The Annals of Probability
1998, Vol. 26, No. 3, 1232–1250

EXACT LIMITING SHAPE FOR A SIMPLIFIED MODEL OF FIRST-PASSAGE PERCOLATION ON THE PLANE

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We derive the limiting shape for the following model of first-passage bond percolation on the two-dimensional integer lattice: the percolation is directed in the sense that admissible paths are nondecreasing in both coordinate directions. The passage times of horizontal bonds are Bernoulli distributed, while the passage times of vertical bonds are all equal to a deterministic constant. To analyze the percolation model, we couple it with a one-dimensional interacting particle system. This particle process has nonlocal dynamics in the sense that the movement of any given particle can be influenced by far-away particles. We prove a law of large numbers for a tagged particle in this process, and the shape result for the percolation is obtained as a corollary.

1. Introduction. Among the central challenges of percolation theory is the rigorous derivation of properties of the asymptotic shape of first-passage percolation. This is a model for the passage of fluid through a porous medium, introduced by Hammersley and Welsh [10]. Let us restrict the discussion to two-dimensional models. Imagine that the origin of the plane is a source of fluid that is allowed to flow along nearest-neighbor edges of the integer lattice $\mathbf{Z}^2$. Each edge e has a random nonnegative passage time $\tau(e)$ assigned to it, and the process $\{\tau(e)\}$ is i.i.d. The fluid takes time $\tau(e)$ to flow along edge e. In the beginning at time 0, only the origin is wet. At time $t > 0$, a site of $\mathbf{Z}^2$ is wet if it can be reached from the origin along a path whose passage times add up to at most t. Let $\widetilde{B}_t$ denote the set of wet sites at time t, and B_t a solid version of this set on $\mathbf{R}^2$, namely, the union of unit squares located on the sites of $\widetilde{B}_t$.

The basic result of first-passage percolation is the existence of a deterministic limiting set. Suppose

$$E\big[\big(\min\{\tau(e_1), \tau(e_2), \tau(e_3), \tau(e_4)\}\big)^2\big] < \infty$$

for a set $\{e_1, e_2, e_3, e_4\}$ of distinct edges. Then there is a nonrandom convex set B_0 with nonempty interior such that this holds: either B_0 is compact and for any $\varepsilon > 0$,

$$(1-\varepsilon)B_0 \subseteq t^{-1}B_t \subseteq (1+\varepsilon)B_0$$

holds for large enough t, almost surely, or $B_0 = \mathbf{R}^2$ and for any $\varepsilon > 0$,

$$\big\{(x, y)\colon |x| + |y| < \varepsilon^{-1}\big\} \subseteq t^{-1}B_t$$

holds for large enough t, a.s. This was first proved by in [5].

Received December 1996; revised September 1997.

AMS 1991 *subject classifications.* Primary 60K35; secondary 82B43, 82C22.

Key words and phrases. First-passage percolation, hydrodynamic limit, tagged particle, asymptotic shape.

Apart from obvious symmetry properties, not much exact is known about the limiting set B_0. A notable exception is the flat edge result of [6]. Let F denote the common distribution function of the passage times. Suppose the left endpoint $\lambda = \inf\{x: F(x) > 0\}$ of the distribution is positive. Then it is clear that B_0 is contained in the diamond $\{(x, y): |x| + |y| \leq 1/\lambda\}$. Let p_c be the critical probability of two-dimensional oriented percolation. The flat edge result states that if $F(\lambda) > p_c$, the intersection $B_0 \cap \{(x, y): |x| + |y| = 1/\lambda\}$ consists of line segments with positive length. This is brought about by the fast edges percolating to the boundary $|x| + |y| = 1/\lambda$ of the diamond, which explains the appearance of p_c in the statement.

For a general overview of first-passage percolation, the reader is referred to Kesten's lectures [13]. Recent important results focus on the speed of convergence in $t^{-1}B_t \to B_0$ and appear in [1], [2], [14], [18]. A limiting shape continues to exist under appropriate hypotheses when the i.i.d. condition of the passage times is weakened to ergodic stationarity. References [4], [9] and [11] contain results about stationary first passage percolation. For certain first-passage percolation models in continuous space it has been determined that the limiting shape is the sphere. Vahidi-Asl and Wierman [24] and Howard and Newman [12] have derived results of this kind.

In this paper we simplify the lattice percolation model to be able to completely describe the limiting set B_0. Three changes to the basic model are needed for our result: first, we only admit paths that are nondecreasing in both coordinates. This model is called directed first-passage percolation. Second, we let only horizontal passage times be random, while all vertical edges have a fixed, common deterministic passage time, and third, the random horizontal passage times are Bernoulli distributed.

This percolation model is amenable to our approach of coupling a growth model with an interacting particle system through a variational formula. The variational formula is preserved by a passage to a scaling limit, and it turns into an example of the Lax–Oleinik formula from the theory of viscosity solutions of Hamilton–Jacobi equations [3], [7], [17]. This variational formula, convex analysis and knowledge of the steady states of the particle system enable us to determine the limiting set B_0. Additionally, we get a law of large numbers for a tagged particle of the particle system under hydrodynamic Euler scaling.

