tau_k}\int_{\tau}^{s} G(r,x)\,dB_r\,dB_s\Big|_{x=\xi_k} = \tfrac{1}{2}G(\tau,\xi_k)(B_\tau - B_{\tau_k})^{\diamond 2} + o(|\tau_k-\tau|), & \tau < \tau_k.\end{cases}$$

Furthermore, *if* τ *and* τ_k*'s are all deterministic*, *then the extra assumption on* g *can dropped*.

For the proof see the next section.

PROOF OF THEOREM 3.2. In light of the argument for Theorem 2.2, we assume first that $\tau_k\uparrow\tau$; and denote $\Delta_1 = u(\tau_k,\xi_k) - u(\tau,\xi_k)$ and $\Delta_2 = u(\tau,\xi_k) - u(\tau,\xi)$ so that $u(\tau_k,\xi_k) - u(\tau,\xi) = \Delta_1 + \Delta_2$. Again, the standard Taylor formula yields that

$$(3.9)\quad \Delta_2 = \langle Du(\tau,\xi), \xi_k - \xi\rangle + \tfrac{1}{2}\langle D^2u(\tau,\xi)(\xi_k-\xi), \xi_k-\xi\rangle + o(|\xi_k-\xi|^2).$$

Further, writing Δ_1 as

$$\Delta_1 = -\int_{\tau_k}^{\tau} u_1(s, \xi_k)\, ds - \int_{\tau_k}^{\tau} g\big(s, x, u(s, x)\big)\, dB_s\big|_{x=\xi_k} \triangleq \Delta_1^1 + \Delta_1^2, \tag{3.10}$$

where $u_1(t, x) \triangleq f(t, x, u(t, x))$, we also have

$$\Delta_1^1 = -u_1(\tau, \xi_k)(\tau - \tau_k) + o(|\tau_k - \tau|) = u_1(\tau, \xi)(\tau_k - \tau) + o(|\tau_k - \tau|). \tag{3.11}$$

Our main task is to estimate Δ_1^2. To this end we note that for fixed x, $g(\cdot, x, \cdot) \in C^{1,2}([0, T] \times \mathbb{R})$, and $u(\cdot, x)$ is an Itô process, thus we can apply Itô's formula to get

$$\begin{aligned} g\big(s, x, u(s, x)\big) &= g\big(\tau_k, x, u(\tau_k, x)\big) \\ &\qquad + \int_{\tau_k}^{s} \big\{\partial_t g\big(r, x, u(r, x)\big) + \partial_u g\big(r, x, u(r, x)\big) u_1(r, x) \\ &\qquad\qquad + \tfrac{1}{2}\partial_{uu} g \cdot g^2\big(r, x, u(r, x)\big)\big\}\, dr \\ &\qquad + \int_{\tau_k}^{s} (g\partial_u g)\big(r, x, u(r, x)\big)\, dB_r \\ &\triangleq g\big(\tau_k, x, u(\tau_k, x)\big) + \int_{\tau_k}^{s} U_1(r, x)\, dr + \int_{\tau_k}^{s} U_2(r, x)\, dB_r, \end{aligned}$$

where U_1 and U_2 are the integrands of the two integrals, respectively, on the right-hand side above. Thus

$$\begin{aligned} \int_{\tau_k}^{\tau} g\big(s, x, u(s, x)\big)\, dB_s &= -g\big(\tau_k, x, u(\tau_k, x)\big)(B_{\tau_k} - B_\tau) \\ &\quad + \int_{\tau_k}^{\tau}\int_{\tau_k}^{s} U_1(r, x)\, dr\, dB_s \\ &\quad + \int_{\tau_k}^{\tau}\int_{\tau_k}^{s} U_2(r, x)\, dB_r\, dB_s. \end{aligned} \tag{3.12}$$

Using Fubini's theorem we have

$$\int_{\tau_k}^{\tau}\int_{\tau_k}^{s} U_1(r, x)\, dr\, dB_s\big|_{x=\xi_k} = \int_{\tau_k}^{\tau} (B_\tau - B_r) U_1(r, \xi_k)\, dr = o(|\tau_k - \tau|). \tag{3.13}$$

Furthermore, setting $G = U_2$ in Lemma 3.5 we see that

$$\begin{aligned} \int_{\tau_k}^{\tau}\int_{\tau_k}^{s} U_2(r, x)\, dB_r\, dB_s\big|_{x=\xi_k} &= \tfrac{1}{2} U_2(\tau_k, \xi_k)(B_{\tau_k} - B_\tau)^{\diamond 2} + o(|\tau_k - \tau|) \\ &= \tfrac{1}{2}(g\partial_u g)\big(\tau_k, \xi_k, u(\tau_k, \xi_k)\big)(B_{\tau_k} - B_\tau)^{\diamond 2} \\ &\quad + o(|t_k - \tau|). \end{aligned} \tag{3.14}$$

Combining (3.12)–(3.14) we obtain that

$$\begin{aligned}\Delta_1^2 &= -\int_{\tau_k}^{\tau} g\big(s,x,u(s,x)\big)\,dB_s\big|_{x=\xi_k}\\ &= g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)\\ &\quad -\tfrac{1}{2}\big(g\partial_u g\big)\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)^{\diamond 2}+o(|\tau_k-\tau|).\end{aligned}$$

This, together with (3.10) and (3.11), gives

$$\begin{aligned}\Delta_1 &= u_1(\tau,\xi)(\tau_k-\tau)+g(\tau_k,\xi_k,u(\tau_k,\xi_k))(B_{\tau_k}-B_\tau)\\ &\quad -\tfrac{1}{2}(g\partial_u g)(\tau_k,\xi_k,u(\tau_k,\xi_k))(B_{\tau_k}-B_\tau)^{\diamond 2}+o(|\tau_k-\tau|).\end{aligned}\tag{3.15}$$

Now combining (3.15) with (3.9) we see that in order to obtain (3.4) it remains to replace the variables $(\tau_k,\xi_k,u(\tau_k,\xi_k))$ inside the functions g and $g\partial_u g$ in (3.15) by $(\tau,\xi,u(\tau,\xi))$. To this end, let us denote $\Delta=u(\tau_k,\xi_k)-u(\tau,\xi))$ $(=\Delta_1+\Delta_2)$ to simplify notations. Applying the usual Taylor expansion ω-wisely we get

$$\begin{aligned}&g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)\\ &\quad=\big\{g\big(\tau,\xi,u(\tau,\xi)\big)+\partial_t g\big(\tau,\xi,u(\tau,\xi)\big)(\tau_k-\tau)\\ &\qquad+\big\langle Dg\big(\tau,\xi,u(\tau,\xi)\big),\xi_k-\xi\big\rangle\\ &\qquad+\tfrac{1}{2}\big\langle D^2 g\big(\tau,\xi,u(\tau,\xi)\big)(\xi_k-\xi),\xi_k-\xi\big\rangle+\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)\Delta\\ &\qquad+\big\langle\partial_u Dg\big(\tau,\xi,u(\tau,\xi)\big),\xi_k-\xi\big\rangle\Delta+\tfrac{1}{2}\partial^2_{uu}g\big(\tau,\xi,u(\tau,\xi)\big)\Delta^2\\ &\qquad+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2)+o(\Delta^2)\big\}(B_{\tau_k}-B_\tau)\\ &\quad=\big\{g\big(\tau,\xi,u(\tau,\xi)\big)+\big\langle Dg\big(\tau,\xi,u(\tau,\xi)\big),\xi_k-\xi\big\rangle+\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)\Delta\\ &\qquad+\tfrac{1}{2}\partial^2_{uu}g\big(\tau,\xi,u(\tau,\xi)\big)\Delta^2\big\}(B_{\tau_k}-B_\tau)+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).\end{aligned}\tag{3.16}$$

Next, using (3.15) and (3.9) and noting that both $|\xi_k-\xi|^2(B_{\tau_k}-B_\tau)$ and $|\xi_k-\xi|(B_{\tau_k}-B_\tau)^2$ are of the order $o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2)$, thanks to the Young inequality, we have

$$\begin{aligned}&\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\Delta\\ &\quad=\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)(\Delta_1+\Delta_2)\\ &\quad=\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\\ &\qquad\times\big\{\langle Du(\tau,\xi),\xi_k-\xi\rangle+\tfrac{1}{2}\langle D^2u(\tau,\xi)(\xi_k-\xi),\xi_k-\xi\rangle\\ &\qquad\quad+u_1(\tau,\xi)(\tau_k-\tau)+g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)\\ &\qquad\quad-\tfrac{1}{2}(g\partial_u g)\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)\big((B_{\tau_k}-B_\tau)^{\diamond 2}\big)\\ &\qquad\quad+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2\big\}\\ &\quad=\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\\ &\qquad\times\big\{\langle Du(\tau,\xi),\xi_k-\xi\rangle+g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)\big\}\\ &\qquad+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2)\end{aligned}\tag{3.17}$$

and

$$
\begin{aligned}
&\tfrac{1}{2}\partial^2_{uu} g\big(\tau,\xi,u(\tau,\xi)\big)\Delta^2(B_{\tau_k}-B_\tau)\\
&\quad=\tfrac{1}{2}\partial^2_{uu} g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)(\Delta_1+\Delta_2)^2\\
&\quad=\tfrac{1}{2}\partial^2_{uu} g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)(\Delta_1)^2+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2)\\
&\quad=o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).
\end{aligned}\tag{3.18}
$$

Similarly to the above argument one obtains

$$
\begin{aligned}
&g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^2\\
&\qquad=(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^2+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).
\end{aligned}\tag{3.19}
$$

Combining the above we obtain

$$
\begin{aligned}
&g\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)\\
&\quad=g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\\
&\qquad+\big\langle Dg\big(\tau,\xi,u(\tau,\xi)\big)+\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)Du(\tau,\xi),\xi_k-\xi\big\rangle(B_{\tau_k}-B_\tau)\\
&\qquad+(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^2+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).
\end{aligned}\tag{3.20}
$$

Using the similar arguments and noting that [by (3.9), (3.15) and Lemma 3.4 again]

$$
\begin{aligned}
[u(\tau_k,\xi_k)-u(\tau,\xi)](B_{\tau_k}-B_\tau)^{\diamond 2}&=(\Delta_1+\Delta_2)(B_{\tau_k}-B_\tau)^{\diamond 2}\\
&=o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\tfrac{1}{2}(g\partial_u g)\big(\tau_k,\xi_k,u(\tau_k,\xi_k)\big)(B_{\tau_k}-B_\tau)^{\diamond 2}\\
&\qquad=\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^{\diamond 2}+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).
\end{aligned}\tag{3.21}
$$

Plugging (3.20) and (3.21) back into (3.15) [whence (3.10)] we obtain

$$
\begin{aligned}
\Delta_1&=u_1(\tau,\xi)(\tau_k-\tau)+g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\\
&\quad+\big\langle\partial_x g\big(\tau,\xi,u(\tau,\xi)\big)+\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)Du(\tau,\xi),\xi_k-\xi\big\rangle(B_{\tau_k}-B_\tau)\\
&\quad+(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^2\\
&\quad-\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^{\diamond 2}\\
&\quad+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2)\\
&=\big[u_1(\tau,\xi)-\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)\big](\tau_k-\tau)\\
&\quad+g\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)\\
&\quad+\big\langle\partial_x g\big(\tau,\xi,u(\tau,\xi)\big)+\partial_u g\big(\tau,\xi,u(\tau,\xi)\big)Du(\tau,\xi),\xi_k-\xi\big\rangle(B_{\tau_k}-B_\tau)\\
&\quad+\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)(B_{\tau_k}-B_\tau)^2+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2).
\end{aligned}\tag{3.22}
$$

Combining (3.22) with (3.9) we finally derive that

$$
\begin{aligned}
u(\tau_k, \xi_k) = u(\tau, \xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 + \langle p, \xi_k - \xi \rangle \\
+ \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \tfrac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle \\
+ o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2),
\end{aligned}
$$

where (a, b, c, p, q, X) are given exactly by (3.4).

In the case when $\tau_k \downarrow \tau$, the proof is essentially the same but slightly simpler because all the ξ_k's are $\mathcal{F}_\tau^B$-measurable, so that one can simply replace x by ξ_k's in the corresponding stochastic integrals of the types (3.12) and (3.13) without any difficulties. We only point out some main differences and leave the details to the readers. First, we define Δ_1 and Δ_2 in the same way, then (3.9) still holds. Next, similar to (3.11) and (3.12) and noting that ξ_k's are all $\mathcal{F}_\tau$-measurable we have

$$
\Delta_1 = u_1(\tau, \xi_k)(\tau_k - \tau) + \int_\tau^{\tau_k} g\big(s, \xi_k, u(s, \xi_k)\big)\, dB_s + o(|\tau_k - \tau|). \tag{3.23}
$$

Now note that $\tau_k > \tau$, we follow the same arguments as (3.12)–(3.15) and apply Lemma 3.5 accordingly to show that in this case (3.15) should be modified to

$$
\begin{aligned}
\Delta_1 &= u_1(\tau, \xi)(\tau_k - \tau) + g\big(\tau, \xi_k, u(\tau, \xi_k)\big)(B_{\tau_k} - B_\tau) \\
&\quad + \tfrac{1}{2}(g\partial_u g)\big(\tau, \xi_k, u(\tau, \xi_k)\big)(B_{\tau_k} - B_\tau)^{\diamond 2} + o(|\tau_k - \tau|).
\end{aligned} \tag{3.24}
$$

Using (3.19) and (3.21) we then have

$$
\begin{aligned}
\Delta_1 &= u_1(\tau, \xi)(\tau_k - \tau) + g\big(\tau, \xi, u(\tau, \xi)\big)(B_{\tau_k} - B_\tau) \\
&\quad + \big\langle \partial_x g\big(\tau, \xi, u(\tau, \xi)\big) + \partial_u g\big(\tau, \xi, u(\tau, \xi)\big) Du(\tau, \xi), \xi_k - \xi \big\rangle (B_{\tau_k} - B_\tau) \\
&\quad + \tfrac{1}{2}(g\partial_u g)\big(\tau, \xi, u(\tau, \xi)\big)(B_{\tau_k} - B_\tau)^{\diamond 2} + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2) \\
&= \big[u_1(\tau, \xi) - \tfrac{1}{2}\big(g\partial_u g)(\tau, \xi, u(\tau, \xi)\big)\big](\tau_k - \tau) \\
&\quad + g\big(\tau, \xi, u(\tau, \xi)\big)(B_{\tau_k} - B_\tau) \\
&\quad + \big\langle \partial_x g\big(\tau, \xi, u(\tau, \xi)\big) + \partial_u g\big(\tau, \xi, u(\tau, \xi)\big) Du(\tau, \xi), \xi_k - \xi \big\rangle (B_{\tau_k} - B_\tau) \\
&\quad + \tfrac{1}{2}(g\partial_u g)\big(\tau, \xi, u(\tau, \xi)\big)(B_{\tau_k} - B_\tau)^2 + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2).
\end{aligned} \tag{3.25}
$$

