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LONG TIME EXISTENCE FOR THE WAVE EQUATION WITH A NOISE TERM[1]

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We consider the equation $u_{tt} = \Delta u + a(u)N$ for $x \in \mathbf{R}^1$ or $\mathbf{R}^2$. N is a Gaussian noise term, which is white noise if $x \in \mathbf{R}^1$. If $a(u)$ grows no faster than $u(\log u)^{1/2-\varepsilon}$, then there is a unique solution valid for all time.

1. Introduction. We prove long time existence for the wave equation with a noise term, in one and two spatial dimensions,

$$\begin{aligned} u_{tt} &= u_{xx} + a(u)\dot{W}(t,x), \qquad x \in \mathbf{R},\ t \ge 0,\\ u(0,x) &= u_0(x), \\ u_t(0,x) &= u_1(x). \end{aligned} \tag{1.1}$$

$$\begin{aligned} u_{tt} &= \Delta u + a(u)G(t,x), \qquad x \in \mathbf{R}^2,\ t \ge 0,\\ u(0,x) &= u_0(x), \\ u_t(0,x) &= u_1(x). \end{aligned} \tag{1.2}$$

Here, $\dot{W}(t,x)$ is two-parameter white noise. $G(t,x)$ is a generalized Gaussian field with covariance $E[G(t,x)G(s,y)] = \delta(s-t)R(|x-y|)$.

We assume

$$|R(x)| \le 1. \tag{1.3}$$

Of course, if $|R(x)|$ were bounded by a constant greater than 1, then we could reduce to the above case by rescaling u. G must be smoother than white noise, since if $a \equiv 1$ and G is white noise, then the solution to (1.2) is a distribution, not a function [see Walsh (1986a), equation (3.20) and the succeeding comments], and then the meaning of $a(u)$ in the nonlinear case is not clear. We could handle a more general noise term in equation (1.1), but white noise is the canonical case. Our assumption on $a(u)$ is

$$\begin{aligned} &|a(u)| \le c(|u|+1)\log(|u|+2)^{\alpha}, \qquad 0 < \alpha < \tfrac{1}{2},\\ &a(u) \text{ is locally Lipschitz} \end{aligned} \tag{1.4}$$

so that $a(u)$ is close to linear. We do not know if these conditions are sharp.

We also assume

$$u_0, u_1 \in C([0,\infty) \times \mathbf{R}^d), \qquad d = 1 \text{ or } 2. \tag{1.5}$$

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As the dimension rises, the fundamental solution of the wave equation becomes more and more singular. Our methods break down for $x \in \mathbf{R}^d$, $d \geq 3$.

Perhaps it is worth noting that in the deterministic case,

$$v_{tt} = v_{xx} + v(\log v)^{\alpha},$$

where $v(t, x) = v(t)$ does not depend on x, an elementary calculation shows that v can blow up iff $\alpha \geq 2$.

$$\begin{aligned} v_{tt} v_t &= v_t v(\log v)^{\alpha}, \\ \tfrac{1}{2}(v_t)^2 &= O\big(v^2(\log v)^{\alpha}\big) \quad \text{if } v \text{ is large}, \\ \frac{v_t}{v(\log v)^{\alpha/2}} &\approx c, \\ (\log v)^{1-\alpha/2} &\approx c_0 t - c_1, \\ \log v &\approx (c_0 t - c_1)^{1/(1-\alpha/2)}. \end{aligned}$$

So it is possible that $v(t) \to \infty$ in finite time if $\alpha \geq 2$, in the deterministic case.

Many mathematicians have studied long time existence of solutions to nonlinear partial differential equations. If a random noise term is added, much less is known. The noise term could be very rough, so that the usual methods might not apply. In Mueller (1991c) it was shown that solutions to $u_t = u_{xx} + u^{\gamma}\dot{W}(t, x)$, $t > 0$, $x \in [0, J]$ (with Dirichlet boundary conditions) exist for all time, provided $1 \leq \gamma < \frac{3}{2}$. Computer simulations by Terry Lyons give evidence that solutions can blow up in finite time if $\gamma \geq \frac{3}{2}$. Recently, the author and R. Sowers have shown that blow-up can occur for large γ. The main reason for blow-up would be that u^{γ} is not a Lipschitz function of u. In a wide variety of equations with Lipschitz coefficients, long time existence follows from standard techniques [see Walsh (1986a), Chapter 3].

Here is a rough idea of the proof of long time existence for (1.1) and (1.2). We consider light cones $\{(t, x): 0 \leq t \leq T, |x| < T - t\}$. If u is bounded on such a region, then $a(u)$ is Lipschitz, with the Lipschitz constant depending on the bound. This, in turn, allows us to estimate the probability that u is bounded on the region and so bootstrap our way up. We consider a sequence of regions with T growing logarithmically with n, and the bound on u growing exponentially with n.

Now we discuss the rigorous meaning of (1.1) and (1.2).

First, remember that the fundamental solution $S(t, x)$ of the wave equation is as follows; see Treves [(1975), Chapter 1, Sections 7 and 8], for example. Here $\hat{S}(t, \xi)$ is the Fourier transform of $S(t, x)$ in the x variable.

$$\hat{S}(t, \xi) = \frac{\sin(|\xi|\, t)}{|\xi|}. \tag{1.6}$$

This formula is valid in any dimension. For $x \in \mathbf{R}^d$, we have

$$(1.7) \qquad S(t, x) = \begin{cases} \dfrac{1}{2} 1(|x| \le t), & d = 1, \\ \dfrac{1}{2\pi} 1(|x| \le t)\left(t^2 - |x|^2\right)^{-1/2}, & d = 2. \end{cases}$$

We note for later use that

$$\int_{\mathbf{R}^d} S(t, x)\, dx = \hat{S}(t, 0) = t.$$

We reformulate (1.1) and (1.2) in a weak form. To that end, suppose that

$$v_{tt} = v_{xx}$$
$$v(0, x) = u_0(x)$$
$$v_t(0, x) = u_1(x)$$

and that $\hat{v}(t, \xi)$ denotes the Fourier transform of $v(t, x)$ in the x-variable: $\hat{v}(t, \xi) = \int_{\mathbf{R}^d} e^{-i\xi \cdot x} v(t, x)\, dx$. Then we easily see that, under appropriate conditions,

$$\hat{v}(t, \xi) = \hat{u}_0(\xi)\cos(t|\xi|) + \hat{u}_1(\xi)\frac{\sin(t|\xi|)}{|\xi|}.$$