2. The percolation result.

Consider the following first-passage percolation model on the lattice $\mathbb{Z}_+^2 = \{0, 1, 2, \ldots\}^2$: each edge e between nearest-neighbor sites of $\mathbb{Z}_+^2$ has a passage time $\tau(e)$ attached to it. If e is vertical, then $\tau(e) = \tau_0$, a positive constant. If e is horizontal, $\tau(e)$ is a Bernoulli random variable with

$$P(\tau(e) = \lambda) = p \quad \text{and} \quad P(\tau(e) = \kappa) = q = 1 - p,$$

where λ and κ are constants satisfying $\kappa > \lambda \geq 0$. All passage times are independent of each other.

The set $\widetilde{B}(t)$ is defined as the set of sites of $\mathbf{Z}_+^2$ that can be reached from the origin by a nondecreasing nearest-neighbor path with passage time at most t. A nondecreasing path is a sequence of sites $(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m)$ that satisfies $x_0 \le x_1 \le \cdots \le x_m$ and $y_0 \le y_1 \le \cdots \le y_m$. It is a nearest-neighbor path if (x_i, y_i) and (x_{i+1}, y_{i+1}) are nearest-neighbors for each i in the sense that $|x_{i+1} - x_i| + |y_{i+1} - y_i| = 1$. The passage time of the path is the sum

$$\sum_{i=0}^{m-1} \tau(e_i),$$

where e_i is the edge from (x_i, y_i) to (x_{i+1}, y_{i+1}).

Let $T(k, l)$ be the time t when site (k, l) first joins the set $\widetilde{B}(t)$. It is equal to the minimal passage time of nondecreasing paths from $(0, 0)$ to (k, l). For points $(x, y) \in \mathbf{R}_+^2$, the time constant is defined by

$$\mu(x, y) = \lim_{n\to\infty} \frac{1}{n} T([nx], [ny]).$$

The existence of the finite, deterministic limit $\mu(x, y)$ is under our assumptions a standard exercise in subadditive ergodic theory. The asymptotic shape is defined by

$$B_0 = \{(x, y) \in \mathbf{R}_+^2 \colon \mu(x, y) \le 1\}.$$

It is the limit of the random sets $\widetilde{B}(t)$ in the following sense: let

$$B(t) = \cup\big\{[k, k+1] \times [l, l+1] \colon (k, l) \in \widetilde{B}(t)\big\}$$

be a solidified version of $\widetilde{B}(t)$. Then, for any $\varepsilon > 0$,

$$(1-\varepsilon)B_0 \subseteq t^{-1}B(t) \subseteq (1+\varepsilon)B_0 \quad \text{for large enough } t, \text{ a.s.}$$

Our result is an explicit formula for the limit.

THEOREM 1. *The time constant is given by*

$$\mu(x, y) = \begin{cases} \lambda x + \tau_0 y, & \text{if } py > qx, \\ \lambda x + \tau_0 y + (\kappa - \lambda)\big(\sqrt{qx} - \sqrt{py}\big)^2, & \text{if } py \le qx. \end{cases}$$

The function $\mu(x, y)$ is C^1-smooth but not C^2 across the line $py = qx$. Thus the boundary of B_0 is C^1. See Figure 1 for an example.

Note that, trivially, $T(k, l) \ge \lambda k + \tau_0 l$. The two-case formula for $\mu(x, y)$ is explained by this observation: if $y > qx/p$ and n is large enough, then with high probability a path from $(0, 0)$ to $([nx], [ny])$ of minimal passage time $\lambda[nx] + \tau_0[ny]$ can be constructed simply by moving rightward along runs of λ-edges and taking a step up whenever the next horizontal edge has passage time κ. Since each run of λ-edges has expected length p/q, this strategy gives us a path with $[nx] + o(n)$ horizontal λ-edges before forcing us up to level $[ny]$.

In the case $y \le qx/p$, the lattice paths from $(0, 0)$ to $([nx], [ny])$ of minimal passage time $T([nx], [ny])$ concentrate asymptotically around the straight line

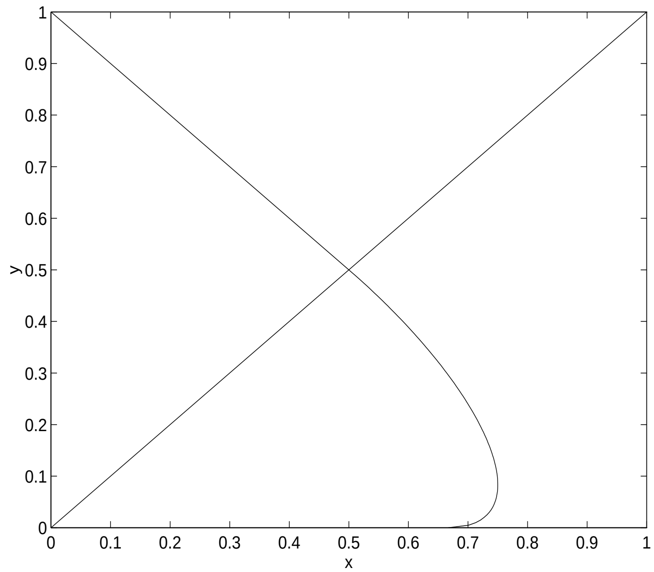

FIG. 1. *The boundary of* B_0 *for the case* $p = q = 1/2$, $\tau_0 = \lambda = 1$, *and* $\kappa = 2$. *Also shown is the line* $py = qx$.

segment from $(0, 0)$ to $([nx], [ny])$. This is an easy consequence of the subadditivity of $\mu(x, y)$: given $\delta > 0$, there is an $\varepsilon > 0$ such that

$$\mu(x_1, y_1) + \mu(x - x_1, y - y_1) \geq \mu(x, y) + \varepsilon$$

for all (x_1, y_1) contained in $[0, x] \times [0, y]$ but outside the δ-neighborhood U_δ of the line segment from $(0, 0)$ to (x, y). Thus the minimal passage time of lattice paths from $(0, 0)$ to $([nx], [ny])$ that are not contained in nU_δ is, for large n, with high probability at least $n(\mu(x, y) + \varepsilon/2)$.

