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STRONG APPROXIMATION THEOREMS FOR GEOMETRICALLY WEIGHTED RANDOM SERIES AND THEIR APPLICATIONS[1]

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Let $\{X_n; n \geq 0\}$ be a sequence of random variables. We consider its geometrically weighted series $\xi(\beta) = \sum_{n=0}^{\infty} \beta^n X_n$ for $0 < \beta < 1$. This paper proves that $\xi(\beta)$ can be approximated by $\sum_{n=0}^{\infty} \beta^n Y_n$ under some suitable conditions, where $\{Y_n;\ n \geq 0\}$ is a sequence of independent normal random variables. Applications to the law of the iterated logarithm for $\xi(\beta)$ are also discussed.

1. Introduction and main results. Let $\{X_n;\ n \geq 0\}$ be a sequence of random variables; one can consider its geometrically weighted series $\xi(\beta) = \sum_{n=0}^{\infty} \beta^n X_n$, $0 < \beta < 1$. The following type of the law of the iterated logarithm (LIL) was obtained by Bovier and Picco [1]:

$$\limsup_{\beta \nearrow 1} \frac{1}{(2 \operatorname{Var} \xi(\beta) \log \log \operatorname{Var} \xi(\beta))^{1/2}} |\xi(\beta)| = 1 \quad \text{a.s.,} \tag{1.1}$$

where $\{X_n;\ n \geq 0\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and variance one. Recently, in a long paper, Picco and Vares [9] proved this kind of LIL for a stationary ergodic martingale difference sequence with finite second moments. If $\{X_n;\ n \geq 0\}$ is not an i.i.d. stochastic sequence, for example, a mixing sequence or a sequence of independent random variables satisfying the conditions of the Kolmogorov LIL, to prove a LIL of the type (1.1) will be very complicated if we use the methods of [1] and [9], and we don't know whether their methods are effective or not. It is the purpose of the present paper to look for a general and simple way to get the LIL of type (1.1). Indeed, we will show that under some suitable conditions, $\xi(\beta)$ can be approximated by $\sum_{n=0}^{\infty} \beta^n Y_n$, where $\{Y_n;\ n \geq 0\}$ is a sequence of independent normal variables. Then we establish some results on the LIL of type (1.1) for $\{X_n; n \geq 0\}$ by proving the same results for normal variables. The results we get on the LIL include not only those of [1] and [9] with a simple proving method, but also the laws of the iterated logarithm for the geometrically weighted series of dependent random variables, independent but not necessarily identically distributed random variables and i.i.d. random variables with possibly infinite variances. This section discusses strong approximations. The laws of the iterated logarithm will be presented in Section 2. Throughout this paper, $C, C_0, C_1, c, c_0, c_1, \ldots$ will denote positive

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constants whose values are uninteresting and may vary from line to line. The expression $a_n \sim b_n$ means $a_n/b_n \to 1$ $(n \to \infty)$; $a_n \approx b_n$ means that there exist $C_1, C_2 > 0$ such that $C_1 \le a_n/b_n \le C_2$ for n large enough.

The following theorem gives a general result on strong approximations for random geometric series.

THEOREM 1.1. *Let $H(x)$ $(x \ge 0)$ be a monotone nondecreasing positive function with $H(x) \to \infty$ $(x \to \infty)$ and $H(2n) \le CH(n)$, $(n \ge 0)$ and let $\{\xi_n;\ n \ge 0\}$, $\{\eta_n;\ n \ge 0\}$ be two sequences of random variables with $E|\xi_n|^p \le Cn^q$, $E|\eta_n|^p \le Cn^q$, $(n \ge 0)$ for some $p, q > 0$. If*

$$\left|\sum_{k=0}^{n} \xi_k - \sum_{k=0}^{n} \eta_k\right| = O(H(n)) \text{ (or } o(H(n))) \quad \text{a.s. } (n \to \infty), \tag{1.2}$$

then

$$\begin{aligned} &\left|\sum_{n=0}^{\infty} \beta^n \xi_n - \sum_{n=0}^{\infty} \beta^n \eta_n\right| \\ &\qquad = O\left(H\left(\frac{1}{1-\beta^2}\right)\right)\left(\text{or } o\left(H\left(\frac{1}{1-\beta^2}\right)\right)\right) \quad \text{a.s. } (\beta \nearrow 1). \end{aligned} \tag{1.3}$$

The following corollary comes from Theorem 1.1 immediately.

COROLLARY 1.1. *Suppose $\{X_n;\ n \ge 0\}$ is a sequence of i.i.d. random variables, or more generally, a stationary ergodic martingale difference with $EX_0 = 0$, $EX_0^2 = \sigma^2$, $0 < \sigma < \infty$. Then there exists a sequence of i.i.d. normal random variables $\{Y_n;\ n \ge 0\}$ with $Y_n =_{\mathscr{D}} N(0, \sigma^2)$ such that*

$$\lim_{\beta \nearrow 1} \frac{\sqrt{1-\beta^2}}{\sqrt{2 \log\log(1/(1-\beta^2))}} \left|\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n\right| = 0 \quad \text{a.s.} \tag{1.4}$$

By Theorem 1.1 and Theorems 1.1, 1.2 of Shao [11], we have the following corollary.

COROLLARY 1.2. *Let $\{X_n;\ n \ge 0\}$ be a stationary stochastic sequence with $EX_0 = 0$, $EX_0^2 < \infty$ and $B_n = E(\sum_{k=0}^n X_k)^2 \to \infty$ $(n \to \infty)$ satisfying one of the following conditions:*

(i) *$\{X_n;\ n \ge 0\}$ is ρ-mixing. The mixing coefficients satisfy*

$$\rho(n) \le \log^{-r} n \quad \text{for some } r > 1;$$

(ii) *$\{X_n;\ n \ge 0\}$ is ϕ-mixing. The mixing coefficients satisfy*

$$\sum_{n=0}^{\infty} \phi^{1/2}(2^n) < \infty.$$

Then $\lim_{n\to\infty}(B_n/n) = \sigma^2$ for some $0 < \sigma < \infty$, and the conclusion of Corollary 1.1. holds true.