This, together with (3.9), shows that (3.4) and (3.5) still hold. The proof is now complete. □

4. Proof of Lemma 3.5. We now provide the proof of Lemma 3.5. First we should note that under assumption (3.2) it is not hard to show that for any $p > 1$, there exist $K_p > 1$ (indeed, $K_p = p$ in this case) and $C_p > 0$, such that the solution u of satisfies

$$E\left\{\sup_{0\le t\le T}\big(|u(t,x)|^p + |Du(t,x)|^p\big)\right\} \le C_p(1+|x|^{K_p}), \qquad x\in\mathbb{R}^n. \tag{4.1}$$

To simplify presentation we first prove the latter half of the lemma, that is, we first assume that all τ_k's and τ are deterministic, such that $\tau_k < \tau$ and $\tau_k \uparrow \tau$. Denote

$$\delta G(t,s,x) = G(t,x) - G(s,x) \qquad \forall (t,s,x)\in[0,T]^2\times\mathbb{R}^n,$$

and $\Phi_k(x) = \int_{\tau_k}^{\tau}\int_{\tau_k}^{s}[\delta G(r,\tau_k,x)]\,dB_r\,dB_s$. Then it is clear that

$$\int_{\tau_k}^{\tau}\int_{\tau_k}^{s} G(r,x)\,dB_r\,dB_s\big|_{x=\xi_k} = \tfrac{1}{2}G(\tau_k,\xi_k)(B_\tau - B_{\tau_k})^{\diamond 2} + \Phi_k(\xi_k).$$

[Warning: here $\Phi_k(\xi_k)$ should be understood as $\Phi_k(x)|_{x=\xi_k}$, as one cannot simply replace x by ξ_k in the integrand of the double stochastic integral because ξ_k is only $\mathcal{F}_\tau$-measurable!] We are to show that $\Phi_k(\xi_k)/|\tau_k - \tau| \to 0$ in probability, as $k\to\infty$.

To this end, let $\varepsilon > 0$ be an arbitrary constant, and $m \ge 1$ be an integer. Then, using Chebyshev's inequality one has

$$P\left\{\frac{1}{\tau-\tau_k}|\Phi_k(\xi_k)| > \varepsilon,\ |\xi_k|\le m\right\} \le \frac{1}{\varepsilon^2(\tau-\tau_k)^2}E\left\{\sup_{|x|\le m}|\Phi_k(x)|^2\right\}. \tag{4.2}$$

Now we apply the well-known Sobolev's imbedding theorem (cf., e.g., Adams [1], Theorem 4.4) to get

$$\sup_{|x|\le m}|\Phi_k(x)| \le C_{n,m}\left\{\int_{|x|\le m}\big(|\Phi_k(x)| + |D\Phi_k(x)|\big)^{n+1}dx\right\}^{1/(n+1)}. \tag{4.3}$$

Here and in the sequal, n is always the dimension of x, and $C_{n,m}$ denotes a generic constant depending only on n and m, and may vary from line to line. Therefore, applying Hölder's and Jensen's inequalities repeatedly, one shows fairly easily from (4.2) and (4.3), with $p \triangleq \frac{(n+1)}{2}$, that

$$\begin{aligned}
&P\left\{\frac{1}{\tau-\tau_k}|\Phi_k(\xi_k)| > \varepsilon,\ |\xi_k|\le m\right\}\\
&\quad\le \frac{1}{\varepsilon^2(\tau-\tau_k)^2}E\left\{\sup_{|x|\le m}|\Phi_k(x)|^2\right\}\\
&\quad\le \frac{C_{n,m}}{\varepsilon^2(\tau-\tau_k)^2}\left\{E\int_{|x|\le m}\big(|\Phi_k(x)|+|D\Phi_k(x)|\big)^{2p}dx\right\}^{1/p}\\
&\quad\le \frac{C_{n,m}}{\varepsilon^2(\tau-\tau_k)^2}\left\{\int_{|x|\le m}\big(E[|\Phi_k(x)|^{2p}] + E[|D\Phi_k(x)|^{2p}]\big)dx\right\}^{1/p}.
\end{aligned}$$

Since τ, τ_k are deterministic, if we denote

$$\|\delta G(t,s,x)\|^2 \triangleq |\delta G(t,s,x)|^2 + |D(\delta G)(t,s,x)|^2,$$

then we have

$$\begin{aligned} E[|\Phi_k(x)|^{2p}] &= E\left\{\left|\int_{\tau_k}^{\tau}\int_{\tau_k}^{s} \delta G(r,\tau_k,x)\,dB_r\,dB_s\right|^{2p}\right\} \\ &\le C_n E\left[\left\{\int_{\tau_k}^{\tau}\left|\int_{t_k}^{s}\delta G(r,\tau_k,x)\,dB_r\right|^2 ds\right\}^p\right] \\ &\le C_n(\tau-\tau_k)^{p-1}\int_{\tau_k}^{\tau} E\left\{\left|\int_{t_k}^{s}\delta G(r,\tau_k,x)\,dB_r\right|^{2p}\right\} ds \\ &\le C_n(\tau-\tau_k)^{p-1}\int_{\tau_k}^{\tau}(s-\tau_k)^{p-1}\left\{\int_{\tau_k}^{\tau} E|\delta G(r,\tau_k,x)|^{2p}\,dr\right\} ds \\ &\le C_n(\tau-\tau_k)^n \int_{\tau_k}^{\tau} E|\delta G(r,\tau_k,x)|^{2p}\,dr, \end{aligned}$$

where C_n is some constant depending only on n. Similarly, one shows that

$$E[|D\Phi_k(x)|^{2p}] \le C_n(\tau-\tau_k)^n \int_{\tau_k}^{\tau} E|D(\delta G)(r,\tau_k,x)|^{2p}\,dr.$$

Therefore (recall that $p = \frac{n+1}{2}$),

$$\begin{aligned} &P\left\{\frac{1}{\tau-\tau_k}|\Phi_k(\xi_k)| > \varepsilon,\ |\xi_k| \le m\right\} \\ &\quad\le \frac{C_{n,m}}{\varepsilon^2(\tau-\tau_k)^2}\left\{(\tau-\tau_k)^n\int_{\tau_k}^{\tau}\int_{|x|\le m} E\|\delta G(r,\tau_k,x)\|^{2p}\,dx\,dr\right\}^{1/p} \\ &\quad\le \frac{C_{n,m}}{\varepsilon^2(\tau-\tau_k)^2}(\tau-\tau_k)^{(n+1)/p}\left\{\sup_{\tau_k\le t\le\tau}\int_{|x|\le m} E\|\delta G(t,\tau_k,x)\|^{2p}\,dx\right\}^{1/p} \\ &\quad= \frac{C_{n,m}}{\varepsilon^2}\sup_{\tau_k\le t\le\tau}\left\{E\int_{|x|\le m}\|\delta G(t,\tau_k,x)\|^{2p}\,dx\right\}^{1/p}. \end{aligned} \tag{4.5}$$

Since $G = (g\partial_u g)(t,x,u(t,x))$, one has

$$DG(t,x) = D(g\partial_u g)\big(t,x,u(t,x)\big) + \partial_u(g\partial_u g)\big(t,x,u(t,x)\big)Du(t,x).$$

Therefore, thanks to (3.2) and (4.1), a simple application of dominated convergence theorem then shows that the mapping $(t,s) \mapsto \{E\{\int_{|x|\le m}\|\delta G(t,s,x)\|^{2p}\,dx\}\}^{1/p}$, $(t,s) \in [0,T]^2$, is continuous. Namely,

$$\sup_{\tau_k\le t\le\tau} E\left\{\int_{|x|\le m}\|\delta G(t,\tau_k,x)\|^{2p}\,dx\right\}^{1/p} \to 0 \qquad \text{as } k\to\infty.$$

Consequently, we derive from (4.5) that for each fixed $m > 0$,

$$P\left\{\frac{1}{\tau - \tau_k}|\Phi_k(\xi_k)| > \varepsilon,\ |\xi_k| \le m\right\} \to 0 \qquad \text{as } k \to \infty. \tag{4.6}$$

Finally, noting that $\lim_{k\to\infty} \xi_k = \xi$ in probability, and that $\xi \in L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, we have $\sup_k P\{|\xi_k| > m\} \to 0$, as $m \to \infty$. Thus, for $\varepsilon, \delta > 0$, we can first choose m large enough so that $\sup_k P\{|\xi_k| > m\} < \frac{\delta}{2}$, and then for fixed m we can find $N_{\varepsilon,\delta} > 0$ such that for all $k > N_{\varepsilon,\delta}$ it holds that

$$\begin{aligned} P\left\{\frac{1}{\tau - \tau_k}|\Phi_k(\xi_k)| > \varepsilon\right\} &\le P\left\{\frac{1}{\tau - \tau_k}|\Phi_k(\xi_k)| > \varepsilon, |\xi_k| \le m\right\} + P\{|\xi_k| > m\} \\ &\le \frac{\delta}{2} + P\left\{\frac{1}{\tau - \tau_k}|\Phi_k(\xi_k)| > \varepsilon, |\xi_k| \le m\right\} < \delta, \end{aligned}$$

thanks to (4.6). Thus, $\lim_{k\to\infty} \Phi_k(\xi_k)/|\tau_k - \tau| = 0$ in probability. The case when $\tau_k \downarrow \tau$ is similar. But since all ξ_k's are $\mathcal{F}_\tau$-measurable, so there is no technical difficulty. Thus we proved (ii).

We now turn to the general case where $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and $\{(\tau_k, \xi_k)\}$ is any (τ, ξ)-approximating sequence. The proof essentially follows the same line as the previous case, but since now both τ_k and τ are stopping times, the estimates (4.2) and (4.5) are no longer valid. In fact, much more careful consideration is needed here.

Again, let us consider only the case when $\tau_k < \tau$, and $\tau_k \uparrow \tau$. Recall that now $g \in C_b^{1,2,3}([0, T] \times \mathbb{R}^n \times \mathbb{R})$, one has $g\partial_u g \in C_b^{1,2,2}([0, T] \times \mathbb{R}^n \times \mathbb{R})$. Replacing g by $g\partial_u g$ and τ by s in (3.12) we get (modulo a negative sign)

$$\begin{aligned} \int_{\tau_k}^s G(r, x)\, dB_r &= G(\tau_k, x)(B_s - B_{\tau_k}) \\ &\quad + \int_{\tau_k}^s \int_{\tau_k}^r G_1(\theta, x)\, d\theta\, dB_r + \int_{\tau_k}^s \int_{\tau_k}^r G_2(\theta, x)\, dB_\theta\, dB_r, \end{aligned}$$

where G_1 and G_2 are the same as U_1 and U_2 in (3.12), with g being replaced by $g\partial_u g$. Therefore,

$$\begin{aligned} \int_{\tau_k}^\tau \int_{\tau_k}^s G(r, x)\, dB_r\, dB_s\big|_{x=\xi_k} &= \tfrac{1}{2} G(\tau_k, \xi_k)(B_\tau - B_{\tau_k})^{\diamond 2} \\ &\quad + \int_{\tau_k}^\tau \int_{\tau_k}^s \int_{\tau_k}^r G_1(\theta, x)\, d\theta\, dB_r\, dB_s\big|_{x=\xi_k} \\ &\quad + \int_{\tau_k}^\tau \int_{\tau_k}^s \int_{\tau_k}^r G_2(\theta, x)\, dB_\theta\, dB_r\, dB_s\big|_{x=\xi_k}. \end{aligned}$$

Clearly, to prove (3.8) it suffices to show that

$$\int_{\tau_k}^\tau \int_{\tau_k}^s \int_{\tau_k}^r G_1(\theta, x)\, d\theta\, dB_r\, dB_s\big|_{x=\xi_k} = o(|\tau_k - \tau|) \tag{4.6}$$

and

$$\int_{\tau_k}^{\tau}\int_{\tau_k}^{s}\int_{\tau_k}^{r} G_2(\theta,x)\,dB_\theta\,dB_r\,dB_s\big|_{x=\xi_k} = o(|\tau_k-\tau|). \tag{4.7}$$

We shall prove only (4.7), as the proof of (4.6) is much simpler. To simplify notation, let us denote $U = G_2$, and $f^k(s,x) = \int_{\tau_k}^{s}\int_{\tau_k}^{r} U(\theta,x)\,dB_\theta\,dB_r$, for $\tau_k \le s \le T$. Then (4.7) is equivalent to that

$$\frac{1}{|\tau_k-\tau|}\int_{\tau_k}^{\tau} f^k(s,x)\,dB_s\big|_{x=\xi_k} \to 0 \qquad \text{as } k\to\infty \text{ in probability.} \tag{4.8}$$

(By now it should be clear that the usual estimate for the stochastic integral do not work because both τ_k and τ are random, and ξ_k is only $\mathcal{F}_\tau$-measurable.)