3. The coupling. The paths of minimal passage time for the percolation model are those with a maximal number of fast horizontal edges. Thus in the proof we need only count horizontal edges with passage time λ. This will be achieved with the help of an interacting particle process on Z, constructed from nondecreasing paths on the planar lattice. For this purpose we set, for each nearest-neighbor edge e of the lattice $\mathrm{Z} \times \mathrm{Z}_+$, independently of all other edges,

$$\sigma(e) = \begin{cases} 0, & \text{if } e \text{ is vertical,} \\ 1, & \text{with probability } p \text{ if } e \text{ is horizontal,} \\ 0, & \text{with probability } q = 1 - p \text{ if } e \text{ is horizontal.} \end{cases}$$

An edge e with $\sigma(e) = 1$ will be called a *marked edge*. The right endpoint of a marked edge will be called a *marked site*. The counting of marked edges is done through these random variables: For $a < b \in \mathbb{Z}$ and $s \le t \in \mathbb{Z}_+$, set

$$L((a,s),(b,t)) = \max\bigl\{k\colon \text{there is a nondecreasing path that uses } k \text{ marked edges and connects site } (a,s) \text{ to site } (b,t)\bigr\}. \tag{3.1}$$

An inverse of $L((a,s),(b,t))$ is defined for $a \in \mathbb{Z}$, $s < t \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$ by

$$\Gamma((a,s),k,t) = \min\bigl\{l \in \mathbb{Z}_+\colon L((a,s+1),(a+l,t)) \ge k\bigr\}. \tag{3.2}$$

The particle process is a totally asymmetric exclusion process on $\mathbb{Z}$, running in discrete time, where particles jump only to the left, never passing each other. The state of the process is a configuration $(z_i)_{i\in\mathbb{Z}}$ of labeled particles, satisfying $z_i \in \mathbb{Z}$ and

$$z_{i-1} \le z_i - 1 \quad \text{for all } i. \tag{3.3}$$

Given an initial configuration $z = (z_i)_{i\in\mathbb{Z}}$ of this type, the configuration $z(t) = (z_k(t))_{k\in\mathbb{Z}}$ at time $t = 1, 2, 3, \dots$ is defined by

$$z_k(t) = \inf_{i\le k}\bigl\{z_i + \Gamma((z_i,0),k-i,t)\bigr\}. \tag{3.4}$$

The first task is to identify a subset of initial configurations for which $z(t)$ is well defined. The right choice of state space turns out to be

$$Z = \Bigl\{z \in \mathbb{Z}^{\mathbb{Z}}\colon z \text{ satisfies (3.3) and } \liminf_{i\to-\infty} |i|^{-1} z_i > -1/p\Bigr\}, \tag{3.5}$$

in the sense that if $z \in Z$, then $z(t) \in Z$ a.s. Once it has been established that $z_k(t) > -\infty$ for all k, it is easy to see that rule (3.4) preserves the ordering and exclusion property (3.3).

Suppose the configurations $z(t)$, $t \in \mathbb{N}$, have been computed by equation (3.4) from an initial configuration $z \in Z$, and all $z(t)$ are well-defined elements of Z. Since the location $z_k(t)$ is a finite integer, the infimum in (3.4) is achieved at some i, and consequently, for some i, there is an increasing path with $k-i$ marked edges that connects $(z_i, 1)$ to $(z_k(t), t)$. By splitting and combining such paths between times 1 and s and between times $s+1$ and t, one can show that a semigroup property holds: for any $0 < s < t \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$z_k(t) = \inf_{i\le k}\bigl\{z_i(s) + \Gamma((z_i(s),s),k-i,t)\bigr\} \quad \text{a.s.} \tag{3.6}$$

This implies that $z(\cdot)$ is a time-homogeneous Markov chain on the state space Z. These existence questions will be settled in Section 5.

The description of the dynamics of the z-process is not very illuminating. There is no local rule that gives the new location $z_k(1)$ depending only on the initial configuration around z_k before the jump. However, if the new location $z_{k-1}(1)$ of the left neighbor is known, then $z_k(1)$ can be computed easily: let $(b_{k-1}(1), 1)$ be the next marked site to the right of $(z_{k-1}(1), 1)$. Then

$$z_k(1) = \min\bigl\{z_k, b_{k-1}(1)\bigr\}. \tag{3.7}$$

In other words, z_k either remains in its original position or jumps to a marked site without violating (3.3), and it chooses the leftmost location admissible under this prescription. This is fairly easy to deduce from (3.4). In Section 6 this observation is the basis for describing the steady-state behavior of the z-process.