When $\{X_n;\ n \ge 0\}$ is a sequence of i.i.d. random variables with higher than second moments by Theorem 1.1, Theorem 1 of Zhang [14] and the results of Komlos, Major and Tusnady [6, 7], we have the following conclusion.

COROLLARY 1.3. *Suppose that $\{X_n;\ n \ge 0\}$ is a sequence of i.i.d. random variables with $EX_0 = 0$, $EX_0^2 = 1$. Let $H(x)$ $(x \ge 0)$ be a nondecreasing positive continuous function such that for some $\gamma > 0$, $x_0 > 0$, $x^{-2-\gamma}H(x)$ $(x \ge x_0)$ is nondecreasing and $x^{-1}\log H(x)$ $(x \ge x_0)$ is nonincreasing.*

(a) *If $EH(|X_0|) < \infty$, then there exists a sequence of i.i.d. standard normal random variables $\{Y_n;\ n \ge 0\}$ such that*

$$\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n = O\left(\operatorname{inv} H\left(\frac{1}{1-\beta^2}\right)\right) \quad \text{a.s. } (\beta \nearrow 1).$$

(b) *If $x^{-1}\log H(x) \to 0$ $(x \to \infty)$ and $EH(t|X_0|) < \infty$ for any $t > 0$, then there exists a sequence of i.i.d. standard normal random variables $\{Y_n;\ n \ge 0\}$ such that*

$$\sum_{n=0}^{\infty} \beta^n X_n - \sum_{n=0}^{\infty} \beta^n Y_n = o\left(\operatorname{inv} H\left(\frac{1}{1-\beta^2}\right)\right) \quad \text{a.s. } (\beta \nearrow 1).$$

The following theorem deals with the sequence of independent but not necessarily identically distributed random variables.

THEOREM 1.2. *Let $\{X_n;\ n \ge 0\}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$ $(n \ge 0)$. Set $B_n = \sum_{k=0}^n EX_k^2$ and $\tau(\beta) = \sum_{n=0}^\infty \beta^{2n} EX_n^2$. Suppose $B_n \to \infty$ $(n \to \infty)$, $\limsup_{n\to\infty} B_{2n}/B_n < \infty$ and for some $p \ge 2$,*

$$\sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon(B_n/\log\log B_n)^{1/2}\}}{(B_n \log\log B_n)^{p/2}} < \infty \quad \text{for any } \varepsilon > 0. \tag{1.5}$$

Then there exists a sequence of independent normal random variables $\{Y_n;\ n \ge 0\}$ with $Y_n =_{\mathscr{D}} N(0, EX_n^2)$ such that

$$\lim_{\beta \nearrow 1} \frac{|\sum_{n=0}^\infty \beta^n X_n - \sum_{n=0}^\infty \beta^n Y_n|}{(2\tau(\beta)\log\log\tau(\beta))^{1/2}} = 0 \quad \text{a.s.} \tag{1.6}$$

If (1.5) is replaced by

$$\sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon(B_n/\log\log B_n)^{1/2}\}}{(B_n \log\log B_n)^{p/2}} < \infty \quad \text{for some } \varepsilon > 0, \tag{1.5$'$}$$

then there exists a sequence of independent normal random variables $\{Y_n;\ n \ge 0\}$ with $Y_n =_{\mathscr{D}} N(0, EX_n^2)$ such that

$$\limsup_{\beta \nearrow 1} \frac{|\sum_{n=0}^\infty \beta^n X_n - \sum_{n=0}^\infty \beta^n Y_n|}{(2\tau(\beta)\log\log\tau(\beta))^{1/2}} \le \Gamma\varepsilon \quad \text{a.s.,} \tag{1.6$'$}$$

where Γ is a numerical constant.

PROOF OF THEOREM 1.1. Without loss of generality, we can assume $0 < p \le 1$. Note that for any $0 < \beta < 1$,

$$E\left(\sum_{n=0}^{\infty} \beta^n |\xi_n|\right)^p \le \sum_{n=0}^{\infty} \beta^{np} E|\xi_n|^p \le C \sum_{n=0}^{\infty} \beta^{np} n^q < \infty.$$

We know that $\sum_{n=0}^{\infty} \beta^n \xi_n$ is a.s. absolutely convergent for any $0 < \beta < 1$. Similarly, so is $\sum_{n=0}^{\infty} \beta^n \eta_n$. Let $N(\beta) = [1/(1-\beta^2)]$. (We keep this in mind in the remainder of this paper.) Note that $H(2n) \le CH(n)$ $(n \ge 0)$ implies $H(kn)/H(n) \le C_0 k^Q$ $(n \ge 0,\ k \ge 1)$ for some $C_0, Q > 0$; we have proved Theorem 1.1 by the following lemma immediately.

LEMMA 1.1. *Let $\{a_n;\ n \ge 0\}$ be a sequence of real numbers, $\{A_n;\ n \ge 0\}$ a sequence of monotonous nondecreasing positive numbers satisfying $|\sum_{k=0}^{n} a_k| \le A_n$ for n large enough and $A_{kn}/A_n \le C_0 k^Q$ $(k \ge 1,\ n \ge 0)$. Then for any $r > 0$, $N_0 \ge 1$, we have*

$$\limsup_{\beta \nearrow 1} A_{N(\beta)}^{-1} \left| \sum_{n=0}^{\infty} \beta^{nr} a_n \right| \le \frac{r}{2} + \frac{rC_0}{2} \int_1^{\infty} \exp\left(-\frac{rx}{2}\right) x^Q\, dx, \tag{1.7}$$

$$\begin{aligned} &\limsup_{\beta \nearrow 1} A_{N(\beta)}^{-1} \left| \sum_{n=N_0 N(\beta)}^{\infty} \beta^{nr} a_n \right| \\ &\qquad \le \frac{rC_0}{2} \int_{N_0}^{\infty} \exp\left(-\frac{rx}{2}\right) x^Q\, dx + C_0 \exp\left(-\frac{rN_0}{2}\right) N_0^Q. \end{aligned} \tag{1.8}$$