In order to analyze the left-hand side in (4.8), we define, for each k, an $\{\mathcal{F}_{\tau_k+t}\}$-martingale $F^k(t,x) \triangleq \int_{\tau_k}^{\tau_k+t} f^k(s,x)\,dB_s$, where $\{B_{\tau_k+t}-B_{\tau_k}\}_{t\ge 0}$ is a "shifted" Brownian motion. Further, for a random field $\{\varphi(t,x,\omega)\}$, and any $k,m\in\mathbb{N}$, and $p\ge 1$, we define the following quantities:

$$\begin{cases} \|\varphi\|_{m,p} \triangleq \left\{\int_{|x|\le m} E\left\{\sup_{0\le t\le T}[|\varphi(t,x)|+|D\varphi(t,x)|]^p\right\}dx\right\}^{1/p}, \\ I_s^{k,m}(\varphi) \triangleq \left\{\int_{|x|\le m} E\left\{\int_{\tau_k}^{\tau_k+s}[|\varphi(r,x)|^2 \right.\right. \\ \qquad\qquad \left.\left. +|D\varphi(r,x)|^2]\,dr\right\}^{(n+1)/2}dx\right\}^{1/(n+1)}, \qquad s>0, \end{cases} \tag{4.9}$$

where, again, n is the dimension of x. Note that $dF^k(t,x) = f^k(\tau_k+t,x)\,dB_{\tau_k+t}$, the following estimate holds:

$$\begin{aligned} &E\left\{\sup_{0\le t\le\delta,|x|\le m}|F^k(t,x)|\right\} \\ &\quad\le C_{n,m}E\left\{\sup_{0\le t\le\delta}\left\{\int_{|x|\le m}[|F^k(t,x)|^{n+1}+|DF^k(t,x)|^{n+1}]\,dx\right\}^{1/(n+1)}\right\} \\ &\quad\le C_{n,m}\left\{\int_{|x|\le m}E\left\{\sup_{0\le t\le\delta}[|F^k(t,x)|^{n+1}+|DF^k(t,x)|^{n+1}]\right\}dx\right\}^{1/(n+1)} \\ &\quad\le C_{n,m}I_\delta^{k,m}(f^k). \end{aligned} \tag{4.10}$$

In the above, the first inequality is due to the Sobolev imbedding Theorem, the second inequality is due to Hölder, and the last inequality is the consequence of Burkholder–Davis–Gundy's inequality. In particular, one has

$$E\left\{\sup_{|x|\le m}|F^k(t,x)|\right\} \le C_{n,m}I_t^{k,m}(f^k) \qquad \forall t\in[0,T]. \tag{4.11}$$

We now establish an estimate for $I^{k,m}(f^k)$. To do this, we apply the Burkholder and Hölder inequalities repeatedly to get

$$
\begin{aligned}
&E\{|f^k(s,x)|^{n+1}|\mathcal{F}_{\tau_k}\} \\
&\quad\le C_n E\biggl\{\biggl[\int_{\tau_k}^s\biggl|\int_{\tau_k}^r U(v,x)\,dB_v\biggr|^2 dr\biggr]^{(n+1)/2}\biggl|\mathcal{F}_{\tau_k}\biggr\} \\
&\quad\le C_n(s-\tau_k)^{(n+1)/2-1}\int_{\tau_k}^s E\biggl\{\biggl|\int_{\tau_k}^r U(v,x)\,dB_v\biggr|^{n+1}\biggl|\mathcal{F}_{\tau_k}\biggr\}dr \\
&\quad\le C_n(s-\tau_k)^{(n-1)/2}\int_{\tau_k}^s E\biggl\{\biggl[\int_{\tau_k}^r |U(v,x)|^2\,dv\biggr]^{(n+1)/2}\biggl|\mathcal{F}_{\tau_k}\biggr\}dr \qquad (4.12)\\
&\quad\le C_n(s-\tau_k)^{(n-1)/2}\int_{\tau_k}^s (r-\tau_k)^{n/2} \\
&\qquad\qquad\times E\biggl\{\biggl[\int_{\tau_k}^{\tau_k+\delta}|U(v,x)|^{2(n+1)}\,dv\biggr]^{1/2}\biggl|\mathcal{F}_{\tau_k}\biggr\}dr \\
&\quad= C_n(s-\tau_k)^{n+1/2}E\biggl\{\biggl[\int_{\tau_k}^{\tau_k+\delta}|U(v,x)|^{2(n+1)}\,dv\biggr]^{1/2}\biggl|\mathcal{F}_{\tau_k}\biggr\}.
\end{aligned}
$$

Replacing f^k by Df^k and U by DU in (4.12), we can have a similar estimate for the quantity $E\{|Df^k(s,x)|^{n+1}|\mathcal{F}_{\tau_k}\}$. Some simple computation then shows that

$$
\begin{aligned}
&E\{|f^k(s,x)|^{n+1}+|Df^k(s,x)|^{n+1}|\mathcal{F}_{\tau_k}\} \\
&\quad\le C_{n,m}(s-\tau_k)^{n+1/2} \qquad (4.13)\\
&\qquad\times E\biggl\{\biggl\{\int_{\tau_k}^{\tau_k+s}[|U(v,x)|^{2(n+1)}+|DU(v,x)|^{2(n+1)}]\,dv\biggr\}^{1/2}\biggl|\mathcal{F}_{\tau_k}\biggr\}.
\end{aligned}
$$

Consequently we derive the following estimate for $I_\delta^{k,m}(f^k)$: let $p\triangleq n+1$, $0\le s\le\delta$,

$$
\begin{aligned}
&I_s^{k,m}(f^k) \\
&\quad=\biggl\{\int_{|x|\le m}E\biggl\{\int_{\tau_k}^{\tau_k+s}\bigl(|f^k(r,x)|^2+|Df^k(r,x)|^2\bigr)\,dr\biggr\}^{p/2}dx\biggr\}^{1/p} \\
&\quad\le C_{n,m}\biggl\{\int_{|x|\le m}E\biggl\{s^{p/2-1}\int_{\tau_k}^{\tau_k+s}\bigl(|f^k(r,x)|^p+|Df^k(r,x)|^p\bigr)\,dr\biggr\}dx\biggr\}^{1/p} \\
&\quad= C_{n,m}\biggl\{\int_{|x|\le m}s^{p/2-1}E\biggl\{\int_{\tau_k}^{\tau_k+s}E\{|f^k(r,x)|^p \\
&\qquad\qquad+|Df^k(r,x)|^p|\mathcal{F}_{\tau_k}\}\,dr\biggr\}dx\biggr\}^{1/p} \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{n,m}\bigg\{\int_{|x|\le m} s^{(p/2-1)+(p+1/2)} E\bigg\{\int_{\tau_k}^{\tau_k+s}[|U(v,x)| \\
&\qquad\qquad\qquad\qquad + |DU(v,x)|]^{2p}\,dv\bigg\}^{1/2} dx\bigg\}^{1/p} \\
&\le C_{n,m}\bigg\{E\bigg\{\int_{|x|\le m}\int_{\tau_k}^{\tau_k+s}[|U(v,x)| \\
&\qquad\qquad + |DU(v,x)|]^{2p}\,dv\,dx\bigg\}^{1/2} s^{(3p/2)-1/2}\bigg\}^{1/p} \\
&\le C_{n,m}\bigg\{\int_{|x|\le m} E\bigg\{\sup_{0\le t\le T}[|U(t,x)|+|DU(t,x)|]^{2p}\bigg\}dx\bigg\}^{1/2p} s^{3/2} \\
&= s^{3/2}\|U\|_{m,2p} = s^{3/2}\|U\|_{m,2(n+1)}.
\end{aligned}
$$

We are now ready to prove (4.7). First note that by an integration by parts [considered on the space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_{\tau_k+t}\})$] we have

$$
\frac{1}{t}F^k(t,x) = \frac{1}{\delta}F^k(\delta,x) + \int_t^\delta \frac{1}{s^2}F^k(s,x)\,ds - \int_{\tau_k+t}^{\tau_k+\delta}\frac{1}{s-\tau_k} f^k(s,x)\,dB_s.
$$

Therefore,

$$
\begin{aligned}
\sup_{0<t\le\delta}\bigg\{\frac{1}{t}|F^k(t,x)|\bigg\} &\le \frac{1}{\delta}|F^k(\delta,x)| + \int_0^\delta \frac{1}{s^2}|F^k(s,x)|\,ds \\
&\quad + 2\sup_{0<t\le\delta}\bigg|\int_{\tau_k}^{\tau_k+t}\frac{1}{s-\tau_k}f^k(s,x)\,dB_s\bigg|.
\end{aligned}
\tag{4.15}
$$

Without loss of generality, we assume that $\xi_k \to \xi$, P-a.s. For any $\varepsilon, \delta, m > 0$, and $k \in \mathbb{N}$, a simple computation using Chebyshev's inequality and (4.15) leads to

$$
\begin{aligned}
&P\bigg\{\frac{1}{\tau-\tau_k}\int_{\tau_k}^{\tau} f^k(s,x)\,dB_s\Big|_{x=\xi_k} > \varepsilon\bigg\} \\
&\quad\le P\{|\tau_k-\tau|>\delta\} + P\{|\xi_k|>m\} \\
&\qquad + P\bigg\{\frac{1}{\tau-\tau_k}\int_{\tau_k}^{\tau} f^k(s,x)\,dB_s\Big|_{x=\xi_k} > \varepsilon,\ \tau-\tau_k\le\delta,\ |\xi_k|\le m\bigg\} \\
&\quad\le P\{|\tau_k-\tau|>\delta\} + P\{|\xi_k|>m\} + \frac{1}{\varepsilon}E\bigg\{\sup_{0<t\le\delta,|x|\le m}\bigg|\frac{1}{t}F^k(t,x)\bigg|\bigg\}
\end{aligned}
\tag{4.16}
$$

$$\begin{aligned}
&\le P\{|\tau_k-\tau|>\delta\}+P\{|\xi_k|>m\}+\frac{1}{\varepsilon\delta}E\Big\{\sup_{|x|\le m}|F^k(\delta,x)|\Big\}\\
&\quad+\frac{1}{\varepsilon}E\Big\{\sup_{|x|\le m}\int_0^\delta\frac{1}{s^2}|F^k(s,x)|\,ds\Big\}\\
&\quad+\frac{2}{\varepsilon}E\Big\{\sup_{\substack{0<t\le\delta,\\|x|\le m}}\Big|\int_{\tau_k}^{\tau_k+t}\frac{1}{s-\tau_k}f^k(s,x)\,dB_s\Big|\Big\}\\
&\le P\{|\tau_k-\tau|>\delta\}+P\{|\xi_k|>m\}+I_1+I_2+I_3,
\end{aligned}$$

where I^i, $i=1,2,3$, are three $\{\cdots\}$'s on the right-hand side of (4.16). Combining (4.10) and (4.14) we have

$$I_1=\frac{1}{\varepsilon\delta}E\Big\{\sup_{|x|\le m}|F^k(\delta,x)|\Big\}\le\frac{C_{n,m}}{\varepsilon\delta}I_\delta^{k,m}(f^k)\le\frac{C_{n,m}}{\varepsilon}\| U\|_{m,2(n+1)}\sqrt{\delta};\tag{4.17}$$

and by using (4.11) and (4.14) we obtain

$$\begin{aligned}
I_2&=\frac{1}{\varepsilon}E\Big\{\sup_{|x|\le m}\int_0^\delta\frac{1}{s^2}|F^k(s,x)|\,ds\Big\}\le\frac{1}{\varepsilon}\int_0^\delta\frac{1}{s^2}E\Big\{\sup_{|x|\le m}|F^k(s,x)|\Big\}\,ds\\
&\le\frac{C_{n,m}}{\varepsilon}\int_0^\delta\frac{1}{s^2}I_s^{k,m}(f^k)\,ds\le\frac{C_{n,m}}{\varepsilon}\| U\|_{m,2(n+1)}\int_0^\delta s^{-1/2}\,ds\\
&=\frac{C_{n,m}}{\varepsilon}\sqrt{\delta}\| U\|_{m,2(n+1)}.
\end{aligned}\tag{4.18}$$

Finally, we replace f^k by $\widetilde{f}^k(t,x)=(t-\tau_k)^{-1}f^k(t,x)$ in the definition of F^k, then by (4.10) and definition (4.9) we have

$$\begin{aligned}
I_3&=\frac{2}{\varepsilon}E\Big\{\sup_{|x|\le m,0<t\le\delta}\Big|\int_{\tau_k}^{\tau_k+s}\widetilde{f}^k(s,x)\,dB_s\Big|\Big\}\le\frac{C_{n,m}}{\varepsilon}I_\delta^{k,m}(\widetilde{f}^k)\\
&=\frac{C_{n,m}}{\varepsilon}\Big\{\int_{|x|\le m}E\Big\{\int_{\tau_k}^{\tau_k+\delta}[|\widetilde{f}^k(s,x)|^2+|D\widetilde{f}^k(s,x)|^2]\,ds\Big\}^{(n+1)/2}dx\Big\}^{1/(n+1)}.
\end{aligned}$$

Now using (4.13) and doing some similar computations as before we obtain

$$\begin{aligned}
I_3&\le\frac{C_{n,m}}{\varepsilon}\delta^{1/2}-\frac{1}{n+1}\Big\{\int_{|x|\le m}E\Big\{\int_{\tau_k}^{\tau_k+\delta}\frac{1}{(s-\tau_k)^{n+1}}(s-\tau_k)^{n+1/2}\,ds\\
&\quad\times E\Big\{\Big(\int_{\tau_k}^{\tau_k+\delta}\big(|U(r,x)|+|DU(r,x)|\big)^{2(n+1)}\,dr\Big)^{1/2}\Big|\mathcal{F}_{\tau_k}\Big\}\Big\}\,dx\Big\}^{1/(n+1)}\\
&\le\frac{C_{n,m}}{\varepsilon}\sqrt{\delta}\| U\|_{m,2(n+1)}.
\end{aligned}\tag{4.19}$$

Combining (4.16)–(4.19) we deduce that for any $\varepsilon, \delta, m > 0$, and $k \in \mathbb{N}$,

$$P\left\{\frac{1}{\tau - \tau_k}\int_{\tau_k}^{\tau} f^k(s,x)\,dB_s\big|_{x=\xi_k} > \varepsilon\right\} \le P\{\tau - \tau_k > \delta\} + P\{|\xi_k| > m\} + \frac{C_{n,m}}{\varepsilon}\sqrt{\delta}\,[\![U]\!]_{m,2(n+1)}.$$

Thus, for each $\varepsilon, \varepsilon' > 0$ we can first choose m large enough so that $\sup_k P\{|\xi_k| > m\} < \varepsilon'/3$, since $\{\xi_k\}$ converges in probability; and for the fixed m and ε we choose δ small enough so that the third term on the right-hand side above is $< \varepsilon'/3$, since $[\![U]\!]_{m,2(n+1)} < \infty$ by assumption; and finally we can find a $K = K(\varepsilon, \varepsilon', \delta, m) > 0$ such that $P\{\tau - \tau_k > \delta\} < \varepsilon'/3$, $\forall k > K$, since $\tau_k \uparrow \tau$ in probability. Consequently we have $(\tau - \tau_k)^{-1}\int_{\tau_k}^{\tau} f^k(s,x)\,dB_s|_{x=\xi_k} \to 0$, as $k \to \infty$, in probability. Namely (4.7) holds, and hence the lemma. □

5. Application to stochasic PDEs and stochastic viscosity solutions. Having worked so hard to prove the pathwise Taylor expansion for solutions to SDEs, it is now quite clear that all the technicalities come from the martingale term in the equation, that is, the term $\int g(s,x,u(s,x))\,dB_s$. Indeed, the method of the previous sections can be used to generalize the result to solutions to a class of *fully nonlinear* second-order Stochastic PDEs (SPDEs). To be more precise, let us consider the following SPDE:

$$\begin{cases} du = F(t,x,u,Du,D^2u)\,dt + g(t,x,u)\,dB_t, \\ u(0,x) = \varphi(x), \end{cases} \tag{5.1}$$

or in an integral form:

$$\begin{aligned} u(t,x) = \varphi(x) &+ \int_0^t F\big(s,x,u(s,x),Du(s,x),D^2u(s,x)\big)\,ds \\ &+ \int_0^t g\big(s,x,u(s,x)\big)\,dB_s, \end{aligned} \tag{5.2}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Throughout this section we assume that

$$F \in C_b^{0,2}([0,T] \times \mathbb{E}), \qquad g \in C^{0,2}([0,T] \times \mathbb{E}_0), \qquad \varphi \in C^2(\mathbb{R}^n), \tag{5.3}$$

where $\mathbb{E} \triangleq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ and $\mathbb{E}_0 \triangleq \mathbb{R}^n \times \mathbb{R}$.