The interparticle distance process $\eta(t) = (\eta_i(t))_{i\in\mathrm{Z}}$ is defined by

$$\eta_i(t) = z_{i+1}(t) - z_i(t) \in \{1, 2, 3, \ldots\}, \qquad i \in \mathrm{Z},\ t \in \mathrm{Z}_+. \tag{3.8}$$

The state space for the η-process, corresponding to the space Z above, is

$$Y = \left\{\eta \in \mathrm{N}^{\mathrm{Z}}\colon \liminf_{n\to\infty} \frac{1}{n}\sum_{i=-n}^{0} \eta_i < \frac{1}{p}\right\}. \tag{3.9}$$

A shifted geometric distribution $\alpha = (\alpha_n)_{n\in\mathrm{N}}$ with expectation $u \geq 1$ is defined on N by

$$\alpha_n = u^{-1}(1-u^{-1})^{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{3.10}$$

The following fact is instrumental for our explicit calculations.

PROPOSITION 1. *Suppose the process $z(\cdot)$ is defined by* (3.4) *from an initial configuration $z \in Z$. Then the interparticle distance process $\eta(\cdot)$ defined by* (3.8) *is a Markov chain on the state space Y. It has a one-parameter family of invariant distributions parametrized by $u \in [1, 1/p)$, specified by letting $(\eta_i)_{i\in\mathrm{Z}}$ be i.i.d. with common distribution α so that $E[\eta_i] = u$.*

The strategy of the proof of Theorem 1 can now be outlined. To obtain the time constant $\mu(x, y)$ we seek to identify the limit

$$\Psi(x, y) = \lim_{n\to\infty} \frac{1}{n} L\big((0,0), ([nx], [ny])\big) \quad \text{a.s., for all } (x, y) \in \mathrm{R}_+. \tag{3.11}$$

The definition (3.2) of the random variables $\Gamma((a, s), k, t)$ and elementary properties of Ψ enable us to transform $\Psi(x, y)$ (bijectively) into a single variable function $g(x)$ that satisfies

$$tg\left(\frac{x}{t}\right) = \lim_{n\to\infty} \frac{1}{n}\Gamma\big((i_n, 0), [nx], [nt]\big) \quad \text{in probability,} \tag{3.12}$$

for any sequence $\{i_n\}$ of sites. The task is now to identify g. To this end we study the evolution of particle z_0 in the z-process. With $z_0 = 0$ initially, and the η-process permanently in u-equilibrium, we can calculate explicitly a function $f(u)$ such that

$$\lim_{n\to\infty} \frac{1}{n} z_0(nt) = -tf(u) \quad \text{in probability,}$$

and obtain

$$f(u) = (1-up)^{-1} pu(u-1), \qquad 1 \leq u < 1/p. \tag{3.13}$$

In the scaling limit (3.4) turns into a convex duality relation between g and f, which leads us from (3.13) to an expression for g:

$$g(z) = \begin{cases} z, & \text{for } 0 \le z \le p/q, \\ p^{-1}\left(\sqrt{1+z} - \sqrt{q}\right)^2, & \text{for } z > p/q. \end{cases} \tag{3.14}$$

From g we can deduce Ψ, as will be done in (7.17) at the very end of the paper. A comparison of the definitions of $\sigma(e)$ and $\tau(e)$ shows that

$$\mu(x, y) = \lambda\Psi(x, y) + \kappa(x - \Psi(x, y)) + \tau_0 y, \tag{3.15}$$

and consequently the explicit time constant $\mu(x, y)$ of Theorem 1 will follow from knowing Ψ.

4. The tagged particle result. In addition we prove a more general law of large numbers for a tagged particle in the z-process. Let $v_0(x)$ be an increasing function on $\mathbf{R}$. Suppose we have a sequence $z^n = (z_i^n)_{i \in \mathbf{Z}}$ of possibly random initial configurations in Z that satisfy

$$\lim_{n\to\infty} \frac{1}{n} z^n_{[ny]} = v_0(y) \quad \text{in probability, for all } y \in \mathbf{R}. \tag{4.1}$$

Furthermore, make the following uniformity assumption:

(4.2) For any $\varepsilon > 0$, there exist $B > 0$ and $\delta > 0$ such that $\inf_n P\{z_i^n \ge i(1-\delta)/p \text{ for } i \le -Bn\} \ge 1 - \varepsilon$.

From the exclusion property (3.3) and assumption (4.2) it follows that a function v_0 for which (4.1) can be valid must satisfy

$$v_0'(x) \ge 1 \quad \text{whenever the derivative exists, and } \liminf_{y\to-\infty} \frac{v_0(y)}{|y|} > -\frac{1}{p}. \tag{4.3}$$

Let $z^n(t)$ be the process defined by (3.4) from initial configuration z^n. The probability space of the process is constructed so that the initial configuration z^n and the marks on the edges of the space-time lattice $\mathbf{Z} \times \mathbf{Z}_+$ that determine the variables $\Gamma((a, s), k, t)$ are independent. We write P for the measure on this product space that gives probabilities for events concerning both the process and the nondecreasing paths on the lattice.