PROOF. The proof is based on Abel's partial summation formula. An analogous idea was used by Horvath [5]. Let $S_n = \sum_{k=0}^{n} a_k$ $(n \ge 0)$, $S_{-1} = 0$. By the Abel lemma we have

$$\sum_{n=l}^{p} \beta^{nr} a_n = (1-\beta^r) \sum_{n=l}^{p} \beta^{nr} S_n - \beta^{lr} S_{l-1} + \beta^{r(p+1)} S_p. \tag{1.9}$$

Note $A_n \le C_0 A_1 n^Q$ $(n \ge 1)$, we know that

$$\sum_{n=0}^{\infty} \beta^{nr} A_n \le A_0 + C_0 A_1 \sum_{n=0}^{\infty} \beta^{nr} n^Q < \infty.$$

It follows that $\sum_{n=0}^{\infty} \beta^{nr} S_n$ is absolutely convergent. So, $\sum_{n=0}^{\infty} \beta^{nr} a_n$ is also absolutely convergent. Hence, by letting $p \to \infty$ in (1.9), we have, for any $l \ge 0$,

$$\sum_{n=l}^{\infty} \beta^{nr} a_n = (1-\beta^r) \sum_{n=l}^{\infty} \beta^{nr} S_n - \beta^{lr} S_{l-1}.$$

Then, note that for β near enough to 1,

$$
\begin{aligned}
&(1-\beta^r)A_{N(\beta)}^{-1}\sum_{n=0}^{\infty}\beta^{nr}|S_n| \\
&\quad= (1-\beta^r)A_{N(\beta)}^{-1}\sum_{n=0}^{N(\beta)}\beta^{nr}|S_n| + (1-\beta^r)A_{N(\beta)}^{-1}\sum_{n=N(\beta)+1}^{\infty}\beta^{nr}|S_n| \\
&\quad\le (1-\beta^r)(N(\beta)+1) \\
&\qquad + (1-\beta^r)N(\beta)\frac{1}{N(\beta)}\sum_{n=N(\beta)+1}^{\infty}\exp\left(-\frac{nr}{2N(\beta)}\right)\frac{A_n}{A_{N(\beta)}}\frac{|S_n|}{A_n} \\
&\quad\le (1-\beta^r)(N(\beta)+1) \\
&\qquad + (1-\beta^r)N(\beta)\frac{1}{N(\beta)}\sum_{n=N(\beta)+1}^{\infty}\exp\left(-\frac{nr}{2N(\beta)}\right)C_0\left(\frac{n}{N(\beta)}\right)^Q \\
&\quad\to \frac{r}{2}+\frac{rC_0}{2}\int_1^{\infty}\exp\left(-\frac{rx}{2}\right)x^Q\,dx, \qquad \text{as } \beta\nearrow 1.
\end{aligned}
$$

We have proved (1.7). The proof of (1.8) is similar.

Proof of Theorem 1.2. If $\limsup_{n\to\infty} B_{2n}/B_n < \infty$ then $B_{2n}/B_n \le C$ for some $C > 0$. Then there exist $C_0, Q > 0$ such that $B_{kn}/B_n \le C_0 k^Q$, $B_n \le C_0 n^Q$ ($n \ge 0$, $k \ge 1$). Hence $EX_n^2 \le C_0 n^Q$ ($n \ge 0$). By Theorem 1.1, we need only to show that

$$\tau(\beta) \approx B_{N(\beta)}, \qquad \beta \nearrow 1, \tag{1.10}$$

that there exists a sequence of independent normal random variables $\{Y_n;\ n \ge 0\}$ with $Y_n =_{\mathscr{D}} N(0, EX_n^2)$ such that

$$\left|\sum_{k=0}^{n} X_k - \sum_{k=0}^{n} Y_k\right| = o\big((B_n \log\log B_n)^{1/2}\big) \quad \text{a.s. } (n\to\infty) \tag{1.11}$$

whenever (1.5) holds, and that there exists a sequence of independent normal random variables $\{Y_n;\ n \ge 0\}$ with $Y_n =_{\mathscr{D}} N(0, EX_n^2)$ such that for some numerical constant Γ,

$$\limsup_{n\to\infty} \frac{|\sum_{k=0}^{n} X_k - \sum_{k=0}^{n} Y_k|}{(B_n \log\log B_n)^{1/2}} \le \Gamma\varepsilon \quad \text{a.s.} \tag{1.11$'$}$$

whenever (1.5) is replaced by (1.5′).

We prove (1.10) first. First, $\tau(\beta) \ge \sum_{n=0}^{N(\beta)} \beta^{2n} EX_n^2 \ge \beta^{2N(\beta)} B_{N(\beta)}$ implies

$$\liminf_{\beta\nearrow 1} \tau(\beta)/B_{N(\beta)} \ge e^{-1}. \tag{1.12}$$

On the other hand, it follows from Lemma 1.1 that

$$\limsup_{\beta \nearrow 1} \frac{\tau(\beta)}{B_{N(\beta)}} \le 1 + C_0 \int_1^\infty \exp(-x) x^Q \, dx < \infty. \tag{1.13}$$

Hence we have proved (1.10).