Again, as before our main purpose is to derive the pathwise Taylor expansion, thus we shall postulate upon the existence of a certain type of "smooth solution" or "classical solution" which we now describe.

DEFINITION 5.1. A random field $u = \{u(t,x,\omega) : (t,x,\omega) \in [0,T] \times \mathbb{R}^n \times \Omega\}$ is called a classical solution to SPDE (5.1) if:

(i) $u \in C^{0,2}(\mathbf{F}; [0,T] \times \mathbb{R}^n)$;

(ii) u is a random field of Itô-type:

$$u(t,x)=\varphi(x)+\int_0^t u_1(s,x)\,ds+\int_0^t u_2(s,x)\,dB_s, \qquad (t,x)\in[0,T]\times\mathbb{R}^n,$$

where

$$u_1(t,x)=F\bigl(t,x,u(t,x),Du(t,x),D^2u(t,x)\bigr) \qquad u_2(t,x)=g\bigl(t,x,u(t,x)\bigr),$$

for all $(t,x)\in[0,T]\times\mathbb{R}^n$, P-a.s.

An analogy of Theorem 3.2 is the following pathwise stochastic Taylor expansion for solutions to SPDE (5.1). Since the proof of this result is almost identical to that of Theorem 3.2, we omit it.

THEOREM 5.2. *Assume* (5.3), *and assume further that* $g\in C_b^{1,2,3}([0,T]\times\mathbb{R}^n\times\mathbb{R})$. *Let* u *be a classical solution of SPDE* (5.1), *satisfying the following estimate*: *for any* $p>1$, *there exist* $K_p>1$ *and* $C_p>0$ *such that*

$$E\Bigl\{\sup_{0\le t\le T}\bigl(|u(t,x)|^p+|Du(t,x)|^p\bigr)\Bigr\}\le C_p(1+|x|^{K_p}), \qquad x\in\mathbb{R}^n. \tag{5.4}$$

Then, for any $(\tau,\xi)\in\mathcal{M}_{0,T}\times L^2(\mathcal{F}_\tau;\mathbb{R}^n)$ *and any* (τ,ξ)-*approximating sequence* $\{(\tau_k,\xi_k)\}$, *it holds that*

$$\begin{aligned}u(\tau_k,\xi_k)=u(\tau,\xi)&+a(\tau_k-\tau)+b(B_{\tau_k}-B_\tau)+\frac{c}{2}(B_{\tau_k}-B_\tau)^2\\&+\langle p,\xi_k-\xi\rangle+\langle q,\xi_k-\xi\rangle(B_{\tau_k}-B_\tau)+\tfrac12\langle X(\xi_k-\xi),\xi_k-\xi\rangle\\&+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2),\end{aligned} \tag{5.5}$$

where

$$\begin{cases}a=F\bigl(\tau,\xi,u(\tau,\xi),Du(\tau,\xi),D^2u(\tau,\xi)\bigr)-\frac12(g\partial_u g)\bigl(\tau,\xi,u(\tau,\xi)\bigr),\\ b=g\bigl(\tau,\xi,u(\tau,\xi)\bigr), \qquad c=(g\partial_u g)\bigl(\tau,\xi,u(\tau,\xi)\bigr),\\ p=Du(\tau,\xi), \qquad q=\partial_x g\bigl(\tau,\xi,u(\tau,\xi)\bigr)+\partial_u g\bigl(\tau,\xi,u(\tau,\xi)\bigr)Du(\tau,\xi),\\ X=D^2u(\tau,\xi).\end{cases} \tag{5.6}$$

In particular, *if* τ *and* τ_k*'s are all deterministic*, *then the extra assumption on* g *can be dropped*.

In the rest of this paper we shall turn our attention to a new definition of the *stochastic viscosity solution* for a class of quasilinear stochastic PDEs, using the pathwise stochastic Taylor expansion that we established in this paper. First let us translate some basic notions from the (deterministic) viscosity solution theory to their stochastic version. We begin with the following definition.

DEFINITION 5.3. We say that a random field of Itô-type $u \in C^{0,2}(\mathbf{F}; [0, T] \times \mathbb{R}^n)$ is of class $\mathcal{S}^{1,2}$ if it has an expansion in the form of (5.5) for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and any (τ, ξ)-approximating sequence.

Further, we call the quantities (a, b, c, p, q, X) appearing in (5.5) a *set of quasi-derivatives* of u at (τ, ξ).

Clearly, Theorems 2.2, 3.2 and 5.2 show that, unlike the deterministic case, not every $C^{0,2}$-Itô type random field is of class $\mathcal{S}^{1,2}$; but on the other hand, we see that the regular solution of SDE (3.1) and classical solution of SPDE (5.1), whenever exists, do belong to $\mathcal{S}^{1,2}$, provided that the coefficient g is regular enough. Bearing these facts in mind, we now try to define a certain type of "subdifferentials" of a $\mathcal{S}^{1,2}$-random field, in the spirit of the so-called "parabolic superjets/subjets" in the theory of viscosity solutions.

We should remark here that one of the main features that a sensible "subdifferential" has to possess is that whenever u is "differentiable," the subdifferential should coincide with the true "derivatives," and the true "derivatives" should be unique. However, at this point we have not been able to prove such results for general $\mathcal{S}^{1,2}$-random fields. Instead, we shall restrict ourselves to a slightly smaller subset of $\mathcal{S}^{1,2}$, characterized by the function g in the martingale term. To be more precise, for any $g \in C^{0,1}([0, T] \times \mathbb{R}^{n+1})$ we define

$$
\begin{aligned}
\mathcal{S}^{1,2}(g) \triangleq \{u \in \mathcal{S}^{1,2} : \exists u_1(\cdot,\cdot),\ \text{such that} \\
du(t, x) = u_1(t, x)\, dt + g(t, x, u(t, x))\, dB_t\}.
\end{aligned}
\tag{5.7}
$$

Further, for $u \in \mathcal{S}^{1,2}(g)$, we call a set of quasi-derivatives (a, b, c, p, q, X) of u a set of *g-quasi-derivatives* if it holds that $c = g\partial_u g(\tau, \xi, u(\tau, \xi))$.

It is worth noting that, in light of the results of Lions–Souganidis [7] and [8], as well as our previous work [2] and [3] on the stochastic viscosity solutions, the function g determines the stochastic characteristics of an SPDE, therefore it is not too surprising that it should be treated differently. The following definition generalizes the notion of "super/subjets" to random field $u \in \mathcal{S}^{1,2}(g)$ along the same line.

DEFINITION 5.4. Assume $g \in C^{0,1}([0, T] \times \mathbb{R}^{n+1})$. Let $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, and $u \in C(\mathbf{F}; [0, T] \times \mathbb{R}^n)$. A triplet (a, p, X) is called a *stochastic g-superjet* of u at (τ, ξ), if the following hold:

(i) (a, b, c, p, q, X) is an $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}(n)$-valued $\mathcal{F}_\tau$-measurable random vector;

(ii) denoting

$$
\begin{cases}
b = g(\tau, \xi, u(\tau, \xi)), \qquad c = (g\partial_u g)(\tau, \xi, u(\tau, \xi)), \\
q = \partial_x g(\tau, \xi, u(\tau, \xi)) + \partial_u g(\tau, \xi, u(\tau, \xi))p,
\end{cases}
$$

then for any (τ, ξ)-approximating sequence $\{(\tau_k, \xi_k)\}$, it holds that

$$
\begin{aligned}
u(\tau_k, \xi_k) \le u(\tau, \xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 \\
+ \langle p, \xi_k - \xi \rangle + \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \tfrac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle \\
+ o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2).
\end{aligned} \tag{5.8}
$$

We denote the set of all stochastic g-superjets of u at (τ, ξ) by $\mathcal{J}_g^{1,2,+}u(\tau, \xi)$.

Similarly, we say that the six-tuple (a, b, c, p, q, X) is a *stochastic g-subjet* of u at (τ, ξ) if (i) holds and the inequality in (5.8) is reversed; and we denote the set of stochastic g-subjets by $\mathcal{J}_g^{1,2,-}u(\tau, \xi)$.

We now establish the relation between the stochastic g-superjets/subjets and the g-quasi-derivatives of a given $u \in \mathcal{S}^{1,2}(g)$. The following lemma is essential.

LEMMA 5.5. *Let $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ be given. Suppose that there exist $a, b \in L^2(\mathcal{F}_\tau; \mathbb{R})$, $p, q \in L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, and $X \in L^2(\mathcal{F}_\tau; \mathcal{S}(n))$, such that for any (τ, ξ)-approximating sequence (τ_k, ξ_k) it holds that*

$$
\begin{aligned}
0 \le (\text{resp. } \ge)\, a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) \\
+ \langle p, \xi_k - \xi \rangle + \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) \\
+ \tfrac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2),
\end{aligned} \tag{5.9}
$$

then $b = 0$, P-a.s.; and $a = 0$, $p = 0$, $q = 0$, and $X \ge$ (resp. $\le$) 0, P-a.s. on the set $\{\tau > 0\}$.

PROOF. We prove only the case when inequality "$\le$" holds in (5.9), as the other direction is virtually identical.

First we choose $\tau_k = \tau + \frac{1}{k}$, and $\xi_k \equiv \xi$. Then clearly $\tau_k \in \mathcal{M}_{0,\infty}$, $\forall k$, and $\{(\tau_k, \xi_k)\}$ is a (τ, ξ)-approximating sequence. Since (5.9) holds true for all (τ, ξ)-approximating sequences, we have

$$
0 \le \frac{a}{k} + b(B_{\tau_k} - B_\tau) + \frac{1}{k}\zeta_k, \tag{5.10}
$$

where $\zeta_k = o(1) \to 0$ in probability, as $k \to \infty$.

For each $k > 0$ we define $\eta_k = \sqrt{k}(B_{\tau_k} - B_\tau)$. It is easily seen that $\{\eta_k\}$ is a sequence of $\mathcal{N}(0, 1)$-random variables, independent of $\mathcal{F}_\tau$. Now for each $\varepsilon > 0$ and $k > 0$ we define

$$
I_k^\varepsilon \triangleq P\{-a - kb(B_{\tau_k} - B_\tau) \ge \varepsilon\}.
$$

Then from (5.10) and the fact that $\zeta_k \to 0$ in probability we have

$$
I_k^\varepsilon \le P\{\zeta_k \ge \varepsilon\} \to 0 \qquad \text{as } k \to \infty. \tag{5.11}
$$

Further, note that both a and b are $\mathcal{F}_\tau$-measurable, using elementary conditional probability argument we obtain that

$$I_k^\varepsilon = \int_{\mathbb{R}} P\{-a - \sqrt{k}yb \geq \varepsilon\} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy, \qquad k = 1, 2, \ldots.$$

Now (5.11) implies that for all $\varepsilon > 0$ (possibly along a subsequence may assume itself), one has $P\{-a - \sqrt{k}yb \geq \varepsilon\} \to 0$, as $k \to \infty$, for a.e. $y \in \mathbb{R}$. This leads easily to that $b = 0$ and $a \geq 0$, P-a.s.

Next we show that $a \leq 0$, P-a.s. To this end we note that $\mathbf{F}$ is a Brownian filtration, thus the stopping time τ is predictable. Let $\{\tau_k\}$ be any sequence of $\mathbf{F}$-stopping times such that $\tau_k \uparrow \tau$ on $\{\tau > 0\}$; and let $\xi_k = \xi$, $\forall k$. Then again $\{(\tau_k, \xi_k)\}$ is a (τ, ξ)-approximating sequence. Recall that now $b = 0$, (5.9) should read

$$0 \leq -a(\tau - \tau_k) + \zeta_k(\tau - \tau_k), \tag{5.12}$$

where $\zeta_k = o(1)$ in probability, as $k \to \infty$. Since $\tau_k < \tau$ on $\{\tau > 0\}$, one has $a \leq \zeta_k \to 0$ on $\{\tau > 0\}$. Consequently, we see that $a = 0$, P-a.e. on $\{\tau > 0\}$.

Next, we take $\tau_k = \tau + \frac{1}{k}$ again, but let $\xi_k = \xi + \frac{1}{k}x$, where $x \in \mathbb{R}^n$ is arbitrary. Since on the set $\{\tau > 0\}$ we now have $a = b = 0$, (5.9) yields that

$$0 \leq \frac{1}{k}\langle p, x\rangle + \frac{1}{k}\langle q, x\rangle(B_{\tau_k} - B_\tau) + \frac{1}{2k^2}\langle Xx, x\rangle + o\left(\frac{1}{k^2}\right) \quad \text{on } \{\tau > 0\},$$

or equivalently,

$$0 \leq \langle p, x\rangle + \langle q, x\rangle(B_{\tau_k} - B_\tau) + \frac{1}{2k}\langle Xx, x\rangle + o\left(\frac{1}{k}\right) \quad \text{on } \{\tau > 0\}. \tag{5.13}$$

Letting $k \to \infty$ we derive that $\langle p, x\rangle \geq 0$, on $\{\tau > 0\}$. Since $x \in \mathbb{R}^n$ is arbitrary, we obtain that $p = 0$, a.s. on $\{\tau > 0\}$.

Finally, note that with $\tau_k = \tau + \frac{1}{k}$ and $\xi_k = \xi + \frac{1}{k}x$ again (5.13) is reduced, on the set $\{\tau > 0\}$, to that

$$0 \leq \langle q, x\rangle(B_{\tau_k} - B_\tau) + \frac{1}{2k}\langle Xx, x\rangle + o\left(\frac{1}{k}\right).$$

Replacing $a = \frac{1}{2}\langle Xx, x\rangle$ and $b = \langle q, x\rangle$ in (5.10) and using the same argument as we did in the first part of the proof we see that $\langle q, x\rangle = 0$, and $\langle Xx, x\rangle \geq 0$ must hold P-a.s. on $\{\tau > 0\}$. Since x is arbitrary, we derive that $q = 0$ and $X \geq 0$, P-a.s. on the set $\{\tau > 0\}$. The proof is now complete. $\square$

Our main result of this section is then the direct consequence of Lemma 5.5. Since the proofs are now very straight forward, we leave them to the interested readers.