THEOREM 2. *Under assumptions* (4.1) *and* (4.2), *the deterministic limit*

$$v(x, t) = \lim_{n\to\infty} \frac{1}{n} z^n_{[nx]}([nt])$$

exists in probability, for all $x \in \mathbf{R}$ *and* $t > 0$. *The limit* $v(x, t)$ *is macroscopically given by*

$$v(x, t) = \inf_{y \le x} \left\{ v_0(y) + tg\left(\frac{x-y}{t}\right) \right\}, \qquad x \in \mathbf{R},\ t > 0, \tag{4.4}$$

where g *is the function of (3.14).*

To see why this theorem should follow, rewrite (3.4) with the scaling included,

$$\frac{1}{n} z^n_{[nx]}([nt]) = \inf_{y\le x}\left\{\frac{1}{n} z^n_{[ny]} + \frac{1}{n}\Gamma\big((z^n_{[ny]}, 0), [nx]-[ny], [nt]\big)\right\},$$

and note that by (3.12) and (4.1) the random variable inside the braces converges to the quantity inside the braces in (4.4). All that is needed are estimates that justify passing the limit through the infimum. For Theorem 1 we take the special case $x = 0$, $t = 1$, $v_0(y) = uy$, $v(0,t) = -f(u)$, and then (4.4) becomes the convex duality of the known function f and the unknown g.

Laws of large numbers for tagged particles in asymmetric particle systems have been previously proved in [16], [19] and [21], and a review of tagged particle results can be found in [8]. Theorem 2 implies that the dynamics of $z^n_{[nx]}([nt])$ obeys a differential equation in the scaling limit: extend f to all of R by setting $f(u) = 0$ for $u < 1$ and $f(u) = \infty$ for $u \ge 1/p$. By Theorem 2.1 of [3], $v(x,t)$ is the unique viscosity solution of the Hamilton–Jacobi equation

$$\begin{aligned} v_t + f(v_x) &= 0,\\ v(x,0) &= v_0(x). \end{aligned}$$

From this follows that the derivative $u(x,t) = v_x(x,t)$ satisfies a conservation law with flux function $f(u)$. With this observation, Theorem 2 can be turned into a hydrodynamic scaling limit for the empirical profile of the η-process. Results of this type can be found in [22, 23], and we leave the details to the reader.

Before beginning the proofs, we indicate the necessity of the uniformity assumption (4.2) with a simple example.

EXAMPLE 1. Pick and fix $u \in [1, 1/p)$, $t > 0$, and a sequence $c_j \nearrow \infty$. By (3.12) it is possible to pick a sequence $n_j \nearrow \infty$ such that

$$\sum_j P\big\{n_j^{-1}\Gamma((0,0), [c_j n_j], [n_j t]) > tg(c_j/t) + 1\big\} < \infty. \tag{4.5}$$

Define a sequence $\{z^n\}$ of deterministic initial configurations as follows: For $n \notin \{n_j\}$, set $z^n_i = [ui]$. For $n = n_j$, set

$$z^{n_j}_{[-c_j n_j]+k} = \begin{cases} \min\{[-c_j n_j(1-\delta_j)/p] + [2uk/p],\ [ku - c_j n_j u]\}, & k \ge 0,\\ [-c_j n_j(1-\delta_j)/p] + k, & k < 0,\end{cases}$$

where we take $\delta_j = c_j^{-1} \searrow 0$ and consider j large enough so that $u < (1-\delta_j)/p$. Each z^n lies in Z and assumption (4.1) is satisfied with $v_0(y) = uy$. But (4.2) fails because for any $B > 0$ and $\delta > 0$, eventually $c_j > B$ and $\delta_j < \delta$ so

$$z^{n_j}_{[-c_j n_j]} = [-c_j n_j(1-\delta_j)/p] < [-c_j n_j](1-\delta)/p$$

for arbitrarily large n_j. Furthermore, the conclusion of Theorem 2 fails also: by (3.4), (3.14) and (4.5), for large enough n_j,

$$\begin{aligned} n_j^{-1} z_0^{n_j}([n_j t]) &\le n_j^{-1} z^{n_j}_{[-c_j n_j]} + n_j^{-1} \Gamma((z^{n_j}_{[-c_j n_j]}, 0), [c_j n_j], [n_j t]) \\ &\le -2tp^{-1}\sqrt{q(1+c_j/t)} + \text{Constant}, \end{aligned}$$

which tends to $-\infty$, while Theorem 2 asserts that $n^{-1} z_0^n([nt]) \to v(0,t) = -tf(u) > -\infty$.

5. The existence of the process.

We begin the proofs with a simple moment bound for $L((a,s),(b,t))$.

LEMMA 5.1. *For $\delta > 0$ there is a constant $C(\delta) > 0$ such that for all $n, t \in \mathbb{N}$,*

$$P\{L((0,1),(n,t)) \ge n(p+\delta)\} \le t\, n^{t-1} \exp[-C(\delta)n]. \tag{5.1}$$

PROOF. A nondecreasing path from $(0,1)$ to (n,t) has $n+t-1$ edges, of which n must be horizontal. The $\sigma(e)$-variables over the edges of a path sum to a Bin(n,p)-distributed random variable, and consequently

$$\begin{aligned} P\{L((0,1),(n,t)) \ge n(p+\delta)\} &\le \binom{n+t-1}{n} \text{Prob}\{\text{Bin}(n,p) \ge n(p+\delta)\} \\ &\le \frac{(n+1)\cdots(n+t-1)}{(t-1)!} \exp[-C(\delta)n], \end{aligned}$$

from which the conclusion follows. □

With this bound we can prove Lemma 5.2 about the well-definedness of the dynamics $z(\cdot)$.