If $K(t, x)$ is a kernel satisfying $\hat{K}(t, \xi) = \cos(t|\xi|)$, then

$$v(t, x) = \int_{\mathbf{R}^d} S(t, x - y) u_1(y)\, dy + \int_{\mathbf{R}^d} K(t, x - y) u_0(y)\, dy.$$

This motivates the weak form for (1.1):

$$(1.8) \quad u(t, x) = v(t, x) + \int_0^t \int_{\mathbf{R}^d} S(t - s, x - y) a(u(s, y)) T(dy\, ds),$$

where $T(s, y)$ is the noise term

$$T(s, y) = \begin{cases} \dot{W}(s, y), & \text{if } d = 1, \\ G(s, y), & \text{if } d = 2. \end{cases}$$

The integral involving T should be interpreted in the sense of Walsh's theory of martingale measures. See Walsh [(1986a), Chapter 2]. The reader can check that $\dot{W}$ is an orthogonal martingale measure and that G is a martingale measure with nuclear covariance. For future use, we note the following. By Theorem 13.2 of Treves (1975), since u_0 and u_1 are C^∞ functions, we know that $v(t, x)$ is a C^∞ function of (t, x), and hence bounded on finite regions. Here is our main theorem.

THEOREM 1. *Suppose that conditions* (1.3), (1.4) *and* (1.5) *are satisfied, and* $d = 1$ *or* 2. *Then* (1.8) *has a unique solution* $u(t, x)$ *defined for* $t \ge 0$, $x \in \mathbf{R}^d$.

For $d = 1$, the proof is easier, since we can use two-parameter martingales, after a change of variables. This technique allows us to estimate the maximum of $u(t, x)$ over certain regions in the (t, x) plane. To make such estimates for $d = 2$, we must estimate the modulus of continuity of $u(t, x)$, using an argument reminiscent of Kolmogorov's criterion for the continuity of processes [see McKean (1969), page 16, or Walsh (1986a), Corollary 1.2].

In Section 2 we give the proof of Theorem 1. Section 3 contains the proof of existence and uniqueness when $a(u)$ is bounded and Lipschitz.

2. Maximum estimates and the proof of Theorem 1. First we prove Theorem 1 for $x \in \mathbf{R}$, where we can use two-parameter martingales to give a relatively simple proof. Let

$$C(t, x) = \{(s, y) \in \mathbf{R}^{d+1}: |x - y| \le t - s,\ s \ge 0\}$$

be the light cone with apex (t, x). The strategy is to replace $a(u(s, x))$ by $a(u(s, x) \wedge L)$, and to show that the new solution $\bar{u}(t, x)$ satisfies $|\bar{u}(t, x)| < L$ with high probability, at least for (t, x) in some light cone $C(t(L), 0)$. Then $u = \bar{u}$ inside this cone, and we have a bound on u. Letting $L \to \infty$, we get a bound for u over all space. This strategy is easiest to carry out with equation (1.1), since we can use maximal estimates for two-parameter martingales. We give this argument first.

For equation (1.2), with $d = 2$, we estimate the maximum of $u(t, x)$ by considering differences $u(t_1, x_1) - u(t_2, x_2)$ over nearest neighbor points on a grid. This general idea goes back to Kolmogorov's criterion for the continuity of processes, and Lévy's modulus of continuity for Brownian motion.

Now consider equation (1.1). It is convenient to stop the solution when it becomes too large. Let

$$\tilde{u}(s, x) = \begin{cases} u(s, x), & \text{if } |u(r, z)| < L \text{ for all } (r, z) \in C(s, x), \\ 0, & \text{if } |u(r, z)| \ge L \text{ for some } (r, z) \in C(s, x), \end{cases}$$

and let

$$F(t, x) = \sigma\left\{\iint h(s, y) W(dy\, ds): h \text{ is supported on } C(t, x)\right\}.$$

Note that $\tilde{u}(t, x)$ is $F(t, x)$-measurable, and that $|\tilde{u}(t, x)| \le L$.

Let $\bar{u}(t, x)$ satisfy

$$\begin{aligned} \bar{u}_{tt}(t, x) &= \bar{u}_{xx}(t, x) + a(\tilde{u}(t, x))\dot{W}(t, x), \\ \bar{u}(0, x) &= u_0(x), \\ \bar{u}_t(0, x) &= u_1(x). \end{aligned} \tag{2.1}$$

Note that $\bar{u}(t, x) = u(t, x)$ if $|u| < L$ on $C(t, x)$. Recall that $v(t, x)$ was the solution to $v_{tt} = v_{xx}$, $v(0, x) = u_0(x)$ and $v_t(0, x) = u_1(x)$. Recall also that $v(t, x)$ is a C^∞ function of (t, x), so that for $T > 0$,

$$V \equiv \sup_{(t, x) \in C(T, 0)} |v(t, x)| < \infty.$$

We prove that $u(t, x)$ has a unique solution in $C(T, 0)$ with probability 1. Since T is arbitrary, the theorem will be true for the entire half-space.

Let

$$(2.2)\qquad \begin{aligned} N(t, x) &= \int_0^t \int_{-s}^{s} a(\tilde{u}(s, x - y)) W(dy\, ds) \\ N(t_0, t, x) &= \iint_{C(t, x)\setminus C(t_0, 0)} a(\tilde{u}(s, x - y)) W(dy\, ds). \end{aligned}$$

Here is the key lemma.

LEMMA 1. *If $\lambda > 0$, $0 < t_0 < t$, then*

$$P\left\{ \sup_{(s, x)\in C(t, 0)} |N(t_0, s, x)| \geq \lambda \,\middle|\, F(t_0, 0) \right\} \leq c_0 \exp\left(\frac{-c_1 \lambda^2}{L^2 \log(L + 2)^{2\alpha} t_0(t - t_0)} \right).$$

Here, c_0 and c_1 depend only on t.

PROOF. We will need the following inequality.

$$\begin{aligned} \langle N(t_0, t, 0)\rangle &\leq \iint_{C(t, 0)\setminus C(t_0, 0)} L^2 \log(L + 2)^{2\alpha}\, dy\, ds \\ &\leq c_2 L^2 \log(L + 2)^{2\alpha} t_0(t - t_0). \end{aligned}$$

Here, $\langle N(t_0, t, 0)\rangle$ denotes the compensator of the martingale N, up to time t. Observe, if $(s, x) \in C(t, 0)$, then $N(t_0, s, x)$ is a time-changed Brownian motion with time scale bounded by $L \equiv c_2 L^2 \log(L + 2)^{2\alpha} t_0(t^2 - t_0^2)$. Therefore, for $\lambda > 0$,

$$(2.3)\qquad \begin{aligned} &\sup_{(s, x)\in C(t, 0)} P\{|N(t_0, s, x)| > \lambda\} \\ &\quad \leq P\left\{\left| B\left(c_2 L^2 \log(L + 2)^{2\alpha} t_0(t - t_0)\right)\right| > \lambda\right\} \\ &\quad \leq \frac{c L^{1/2}}{\lambda} \exp\left[-\lambda^2 L^{-1}/2\right] \end{aligned}$$

by standard estimates of the Normal.