THEOREM 5.6. *Suppose that* $g \in C^{0,1}([0,T] \times \mathbb{R}^{n+1})$, *and* $u \in \mathcal{S}^{1,2}(g)$. *Let* $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$. *Then*:

(i) *the set of* g-*quasi-derivatives of* u *at* (τ, ξ) *is unique*;

(ii) *if* (a, b, c, p, q, X) *is a set of* g-*quasi-derivatives of* u *at* (τ, ξ), *and* $(\widehat{a}, \widehat{p}, \widehat{X}) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$ [*resp.* $\mathcal{J}_g^{1,2,-}u(\tau, \xi)$], *then it holds*, P-*a.s. on the set* $\{0 < \tau < T\}$, *that* $a = \widehat{a}$, $p = \widehat{p}$, $X \le \widehat{X}$ (*resp.* $X \ge \widehat{X}$) *and*

$$b = g\big(\tau, \xi, u(\tau, \xi)\big), \qquad c = (g\partial_u g)\big(\tau, \xi, u(\tau, \xi)\big),$$
$$q = Dg\big(\tau, \xi, u(\tau, \xi)\big) + \partial_u g\big(\tau, \xi, u(\tau, \xi)\big)p.$$

We now give the new definition of the *stochatic viscosity solution* of the fully nonlinear SPDE (5.1). We note that in the deterministic case such a definition is equivalent to the one we defined in our previous works [2] and [3].

DEFINITION 5.7. A random field $u \in C(\mathbf{F}; [0,T] \times \mathbb{R}^n)$ is called a *stochastic viscosity subsolution* of (5.1) if $u(0,x) \le \varphi(x)$, $\forall x \in \mathbb{R}^n$, and for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and any $(a, p, X) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$, it holds that

$$a \le F\big(\tau, \xi, u(\tau, \xi), p, X\big) - \tfrac{1}{2}(g\partial_u g)\big(\tau, \xi, u(\tau, \xi)\big); \tag{5.14}$$

u is said to be a *stochastic viscosity supersolution* if $u(0,x) \ge \varphi(x)$, $\forall x \in \mathbb{R}^n$, and for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and any $(a, p, X) \in \mathcal{J}_g^{1,2,-}u(\tau, \xi)$, (5.14) holds with the inequality being reversed.

If u is both a stochastic viscosity subsolution and supersolution, we say that u is a stochastic viscosity solution of (5.1).

To conclude this section, we give a theorem which in a sense justifies our definition of stochastic viscosity solutions. Let us recall from [4] that the SPDE (5.1) is called *parabolic* if the function F is "*degenerate elliptic*," that is, for any $X, Y \in \mathcal{S}(n)$, the set of all symmetric $n \times n$ matrices, such that $X \le Y$, then

$$F(t, x, u, p, X) \le F(t, x, u, p, Y) \qquad \forall (t, x, u, p). \tag{5.15}$$

(Note that our F and that in [4] differ by a sign!)

THEOREM 5.8. *Suppose that the assumptions of Theorem* 5.2 *hold*, *and that the function* F *is degenerate elliptic*. *Then*, *a classical solution to SPDE* (5.1) *must be a stochastic viscosity solution of* (5.1).

PROOF. Assume that u is a classical solution of (5.1). Then Theorem 5.2 tells us that $u \in \mathcal{S}^{1,2}(g)$. Furthermore, by Theorems 5.2 and 5.6(i) we know that for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, the unique set of g-quasi-derivatives of u at (τ, ξ), denoted by (a, b, c, p, q, X), is given by (5.6). Now let $(\widehat{a}, \widehat{p}, \widehat{X})$ be any element of $\mathcal{J}_g^{1,2,+}u(\tau,\xi)$. By Theorem 5.6(ii) we have $\widehat{a}=a$, $\widehat{p}=p$, and $\widehat{X}\ge X$. This, together with the monotonicity of F (5.15), yields that

$$\begin{aligned}\widehat{a}=a&=F\big(\tau,\xi,u(\tau,\xi),p,X\big)-\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big)\\&\le F\big(\tau,\xi,u(\tau,\xi),\widehat{p},\widehat{X}\big)-\tfrac{1}{2}(g\partial_u g)\big(\tau,\xi,u(\tau,\xi)\big),\end{aligned}$$

verifying (5.14), namely u is a stochastic viscosity subsolution. That u is a viscosity supersolution can be proved in a same way. □

6. Stochastic viscosity solutions for quasi-linear stochastic PDEs. In this section we apply the results of the previous section to a special class of quasi-linear parabolic SPDEs:

$$\begin{aligned}u(t,x)=\varphi(x)&+\int_0^t\big\{\mathcal{L}u(s,x)+f\big(s,x,u,\sigma^T(x)Du\big)\big\}\,ds\\&+\int_0^t g\big(s,x,u(s,x)\big)\,dB_s,\end{aligned}\tag{6.1}$$

where

$$\mathcal{L}u\triangleq\tfrac{1}{2}\mathrm{tr}\,\{\sigma\sigma^T(x)D^2u\}+\langle\beta(x),Du\rangle.\tag{6.2}$$

For notational convenience, we shall refer to (6.1) as SPDE(f,g) in the sequel. We should point out here that if the coefficients σ, β, f, g and φ are sufficiently smooth, Pardoux and Peng [10] proved that the SPDE (6.1) will have a classical solution in the sense of Definition 5.1, and the solution can be represented by the solution to a BDSDE. Our purpose here is to study the stochastic viscosity solutions under less regularity on the coefficients.

To simplify discussion, in what follows we consider only the case where the function g takes a simpler form: $g(t,x,u)=g(t,x)u$. We note that in the case when when g is nonlinear in u, we can still apply the so-called *Doss transformation* as we did in [2] and [3] to carry out our argument, with more complicated expressions. We prefer not to pursue the full complexity here because of the length of the paper.

We shall make use of the following assumptions:

(A1) σ: $\mathbb{R}^n\mapsto\mathbb{R}^k$ and β: $\mathbb{R}^n\mapsto\mathbb{R}^n$ are uniform Lipschitz continuous, with common Lipschitz constant $K>0$.

(A2) f: $\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\mapsto\mathbb{R}$ is continuous, such that for fixed (x,u,p), the process $(\omega,t)\mapsto f(\omega,t,x,u,\sigma^T(x)p)$ is $\mathbf{F}$-progressively measurable; and for some constant K, $p>0$,

$$\begin{cases}|f(\omega,t,x,0,0)|\le K(1+|x|^p), \qquad (\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^n;\\ |f(\omega,t,x,y,z)-f(\omega,t,x,y',z')|\le K(|y-y'|+|z-z'|),\\ \qquad\qquad (\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^n;\ y,\ y'\in\mathbb{R};\ z,\ z'\in\mathbb{R}^k.\end{cases}$$

(A3) $\varphi \in C_p(\mathbb{R}^n)$; and $g \in C_b^{1,2}([0,T]\times\mathbb{R}^n)$.

Let us consider the following "*Doss transformation*" for the SPDE (6.1) which we introduced in our previous works [2] and [3]. For fixed $(x,y)\in\mathbb{R}^n\times\mathbb{R}$, let $\eta(\omega,t,x,y)$ be the solution to the SDE:

$$\eta(t,x,y) = y + \frac{1}{2}\int_0^t g^2(s,x)\eta(s,x,y)\,ds + \int_0^t g(s,x)\eta(s,x,y)\,dB_s. \tag{6.3}$$

Then, $\eta(t,x,y) = y\exp\{\int_0^t g(s,x)\,dB_s\} \triangleq y\widehat{\eta}(t,x)$, where $\widehat{\eta}(t,x) \triangleq \exp\{\int_0^t g(s,x)\,dB_s\}$; and the y-inverse of η is

$$\begin{aligned}\mathcal{E}(t,x,y) &= y\exp\biggl\{-\int_0^t g(s,x)\,dB_s\biggr\}\\ &= y\widehat{\eta}^{-1}(t,x), \qquad (t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}.\end{aligned}$$

Now let $u(t,x)$ be a stochastic viscosity solution to (6.1), we define

$$v(t,x) = \mathcal{E}(t,x,u(t,x)) = \widehat{\mathcal{E}}(t,x)u(t,x), \tag{6.4}$$

where $\widehat{\mathcal{E}}(t,x) \triangleq \exp\{-\int_0^t g(s,x)\,dB_s\} = \widehat{\eta}^{-1}(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^n$.

REMARK 6.1. We note that the traditional Doss transformation (see, e.g., Bensoussan [1]) is of the form: $v(t,x) = \exp\{-g(t,x)B_t\}u(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^n$, which is slightly different from ours. But note that if $g\in C^{1,2}$, then

$$g(t,x)B_t = \int_0^t g(s,x)\,dB_s + \int_0^t \partial_t g(s,x)B_s\,ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

By slightly changing the computation below, we see that the two transformations are essentially the same.

To simplify notation from now on, we denote $\widehat{\mathcal{E}}(t,x)$ as $\mathcal{E}(t,x)$, and $\widehat{\eta}(t,x)$ and $\eta(t,x)$ (not to confused with $\mathcal{E}(t,x,y)$ and $\eta(t,x,y)$!). Then, $\mathcal{E}(t,x)$ satisfies the following SDE:

$$\mathcal{E}(t,x) = 1 + \frac{1}{2}\int_0^t g^2(s,x)\mathcal{E}(s,x)\,ds - \int_0^t g(s,x)\mathcal{E}(s,x)\,dB_s. \tag{6.5}$$

Thus, using integration by parts we have

$$\begin{aligned}dv(t,x) &= \mathcal{E}(t,x)\{\mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^T(x)Du(t,x))\}\,dt\\ &\quad + \mathcal{E}(t,x)g(t,x)u(t,x)\,dB_t + \tfrac{1}{2}u(t,x)\mathcal{E}(t,x)g^2(t,x)\,dt\\ &\quad - u(t,x)g(t,x)\mathcal{E}(t,x)\,dB_t - g^2(t,x)\mathcal{E}(t,x)u(t,x)\,dt\\ &= \mathcal{E}(t,x)\{\mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^T(x)Du(t,x))\}\,dt\\ &\quad - \tfrac{1}{2}v(t,x)g^2(t,x)\,dt,\end{aligned}$$

or equivalently,

$$\partial_t v(t,x) = \mathcal{E}(t,x)\{\mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^T(x)Du(t,x))\} - \tfrac{1}{2}v(t,x)g^2(t,x). \tag{6.6}$$

Since $u(t,x) = \mathcal{E}(t,x)^{-1}v(t,x) = \eta(t,x)v(t,x)$, denoting $G(t,x) \triangleq \int_0^t g(s,x)\,dB_s$, we have

$$DG(t,x) = \int_0^t Dg(s,x)\,dB_s, \qquad D^2G(t,x) = \int_0^t D^2g(s,x)\,dB_s \tag{6.7}$$

and

$$\begin{aligned} Du(t,x) &= \eta(t,x)Dv(t,x) + v(t,x)\eta(t,x)DG(t,x),\\ D^2u(t,x) &= \eta(t,x)D^2v(t,x) + 2\eta(t,x)Dv(t,x)\otimes DG(t,x)\\ &\quad + v(t,x)\eta(t,x)[DG(t,x)]^{\otimes 2} + v(t,x)\eta(t,x)D^2G(t,x), \end{aligned} \tag{6.8}$$

where $a\otimes b \triangleq ab^T$ and $a^{\otimes 2} \triangleq a\otimes a$, for $a, b \in \mathbb{R}^n$. We obtain easily that

$$\begin{aligned} \mathcal{L}u(t,x) = \eta(t,x)\{&\mathcal{L}v(t,x) + \langle(\sigma^T Dv)(t,x), (\sigma^T DG)(t,x)\rangle\\ &+ \tfrac{1}{2}v(t,x)|(\sigma^T DG)(t,x)|^2 + v(t,x)\mathcal{L}G(t,x)\}. \end{aligned} \tag{6.9}$$

Now define a new (random) function: $\widetilde{f}\colon \Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \mapsto \mathbb{R}$ by

$$\begin{aligned} \widetilde{f}(\omega,t,x,y,z) \triangleq{}& \mathcal{E}(t,x,\omega)\\ &\times f(\omega,t,x,\eta(t,x,\omega)y, \eta(t,x,\omega)[z + (\sigma^T DG)(t,x,\omega)y])\\ &+ \{\mathcal{L}G(t,x,\omega) + \tfrac{1}{2}|(\sigma^T DG)(t,x,\omega)|^2 - \tfrac{1}{2}g^2(t,x)\}y\\ &+ \langle(\sigma^T DG)(t,x,\omega), z\rangle. \end{aligned} \tag{6.10}$$

Then combining (6.7)–(6.10) we see that (6.6) can be written as

$$\begin{cases} \partial_t v(t,x) = \mathcal{L}v(t,x) + \widetilde{f}(t,x,v(t,x),\sigma(t,x)Dv(t,x)),\\ v(0,x) = \varphi(x). \end{cases} \tag{6.11}$$

REMARK 6.2. The random function $\widetilde{f}$ in (6.10) is much more "regular" than that in [2] and [3], as it does not have a quadratic growth in the variable z. This is due to the special form we take for $g(t,x,u)$.

Now let us see what will happen for the stochastic super(sub)jets under the same transformation. Note that in the present case we have $g(t,x,u) = g(t,x)u$, thus the identities in Definition 5.4(ii) should now read

$$\begin{aligned} &b = g(\tau,\xi)u(\tau,\xi), \qquad c = g^2(\tau,\xi)u(\tau,\xi),\\ &q = Dg(\tau,\xi)u(\tau,\xi) + g(\tau,\xi)p. \end{aligned} \tag{6.12}$$

We first prove the following lemma.