LEMMA 5.2. *Suppose $z = (z_i)_{i\in\mathbb{Z}}$ satisfies (3.3) and*

$$\liminf_{i\to-\infty} \frac{z_i}{|i|} \ge -c_0$$

for some $c_0 \in [1, 1/p)$. Pick $t \in \mathbb{N}$ and define $(z_k(t))_{k\in\mathbb{Z}}$ by (3.4). Then

$$\liminf_{k\to-\infty} \frac{z_k(t)}{|k|} \ge -c_0 \quad \text{a.s.} \tag{5.2}$$

In particular, $z(t) \in Z$ a.s. whenever $z \in Z$, so the process $z(\cdot)$ is well defined on the state space Z by (3.4).

PROOF. Let $\varepsilon > 0$ be small enough so that

$$c_0(1+\varepsilon) < 1/p. \tag{5.3}$$

Pick $k_0 < 0$ so that

$$z_i \ge c_0(1+\varepsilon)i \quad \text{whenever } i \le k_0. \tag{5.4}$$

Fix $k \le k_0$ for the moment. To estimate the probability that

$$z_k(t) \le c_0(1+2\varepsilon)k,$$

we need to estimate the probability that, for some $i < k$, there is a nondecreasing path from site $(z_i, 1)$ to site $([c_0(1+2\varepsilon)k], t)$ with at least $k-i$ marked edges. This cannot happen unless

$$z_i \le c_0(1+2\varepsilon)k - (k-i),$$

which by (5.4) implies that

$$i \le c_1 k \quad \text{with } c_1 = \frac{(1+2\varepsilon)c_0 - 1}{(1+\varepsilon)c_0 - 1}. \tag{5.5}$$

If $i \le c_1 k$, then with $n = [c_0(1+2\varepsilon)k] - [c_0(1+\varepsilon)i]$ and $\delta = c_0^{-1}(1+\varepsilon)^{-1} - p > 0$, (5.1) implies that for some constants $a_1, a_2 > 0$,

$$P\{L((z_i, 1), ([c_0(1+2\varepsilon)k], t) \ge k-i\} \le a_1 \exp\bigl[-a_2\bigl((1+\varepsilon)|i| - (1+2\varepsilon)|k|\bigr)\bigr].$$

Thus, summing over $i \le c_1 k$, for two further constants $a_3, a_4 > 0$,

$$P\{z_k(t) \le c_0(1+2\varepsilon)k\} \le a_3 \exp[-a_4|k|].$$

This is valid for all $k \le k_0$; thus by Borel–Cantelli there is a.s. a (random) k_1 such that $z_k(t) \ge c_0(1+2\varepsilon)k$ for $k \le k_1$. Since ε can be made arbitrarily small, this proves the lemma. $\square$

Further arguments of the kind used in the proof of Lemma 5.2 show that the dynamics preserves asymptotic slope in the sense that if

$$\lim_{|k|\to\infty} \left|\frac{z_k(t)}{k}\right| = c_0 \quad \text{a.s.} \tag{5.6}$$

holds at time $t = 0$ for some $c_0 \in [1, 1/p)$, then it continues to hold at all successive times t. Next we indicate why the semigroup property holds.

LEMMA 5.3. *Suppose the initial configuration $z \in Z$ and let $z(s)$ and $z(t)$ be defined by* (3.4) *for two further times $s < t \in \mathrm{N}$. Then*

$$z_k(t) = \inf_{j \le k}\{z_j(s) + \Gamma((z_j(s), s), k-j, t)\} \quad \text{a.s.} \tag{5.7}$$

for all $k \in \mathrm{Z}$. In particular, $z(\cdot)$ is a time-homogeneous Markov chain whose transition probability is defined by (3.4) *with $t = 1$.*

PROOF. Fix a realization of the marked edges such that $z_k(t) \in \mathrm{Z}$ for all k, as can be done by the Lemma 5.2. It follows that $z_k(s) \in \mathrm{Z}$ for all k too, because $\Gamma((a,0), t, k)$ is decreasing in t. Since the infimum in (3.4) is then attained, we can find integers $i(j)$ such that $z_j(s) = z_{i(j)} + \Gamma((z_{i(j)}, 0), j - i(j), s)$ for all j. Then the right-hand side of (5.7) becomes

$$\inf_{j \le k}\bigl\{z_{i(j)} + \Gamma((z_{i(j)}, 0), j - i(j), s) + \Gamma((z_j(s), s), k-j, t)\bigr\}.$$

This is greater than or equal to $z_k(t)$ because of (3.4) and because of the inequality

$$\Gamma((z_{i(j)},0),\, j-i(j),\, s)+\Gamma((z_j(s),s),\, k-j,\, t)\geq \Gamma((z_{i(j)},0),\, k-i(j),\, t),$$

which is true because the paths from $(z_{i(j)},1)$ to $(z_j(s),s)$ and from $(z_j(s),s+1)$ to $(z_k(t),t)$ together form one possible path of $k-i(j)$ marked edges starting from $(z_{i(j)},1)$.