To prove the lemma, we use Cairoli's maximum inequality for multiparameter submartingales [see Walsh (1986b), Theorem 2.2]. This theorem is stated for a countable index set, but it easily extends to continuous parameter martingales. It states that, if X_t is a positive n-parameter submartingale on a parameter set I, and if the σ-fields satisfy certain conditions, then for $\mu > 0$,

$$\mu P\left\{ \sup_I X_t \geq \mu \right\} \leq c + c \sup_I E\left\{ X_t (\log^+ X_t)^{n-1} \right\}.$$

To apply this inequality to $N(t_0, s, x)$ we rotate the (s, x) plane by 45 degrees and dilate by $\sqrt{2}$. Let $t_1 = s + x$, $t_2 = s - x$, and let

$$I = \{(t_1, t_2): 0 \le t_1 + t_2, |t_1 - t_2| \le 2t - t_1 - t_2\}.$$

Thus, I corresponds to $(s, x) \in C(t, 0)$. It is not hard to check that Cairoli's inequality still holds with the conditioning on $F(t_0, 0)$, as in the statement of Lemma 1.

For the duration of the proof, let

$$f(x) = \exp[x^2 L^{-1}/4].$$

Since $f(x)$ is a convex function,

$$X(t_1, t_2) \equiv f(N(t_0, s, x))$$

is a two-parameter submartingale. Let $\mu = f(\lambda)$. Then, by Cairoli's inequality (with conditioning), we have

$$\begin{aligned} P\left\{ \sup_{(s,x)\in C(t,0)} |N(t_0, s, x)| \ge \lambda \,\middle|\, F(t_0, 0)\right\} \\ = P\left\{ \sup_I X(t_1, t_2) \ge \mu \,\middle|\, F(t_0, 0)\right\} \\ \le \frac{1}{\mu}\left[c + c \sup_I E\{X(t_1, t_2)\log^+ X(t_1, t_2)\}\right]. \end{aligned} \tag{2.4}$$

Using inequality (2.3), we find

$$\begin{aligned} \sup_I E\{X(t_1, t_2)\log^+ X(t_1, t_2)\} \\ = \sup_{(s,x)\in C(t,0)} \int_0^\infty P\{f(N(t_0, s, x))\log^+(f(N(t_0, s, x))) \\ \ge f(\lambda)\log^+ f(\lambda)\} \\ \times d[f(\lambda)\log^+ f(\lambda)] \\ = \sup_{(s,x)\in C(t,0)} \int_0^\infty P\{|N(t_0, s, x)| \ge \lambda\} \\ \times \exp\left[\frac{\lambda^2 L^{-1}}{4}\right]\left[\frac{2\lambda L^{-1}}{4} + \frac{2\lambda^3 L^{-2}}{16}\right] d\lambda \\ \le c\int_0^\infty \frac{L^{1/2}}{\lambda} \exp\left[\frac{-\lambda^2 L^{-1}}{2}\right]\exp\left[\frac{\lambda^2 L^{-1}}{4}\right]\left[\frac{\lambda L^{-1}}{2} + \frac{\lambda^3 L^{-2}}{8}\right] d\lambda \\ \le c. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we find

$$P\left\{ \sup_{(s,x)\in C(t,0)} |N(t_0, s, x)| \ge \lambda \,\middle|\, F(t_0, 0)\right\} \le \frac{c}{\mu}.$$

Since $\mu = f(\lambda) = \exp[\lambda^2 L^{-1}/4]$, and recalling $L = c_1 L^2 \log(L+2)^{2\alpha} t_0(t - t_0)$, we see that Lemma 1 is established. □

Now we finish the proof of Theorem 1, in the case $d = 1$. Let $t_n = (1/K)\sum_{k=1}^{n} 1/k$, $t_0 = 0$. Consider the solution $\bar{u}_n(t, x)$ to (2.1) with cut-off $L_n = V + K2^{n+1}$, and let $\overline{N}_n$ be the corresponding noise term.

Let $E_n = E_n(K)$ be the event that

$$\sup_{(t,x)\in C(t_n,0)} \left|\overline{N}_n(t,x)\right| \le K2^n$$

and $F_n = F_n(K)$ be the event that

$$\sup_{(t,x)\in C(t_n,0)\setminus C(t_{n-1},0)} \left|\overline{N}_n(t_{n-1},t,x)\right| \le K2^{n-1}.$$

Using Lemma 1, we have, for large K,

$$\begin{aligned} &P\{F_n^c \mid E_{n-1}\} \\ &\quad\le \exp\left(-\frac{cK^2 2^{2n-2}}{(V+K2^{n+1})^2 \log(V+2+K2^{n-1})^{2\alpha} \log(n+1)/K^2 n}\right) \\ &\quad\le \exp\left(-\frac{cn^{1-2\alpha}K^2}{\log(n+1)}\right), \end{aligned}$$

where c does not depend on K, and may change from line to line.

Thus,

$$\sum_{n=1}^{\infty} P\{F_n^c \mid E_{n-1}\} \le c_1 \exp(-cK^2).$$

Next, note that if E_{n-1} and F_n occur, then E_n occurs. Also, E_0 trivially occurs. Thus,

$$\begin{aligned} P\left\{\bigcup_{n=1}^{\infty} F_n^c\right\} &\le \sum_{n=1}^{\infty} P\{F_n^c \cap F_{n-1} \cap \cdots \cap F_1\} \\ &\le \sum_{n=1}^{\infty} P\{F_n^c \cap F_{n-1} \cap \cdots \cap F_1 \cap E_0\} \\ &\le \sum_{n=1}^{\infty} P\{F_n^c \cap E_{n-1}\} \\ &\le \sum_{n=1}^{\infty} P\{F_n^c \mid E_{n-1}\} \\ &\le c_1 \exp(-cK^2). \end{aligned}$$

However, if $F_1 \cap \cdots \cap F_n$ occurs, then for $(t,x) \in [C(t_n,0)\setminus C(t_{n-1},0)] \cap C(T,0)$ we have $\bar{u}(t,x) \le V + K2^{n+1}$ and therefore $\bar{u}(t,x) = u(t,x)$ on $[C(t_n,0)\setminus C(t_{n-1},0)] \cap C(t,0)$. If all F_n occur, then $u(t,x)$ is defined for all $(t,x) \in C(T,0)$. Since T is arbitrary, long time existence is established. This proves Theorem 1, in the case $d = 1$ (modulo the proof of uniqueness).