Now we will prove (1.11). If (1.5) holds, then there exists a sequence of nonincreasing positive numbers $\{\varepsilon_n; n \ge 0\}$ satisfying $1 > \varepsilon_n \to 0$, $\varepsilon_n^{p-2} \log\log B_n \nearrow \infty$, $\varepsilon_n (B_n/\log\log B_n)^{1/2} \nearrow \infty$ $(n \to \infty)$ such that

$$\sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon_n (B_n/\log\log B_n)^{1/2}\}}{(B_n \log\log B_n)^{p/2}} < \infty. \tag{1.14}$$

Let

$$\begin{aligned} \xi_n &= X_n I\left\{|X_n| \le \varepsilon_n \left(\frac{B_n}{\log\log B_n}\right)^{1/2}\right\} \\ &\qquad - E X_n I\left\{|X_n| \le \varepsilon_n \left(\frac{B_n}{\log\log B_n}\right)^{1/2}\right\}, \\ \tilde{\xi}_n &= X_n - \xi_n. \end{aligned} \tag{1.15}$$

Then $\xi_n I\{|\xi_n| \le (B_n/\log\log B_n)^{1/2}\} = \xi_n$ and

$$\sum_{n=0}^{\infty} P\{|\xi_n| > \varepsilon (B_n/\log\log B_n)^{1/2}\} < \infty$$

for any $\varepsilon > 0$. By Theorem 1.1 of Shao [12] (see also [10]), there exists a sequence of i.i.d. normal random variables $\{\eta_n;\ n \ge 0\}$ with $\eta_n =_{\mathscr{D}} N(0,1)$ such that

$$\begin{aligned} &\sum_{i=0}^n \xi_i - \sum_{i=0}^n \eta_i (\operatorname{Var} \xi_i)^{1/2} \\ &\qquad = o\left(\left(\frac{B_n}{\log\log B_n}\right)^{1/2} \log \sum_{i=0}^n \frac{(\log\log B_i) E\xi_i^2}{B_i}\right) \quad \text{a.s. } (n \to \infty), \end{aligned}$$

which together with

$$\sum_{i=0}^n \frac{(\log\log B_i) E\xi_i^2}{B_i} \le C \log B_n \log\log B_n$$

implies that

$$\sum_{i=0}^n \xi_i - \sum_{i=0}^n \eta_i (\operatorname{Var} \xi_i)^{1/2} = o\big((B_n \log\log B_n)^{1/2}\big) \quad \text{a.s. } (n \to \infty). \tag{1.16}$$

Set $Y_n = (\operatorname{Var} X_n)^{1/2} \eta_n$ $(n \ge 0)$. According to (1.15) and (1.16), in order to prove (1.11) we need only to show that

$$\sum_{i=0}^n \tilde{\xi}_i = o\big((B_n \log\log B_n)^{1/2}\big) \quad \text{a.s. } (n \to \infty) \tag{1.17}$$

and

$$\sum_{i=0}^{n}\big((\mathrm{Var}\ X_i)^{1/2}-(\mathrm{Var}\ \xi_i)^{1/2}\big)\eta_i = o\big((B_n \log\log B_n)^{1/2}\big) \quad \text{a.s. } (n\to\infty). \tag{1.18}$$

First, we apply Proposition 2.2. of [2] to prove (1.17). Let $S_n = \sum_{k=0}^{n}\tilde{\xi}_k$, $a_n = (2B_n \log\log B_n)^{1/2}$. It is easy to see that hypothesis (2.9) of [2] is fulfilled, since

$$P\left(\frac{|S_n|}{a_n}\ge \varepsilon\right) \le \frac{1}{\varepsilon^2}\frac{ES_n^2}{a_n^2} \le \frac{1}{\varepsilon^2}\frac{1}{2\log\log B_n}.$$

From (1.14), it follows that hypothesis (2.3) of [2] is fulfilled. Now, let $\{n_k\}$ satisfy (2.2) of [2]. That is,

$$\lambda a_{n_k} \le a_{n_{k+1}} \le \lambda^3 a_{n_k+1}$$

for some $\lambda > 1$. Let $I(k) = \{n_k+1,\dots,n_{k+1}\}$ and

$$N_1 = \left\{k\in N;\ \sum_{j\in I(k)} \frac{E|X_j|^p I\{|X_j| > \varepsilon_j (B_j/\log\log B_j)^{1/2}\}}{a_j^p} \le (2\log\log B_{n_{k+1}})^{-p}\right\}.$$

For each $k\in N_1$, we have

$$\begin{aligned}
&\frac{1}{B_{n_{k+1}}}\sum_{j\in I(k)} EX_j^2 I\{|X_j| > \varepsilon_j (B_j/\log\log B_j)^{1/2}\}\\
&\quad\le \sum_{j\in I(k)} 2^{p/2}\frac{B_j}{B_{n_{k+1}}}\frac{(\log\log B_j)^p}{\varepsilon_j^{p-2}\log\log B_j} EX_j^2 I\{|X_j| > \varepsilon_j (B_j/\log\log B_j)^{1/2}\}/a_j^p\\
&\quad\le \frac{2^{p/2}(\log\log B_{n_{k+1}})^p}{\varepsilon_{n_k}^{p-2}\log\log B_{n_k}} \sum_{j\in I(k)} EX_j^2 I\{|X_j| > \varepsilon_j (B_j/\log\log B_j)^{1/2}\}/a_j^p\\
&\quad\le \frac{2^{-p/2}}{\varepsilon_{n_k}^{p-2}\log\log B_{n_k}} \to 0,
\end{aligned}$$

which together with (1.14) implies (see [2])

$$\begin{aligned}
&\sum_k \exp\left\{-\frac{\delta a_{n_{k+1}}^2}{\sum_{j\in I(k)} E\tilde{\xi}_j^2}\right\}\\
&\quad\le \sum_k \exp\left\{-\frac{\delta a_{n_{k+1}}^2}{\sum_{j\in I(k)} EX_j^2 I\{|X_j| > \varepsilon_j (B_j/\log\log B_j)^{1/2}\}}\right\} < \infty
\end{aligned} \tag{1.19}$$

for every $\delta > 0$.

It follows that hypothesis (2.8) of [2] is fulfilled. Thus, by Proposition 2.2 of [2] we have proved (1.17).