For the converse, pick i_0 so that $z_k(t)=z_{i_0}+\Gamma((z_{i_0},0),k-i_0,t)$. Let (l,s) be the last site on the path from $(z_{i_0},1)$ to $(z_k(t),t)$ at or below time s. Split the path into two pieces: a path with $j-i_0$ marked edges up to site (l,s), and the remaining path with $k-j$ marked edges from $(l,s+1)$ to $(z_k(t),t)$. This determines the number $j\in\{i,i+1,\dots,k\}$. Then $z_j(s)\leq l$, and since j is a potential minimizer in (5.7), it is seen that the right-hand side of (5.7) is less than or equal to $z_k(t)$. □

6. The steady state. In this section we prove Proposition 1 about the interparticle distance process $\eta(\cdot)$ defined by (3.8). Adopt the following conventions: given an initial configuration $z\in Z$, let z' denote the configuration obtained by an application of formula (3.4) over a single time step, that is, $z'=z(1)$. To calculate z' we need a realization of marked sites on the horizontal row $\{(i,1)\colon i\in \mathbf{Z}\}$ of the lattice. Let this be denoted by $\omega=(\omega_i)_{i\in\mathbf{Z}}$ with i.i.d. Bernoulli distribution $\nu(d\omega)$ so that

$$\nu[\omega_i=1]=p=1-\nu[\omega_i=0].$$

For $a\in\mathbf{Z}$, $k\in\mathbf{Z}_+$, and a realization of ω, set

$$\Gamma(a,k,\omega)=\min\left\{l\in\mathbf{Z}_+\colon \sum_{i=a+1}^{a+l}\omega_i\geq k\right\}.$$

This of course corresponds to $\Gamma((a,0),k,1)$ defined in (3.2). The sum above starts at $a+1$ instead of a because marked sites are the *right* endpoints of marked edges. Given $z\in Z$ and ω, let $x_k=x_k(z,\omega)$ be the amount by which particle z_k jumped, or

$$x_k(z,\omega)=z_k-z'_k=z_k-\inf_{i\leq k}\{z_i+\Gamma(z_i,k-i,\omega)\}.$$

Similarly let η and η' denote the initial and later interparticle distances. We have the identity $\eta'_k=\eta_k+x_k-x_{k+1}$, or

$$\eta'=\eta+x-\theta x, \tag{6.1}$$

if we write $x=(x_k)_{k\in\mathbf{Z}}$ for the configuration of x_k's and θ for the shift operator $(\theta x)_k=x_{k+1}$.

Given η, the distribution of η' can be determined from (6.1) once the location of z_0 is chosen. One can check that even if z_0 is chosen random and dependent on η, this choice does not influence the distribution of η' due to the translation-invariance of the marked sites ω. Thus (6.1) in fact defines a transition probability $P(\eta, d\eta')$ for the interparticle distances. From the Markov property of the larger process $z(t)$ it then follows that the process $\eta(t)$, defined by (3.8) in terms of $z(t)$, is itself an autonomous Markov chain with transition probability $P(\eta, d\eta')$. This proves the first part of Proposition 1.

Next we turn to the invariance claim of Proposition 1. Let $A(d\eta) = \bigotimes_{i\in \mathbf{Z}} \alpha(d\eta_i)$ denote the i.i.d. distribution on Y with marginal α defined by (3.10). Assume that η is A-distributed. The first step is to find the distribution of the joint process $(\eta_i, x_i)_{i\in \mathbf{Z}}$.

In terms of our present notation, we can rewrite rule (3.7) as follows:

$$z'_{k+1} = \min\bigl[z_{k+1},\ \min\{i > z'_k\colon \omega_i = 1\}\bigr]. \tag{6.2}$$

From this it follows that, given $(\eta_i)_{i\in \mathbf{Z}}$ and $(x_i)_{i\le k}$, the probability distribution of x_{k+1} depends only on (η_k, x_k). Let us denote this distribution by $P^{\eta_k}_{x_k, x_{k+1}}$. It can be derived from the distribution of ω, and one gets, for $m \in \mathbf{N}$ and $x, y \in \mathbf{Z}_+$,

$$P^m_{x,y} = \begin{cases} q^{m+x-1}, & \text{if } y = 0,\\ pq^{m+x-y-1}, & \text{if } 1 \le y \le m+x-1. \end{cases} \tag{6.3}$$

Since the η_i are i.i.d. α-distributed, it follows now that the process $(\eta_i, x_i)_{i\in \mathbf{Z}}$ is a Markov chain on the state space $\mathbf{N}\times \mathbf{Z}_+$, with transition probability matrix Q given by

$$Q((m,x),(n,y)) = \alpha_n P^m_{x,y}. \tag{6.4}$$

One checks that Q is irreducible, aperiodic, and has invariant distribution $\alpha\otimes\beta$ where the distribution β on $\mathbf{Z}_+$ is defined by

$$\beta_x = \begin{cases} (1-up)q^{-1}, & x = 0,\\ p(1-up)q^{-1}r^x, & x = 1,2,3,\dots \end{cases} \tag{6.5}$$

with $r = (u-1)(uq)^{-1} \in [0,1)$. (Recall that the range of the parameter u is $u \in [1, 1/p)$, so the formula for β does make sense.) Finally we observe that the Markov chain $(\eta_i, x_i)_{i\in \mathbf{Z}}$ must be in equilibrium, because the distribution of any fixed pair (η_i, x_i) is the limit distribution of a chain that started in the infinite past. Thus we conclude that the joint process $(\eta_i, x_i)_{i\in \mathbf{Z}}$ is the stationary Markov chain with transition Q and marginal distribution $\alpha\otimes\beta$.