Now we turn to the case $d = 2$. Define $C(t, x)$ as in the case $d = 1$. Let $\tau = \tau(T, L)$ be the first time $t \in [0, T]$ such that

$$\sup_{x \in C(t,0)} |u(t, x)| \geq L.$$

If there is no such time, let $\tau = T$. Let $\bar{u}(t, x)$ satisfy

$$\begin{aligned} \bar{u}_{tt}(t, x) &= \Delta \bar{u}(t, x) + a(u(t \wedge \tau, x))G(t, x), \\ \bar{u}(0, x) &= u_0(x), \\ \bar{u}_t(0, x) &= u_1(x). \end{aligned} \tag{2.6}$$

Of course, this is shorthand for an integral equation similar to (1.8).

We wish to estimate the maximum of $\bar{u}(t, x)$ over a region $C(t, 0) \setminus C(s, 0)$. We shall do this by giving an estimate on the modulus of continuity of $\bar{u}(t, x)$. Related estimates in the case of the heat equations were given in Sowers (1992a, b) and Mueller (1991a, b, c). Again, let

$$N(t, x) = \int_0^t \int_{\mathbf{R}^2} S(t - s, x - y) a(u(s \wedge \tau, y)) G(dy\, ds).$$

This integral is well defined by Lemma 2.4 of Walsh (1986a).

LEMMA 2. *Suppose* $0 \leq s \leq t$ *and* $x, y \in C(T, 0)$. *Then, for* Δ *sufficiently large, we have*

(A) $$P\{|N(t, x) - N(t, y)| > \Delta\} \leq c \frac{T_A}{\Delta} \exp\left(-\frac{\Delta^2}{2T_A}\right)$$

(B) $$P\{|N(s, x) - N(t, x)| > \Delta\} \leq c \frac{T_B}{\Delta} \exp\left(-\frac{\Delta^2}{2T_B}\right),$$

where

$$T_A = cL^2 \log(L + 2)^{2\alpha}(t^3 + 0 \vee \log t - 0 \wedge \log|x|)(|x - y|^3 + |x - y|),$$
$$T_B = cL^2 \log(L + 2)^{2\alpha}(1 + t^2 \log(t + 1))\left[(t - s)^3 + t - s\right].$$

The proof of Lemma 2 relies on the following result. Let $S(t, x) = 0$ if $t < 0$.

LEMMA 3.

(A) $$\int_0^t \left[\int_{\mathbf{R}^2} |S(r, x - z) - S(r, y - z)|\, dz\right]^2 dr \leq c(t^3 + 0 \vee \log t - 0 \wedge \log|x|)(|x|^3 + |x|),$$

(B) $$\int_0^t \left[\int_{\mathbf{R}^2} |S(t - r, z) - S(s - r, z)|\, dz\right]^2 dr \leq c(1 + t^2 \log t)\left[(t - s)^3 + (t - s)\right].$$

PROOF OF LEMMA 3(A). For ease of notation, we let $y = -x$, so $|x - y| = 2|x|$. Then,

$$\int_0^t \left[\int_{\mathbf{R}^2} |S(r, x-z) - S(r, -x-z)|\, dz \right]^2 dr$$
$$\leq 4 \int_0^{2|x| \wedge t} \left(\int_{\mathbf{R}^2} S(r, z)\, dz \right)^2 dr$$
$$+ 2 \int_{2|x| \wedge t}^t \left(\int_{|z| \leq r - 2|x|} |S(r, z+x) - S(r, z-x)|\, dz \right)^2 dr$$
$$+ 2 \int_{2|x| \wedge t}^t \left(\int_{|z| > r - 2|x|} |S(r, z+x) - S(r, z-x)|\, dz \right)^2 dr$$
$$= (I) + (II) + (III).$$

By Plancherel's theorem, since $\hat{S}(t, \xi) = (\sin(t|\xi|))/|\xi|$,

$$\int_{\mathbf{R}^2} S(r, z)\, ds = \lim_{\xi \to 0} \frac{\sin(r|\xi|)}{|\xi|} = r,$$

and thus,

$$(I) = 2 \int_0^{2|x| \wedge t} r^2\, dr = \tfrac{16}{3}(|x|^3 \wedge t^3).$$

By the mean-value theorem,

$$(II) = \int_{2|x| \wedge t}^t \left(\int_{|z| \leq r - 2|x|} (2\pi)^{-1/2} \frac{|\bar{z}|}{(r^2 - |\bar{z}|^2)^{3/2}}\, 2|x|\, dz \right)^2 dr,$$

where $\bar{z} = \bar{z}(z) \in \mathbf{R}^2$ lies on the line segment joining $z + x$ and $z - x$. Thus, $|\bar{z}| \leq |z| + |x|$, and in the domain of integration,

$$r^2 - |z|^2 = (r + |z|)(r - |z|) \geq r(r - |z| - |x|).$$

We may assume $2|x| \wedge t = 2|x|$, or else $(II) = 0$. Using polar coordinates, we have

$$(II) \leq \int_{2|x|}^t \left(\int_{|z| \leq r - 2|x|} \frac{2}{\sqrt{2\pi}} \frac{|z| + |x|}{r^{3/2}(r - |z| - |x|)^{3/2}} |x|\, dz \right)^2 dr$$
$$\leq c \int_{2|x|}^t \frac{|x|^2}{r^3} \left(\int_0^{r - 2|x|} \frac{v^2\, dv}{(r - |x| - v)^{3/2}} \right)^2 dr$$

$$
\begin{aligned}
&+ c\int_{2|x|}^{t} \frac{|x|^4}{r^3}\left(\int_0^{r-2|x|} \frac{v\,dv}{(r-|x|-v)^{3/2}}\right) dr \\
&\le c\int_{2|x|}^{t} |x|^2\, r\left(\int_0^{r-2|x|} \frac{dv}{(r-|x|-v)^{3/2}}\right)^2 dr \\
&\quad + c\int_{2|x|}^{t} |x|^4\, r^{-1}\left(\int_0^{r-2|x|} \frac{dv}{(r-|x|-v)^{3/2}}\right)^2 dr \quad (\text{since } v \le r) \\
&\le c\int_0^{t} |x|\, r\, dr + c\int_{2|x|}^{t} |x|^3\, r^{-1}\, dr \\
&\le ct^2\,|x| + c\,|x|^3 \log t - c\,|x|^3 \log |x| \\
&\le ct^2\,|x| + c\,|x|^3 (0 \vee \log t) - c\,|x|^3 (0 \wedge \log |x|).
\end{aligned}
$$