LEMMA 6.3. *Assume* (A1)–(A3). *Let* $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R})$, *and* $(a, p, X) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$ [*resp.* $\mathcal{J}_g^{1,2,-}u(\tau, \xi)$]. *Define* $v(t, x) \triangleq \mathcal{E}(t, x)u(t, x)$, *then for any* (τ, ξ)*-approximating sequence* $\{(\tau_k, \xi_k)\}$, *and for* P*-a.e.* $\omega \in \Omega$, *it holds that*

$$v(\tau_k, \xi_k) \le (\text{resp. } \ge)\, v(\tau, \xi) + a_v(\tau_k - \tau) + \langle p_v, \xi_k - \xi \rangle + \tfrac{1}{2} \langle X_v(\xi_k - \xi), \xi_k - \xi \rangle + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2), \tag{6.14}$$

where

$$\begin{cases} a_v = \mathcal{E}(\tau, \xi)a, \\ p_v = \mathcal{E}(\tau, \xi)p - v(\tau, \xi)DG(\tau, \xi), \\ X_v = \mathcal{E}(\tau, \xi)X + v(\tau, \xi)[DG(\tau, \xi)]^{\otimes 2} - v(\tau, \xi)D^2G(\tau, \xi) \\ \qquad - \mathcal{E}(\tau, \xi)\{p \otimes DG(\tau, \xi) + DG(\tau, \xi) \otimes p\}. \end{cases} \tag{6.15}$$

Namely, $(a_v, p_v, X_v) \in \mathcal{J}_0^{1,2,+}v(\tau, \xi)$ [*resp.* $\mathcal{J}_0^{1,2,-}v(\tau, \xi)$].

Conversely, if $(a, p, X) \in \mathcal{J}_0^{1,2,+}v(\tau, \xi)$ [*resp.* $\mathcal{J}_0^{1,2,-}v(\tau, \xi)$], *and define* $u(t, x) = \mathcal{E}(t, x)^{-1}v(t, x) = \eta(t, x)v(t, x)$, *then the triplet* (a_u, p_u, X_u) *given by*

$$\begin{cases} a_u = \eta(\tau, \xi)a, \\ p_u = \eta(\tau, \xi)p + u(\tau, \xi)DG(\tau, \xi), \\ X_u = \eta(\tau, \xi)X + u(\tau, \xi)[DG(\tau, \xi)]^{\otimes 2} + u(\tau, \xi)D^2G(\tau, \xi) \\ \qquad + \eta(\tau, \xi)\{p \otimes DG(\tau, \xi) + DG(\tau, \xi) \otimes p\}, \end{cases} \tag{6.16}$$

will satisfy $(a_u, p_u, X_u) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$ [*resp.* $\mathcal{J}_g^{1,2,-}u(\tau, \xi)$].

PROOF. First note that the random field $\mathcal{E}(t, x) = \exp\{-G(t, x)\} = \exp\{-g(t, x)B_t\}$ is a "regular" solution of the SDE (6.5), in the sense of Definition 3.1. Thus applying Theorem 3.2 we see that for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R})$ and any (τ, ξ) approximating sequence $\{(\tau_k, \xi_k)\}$, it holds that

$$\begin{aligned} \mathcal{E}(\tau_k, \xi_k) = {} & \mathcal{E}(\tau, \xi) + a_0(\tau_k - \tau) + b_0(B_{\tau_k} - B_\tau) \\ & + \frac{c_0}{2}(B_{\tau_k} - B_\tau)^2 + \langle p_0, \xi_k - \xi \rangle \\ & + \langle q_0, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \tfrac{1}{2} \langle X_0(\xi_k - \xi), \xi_k - \xi \rangle \\ & + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2), \end{aligned} \tag{6.17}$$

where

$$\begin{cases} a_0 = \frac{1}{2}g^2(\tau, \xi)\mathcal{E}(\tau, \xi) - \frac{1}{2}g^2(\tau, \xi)\mathcal{E}(\tau, \xi) = 0, \\ b_0 = -g(\tau, \xi)\mathcal{E}(\tau, \xi), \qquad c_0 = g^2(\tau, \xi)\mathcal{E}(\tau, \xi), \\ p_0 = D\mathcal{E}(\tau, \xi) = -\mathcal{E}(\tau, \xi)DG(\tau, \xi), \\ q_0 = -\mathcal{E}(\tau, \xi)Dg(t, \xi) - g(\tau, \xi)p_0, \\ X_0 = D^2\mathcal{E}(\tau, \xi) = \mathcal{E}(\tau, \xi)[DG(\tau, \xi)]^{\otimes 2} - \mathcal{E}(\tau, \xi)D^2G(\tau, \xi). \end{cases} \tag{6.18}$$

Therefore, (6.17) becomes

$$
\begin{aligned}
\mathcal{E}(\tau_k,\xi_k) = \mathcal{E}(\tau,\xi)\big\{&1 - g(\tau,\xi)(B_{\tau_k}-B_\tau) + \tfrac{1}{2}g^2(\tau,\xi)(B_{\tau_k}-B_\tau)^2 \\
&- \langle DG(\tau,\xi), \xi_k - \xi\rangle \\
&+ \langle [gDG - Dg](\tau,\xi), \xi_k - \xi\rangle (B_{\tau_k}-B_\tau) \\
&+ \tfrac{1}{2}|\langle DG(\tau,\xi), \xi_k-\xi\rangle|^2 \\
&- \tfrac{1}{2}\langle D^2G(\tau,\xi)(\xi_k-\xi), \xi_k-\xi\rangle \\
&+ o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2)\big\}.
\end{aligned} \tag{6.19}
$$

Now let $(a,p,X)\in\mathcal{J}_g^{1,2,+}u(\tau,\xi)$. By definition we must have

$$
\begin{aligned}
u(\tau_k,\xi_k) \le u(\tau,\xi) &+ a(\tau_k-\tau) + b(B_{\tau_k}-B_\tau) \\
&+ \frac{c}{2}(B_{\tau_k}-B_\tau)^2 + \langle p, \xi_k-\xi\rangle \\
&+ \langle q, \xi_k-\xi\rangle(B_{\tau_k}-B_\tau) + \tfrac{1}{2}\langle X(\xi_k-\xi), \xi_k-\xi\rangle \\
&+ o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2),
\end{aligned} \tag{6.20}
$$

where $b = g(\tau,\xi)u(\tau,\xi)$; $c = g^2(\tau,\xi)u(\tau,\xi)$, and $q = Dg(\tau,\xi)u(\tau,\xi) + g(\tau,\xi)p$. Since $\mathcal{E}(t,x)\ge 0$, by some computation combining (6.19) and (6.20), one obtains fairly easily that

$$
\begin{aligned}
v(\tau_k,\xi_k) &= \mathcal{E}(\tau_k,\xi_k)u(\tau_k,\xi_k) \\
&\le v(\tau,\xi) + \mathcal{E}(\tau,\xi)a(\tau_k-\tau) \\
&\quad + \{\mathcal{E}(\tau,\xi)b - v(\tau,\xi)g(\tau,\xi)\}(B_{\tau_k}-B_\tau) \\
&\quad + \left\{\mathcal{E}(\tau,\xi)\frac{c}{2} + \frac{1}{2}v(\tau,\xi)g^2(\tau,\xi) - b\mathcal{E}(\tau,\xi)g(\tau,\xi)\right\}(B_{\tau_k}-B_\tau)^2 \\
&\quad + \langle \mathcal{E}(\tau,\xi)p - v(\tau,\xi)DG(\tau,\xi), \xi_k-\xi\rangle \\
&\quad + \big\langle \mathcal{E}(\tau,\xi)q + v(\tau,\xi)[gDG(\tau,\xi) - Dg(\tau,\xi)] \\
&\qquad - b\mathcal{E}(\tau,\xi)DG(\tau,\xi) - \mathcal{E}(\tau,\xi)g(\tau,\xi)p, \xi_k-\xi\big\rangle(B_{\tau_k}-B_\tau) \\
&\quad + \tfrac{1}{2}\big\langle[\mathcal{E}(\tau,\xi)X + v(\tau,\xi)Dg(\tau,\xi)^{\otimes 2} - v(\tau,\xi)D^2G(\tau,\xi) \\
&\qquad\quad - \mathcal{E}(\tau,\xi)p\otimes DG(\tau,\xi)](\xi_k-\xi), \xi_k-\xi\big\rangle \\
&\quad + o(|\tau_k-\tau|) + o(|\xi_k-\xi|^2).
\end{aligned}
$$

Now using the definition of b, c, q, and doing some cancellations we see that the above is nothing but (6.14).

The second half of the lemma can be proved in a similar way, we omit it. $\square$

Lemma 6.3 has the following easy but important consequence:

COROLLARY 6.4. *Assume* (A1)–(A3). *Let* u, v *be two random fields such that* $v(t,x) = \mathcal{E}(t,x)u(t,x)$, $\forall (t,x)$, P*-a.s. Then* u *is a stochastic viscosity solution to SPDE*(f, g) *if and only if* v *is stochastic viscosity solution to SPDE*$(\widetilde{f}, 0)$, *where* $\widetilde{f}$ *is given by* (6.10).

PROOF. Let us first assume that $u(\cdot,\cdot)$ is a stochastic viscosity solution. Define $v(t,x) = \mathcal{E}(t,x)u(t,x)$. In order to show that v is a stochastic viscosity solution to the SPDE$(\widetilde{f}, 0)$ we let $\tau \in \mathcal{M}_{0,T}$ and $\xi \in L^2(\mathbf{F}; \mathbb{R}^n)$ be given and let $(a, p, X) \in \mathcal{J}_0^{1,2,+}v(\tau, \xi)$, in the sense of Definition 5.4. Using the Doss transformation and applying Lemma 6.3 we derive the set $(a_u, p_u, X_u) \in \mathcal{J}_g^{1,2,+}u(\tau, \xi)$, defined by (6.16). Since u is a stochastic viscosity solution, it holds that (recall Definition 5.7)

$$\begin{aligned}a_u &\le \tfrac{1}{2}\mathrm{tr}\{\sigma\sigma^T(\xi)X_u\} + \langle \beta(\xi), p_u \rangle \\ &\quad + f\big(\tau, \xi, u(\tau,\xi), \sigma(\xi)p_u\big) - \tfrac{1}{2}g^2(\tau,\xi)u(\tau,\xi).\end{aligned} \tag{6.21}$$

But then applying the inverse transformation, we see that (a, p, X) will satisfy the inequality (5.14) with $g \equiv 0$, which is exactly what we need. The other direction can be proved in a similar way, we leave it to the readers. □

7. An equivalence theorem. In this section we prove that, for the quasilinear SPDE (6.1), the definition of stochastic viscosity solution given by Definition 5.7 is indeed equivalent to that of [2] and [3], verifying a featured result in the (deterministic) theory of viscosity solutions. We note that the definition proposed in this paper works for a much larger class of nonlinear SPDEs, it is therefore potentially more useful in applications.

In light of Corollary 6.4 and a similar result in [3], we shall consider only the case when $g = 0$. That is, we need only show that any stochastic viscosity sub(super)solution of SPDE$(f, 0)$ in the sense of Definition 5.7 must also be one under the definition of [2] and [3], and vice versa. Moreover, by the uniqueness result in [3] we know that a stochastic viscosity solution there must coincide with the ω-wise viscosity solution to the PDE

$$\begin{cases}\partial_t u(t,x) = \mathcal{L}u(t,x) + f\big(\omega, t, x, u(t,x), \sigma^T(x)Du(t,x)\big),\\ u(0,x) = \varphi(x),\end{cases} \tag{7.1}$$

for P-a.e. ω, in the usual sense. The following result will be sufficient for our purpose. Recall from [2] and [3] that a random field $u \in C(\mathbf{F}; [0,T] \times \mathbb{R}^n)$ is called "*uniformly stochastically bounded*" if there exists a positive, increasing process $\Theta \in L^0(\mathbf{F}, [0,T])$, such that

$$|u(t,x)| \le \Theta_t \qquad \forall (t,x), \ \ P\text{-a.s.}$$

THEOREM 7.1. *Assume* (A1)–(A3), *and let* $v \in C(\mathbf{F}; [0,T] \times \mathbb{R}^n)$ *be uniformly stochastically bounded. Then*, v *is a stochastic viscosity solution to SPDE* $(f, 0)$ *if and only if for* P*-a.e* $\omega \in \Omega$, $v(\cdot,\cdot,\omega)$ *is a viscosity solution to the PDE*$(f(\omega,\dots), 0)$ *in the usual sense*.

Before we prove the theorem let us first recall some facts regarding "*backward doubly stochastic differential equations*" (BDSDEs) and the associated nonlinear Feynman–Kac formula (see Pardoux and Peng [9, 10] and Buckdahn and Ma [2]). Let W be a k-dimensional Brownian motion defined on a canonical Wiener space $(\Omega', \mathcal{F}', P')$, where $\Omega' = C([0,T]; \mathbb{R}^k)$; and let $\mathcal{F}'_{s,t} \triangleq \sigma\{W_r - W_s, r \in [s,t]\}$, for $0 \le s \le t \le T$. Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ be the completed product space of $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ and denote by $\mathbf{G} = \{\mathcal{G}_s\}_{s\in[0,T]}$ the *backward filtration* in which $\mathcal{G}_s \triangleq \mathcal{F}_T \otimes \mathcal{F}'_{s,T}$, augmented by all the $\overline{P}$-null sets. We note that if ζ and η are two random variables defined on $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$, respectively, then we view them as random variables in $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ by the following usual identification:

$$\zeta(\overline{\omega}) = \zeta(\omega), \qquad \eta(\overline{\omega}) = \eta(\omega'), \qquad \overline{\omega} = (\omega, \omega') \in \overline{\Omega}.$$

For fixed $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^n$, let us consider the following system of SDEs on $(\Omega', \mathcal{F}', P'; \{\mathcal{F}'_{s,T}\}_{s\in[0,T]})$:

$$\begin{cases} X_s = x + \int_s^t b(X_r)\,dr + \int_s^t \sigma(X_r) \downarrow dW_r, \\ Y_s = u_0(X_0) + \int_0^s f(\omega, r, X_r, Y_r, Z_r)\,dr - \int_0^s Z_r \downarrow dW_r, \end{cases} \qquad s \in [0,t]. \tag{7.2}$$

Here "$\downarrow dW$" denotes the *backward Itô integral* with respect to W. Clearly, (7.2) is a special case of the BDSDEs studied in [2] and [3], and is a time-reversed version the BSDE/BDSDE studied in [9] and [10]. Consequently we know that (7.2) admits a unique adapted solution, which we denote by $(X^{\omega,t,x}, Y^{\omega,t,x}, Z^{\omega,t,x})$. Also, if we define $v(\omega,t,x) \triangleq Y_t^{\omega,t,x}$, for $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^n$. Then, the random field $v \in C(\mathbf{F}; [0,T] \times \mathbb{R}^n)$; and the nonlinear Feynman–Kac formula of [9] tells us that for P-a.e. $\omega \in \Omega$, $v(\omega,\cdot,\cdot)$ is a viscosity solution to the PDE (7.1). Furthermore, given any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, for P-a.e. $\omega \in \Omega$, it holds that (see [9] and [10] or [2] and [3] for details)

$$Y_s^{\tau(\omega),\xi(\omega)} = v\big(s, X_s^{\tau(\omega),\xi(\omega)}\big) \qquad \text{for } s \in [0, \tau(\omega)],\ P'\text{-a.s.} \tag{7.3}$$