Now, note that

$$\begin{aligned}\big((\mathrm{Var}\,X_j)^{1/2}-(\mathrm{Var}\,\xi_j)^{1/2}\big)^2 &\le \mathrm{Var}\,X_j-\mathrm{Var}\,\xi_j\\ &\le 3EX_j^2 I\{|X_j|>\varepsilon_j(B_j/\log\log B_j)^{1/2}\}.\end{aligned}$$

We have

$$\begin{aligned}&P\bigg(\max_{i\in I(k)}\bigg|\sum_{j=n_k+1}^{i}\big((\mathrm{Var}\,X_j)^{1/2}-(\mathrm{Var}\,\xi_j)^{1/2}\big)\eta_j\bigg|\ge\delta a_{n_{k+1}}\bigg)\\ &\quad\le 2P\bigg(\bigg|\sum_{j\in I(k)}\big((\mathrm{Var}\,X_j)^{1/2}-(\mathrm{Var}\,\xi_j)^{1/2}\big)\eta_j\bigg|\ge\delta a_{n_{k+1}}\bigg)\\ &\quad\le 2\exp\bigg\{-\frac{\delta^2a_{n_{k+1}}^2}{2\sum_{j\in I(k)}\big((\mathrm{Var}\,X_j)^{1/2}-(\mathrm{Var}\,\xi_j)^{1/2}\big)^2}\bigg\}\\ &\quad\le 2\exp\bigg\{-\frac{\delta^2a_{n_{k+1}}^2}{6\sum_{j\in I(k)}EX_j^2I\{|X_j|>\varepsilon_j(B_j/\log\log B_j)^{1/2}\}}\bigg\}.\end{aligned}\tag{1.20}$$

It follows from (1.19), (1.20) and the Borel–Cantelli lemma that

$$\lim_{k\to\infty}\max_{i\in I(k)}\frac{|\sum_{j=n_k+1}^{i}\big((\mathrm{Var}\,X_j)^{1/2}-(\mathrm{Var}\,\xi_j)^{1/2}\big)\eta_j|}{a_{n_{k+1}}}=0\quad\text{a.s.}$$

which implies (1.18) by the standard methods (cf. [8], page 181, or [13], page 158).

If (1.5′) holds, we define $\{\xi_n\}$ and $\{\tilde{\xi}_n\}$ by (1.15) with ε instead of ε_n. By Remark 2.1 of [12], $\{\eta_n\}$ can be constructed such that for some numerical constant Γ,

$$\limsup_{n\to\infty}\frac{|\sum_{i=0}^{n}\xi_i-\sum_{i=0}^{n}\eta_i(\mathrm{Var}\,\xi_i)^{1/2}|}{(B_n\log\log B_n)^{1/2}}\le\Gamma\varepsilon\quad\text{a.s.}\tag{1.16′}$$

And we also have (1.17) and (1.18). Then (1.11′) holds true. The proof of Theorem 1.2 is complete. □

2. Applications to the law of the iterated logarithm. Using theorems in Section 1, we can establish some results on the law of the iterated logarithm for the geometrically weighted random series.

We start with a preliminary proposition, the proof of which will be stated in the Appendix.

PROPOSITION 2.1. *Let* $\{Y_n;\ n\ge 0\}$ *be a sequence of independent normal random variables with* $EY_n=0$, $B_n=:\sum_{i=0}^{n}EY_i^2\to\infty$ $(n\to\infty)$ *and*

$\limsup_{n\to\infty} B_{2n}/B_n < \infty$. *Set*

$$\tau(\beta) = \sum_{n=0}^{\infty} \beta^{2n} EY_n^2, \qquad 0 < \beta < 1,$$

$$\tilde{\xi}(\beta) = \frac{\sum_{n=0}^{\infty} \beta^n Y_n}{(2\tau(\beta) \log\log \tau(\beta))^{1/2}}, \qquad 0 < \beta < 1.$$

Then:

(i) $$\mathscr{C}(\{\tilde{\xi}(\beta)\}) = [-1, 1] \quad \textit{a.s.;}$$

(ii) $$\lim_{\beta \nearrow 1} d(\tilde{\xi}(\beta), [-1, 1]) = 0 \quad \textit{a.s.},$$

where $\mathscr{C}(\{\tilde{\xi}(\beta)\})$ *denotes the cluster set (set of all limit points) of* $\tilde{\xi}(\beta)$ *as* β *tends to one and* $d(x, A) = \inf_{y\in A} |x - y|$.

From Corollary 1.1, Corollary 1.2 and Proposition 2.1 the following theorem follows immediately.

THEOREM 2.1. *Let* $\{X_n; n \geq 0\}$ *satisfy the conditions in Corollary 1.1 or Corollary 1.2. Set*

$$\tilde{\xi}(\beta) = \frac{\sqrt{1-\beta^2}}{\sqrt{2\log\log(1/(1-\beta^2))}} \sum_{n=0}^{\infty} \beta^n X_n, \qquad 0 < \beta < 1.$$

Then

$$\mathscr{C}(\{\tilde{\xi}(\beta)\}) = [-\sigma, \sigma] \quad \textit{a.s.},$$

$$\lim_{\beta \nearrow 1} d(\tilde{\xi}(\beta), [-\sigma, \sigma]) = 0 \quad \textit{a.s.}$$

By Theorem 1.2 and Proposition 2.1, we have the following theorem.

THEOREM 2.2. *Let* $\{X_n; n \geq 0\}$ *be a sequence of independent random variables with* $EX_n = 0$ *and* $EX_n^2 < \infty$ $(n \geq 0)$. *Set* $B_n = \sum_{k=0}^{n} EX_k^2$ *and* $\tau(\beta) = \sum_{n=0}^{\infty} \beta^{2n} EX_n^2$. *Suppose* $B_n \to \infty$ $(n \to \infty)$, $\limsup_{n\to\infty} B_{2n}/B_n < \infty$ *and for each* $\varepsilon > 0$ *there exists* $p \geq 2$ *such that*

$$\sum_{n=0}^{\infty} \frac{E|X_n|^p I\{|X_n| > \varepsilon(B_n/\log\log B_n)^{1/2}\}}{(B_n \log\log B_n)^{p/2}} < \infty.$$

Let

$$\tilde{\xi}(\beta) = \frac{\sum_{n=0}^{\infty} \beta^n X_n}{(2\tau(\beta) \log\log \tau(\beta))^{1/2}}, \qquad 0 < \beta < 1.$$

Then (i) *and* (ii) *in Proposition 2.1 hold true.*

In particular, we have the following Kolmogorov type law of the iterated logarithm.