Finally, again assuming $2|x| \wedge t = 2|x|$,

$$
\begin{aligned}
(III) &\le 2\int_{2|x|}^{t}\left(\int_{|z|>r-2|x|} S(r, z+x)\, dz\right)^2 dr \\
&\le 2\int_{2|x|}^{t}\left(\int_{|z|>r-3|x|} S(r, z)\, dz\right)^2 dr \\
&\le c\int_{2|x|}^{t}\left(\int_{r-3|x|}^{r} \frac{v\,dv}{[(r+v)(r-v)]^{1/2}}\right)^2 dr \\
&\le c\int_{2|x|}^{t} r\left(\int_{r-3|x|}^{r} \frac{dv}{(r-v)^{1/2}}\right)^2 dr \\
&\le ct^2\,|x|.
\end{aligned}
$$

Putting together these estimates, we get Lemma 3(A). □

PROOF OF LEMMA 3(B).

$$
\begin{aligned}
&\int_0^{\infty}\left(\int_{\mathbf{R}^2} |S(t-r, z) - S(s-r, z)|\, dz\right)^2 dr \\
&\quad = \int_s^t \left(\int_{\mathbf{R}^2} S(t-r, z)\, dz\right)^2 dr \\
&\qquad + \int_0^s \left(\int_{\mathbf{R}^2} |S(t-r, z) - S(s-r, z)|\, dz\right)^2 dr \\
&\quad \le \int_s^t (t-r)^2\, dr + \int_0^{(t-s)\wedge s}\left(\int_{\mathbf{R}^2} |S(t-r, z) - S(s-r, z)|\, dz\right)^2 dr \\
&\qquad + \int_{(t-s)\wedge s}^{s}\left(\int_{\mathbf{R}^2} |S(t-r, z) - S(s-r, z)|\, dz\right)^2 dr \\
&\quad = (I) + (II) + (III).
\end{aligned}
$$

Now

$$(I) \le c(t-s)^3$$

and

$$\begin{aligned}(II) &\le \int_0^{t-s}\left(\int_{\mathbf{R}^2}|S(t-r,z)+S(s-r,z)|\,dz\right)^2 dr\\ &\le c\int_0^{t-s}[t-r+s-r]^2\,dr\\ &\le ct^2(t-s).\end{aligned}$$

For term (III), we may assume $(t-s)\wedge s = t-s$ since otherwise the integral is 0. Setting $\Delta = t-s$, we have

$$\begin{aligned}(III) &\le \int_0^{s-\Delta}\left(\int_{\mathbf{R}^2}|S(r+\Delta,z)-S(r,z)|\,dz\right)^2 dr\\ &\le \int_\Delta^{s}\left(\int_{|z|<r-\Delta}|S(r+\Delta,z)-S(r,z)|\,dz\right)^2 dr\\ &\quad+\int_\Delta^{s}\left(\int_{|z|>r-\Delta}(S(r+\Delta,z)+S(r,z))\,dz\right)^2 dr\\ &=(III.1)+(III.2).\end{aligned}$$

Using the mean-value theorem, we find, for some $b\in[r,r+\Delta]$ that

$$\begin{aligned}|S(r+\Delta,z)-S(r,z)| &= \frac{1}{\sqrt{2\pi}}\,\frac{b}{\left(b^2-|z|^2\right)^{3/2}}\,\Delta\\ &\le \frac{1}{\sqrt{2\pi}}\,\frac{r+\Delta}{\left(r^2-|z|^2\right)^{3/2}}\,\Delta.\end{aligned}$$

Thus, using polar coordinates, we have

$$\begin{aligned}(III.1) &\le c\int_\Delta^s\left(\int_{|z|<r-\Delta}\frac{r+\Delta}{\left(r^2-|z|^2\right)^{3/2}}\,\Delta\,dz\right)^2 dr\\ &\le c\Delta^2\int_\Delta^s\left(\int_0^{r-\Delta}\frac{2r}{r^{3/2}(r-v)}\,v\,dv\right)^2 dr\\ &\le c\Delta^2\int_\Delta^s r\left(\int_0^{r-\Delta}\frac{dv}{r-v}\right)^2 dr\\ &= c\Delta^2\int_\Delta^s r(\log r-\log\Delta)^2\,dr\\ &\le c\Delta^2 s^2(\log s+\log\Delta)^2\\ &\le c\Delta^2 t^2(\log t+\log\Delta)^2\\ &= c(t-s)^2t^2(\log t+\log(t-s))^2.\end{aligned}$$

Finally, again using polar coordinates, we find

$$\begin{aligned}(III.2) &\le c\int_{\Delta}^{s}\left(\int_{r-\Delta}^{r+\Delta}\left[(r+\Delta)^2 - v^2\right]^{-1/2} v\,dv\right)^2 dr \\ &\quad + c\int_{\Delta}^{s}\left(\int_{r-\Delta}^{r}[r^2 - v^2]^{-1/2} v\,dv\right)^2 dr \\ &\le c\int_{\Delta}^{s}\left(\int_{r-2\Delta}^{r} r^{-1/2}[r-v]^{-1/2}(r+\Delta)\,dv\right)^2 dr \\ &\le c\int_{\Delta}^{s} r\Delta\,dr \\ &\le ct^2(t-s).\end{aligned}$$

Putting together (I), (II), $(III.1)$ and $(III.2)$, we have Lemma 3(B). □

Now we apply Lemma 3 to prove Lemma 2. Lemma 2.4 in Walsh (1986a), shows that for appropriate nonanticipating (in r) $f(t, r, y)$ that

$$M_a = \int_0^a \int_{\mathbf{R}^2} f(t, r, y) G(dy\,dr)$$

is a continuous martingale with quadratic variation

$$\begin{aligned}\langle M\rangle_a &= \int_0^a \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} f(t, r, y_1) f(t, r, y_2) R(y_1 - y_2)\,dy_1\,dy_2\,dr \\ &\le \int_0^a \left[\int_{\mathbf{R}^2} |f(t, r, y)|\,dy\right]^2 dr,\end{aligned}$$

since $|R(y_1 - y_2)| \le 1$.