Now let $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ be fixed. For each $\omega \in \Omega$ we consider the solution $(X^{\omega,\tau(\omega),\xi(\omega)}, Y^{\omega,\tau(\omega),\xi(\omega)}, Z^{\omega,\tau(\omega),\xi(\omega)})$ on $[0, \tau(\omega)]$. To simplify notation, we shall denote, for $\Xi = X, Y, Z$, respectively,

$$\Xi_s^{\tau,\xi}(\omega,\omega') = \Xi_s^{\omega,\tau(\omega),\xi(\omega)}(\omega'), \qquad s \in [0, \tau(\omega)].$$

By defining $X_s^{\tau,\xi} = \xi$, $Y_s^{\tau,\xi} = Y_\tau^{\tau,\xi}$, and $Z_s^{\tau,\xi} = 0$ for $s \in [\tau(\omega), T]$, $\overline{P}$-a.s., one can easily check that, for each $s \in [0, T]$, the mapping

$$\overline{\omega} = (\omega, \omega') \mapsto \big(X_s^{\tau,\xi}(\omega, \omega'), Y_s^{\tau,\xi}(\omega, \omega'), Z_s^{\tau,\xi}(\omega, \omega')\big)$$

is $\mathcal{G}_s$-measurable; and that $(X^{\tau,\xi}, Y^{\tau,\xi}, Z^{\tau,\xi}) \in L^2(\mathbf{G}; [0, T] \times \mathbb{R}^{n+1+k})$. Furthermore, using the relation (7.3) one can check that, for any $\mathbf{G}$-stopping time η such that $0 \le \eta \le \tau$, $\overline{P}$-a.s., it holds that

$$\begin{aligned} Y_s^{\tau,\xi} &= v(\eta, X_\eta^{\tau,\xi}) + \int_\eta^s f(r, X_r^{\tau,\xi}, Y_r^{\tau,\xi}, Z_r^{\tau,\xi})\,dr \\ &\quad - \int_\eta^s Z_r^{\tau,\xi} \downarrow dW_r, \qquad s \in [\eta, \tau]. \end{aligned} \tag{7.4}$$

Next, for $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R})$, let $(a, p, A) \in \mathcal{J}_0^{1,2,+} v(\tau, \xi)$ be given. Define a random field on $(\Omega, \mathcal{F}, P)$:

$$\begin{aligned} \varphi(s, x) &\triangleq v(\tau, \xi) + a(s - \tau) + \langle p, x - \xi \rangle \\ &\quad + \tfrac{1}{2} \langle A(x - \xi), x - \xi \rangle, \qquad (s, x) \in [0, T] \times \mathbb{R}^n; \end{aligned} \tag{7.5}$$

and consider the following random variable on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$:

$$\Theta(s, \overline{\omega}) \triangleq \begin{cases} \dfrac{(v - \varphi)^+(s, X_s^{\tau,\xi}(\overline{\omega}))}{|\tau(\omega) - s| + |\xi(\omega) - X_s^{\tau,\xi}(\overline{\omega})|^2}, & s < \tau(\omega), \\ 0, & s \ge \tau(\omega), \end{cases} \tag{7.6}$$

for $(s, \overline{\omega}) \in [0, T] \times \overline{\Omega}$. Since $\mathbf{F}$ is a Brownian filtration, every stopping time is predictable. To wit, there exists a sequence of stopping times $\{\tau_\ell\} \subseteq \mathcal{M}_{0,T}$ such that $\tau_\ell \uparrow \tau$, P-a.e. on $\{\tau > 0\}$, as $\ell \to \infty$; and $\tau_\ell = 0$ on $\{\tau = 0\}$. Let $\xi_\ell^{\omega'}(\omega) \triangleq X_{\tau_\ell(\omega)}^{\tau,\xi}(\omega, \omega')$, $\ell = 1, 2, \ldots$, where $X^{\tau,\xi}$ is defined by (7.2). Clearly, for P'-a.e. $\omega' \in \Omega'$, $\{(\tau_\ell, \xi_\ell^{\omega'})\}_{\ell \ge 1}$ is a (τ, ξ)-approximating sequence on the space $(\Omega, \mathcal{F}, P)$. Thus for P'-a.e. $\omega' \in \Omega'$, by definition of a stochastic superjet we have,

$$v(\tau_\ell, \xi_\ell^{\omega'}) \le \varphi(\tau_\ell, \xi_\ell^{\omega'}) + R_\ell^{\omega'}, \tag{7.7}$$

where $\frac{R_\ell^{\omega'}}{|\tau - \tau_\ell| + |\xi - \xi_\ell^{\omega'}|^2} \to 0$ in probability (P), as $\ell \to \infty$. Consequently, note from (7.7) that $[v(\tau_\ell, \xi_\ell^{\omega'}) - \varphi(\tau_\ell, \xi_\ell^{\omega'})]^+ \le |R_\ell^{\omega'}|$, if we define $\Theta_\ell(\overline{\omega}) \triangleq \Theta(\tau_\ell(\omega), \overline{\omega})$, then for each $\omega' \in \Omega'$,

$$P\{|\Theta_\ell(\cdot, \omega')| > \varepsilon\} \le P\left\{\left|\frac{R_\ell^{\omega'}}{|\tau - \tau_\ell| + |\xi - \xi_\ell^{\omega'}|^2}\right| > \varepsilon\right\} \to 0 \qquad \text{as } \ell \to \infty.$$

Therefore,

$$\overline{P}\{|\Theta_\ell| > \varepsilon\} = \int_{\Omega'} P\{|\Theta_\ell(\cdot, \omega')| > \varepsilon\} P'(d\omega') \to 0 \qquad \text{as } \ell \to \infty, \tag{7.8}$$

thanks to the bounded convergence theorem. That is, $\Theta_\ell \to 0$ in probability ($\overline{P}$), and thus, by extracting a subsequence if necessary, we shall assume in the sequel that $\Theta_\ell \to 0$, $\overline{P}$-a.s., as $\ell \to \infty$.

We remark here that we may assume without loss of generality that Θ_ℓ is bounded. For otherwise we can define a **G**-stopping time

$$\eta(\overline{\omega}) \triangleq \sup\{s \le \tau(\omega) : |\Theta(s, \overline{\omega})| > 1\},$$

and consider the sequence of **G**-stopping times $\widehat{\tau}_\ell \triangleq \eta \vee \tau_\ell$, $\ell = 1, 2, \dots$. Since $\Theta(\tau(\omega)-, \overline{\omega}) = 0$, we must have $\eta < \tau$, thus for each ω, $\widehat{\tau}_\ell = \tau_\ell$ for ℓ large enough, and hence $\widehat{\Theta}_\ell(\overline{\omega}) = \Theta(\widehat{\tau}_\ell(\overline{\omega}), \overline{\omega}) \to 0$, $\overline{P}$-a.s., as $\ell \to \infty$. Since replacing τ_ℓ by $\widehat{\tau}_\ell$ (and Θ_ℓ by $\widehat{\Theta}_\ell$) in our future discussion will not cause any substantial difficulty, we shall assume from now on $|\Theta_\ell| \le 1$, $\overline{P}$-a.s., for all $\ell \ge 1$.

We are now ready to prove Theorem 7.1.

PROOF OF THEOREM 7.1. We first assume that v is a stochastic viscosity subsolution of SPDE$(f, 0)$ in the sense of Definition 5.7. We show that it is also a stochastic viscosity solution of SPDE$(f, 0)$ in the sense of [2] and [3]. Indeed, let $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$, and let $\varphi \in C^{1,2}(\mathcal{F}_\tau; [0, T] \times \mathbb{R}^n)$ be a random field such that

$$v(t, x) - \varphi(t, x) \le (\text{resp. } \ge)\, v(\tau, \xi) - \varphi(\tau, \xi)$$

for all (t, x) in some neighborhood of (τ, ξ), P-a.e. on $\{0 < \tau < T\}$. Then clearly, one has

$$\begin{aligned} v(t, x) - v(\tau, \xi) \le (\text{resp. } \ge)\, & \varphi(t, x) - \varphi(\tau, \xi) \\ & + \partial_t \varphi(\tau, \xi)(t - \tau) + \langle D\varphi(\tau, \xi), x - \xi \rangle \\ & + \tfrac{1}{2} \langle D^2\varphi(\tau, \xi)(x - \xi), x - \xi \rangle \\ & + R(t - \tau, x - \xi), \qquad (t, x) \in [0, T] \times \mathbb{R}^n, \end{aligned}$$

where, P-a.e. on $\{0 < \tau < T\}$, it holds that

$$\frac{R(t - \tau, x - \xi)}{|t - \tau| + |x - \xi|^2} \to 0 \qquad \text{as } (t, x) \to (\tau, \xi).$$

Therefore, one has $(\partial_t \varphi(\tau, \xi), D\varphi(\tau, \xi), D^2\varphi(\tau, \xi)) \in \mathcal{J}_0^{1,2,+} v(\tau, \xi)$ [resp. $\mathcal{J}_0^{1,2,-} v(\tau, \xi)$], and consequently, since v is a stochastic viscosity subsolution of SPDE$(f, 0)$, it follows from Definition 5.7 that

$$\partial_t \varphi(\tau, \xi) \le (\text{resp. } \ge)\, \mathcal{L}\varphi(\tau, \xi) + \widetilde{f}\big(\tau, \xi, v(\tau, \xi), \sigma(\tau) D\varphi(\tau, \xi)\big),$$

P-a.s. on $\{0 < \tau < T\}$. Comparing to the definitions in [2] and [3] we then see that v is indeed a stochastic viscosity sub(resp. super)solution in the that sense. Finally, using the uniform stochastical boundedness assumption and applying the uniqueness result in [3] we conclude that $v(\cdot, \cdot, \omega)$ must coincide with the unique pathwise viscosity solution of PDE (7.1) in the usual sense, proving the assertion.

The proof of the reverse direction is more involved. We shall only argue for the stochastic viscosity subsolution case, as the super solution part is similar. Therefore, in what follows we assume that $v \in C(\mathbf{F}; [0, T] \times \mathbb{R}^n)$ is an ω-wise viscosity subsolution of PDE (7.1) in the usual sense, and try to prove that it is a stochastic viscosity subsolution of SPDE$(f, 0)$ in the sense of Definition 5.7. That is, for any $(\tau, \xi) \in \mathcal{M}_{0,T} \times L^2(\mathcal{F}_\tau; \mathbb{R}^n)$ and $(a, p, A) \in \mathcal{J}_0^{1,2,+} v(\tau, \xi)$, it holds that

$$a \le \frac{1}{2}\mathrm{tr}\{\sigma\sigma^T(\xi)A\} + \langle \beta(\xi), p\rangle + f\big(\tau, \xi, v(\tau, \xi), \sigma^T(\xi)p\big), \tag{7.9}$$

P-a.s. on $\{0 < \tau < T\}$.

We now introduce an intermediate BSDE that will play an important role in our proof. For each $\omega \in \Omega$, let (Y^ℓ, Z^ℓ) be the $\mathbf{G}$-adapted solution to the BSDE

$$\begin{aligned} Y_s^\ell &= \varphi(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) + \int_{\tau_\ell}^s f(r, X_r^{\tau,\xi}, Y_r^\ell, Z_r^\ell)\, dr \\ &\quad - \int_{\tau_\ell}^s \langle Z_r^\ell, \downarrow dW_r\rangle, \qquad s \in [\tau_\ell, \tau]. \end{aligned} \tag{7.10}$$

On the other hand, applying Itô's formula we get (suppressing variables):

$$\begin{aligned} \varphi(s, X_s^{\tau,\xi}) &= \varphi(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) + \int_{\tau_\ell}^s \{\partial_t\varphi - \mathcal{L}\varphi\}(r, X_r^{\tau,\xi})\, dr \\ &\quad - \int_{\tau_\ell}^s \langle(\sigma^T D\varphi)(r, X_r^{\tau,\xi}), \downarrow dW_r\rangle, \end{aligned} \tag{7.11}$$

for $s \in [0, \tau(\omega)]$, P'-a.s. Define

$$\begin{cases} \widehat{Y}_s^\ell = Y_s^\ell - \varphi(s, X_s^{\tau,\xi}); \quad \widehat{Z}_s^\ell = Z_s^\ell - (\sigma^T D\varphi)(s, X_s^{\tau,\xi}), & s \in [\tau_\ell, \tau], \\ \widehat{Y}_s^\ell = 0, \quad \widehat{Z}_s^\ell = 0, & s \in [0, \tau_\ell]. \end{cases} \tag{7.12}$$

Then clearly $(\widehat{Y}^\ell, \widehat{Z}^\ell)$ is the (unique) $\mathbf{G}$-adapted solution to the following BSDE:

$$\begin{aligned} \widehat{Y}_s^\ell = \int_{\tau_\ell}^s &\{f\big(r, X_r^{\tau,\xi}, \widehat{Y}_r^\ell + \varphi(r, X_r^{\tau,\xi}), \widehat{Z}_r^\ell + (\sigma^T D\varphi)(r, X_r^{\tau,\xi})\big) \\ &- (\partial_t\varphi - \mathcal{L}\varphi)(r, X_r^{\tau,\xi})\}\, dr - \int_{\tau_\ell}^s \langle \widehat{Z}_r^\ell, \downarrow dW_r\rangle, \qquad s \in [\tau_\ell, \tau]. \end{aligned} \tag{7.13}$$

Setting $s = \tau$, taking conditional expectation $\overline{E}\{\cdot|\mathcal{G}_\tau\}$ on both sides above, and noting that $\sigma\{W_r - W_\tau, r \in [0, \tau]\}$ is independent of $\mathcal{F}'_{\tau,T}$ given $\mathcal{F}_T$, we obtain that, $\overline{P}$-almost surely,

$$
\begin{aligned}
\widehat{Y}^\ell_\tau(\overline{\omega}) &= \overline{E}\bigg\{\int_{\tau_\ell}^{\tau}\{f\big(r, X^{\tau,\xi}_r, \widehat{Y}^\ell_r + \varphi(r, X^{\tau,\xi}_r),\\
&\qquad \widehat{Z}^\ell_r + (\sigma^T D\varphi)(r, X^{\tau,\xi}_r)\big) - (\partial_t\varphi - \mathcal{L}\varphi)(r, X^{\tau,\xi}_r)\}\,dr\big|\mathcal{G}_\tau\bigg\}(\overline{\omega})\\
&= E'\bigg\{\int_{\tau_\ell(\omega)}^{\tau(\omega)}\{f\big(r, X^{\tau,\xi}_r, \widehat{Y}^\ell_r + \varphi(r, X^{\tau,\xi}_r),\\
&\qquad \widehat{Z}^\ell_r + (\sigma^T D\varphi)(r, X^{\tau,\xi}_r)\big) - (\partial_t\varphi - \mathcal{L}\varphi)(r, X^{\tau,\xi}_r)\}(\omega,\cdot)\,dr\bigg\},\\
&\qquad\qquad s \in [\tau_\ell(\omega), \tau(\omega)].
\end{aligned}
\tag{7.14}
$$