COROLLARY 2.1. *Let* $\{X_n;\ n \geq 0\}$ *be a sequence of independent random variables with* $EX_n = 0$ *and* $EX_n^2 < \infty$ $(n \geq 0)$. *Suppose* $B_n = \sum_{i=0}^n EX_i^2 \to \infty$ $(n \to \infty)$ *and* $\limsup_{n\to\infty} B_{2n}/B_n < \infty$. *Let* $\tau(\beta)$ *and* $\tilde{\xi}(\beta)$ *be defined as in Theorem 2.2. If there exists a sequence of positive numbers* $\{k_n;\ n \geq 0\}$ *with* $k_n \to 0$ $(n \to \infty)$ *such that* $|X_n| \leq k_n(B_n/\log\log B_n)^{1/2}$, *then* (i) *and* (ii) *in Proposition 2.1 hold true.*

For the sequence of i.i.d. random variables with possible infinite variance, we have the following results on the law of the iterated logarithm corresponding to those of Feller [4] (see also [3]).

THEOREM 2.3. *Let* $\{X_n; n \geq 0\}$ *be a sequence of i.i.d. symmetric random variables. Suppose the function* $H(\lambda) = E(X_0^2 I\{|X_0| < \lambda\})$ $(\lambda \geq 0)$ *satisfies*

$$\limsup_{\lambda\to\infty} \frac{H(2\lambda)}{H(\lambda)} < \infty. \tag{2.1}$$

For any $n \geq 1$, *let* a_n *be the largest solution of the equation*

$$\lambda^2 = nH(\lambda)\log\log\lambda \tag{2.2}$$

satisfying $a_n \uparrow \infty$. *Set* $\tau(\beta) = \sum_{n=0}^\infty \beta^{2n} E(X_0^2 I\{|X_0| \leq a_n\})$ $(0 < \beta < 1)$ *and*

$$\tilde{\xi}(\beta) = \frac{\sum_{n=0}^\infty \beta^n X_n}{(2\tau(\beta)\log\log\tau(\beta))^{1/2}}, \qquad 0 < \beta < 1. \tag{2.3}$$

If

$$\int_0^\infty \frac{d\ H(\lambda)}{H(\lambda)\log\log\lambda} < \infty, \tag{2.4}$$

then (i) *and* (ii) *in Proposition 2.1 hold true.*

PROOF. Let $B_n = \sum_{k=0}^n E(X_0^2 I\{|X_0| \leq a_k\})$. From (2.1), it can be shown that

$$\limsup_{n\to\infty} a_{2n}/a_n < \infty. \tag{2.5}$$

It can be shown that (2.4) is equivalent to

$$\sum_{n=0}^\infty P(|X_0| \geq \varepsilon a_n) < \infty \quad \text{for some } \varepsilon > 0 \text{ (or equivalently for any } \varepsilon > 0).$$

Note that X_0 is symmetric. By Corollary 1.3 of [12] there exists a sequence of independent normal variables $\{Y_n;\ n \geq 0\}$ with $Y_n =_{\mathscr{D}} N(0, EX_0^2 I\{|X_0| \leq a_n\})$ such that

$$\sum_{i=0}^n X_i - \sum_{i=0}^n Y_i = o(a_n) \quad \text{a.s. } (n \to \infty). \tag{2.6}$$

It can be also proved that $a_n \approx (B_n \log\log B_n)^{1/2}$. By (2.5), it follows that

$$\limsup_{n\to\infty} \frac{B_{2n}}{B_n} < \infty. \tag{2.7}$$

Hence, by Theorem 1.1 we have

$$\lim_{\beta\nearrow 1} \frac{|\sum_{n=0}^{\infty}\beta^n X_n - \sum_{n=0}^{\infty}\beta^n Y_n|}{(2\tau(\beta)\log\log\tau(\beta))^{1/2}} = 0 \quad \text{a.s.} \tag{2.8}$$

And so, by Proposition 2.1 and (2.8) we have proved Theorem 2.3. □

APPENDIX

Proof of Proposition 2.1. To prove Proposition 2.1, we need a lemma as follows.

LEMMA A.1. *Let $\{u_n;\ n\ge 0\}$ be a nonincreasing sequence of positive numbers and $\{\zeta_n;\ n\ge 0\}$ be a sequence of real numbers. Then for each $n\ge 0$,*

$$\left|\sum_{i=0}^{n} u_i\zeta_i\right| \le u_0 \max_{i\le n}\left|\sum_{j=0}^{i}\zeta_j\right|.$$

The proof follows from the usual Abel transformation and so is omitted here.

To prove Proposition 2.1, we need only to prove that

$$\limsup_{\beta\nearrow 1} |\tilde{\xi}(\beta)| \le 1 \quad \text{a.s.} \tag{A.1}$$

and

$$\mathscr{C}(\{\tilde{\xi}(\beta)\}) \supseteq [-1,1] \quad \text{a.s.} \tag{A.2}$$

We prove (A.1) first. First, $\limsup_{n\to\infty} B_{2n}/B_n < \infty$ implies that $B_{kn}/B_n \le C_0 k^Q$, $B_n \le C_0 n^Q$ $(n\ge 0, k\ge 1)$ for some $C_0, Q > 0$. It is easy to show that $\tau(\beta) = \sum_{n=0}^{\infty}\beta^{2n}EY_n^2$ is a monotonous increasing function of β and $\tau(\beta)\to\infty$ $(\beta\nearrow 1)$. Let

$$\beta_k = \sup\{\beta; 0<\beta<1,\ \tau(\beta)\le \exp(k/\log\log k)\}, \qquad k=1,2,\dots. \tag{A.3}$$