By an abuse of notation, we write

$$\left\langle \int_0^t \int_{\mathbf{R}^2} f(t, r, y) G(dy\,dr)\right\rangle = \langle M\rangle_a\big|_{a=t}.$$

If T is a bound for $\langle M_t\rangle$, we may regard M_t as a time changed Brownian motion, with time scale bounded by T. Using the reflection principle for Brownian motion and standard estimates for the normal, we find

$$P\{|M_t| > \Delta\} \le P\left\{\sup_{0\le t\le T} |B_t| > \Delta\right\} \tag{2.7}$$

$$\le 4P\{B_T > \Delta\}$$

$$\le c\,\frac{T}{\Delta}\exp\left(-\frac{\Delta^2}{2T}\right). \tag{2.8}$$

We apply this estimate to Lemma 2. For part (A), we let $M_t = N(t, x) - N(t, y)$. Since $|a(u)| \le cL\log(L+2)^{\alpha}$, Lemma 3(A) implies that

$$\langle M_t\rangle \le cL^2\log(L+2)^{2\alpha}\left(t^3 + \log t\right)^2\left(|x|^3 + |x|\right)^2.$$

Then, estimate (2.7) implies Lemma 2(A). For part (B), we let $M_t = N(t-r, z) - N(s-r, z)$. We find

$$\langle M_t \rangle \le cL^2 \log(L+2)^{2\alpha}\big(1 + t^2 \log(t+1)\big)\Big[(t-s)^3 + t - s\Big],$$

so estimate (2.7) implies Lemma 2(B), also. □

Now we use Lemma 2 to prove Theorem 1 in the case $x \in \mathbf{R}^2$. The strategy is reminiscent of the proof for $u_{tt} = u_{xx} + a(u)\dot{W}$. As before, we let $t_0 = 0$ and $t_n = (1/K)\sum_{k=1}^{n} 1/k$. Let $L_n = K2^n$. We give an argument similar to Kolmogorov's proof of continuity for processes [see McKean (1969), page 16 or Walsh (1986a), Corollary 1.2] to show that on the dyadic rationals, $u(t, x)$ is bounded with high probability. The "dyadic rationals" are vectors $(t, x) \in \mathbf{R}^3$ with components of the form $k/2^m$.

Again, we restrict attention to $(t, x) \in C(T, 0)$. Recall that $v(t, x)$, defined in the introduction, was the solution to the wave equation without noise. Since $v(t, x)$ is a C^∞ function, $V \equiv \sup_{(t,x)\in C(T,0)} |v(t, x)| < \infty$. Choose $K > V$.

Let E_n be the event that

$$\sup_{(t,x)\in C(t_n,0)} u(t, x) \le \sup_{(t,x)\in C(t_{n-1},0)} u(t, x) + 2V + K2^{n-2}$$

and let E_0 be the event that

$$\sup_{(t,x)\in C(t_1,0)} u(t, x) \le V + \frac{K}{2}.$$

LEMMA 4.

$$P\{E_n^c \mid E_{n-1} \cap \cdots \cap E_1\} \le c_0 \exp\big(-c_1 K^{1-5\varepsilon} n^{1-2\alpha-4\varepsilon}\big)$$

for K large and $1 - 2\alpha - 4\varepsilon > 0$.

PROOF. Fix $\varepsilon > 0$. For $m \ge \log(Kn)$, let $F_{n,m}$ be the event that for all pairs

$$(t, x) = \left(\frac{k_1}{2^m}, \frac{k_2}{2^m}, \frac{k_3}{2^m}\right) \in C(t_n, 0)$$

and

$$(s, y) = \left(\frac{l_1}{2^m}, \frac{l_2}{2^m}, \frac{l_3}{2^m}\right) \in C(t_n, 0),$$

such that $(t, x), (s, y)$ are nearest neighbors (i.e., $|(t, x) - (s, y)| = (1/2^m)$) and such that at least one of $(t, x), (s, y)$ lies in $C(t_n, 0) \setminus C(t_{n-1}, 0)$, we have

$$|N(t, x) - N(s, y)| \le K2^{n-m\varepsilon}.$$

Let $F_n = \bigcap_{m \ge \log(Kn)} F_{n,m}$.

First we show that

$$E_1 \cap \cdots \cap E_{n-1} \cap F_n \subset E_1 \cap \cdots \cap E_n \tag{2.9}$$

so that

$$P\{E_n^c \mid E_{n-1} \cap \cdots \cap E_1\} \le P\{F_n^c \mid E_{n-1} \cap \cdots \cap E_1\}. \tag{2.10}$$

Then we estimate $P\{F_n^c \mid E_{n-1} \cap \cdots \cap E_1\}$.

To show (2.6), suppose that $E_1 \cap \cdots \cap E_{n-1}$ holds and that $(t, x) \in C(t_n, 0) \setminus C(t_{n-1}, 0)$. Assume without loss of generality that $x = (x_1, x_2)$ and that $x_1, x_2 \geq 0$. Let (s, y) be the closest point in $C(t_{n-1}, 0)$ to (t, x). Of course, $y = (y_1, y_2)$ where $0 \leq s \leq t$, $0 \leq y_i \leq x_i$, $i = 1, 2$. Choose m_0 such that $2^{-m_0-2} \leq (1/Kn) \leq 2^{-m_0-1}$. Then

$$|t - s| \leq |(t, x) - (s, y)| \leq \frac{1}{Kn} \leq 2^{-m_0-1}$$

and

$$|x_i - y_i| \leq 2^{-m_0-1}, \qquad i = 1, 2.$$

The reader can check that $k, l_1, l_2 \geq 0$ can be chosen such that

$$\frac{k}{2^{m_0}} \leq s \leq t \leq \frac{k+1}{2^{m_0}}$$

and

$$\frac{l_i}{2^{m_0}} \leq y_i \leq x_i \leq \frac{l_i + t}{2^{m_0}}, \qquad i = 1, 2.$$

As mentioned, we assume that the entries of (t, x) are dyadic rationals, relying on continuity to deal with other cases.