Now let us recall the definition (7.5) and note that

$$
\begin{aligned}
\partial_t\varphi(t,x) &= a, \qquad D\varphi(t,x) = p + A(x-\xi),\\
D^2\varphi(t,x) &= A, \qquad (t,x) \in [0,T] \times \mathbb{R}^n.
\end{aligned}
$$

It is easily seen that for some positive random variable $\zeta \in L^0(\mathcal{F}_\tau; \mathbb{R}_+)$ one has

$$
\begin{aligned}
\varphi(r, X^{\tau,\xi}_r)| + |(\partial_t - \mathcal{L})\varphi(r, X^{\tau,\xi}_r)| &\le \zeta(1 + |X^{\tau,\xi}_r|^2),\\
|D\varphi(r, X^{\tau,\xi}_r)| &\le \zeta(1 + |X^{\tau,\xi}_r|), \qquad r \in [0,\tau].
\end{aligned}
$$

Using some standard estimates for BSDEs we obtain that, for $p \ge 2$ in (A2), there exist some generic constant $C > 0$ and positive random variable, still denoted by $\zeta \in L^0(\mathcal{F}_\tau; \mathbb{R}_+)$, which are allowed to vary from line to line, such that

$$
\begin{aligned}
&|\widehat{Y}^\ell_s|^2 + \tfrac{1}{2}\overline{E}\bigg\{\int_{\tau_\ell}^{s}|\widehat{Z}^\ell_r|^2\,dr\big|\mathcal{G}_s\bigg\}\\
&\quad \le \zeta\overline{E}\bigg\{\int_{\tau_\ell}^{s}(1 + |X^{\tau,\xi}_r|^p)|\widehat{Y}^\ell_r|\,dr\big|\mathcal{G}_s\bigg\} + C\int_{\tau_\ell}^{s}\overline{E}\{|\widehat{Y}^\ell_r|^2|\mathcal{G}_s\}\,dr\\
&\quad \le \zeta(1 + |X^{\tau,\xi}_s|^{2p})(s - \tau_\ell) + C\int_{\tau_\ell}^{s}\overline{E}\{|\widehat{Y}^\ell_r|^2|\mathcal{G}_s\}\,dr, \qquad s \in [\tau_\ell, \tau].
\end{aligned}
\tag{7.15}
$$

Applying Gronwall's inequality we then have

$$
|\widehat{Y}^\ell_s|^2 + \tfrac{1}{2}\overline{E}\bigg\{\int_{\tau_\ell}^{s}|\widehat{Z}^\ell_r|^2\,dr\big|\mathcal{G}_s\bigg\} \le \zeta(1 + |X^{\tau,\xi}_s|^{2p})(s - \tau_\ell), \qquad s \in [\tau_\ell, \tau].
$$

In particular, it holds that

$$
|\widehat{Y}^\ell_s|^2 \le \zeta(1 + |X^{\tau,\xi}_s|^{2p})(s - \tau_\ell), \qquad s \in [\tau_\ell, \tau],\ \ell \ge 1.
\tag{7.16}
$$

Using this in (7.15) we derive further that, for $s \in [\tau_\ell, \tau]$,

$$
\begin{aligned}
\tfrac{1}{2}\overline{E}\Big\{\int_{\tau_\ell}^s |\widehat{Z}_r^\ell|^2\,dr\Big|\mathcal{G}_s\Big\} &\le \overline{E}\Big\{\int_{\tau_\ell}^s \zeta(1+|X_r^{\tau,\xi}|^{2p})\sqrt{r-\tau_\ell}\,dr\Big|\mathcal{G}_s\Big\} \\
&\le \zeta\int_{\tau_\ell}^s \sqrt{r-\tau_\ell}\big(1+\overline{E}\{|X_r^{\tau,\xi}|^{2p}|\mathcal{G}_s\}\big)\,dr \qquad (7.17)\\
&\le \zeta(1+|X_s^{\tau,\xi}|^{2p})(s-\tau_\ell)^{3/2}.
\end{aligned}
$$

By virtue of (7.14) we then have (denoting $X = X^{\tau,\xi}$)

$$
\begin{aligned}
&\bigg|\widehat{Y}_\tau^\ell(\overline{\omega}) - \overline{E}\Big\{\int_{\tau_\ell}^\tau \{f\big(r, X_r, \varphi(r,X_r), (\sigma^T D\varphi)(r,X_r)\big) \\
&\qquad\qquad \times (\partial_t - \mathcal{L})\varphi(r,X_r)\}\,dr\Big|\mathcal{G}_\tau\Big\}(\overline{\omega})\bigg| \\
&\quad\le \zeta(\omega)E'\Big\{\int_{\tau_\ell(\omega)}^{\tau(\omega)}\big(|\widehat{Y}_r^\ell(\omega,\cdot)| + |\widehat{Z}_r^\ell(\omega,\cdot)|\big)\,dr\Big\} \qquad (7.18)\\
&\quad\le \zeta(\omega)\big(1+|\xi(\omega)|^p\big)\big(\tau(\omega)-\tau_\ell(\omega)\big)^{3/2} \\
&\qquad + \zeta(\omega)\big(\tau(\omega)-\tau_\ell(\omega)\big)^{1/2}\Big\{E'\Big[\int_{\tau_\ell(\omega)}^{\tau(\omega)}|\widehat{Z}_r^\ell(\omega,\cdot)|^2\,dr\Big]\Big\}^{1/2} \\
&\quad\le \zeta(\omega)\big(1+|\xi(\omega)|^p\big)\big(\tau(\omega)-\tau_\ell(\omega)\big)^{5/4}, \qquad \overline{P}\text{-a.e. } \overline{\omega}.
\end{aligned}
$$

Consequently, dividing $\tau - \tau_\ell$ on both sides of (7.18), and then letting $\ell \to \infty$ and applying the dominated convergence theorem we obtain that, $\overline{P}$-a.s.,

$$
\begin{aligned}
\lim_{\ell\to 0}\frac{1}{\tau-\tau_\ell}\widehat{Y}_\tau^\ell &= f\big(\tau,\xi,v(\tau,\xi),\sigma^T(\xi)p\big) - a \qquad (7.19)\\
&\quad + \frac{1}{2}\mathrm{tr}\{\sigma\sigma^T(\xi)A\} + \langle\beta(\xi), p\rangle.
\end{aligned}
$$

Now comparing (7.19) to (7.9) we see that it remains to show that the left-hand side of (7.19) is nonnegative. To this end, we first note that

$$
\widehat{Y}_\tau^\ell = Y_\tau^\ell - \varphi(\tau,\xi) = Y_\tau^\ell - v(\tau,\xi) = Y_\tau^\ell - Y_\tau^{\tau,\xi}. \qquad (7.20)
$$

Denote now $\widetilde{Y}_s^\ell = Y_s^\ell - Y_s^{\tau,\xi}$, and $\widetilde{Z}_s^\ell = Z_s^\ell - Z_s^{\tau,\xi}$, for $s \in [\tau_\ell, \tau]$. Then, from (7.4) and (7.10) we see that

$$
\begin{aligned}
\widetilde{Y}_s^\ell &= (\varphi - v)(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) \\
&\qquad + \int_{\tau_\ell}^s \{f(r, X_r^{\tau,\xi}, Y_r^\ell, Z_r^\ell) - f(r, X_r^{\tau,\xi}, Y_r^{\tau,\xi}, Z_r^{\tau,\xi})\}\,dr - \int_{\tau_\ell}^s \langle \widetilde{Z}_r^\ell, \downarrow dW_r\rangle \\
&= (\varphi - v)(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) + \int_{\tau_\ell}^s \{\alpha_r^\ell \widetilde{Y}_r^\ell + \langle\beta_r^\ell, \widetilde{Z}_r^\ell\rangle\}\,dr - \int_{\tau_\ell}^s \langle\widetilde{Z}_r^\ell, \downarrow dW_r\rangle,
\end{aligned}
$$

where $\alpha^\ell \in L^\infty(\mathbf{G};[0,T])$ and $\beta^\ell \in L^\infty(\mathbf{G};[0,T]\times\mathbb{R}^k)$ are two bounded processes, whose bounds depend on the Lipschitz constant K in (A2). Therefore, one has

$$R_s^\ell \widetilde{Y}_s^\ell = R_{\tau_\ell}^\ell(\varphi - v)(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) - \int_{\tau_\ell}^s \langle R_r^\ell \widetilde{Z}_r^\ell, \downarrow d\widetilde{W}_r^\ell \rangle, \qquad s \in [\tau_\ell, \tau].$$

Here we denote $R_s^{\ell,t} = \exp\{\int_s^t \alpha_r^\ell\, dr\}$, $R_s^\ell = R_s^{\ell,T}$, and $\widetilde{W}_s^\ell \triangleq W_s + \int_s^T \beta_r^\ell\, dr$, $s \in [0,T]$. Applying Girsanov's theorem we know that, with $L_s^{\ell,t} \triangleq \exp\{\int_s^t \langle \beta_r^\ell, \downarrow dW_r\rangle - \frac{1}{2}\int_s^t |\beta_r^\ell|^2\, dr\}$, and $d\widetilde{P}^\ell \triangleq L_0^{\ell,T} d\overline{P}$, $\widetilde{W}$ is a $\mathbf{G}$-Brownian motion under $\widetilde{P}^\ell$. Consequently, with $C > 0$ denoting the generic constant as before, we have, for $\overline{P}$-a.e. $\overline{\omega}$,

$$\begin{aligned} \widetilde{Y}_\tau^\ell(\overline{\omega}) &= \overline{E}\{R_{\tau_\ell}^{\ell,\tau} L_0^{\ell,T}(\varphi - v)(\tau_\ell, X_{\tau_\ell}^{\tau,\xi}) | \mathcal{G}_\tau\}(\overline{\omega}) \\ &= E'\{R_{\tau_\ell}^{\ell,\tau}(\omega,\cdot) L_0^{\ell,T}(\varphi - v)(\tau_\ell(\omega), X_{\tau_\ell}^{\tau,\xi})(\omega,\cdot)\} \\ &\ge -C\big(E'\big[(v-\varphi)^+\big(\tau_\ell(\omega), X_{\tau_\ell}^{\tau,\xi}(\omega,\cdot)\big)^2\big]\big)^{1/2}. \end{aligned} \tag{7.21}$$

Since τ, τ_ℓ and ξ depend only on ω, we get that, for P-a.e. $\omega \in \{\tau > 0\}$,

$$\begin{aligned} &-\frac{1}{\tau(\omega) - \tau_\ell(\omega)} \widetilde{Y}_\tau^\ell(\omega,\cdot) \\ &\quad \le C\left\{E'\left[\left((v-\varphi)^+ \frac{(\tau_\ell(\omega), X_{\tau_\ell}^{\tau,\xi}(\omega,\cdot))}{\tau(\omega) - \tau_\ell(\omega)}\right)^2\right]\right\}^{1/2} \\ &\quad \le C\left\{E'\left[\Theta_\ell^2(\omega,\cdot)\left(\frac{(\tau(\omega) - \tau_\ell(\omega)) + |\xi(\omega) - X_{\tau_\ell}^{\tau,\xi}(\omega,\cdot)|^2}{\tau(\omega) - \tau_\ell(\omega)}\right)^2\right]\right\}^{1/2} \\ &\quad \le C\left\{E'[\Theta_\ell^4(\omega,\cdot)] E'\left[1 + \frac{\sup_{s\in[\tau_\ell(\omega),\tau(\omega)]} |\xi(\omega) - X_{\tau_\ell}^{\tau,\xi}(\omega,\cdot)|^8}{|\tau(\omega) - \tau_\ell(\omega)|^4}\right]\right\}^{1/4} \\ &\quad \le C\{E'[\Theta_\ell^4(\omega,\cdot)]\}^{1/4}\big(1 + |\xi(\omega)|^2\big), \qquad P'\text{-a.s.} \end{aligned}$$

Now noting that $|\Theta_\ell| \le 1$ by our assumption and $\Theta_\ell \to 0$, $\overline{P}$-a.s., as we showed before, applying the dominated convergence theorem again we have

$$\begin{aligned} &E\left\{\overline{\lim_{\ell\uparrow\infty}}\left(\frac{1}{\tau - \tau_\ell}(\widetilde{Y}_\tau^\ell)^-\right)^{1/2} \mathbb{1}_{\{\tau > 0\}}\right\} \\ &\quad \le CE\{(E'|\Theta_\ell|^4)^{1/8}(1 + |\xi|^2)^{1/2}\} \\ &\quad \le C(\overline{E}|\Theta_\ell|^4)^{1/8}\big(E(1 + |\xi|^2)\big)^{1/2} \to 0 \qquad \text{as } \ell \to \infty. \end{aligned}$$

This amounts to saying that $\overline{\lim}_{\ell\uparrow\infty} \frac{1}{\tau - \tau_\ell}(\widetilde{Y}_\tau^\ell)^- = 0$, P-a.e. on $\{\tau > 0\}$, or equivalently, $\underline{\lim}_{\ell\uparrow\infty} \frac{1}{\tau - \tau_\ell}\widetilde{Y}_\tau^\ell \ge 0$, P-a.e. on $\{\tau > 0\}$. This, together with (7.20) and (7.21), leads to

$$0 \le \varliminf_{\ell\uparrow\infty} \frac{1}{\tau - \tau_\ell} \widetilde{Y}_\tau^\ell = \varliminf_{\ell\uparrow\infty} \frac{1}{\tau - \tau_\ell} \widehat{Y}_\tau^\ell$$

$$\le f\big(\tau, \xi, v(\tau, \xi), \sigma^T(\xi) p\big) - a + \frac{1}{2} \mathrm{tr}\{\sigma\sigma^T(\xi) A\} + \langle \beta(\xi), p\rangle,$$

P-a.e. on $\{\tau > 0\}$, proving (7.9), hence the theorem. $\square$

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