Then $\beta_k \nearrow 1$. Note that $EY_n^2 \le B_n \le C_0 n^Q$. We have for any $0<\beta_0<1$ and $0\le\beta\le\beta_0$, $\sum_{n=0}^{\infty}\beta^{2n}EY_n^2 \le C_0\sum_{n=0}^{\infty}\beta_0^{2n}n^Q < \infty$. It follows that the series $\sum_{n=0}^{\infty}\beta^{2n}EY_n^2$ is uniformly convergent on $[0,\beta_0)$. And so, $\tau(\beta)$ is a continuous function on $[0,1)$. This implies $\tau(\beta_k) = \exp(k/\log\log k)$. Then

$$\tau(\beta_k)/\tau(\beta_{k-1}) \to 1, \qquad k\to\infty. \tag{A.4}$$

Note that for $\beta_{k-1}\le\beta\le\beta_k$ we have

$$|\tilde{\xi}(\beta)| \le \frac{\sup_{0\le\beta\le\beta_k}|\sum_{n=0}^{\infty}\beta^n Y_n|}{(2\tau(\beta_{k-1})\log\log\tau(\beta_{k-1}))^{1/2}}.$$

To prove (A.1), we need only to show

$$\limsup_{k\to\infty}\frac{\sup_{0\le\beta\le\beta_k}|\sum_{n=0}^{\infty}\beta^n Y_n|}{(2\tau(\beta_k)\log\log\tau(\beta_k))^{1/2}}\le 1\quad\text{a.s.}\tag{A.5}$$

From Lemma A.1, it follows that for any $0\le\beta\le\beta_k$,

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\beta^n Y_n\right| &= \left|\sum_{n=0}^{\infty}\left(\frac{\beta}{\beta_k}\right)^n\beta_k^n Y_n\right|\\ &\le\left(\frac{\beta}{\beta_k}\right)^0\sup_{0\le m\le\infty}\left|\sum_{n=0}^{m}\beta_k^n Y_n\right|\le\sup_{0\le m\le\infty}\left|\sum_{n=0}^{m}\beta_k^n Y_n\right|.\end{aligned}\tag{A.6}$$

This implies

$$\sup_{0\le\beta\le\beta_k}\left|\sum_{n=0}^{\infty}\beta^n Y_n\right|\le\sup_{0\le m\le\infty}\left|\sum_{n=0}^{m}\beta_k^n Y_n\right|.$$

Then

$$\begin{aligned}&P\left(\frac{\sup_{0\le\beta\le\beta_k}|\sum_{n=0}^{\infty}\beta^n Y_n|}{(2\tau(\beta_k)\log\log\tau(\beta_k))^{1/2}}\ge 1+\varepsilon\right)\\ &\quad\le P\left(\sup_{0\le m\le\infty}\left|\sum_{n=0}^{m}\beta_k^n Y_n\right|\ge(1+\varepsilon)(2\tau(\beta_k)\log\log\tau(\beta_k))^{1/2}\right)\\ &\quad\le 2P\left(\left|\sum_{n=0}^{\infty}\beta_k^n Y_n\right|\ge(1+\varepsilon)(2\tau(\beta_k)\log\log\tau(\beta_k))^{1/2}\right)\\ &\quad= 2P(|N(0,1)|\ge(1+\varepsilon)(2\log\log\tau(\beta_k))^{1/2})\\ &\quad\le 2\exp\bigl(-(1+\varepsilon)\log\log\tau(\beta_k)\bigr)=2\left(\frac{k}{\log\log k}\right)^{-(1+\varepsilon)},\end{aligned}\tag{A.7}$$

which together with the Borel–Cantelli lemma implies (A.5). We have proved (A.1).

Now, we show (A.2). Set $S_n=\sum_{k=0}^{n}Y_k$ $(n\ge1)$, $S_{-1}=0$. We have

$$\begin{aligned}\limsup_{n\to\infty}\frac{|S_n|}{\sqrt{2B_n\log\log B_n}}&\le\limsup_{n\to\infty}\frac{|W(B_n)|}{\sqrt{2B_n\log\log B_n}}\\ &\le\limsup_{t\to\infty}\frac{|W(t)|}{\sqrt{2t\log\log t}}\le 1\quad\text{a.s.}\end{aligned}\tag{A.8}$$

where $\{W(t);\ t\ge0\}$ is a standard Wiener process.

From (A.8) and Lemma 1.1, it follows that for any $N_0 \geq 1$,

$$\begin{aligned}
&\limsup_{\beta\nearrow 1} \frac{|\sum_{n=N_0N(\beta)+1}^{\infty} \beta^n Y_n|}{(2\tau(\beta)\log\log\tau(\beta))^{1/2}} \\
&\quad\leq e\limsup_{\beta\nearrow 1} \frac{|\sum_{n=N_0N(\beta)+1}^{\infty} \beta^n Y_n|}{(2B_{N(\beta)}\log\log B_{N(\beta)})^{1/2}} \qquad \text{(A.9)}\\
&\quad\leq \frac{eC_0}{2}\int_{N_0}^{\infty} \exp\biggl(-\frac{x}{2}\biggr)x^Q\,dx + eC_0\exp\biggl(-\frac{N_0}{2}\biggr)N_0^Q \to 0\ (N_0\to\infty).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\limsup_{\beta\nearrow 1} \frac{\sum_{n=N_0N(\beta)+1}^{\infty} \beta^{2n} EY_n^2}{\tau(\beta)} \\
&\quad\leq e\limsup_{\beta\nearrow 1} \frac{\sum_{n=N_0N(\beta)+1}^{\infty} \beta^{2n} EY_n^2}{B_{N(\beta)}} \qquad \text{(A.10)}\\
&\quad\leq eC_0\int_{N_0}^{\infty} \exp(-x)x^Q\,dx + eC_0\exp(-N_0)N_0^Q \to 0\ (N_0\to\infty).
\end{aligned}$$

To prove (A.2), we need only to show that for any $b \in (-1,1)$ and any $\delta > 0$ small enough, there exists a subsequent $\beta_k \nearrow 1$ such that

$$P(\tilde{\xi}(\beta_k) \in (b-2\delta, b+2\delta) \text{ i.o.}) = 1. \qquad \text{(A.11)}$$