We intend to break $N(t, x) - N(s, y)$ into a telescoping sum

$$\sum_{i=0}^{M} [N(s_{i+1}, y_{i+1}) - N(s_i, y_i)],$$

where $(s_0, y_0) = (s, y)$ and $(s_M, y_M) = (t, x)$. Furthermore, we require that $s_i \leq t$ for $i = 0, \dots, M$. Also each difference $(s_{i+1}, y_{i+1}) - (s_i, y_i)$ should be of the form $(2^{-m}, 0, 0)$, $(0, 2^{-m}, 0)$ or $(0, 0, 2^{-m})$, $m \geq m_0$.

Also, each of the above difference vectors should occur only once, so that there are at most three differences $(s_{i+1}, y_{i+1}) - (s_i, y_i)$ of magnitude 2^{-m}. To see that this is possible, we need only express the entries of (t, x) and (s, y) in terms of base 2 notation. For example, in the one dimensional case, if $s = \frac{1}{2}$ and $t = \frac{29}{32}$, we would write $s_0 = \frac{1}{2}$, $s_1 = \frac{3}{4}$, $s_2 = \frac{7}{8}$, $s_3 = \frac{29}{32}$.

Now, assume that F_n occurs, so that if $|(s_{i+1}, y_{i+1}) - (s_i, y_i)| \leq 2^{-m}$ then

$$|N(s_{i+1}, y_{i+1}) - N(s_i, y_i)| \leq K 2^{n - m\varepsilon}.$$

We find that

$$\begin{aligned} |N(t, x) - N(s, y)| &\leq \sum_{i=0}^{M} |N(s_{i+1}, y_{i+1}) - N(s_i, y_i)| \\ &\leq 3 \sum_{k=m_0}^{\infty} K 2^{n - m\varepsilon} \\ &\leq c(\varepsilon) K 2^{n - m_0 \varepsilon} \\ &\leq c(\varepsilon) \frac{K^{1-\varepsilon}}{n^{\varepsilon}} 2^n \end{aligned}$$

since $2^{-m_0-2} \leq (1/Kn)$.

Therefore, since $u(t, x) = v(t, x) + N(t, x)$, and since $(s, y) \in C(t_{n-1}, 0)$ we have

$$\begin{aligned} |u(t, x)| &\le |u(s, y)| + |v(t, x) - v(s, y)| + |N(t, x) - N(s, y)| \\ &\le \sup_{(s, y) \in C(t_{n-1}, 0)} u(s, y) + 2V + c(\varepsilon) \frac{K^{1-\varepsilon}}{n^{\varepsilon}} 2^n \\ &\le \sup_{(s, y) \in C(t_{n-1}, 0)} u(s, y) + 2V + K2^n \end{aligned}$$

for K large.

This shows that $E_1 \cap \cdots \cap E_{n-1} \cap F_n \subset E_1 \cap \cdots \cap E_n$.

Now we estimate $P\{F_n^c \mid E_{n-1} \cap \cdots \cap E_1\}$. Taking into account the definition of $\bar{u}$, we can drop the conditioning on $E_{n-1} \cap \cdots \cap E_1$ provided we define F_n in terms of $\bar{u}(t, x)$, not $u(t, x)$. Note that $C(t_n, 0) \setminus C(t_{n-1}, 0)$ has volume bounded by $ct_n^2(t_n - t_{n-1})$ or $cK^{-2}(\log n)^2 n^{-1}$. Now the event $F_{n,m}$ involves differences of $N(t, x)$ over nearest neighbors on the grid of order m. There are at most $c2^{3m}K^{-3}n^{-1}(\log n)^2$ such differences. Using Lemma 2, we find

$$P\{F_{n,m}^c\} \le c2^{3m}K^{-3}n^{-1}(\log n)^2 \frac{T}{\Delta} \exp\left(-\frac{\Delta^2}{2T}\right),$$

where

$$\Delta = K2^{n-m\varepsilon},$$

$$T = cK^2 2^{2n} \log(K2^n + 2)^{2\alpha}\left[1 + \left(\frac{\log n}{K}\right)^6\right]2^{-m}.$$

Thus, for K large, and therefore m large (since $m > m_0$ where $2^{-m_0-2} \le (1/Kn) \le 2^{-m_0-1}$).

$$\begin{aligned} P\{F_{n,m}^c\} &\le c_0 \exp\left(-c_1 2^{m(1-\varepsilon)}(\log K + n)^{-2\alpha}\right) \\ &\le c_0 \exp\left(-c_1 2^{-m\varepsilon}(Kn)^{1-2\varepsilon}(\log K + n)^{-2\alpha}\right). \end{aligned}$$

Summing from $m = m_0$ to ∞, we find

$$\begin{aligned} P\{F_n^c\} &\le c_0 \exp\left(-c_1 2^{-m_0\varepsilon}(Kn)^{1-2\varepsilon}(\log K + n)^{-2\alpha}\right) \\ &\le c_0 \exp\left(-c_1 (Kn)^{1-4\varepsilon}(\log K + n)^{-2\alpha}\right) \\ &\le c_0 \exp\left(-c_1 K^{1-5\varepsilon} n^{1-2\alpha-4\varepsilon}\right) \end{aligned}$$

if K is large and $1 - 2\alpha - 4\varepsilon > 0$. This proves Lemma 4. □

So finally,

$$\begin{aligned} \sum_{n=1}^{\infty} P\{E_n^c \mid E_{n-1} \cap \cdots \cap E_1\} &\le \sum_{n=1}^{\infty} c_0 \exp\left(-c_1 K^{1-5\varepsilon} n^{1-2\alpha-4\varepsilon}\right) \\ &= o(1) \quad \text{as } K \to \infty. \end{aligned}$$

Thus,

$$P\left\{\left[\bigcap_{n=1}^{\infty} E_n\right]^c\right\} \le \sum_{n=1}^{\infty} P\{E_n^c \cap E_{n-1} \cap \cdots \cap E_1\}$$
$$\le \sum_{n=1}^{\infty} P\{E_n^c \mid E_{n-1} \cap \cdots \cap E_1\}$$
$$= o(1) \quad \text{as } K \to \infty.$$

So $E = \bigcap_{n=1}^{\infty} E_n$ occurs with high probability, if K is large. But if all of the E_n occur, then $\tau = \infty$ and $\bar{u}(t, x) = u(t, x)$ for all $t \ge 0$, $x \in \mathbf{R}$. Thus $u(t, x)$ exists for all time, with high probability. Since K is arbitrary, $u(t, x)$ exists for all time with probability 1. This completes the proof of Theorem 1. □

3. Existence and uniqueness. In this section we prove existence and uniqueness for solutions $u(t, x)$ to

$$\begin{aligned} u_{tt} &= u_{xx} + a(u)\dot{W}, \qquad t \ge 0,\ x \in \mathbf{R}, \\ u(0, x) &= u_0(x), \\ u_t(0, x) &= u_1(x), \end{aligned} \tag{3.1}$$

$$\begin{aligned} u_{tt} &= \Delta u + a(u)G, \qquad t \ge 0,\ x \in \mathbf{R}^d,\ d = 2, \\ u(0, x) &= u_0(x), \\ u_t(0, x) &= u_1(x). \end{aligned} \tag{3.2}$$

Here $\dot{W}$, G, u_0, u_1 are as in Section 2.