Set $\tau_{N_0}(\beta) = \sum_{n=0}^{N_0N(\beta)} \beta^{2n} EY_n^2$. Choose β_k such that $1-\beta_k^2 = \exp(-k\log\log k)$. Then $N(\beta_{k-1})/N(\beta_k) \to 0$ $(k\to\infty)$. Define $\tau_{N_0}^*(\beta_k) = \sum_{n=N_0N(\beta_{k-1})+1}^{N_0N(\beta_k)} \beta_k^{2n} EY_n^2$. Noting (A.9) and (A.10), we need only to show that for N_0 large enough,

$$P\Biggl(\frac{\sum_{n=0}^{N_0N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}(\beta_k)\log\log\tau_{N_0}(\beta_k))^{1/2}} \in (b-\delta, b+\delta) \text{ i.o.}\Biggr) = 1. \qquad \text{(A.12)}$$

From $e^{-1}B_{N(\beta)} \leq \tau_{N_0}(\beta) \leq \tau(\beta) \leq CB_{N(\beta)}$, it follows that

$$\frac{\sum_{n=0}^{N_0N(\beta_{k-1})} \beta_k^{2n} EY_n^2}{\tau_{N_0}(\beta_k)} \leq e\frac{B_{N_0N(\beta_{k-1})}}{B_{N(\beta_k)}} \leq C_0 e\biggl(\frac{N_0N(\beta_{k-1})}{N(\beta_k)}\biggr)^Q \to 0, \qquad k\to\infty.$$

Then

$$\frac{\tau_{N_0}^*(\beta_k)}{\tau_{N_0}(\beta_k)} \to 1\ (k\to\infty),$$

$$\limsup_{\beta\nearrow 1} \frac{|\sum_{n=0}^{N_0N(\beta_{k-1})} \beta_k^n Y_n|}{(2\tau_{N_0}(\beta_k)\log\log\tau_{N_0}(\beta_k))^{1/2}} = 0 \quad \text{a.s.}$$

Hence, we need only to show that for N_0 large enough,

$$P\Biggl(\frac{\sum_{n=N_0N(\beta_{k-1})+1}^{N_0N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}^*(\beta_k)\log\log B_{N(\beta_k)})^{1/2}} \in (b-\delta, b+\delta) \text{ i.o.}\Biggr) = 1. \qquad \text{(A.13)}$$

Note the independence. By the Borel–Cantelli lemma, we need only to prove

$$\sum_{k=1}^{\infty} P\left(\frac{\sum_{n=N_0N(\beta_{k-1})+1}^{N_0N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}^*(\beta_k)\log\log B_{N(\beta_k)})^{1/2}} \in (b-\delta, b+\delta)\right) = \infty. \tag{A.14}$$

Now, it can be shown that for k large enough

$$\begin{aligned} &P\left(\frac{\sum_{n=N_0N(\beta_{k-1})+1}^{N_0N(\beta_k)} \beta_k^n Y_n}{(2\tau_{N_0}^*(\beta_k)\log\log B_{N(\beta_k)})^{1/2}} \in (b-\delta, b+\delta)\right) \\ &\quad = P\big(N(0,1) \in ((b-\delta)(2\log\log B_{N(\beta_k)})^{1/2}, (b+\delta)(2\log\log B_{N(\beta_k)})^{1/2})\big) \\ &\quad \geq \exp\big(-b^2 \log\log B_{N(\beta_k)}\big) \frac{1}{\sqrt{2\pi}} \int_{-\delta(2\log\log B_{N(\beta_k)})^{1/2}}^{\delta(2\log\log B_{N(\beta_k)})^{1/2}} e^{-x^2/2}\,dx \\ &\quad \geq \frac{1}{2}\exp\big(-b^2\log\log C_0(N(\beta_k))^Q\big) \\ &\quad \geq \frac{1}{2}\exp\big(-b^2(1+\varepsilon)\log k\big) = \frac{1}{2}k^{-b^2(1+\varepsilon)}, \end{aligned}$$

which implies (A.14) immediately, where ε satisfies $b^2(1+\varepsilon) < 1$. Hence we have proved (A.2). The proof of Proposition 2.1 is complete. □

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REFERENCES

[1] Bovier, A. and Picco, P. (1993). A law of iterated logarithm for random geometric series. *Ann. Probab.* 21 168–184.
[2] Chen, X. (1993). On the law of the iterated logarithm for independent Banach space valued random variables. *Ann. Probab.* 21 1991–2011.
[3] Chen, X. (1995). The Feller law of the iterated logarithm in Banach spaces. *Chinese Ann. Math. Ser. A* 16 251–258.
[4] Feller, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mechanics* 18 343–355.
[5] Horvath, L. (1985). Approximation for Abel sums of independent, identically distributed random variables. *Statist. Probab. Lett.* 3 221–226.
[6] Komlos, J., Major, P. and Tusnady, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrsch. Verw. Gebiete* 32 111–133.
[7] Komlos, J., Major, P. and Tusnady, G. (1976). An approximation of partial sums of independent RV's and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* 34 33–58.
[8] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces*. Springer, New York.
[9] Picco, P. and Vares, M. E. (1994). A law of iterated logarithm for geometrically weighted martingale difference sequence. *J. Theoret. Probab.* 7 375–415.
[10] Sakhanenko, A. I. (1985). Convergence rate in the invariance principle for nonidentically distributed variables with exponential moment. In *Advances in Probability Theory: Limit Theorems for Sums of Random Variables* (A. A. Borovkov, ed.) 2–73. Springer, New York.
[11] Shao, Q. M. (1993). Almost sure invariance principles for mixing sequences of random variables. *Stochastic Process. Appl.* 48 319–334.

[12] SHAO, Q. M. (1995). Strong approximation theorems for independent random variables and their applications. *J. Multivariate Anal.* 52 107–130.
[13] STOUT, F. (1974). *Almost Sure Convergence*. Academic Press, New York.
[14] ZHANG, L.-X. (1995). The best rates in strong invariance principle. *Science in China* 38A 428–434.

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