We assume that $a(u)$ is a bounded, Lipschitz function of u. Thus, if necessary, we consider $a(u) = \bar{a}((u \wedge L) \vee (-L))$ where $\bar{a}$ is what we called a in Section 2. As in (1.8), we give these equations rigorous meaning by reformulating them as integral equations. The proofs given here use standard Picard iteration arguments, but we have not found them elsewhere.

THEOREM 2. *Under the above conditions,* (3.1) *and* (3.2) *possess unique solutions.*

PROOF. We begin with (3.1). Let $u^{(0)}(t, x)$ satisfy

$$\begin{aligned} u_{tt}^{(0)} &= u_{xx}^{(0)}, \\ u^{(0)}(0, x) &= u_0(x), \\ u_t^{(0)}(0, x) &= u_1(x) \end{aligned}$$

and define $u^{(n)}(t, x)$ by induction:

$$u^{(n)}(t, x) = u^{(0)}(t, x) + \int_0^t \int_{x-t}^{x+t} a\big(u^{(n-1)}(s, y)\big) W(dy\, ds).$$

Let K be the Lipschitz constant of a, and also the bound on a, and let

$$M_n(t) = \sup_{(s, x) \in C(0, t)} E\big[u^{(n+1)}(s, x) - u^{(n)}(s, x)\big]^2.$$

Since a is bounded, we easily compute that $M_1(t) < \infty$ for all $t > 0$.

Then one finds

$$M_n(t) \le K^2 \sup_{(s,x)\in C(0,t)} \int_0^s \int_{x-s}^{x+s} M_{n-1}(r)\, dy\, dr.$$

Now a standard application of Gronwall's lemma implies that for t fixed, $M_n(t)$ is a summable sequence. The reader can follow similar arguments in Walsh [(1986a), page 322], to show that $u(t, x)$, the L^2 limit of $u^{(n)}(t, x)$ as $n \to \infty$, exists a.s. for each (t, x), and that $u(t, x)$ satisfies (3.1). Actually, Walsh leaves the calculation as "an exercise for iteration enthusiasts."

Uniqueness is also standard. Suppose that $u^{(1)}(t, x)$, $u^{(2)}(t, x)$ are two solutions. Let

$$M(t) = \sup_{(s,x)\in C(t,0)} E\left[u^{(1)}(s, x) - u^{(2)}(s, x)\right]^2.$$

Since $|a(u)|$ is bounded, we see from equation (3.1) that $M(t) \le \infty$ for all t. Also,

$$M(t) \le K^2 \sup_{(s,x)\in C(t,0)} \int_0^s \int_{x-s}^{x+s} M(r)\, dr$$

and so by Gronwall's lemma, $M(t) = 0$ for all t. Thus, for each (t, x), $u^{(1)}(t, x) = u^{(2)}(t, x)$ a.s. Now we deal with equation (3.2), using the same basic ideas as for (3.1). Let $u^{(0)}(t, x)$ satisfy

$$\begin{aligned} u_{tt}^{(0)} &= \Delta u^{(0)}, \\ u^{(0)}(0, x) &= u_0(x), \\ u_t^{(0)}(0, x) &= u_1(x) \end{aligned}$$

and define $u^{(n)}(t, x)$ by induction:

$$u^{(n)}(t, x) = u^{(0)}(t, x) + \int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\, a(u(s, y))\, G(dy\, ds).$$

Again, let

$$M_n(t) = \sup_{(s,x)\in C(t,0)} E\left[u^{(n+1)}(s, x) - u^{(n)}(s, x)\right]^2.$$

Then,

$$\begin{aligned} M_1(t) &= E\left[\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\, a(u(s, y))\, G(dy\, ds)\right]^2 \\ &\le K^2 \left[\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y)\, dy\, ds\right]^2 \\ &\le K^2 \frac{t^4}{4}. \end{aligned}$$

Here, as in Section 2, we have used the fact that $|R(x)| \le 1$.

Now,

$$\begin{aligned} M_{n+1}(t) &= E\bigg[\int_0^t \int_{\mathbf{R}^2} S(t-s, x-y) \\ &\qquad \times \big\{a\big(u^{(n+1)}(s,y)\big) - a\big(u^{(n)}(s,y)\big)\big\} G(dy\,ds)\bigg]^2 \\ &\le K^2 \int_0^t \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} S(t-s_1, x-y_1) S(t-s_2, x-y_2) \\ &\quad \times E\Big[\big|u^{(n+1)}(s_1, y_1) - u^{(n)}(s,y)\big| \\ &\qquad \cdot \big|u^{(n+1)}(s_2, y_2) - u^{(n)}(s_2, y_2)\big|\Big]\, dy_1\, dy_2\, ds \\ &\le K^2 \int_0^t \bigg[\int_{\mathbf{R}^2} S(t-s, x-y) M_n^{1/2}(s)\, dy\bigg]^2 ds \\ &\le K^2 \int_0^t (t-s)^2 M_n(s)\, ds. \end{aligned}$$

Here we have used the Cauchy–Schwartz inequality. Then Gronwall's lemma shows that $\sum_{n=1}^{\infty} M_n(t)$ converges for all t. Again, we leave it to the reader to show that $u(t,x)$, the L^2 limit of $u^{(n)}(t,x)$ as $n \to \infty$, exists a.s. for each (t,x), and that $u(t,x)$ satisfies (3.3).

The proof of uniqueness is also easy. Let $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ be two solutions, and let

$$M(t) = \sup_{(s,x)\in C(t,0)} E\big[u^{(1)}(s,x) - u^{(2)}(s,x)\big]^2.$$

Again, equation (3.3) and the fact that a is bounded shows that $M(t) < \infty$ for all t. As in the previous calculation,

$$M(t) \le K^2 \int_0^t (t-s)^2 M^{1/2}(s)\, ds.$$

Then Gronwall's lemma implies that $M(t) = 0$ for all t. This proves uniqueness. □

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