The proof of Proposition 1 is completed by Lemma 6.1.

LEMMA 6.1. *Suppose $(\eta_i, x_i)_{i\in \mathbf{Z}}$ is the stationary Markov chain with transition Q and marginal distribution $\alpha\otimes\beta$. Then $\eta' = \eta + x - \theta x$ is again A-distributed.*

PROOF. The proof follows from one calculation and an induction argument. Write Q for the probability measure of the stationary Markov chain. First check that

$$\begin{aligned} Q\{\eta_i' = m,\ \eta_{i+1} = n,\ x_{i+1} = y\} &= Q\{\eta_i + x_i = m + y,\ \eta_{i+1} = n,\ x_{i+1} = y\} \\ &= \sum_{k=1}^{m+y} \alpha_k \beta_{m+y-k} \alpha_n P^k_{m+y-k,\,y} \\ &= \alpha_m \alpha_n \beta_y. \end{aligned}$$

Now use the Markov property and the above calculation:

$$\begin{aligned} &Q\{\eta_i' = m_i,\ \eta_{i+1}' = m_{i+1}, \dots, \eta_j' = m_j\} \\ &\quad = E^Q\big[I_{\{\eta_i' = m_i\}} Q\{\eta_{i+1}' = m_{i+1}, \dots, \eta_j' = m_j \mid \eta_{i+1}, x_{i+1}\}\big] \\ &\quad = \alpha_{m_i} \sum_{n,\,y} \alpha_n \beta_y Q\{\eta_{i+1}' = m_{i+1}, \dots, \eta_j' = m_j \mid (\eta_{i+1}, x_{i+1}) = (n, y)\} \\ &\quad = \alpha_{m_i} Q\{\eta_{i+1}' = m_{i+1}, \dots, \eta_j' = m_j\}. \end{aligned}$$

This induction step completes the proof of the lemma. □

7. Proofs of the theorems. We begin by discussing the limit in (3.11). For $a < b < c \in \mathbb{Z}$ and $r < s < t \in \mathbb{Z}_+$, we have superadditivity

$$L((a,r),(b,s)) + L((b,s),(c,t)) \le L((a,r),(c,t))$$

and the deterministic bound

$$0 \le L((a,s),(b,t)) \le b - a. \tag{7.1}$$

Furthermore, since edges are marked in an i.i.d. fashion, Kingman's theorem [15] applies to the subadditive process $X_{m,n} = -L((mx,my),(nx,ny))$, $0 \le m < n$, for a fixed pair (x, y) of positive integers, and gives a finite, deterministic limit

$$\Psi(x,y) = -\lim_{n\to\infty} \frac{1}{n} X_{0,n} \quad \text{a.s.}$$

Through homogeneity of $\Psi(x, y)$ this limit is also valid for rational $(x, y) \in \mathbb{Q}_+^2$, and through simple approximations one arrives at the existence of the deterministic limit

$$\Psi(x,y) = \lim_{n\to\infty} \frac{1}{n} L((0,0),([nx],[ny])) \quad \text{a.s.} \tag{7.2}$$

for all $x, y \ge 0$. It is trivial to observe that $\Psi(x, 0) = px$ and $\Psi(0, y) = 0$.

Concurrently with the steps that establish the limit, one also checks that $\Psi(x, y)$ is superadditive and homogeneous. From this follows that $\Psi(x, y)$ is concave and consequently continuous on $(0,\infty)^2$. Inequality (7.1) and the strong law of large numbers applied to a single row of edges gives the bounds $px \le \Psi(x, y) \le x$, and hence Ψ extends continuously to $[0,\infty) \times (0,\infty)$. Let

$$h(x) = \Psi(x, 1), \qquad x \ge 0.$$

By homogeneity, $\Psi(x, y) = yh(x/y)$ for $x \geq 0$, $y > 0$. Since h is concave and satisfies $px \leq h(x) \leq x$, it follows that h must be continuous and strictly increasing. Consequently h has a continuous, convex, strictly increasing inverse function $g(x) = h^{-1}(x)$ that satisfies

$$x \leq g(x) \leq x/p \quad \text{for } x \geq 0. \tag{7.3}$$

From the convergence (7.2) and the continuity and monotonicity of g follows that

$$\lim_{n\to\infty} \frac{1}{n}\Gamma\big((i_n, 0), [nx], [nt]\big) = tg\left(\frac{x}{t}\right) \quad \text{in probability} \tag{7.4}$$

for any $x \geq 0$, $t > 0$, and any sequence $i_n \in \mathbf{Z}$. Here is the argument for one half of (7.4): the base point i_n has no influence on the distribution, so we might as well take it equal to 0. Definition (3.2) implies the equivalence

$$\Gamma((0,0), k, \tau) > l \iff L((0,1),(l,\tau)) < k$$

for integral k, l and τ. Pick $x, t > 0$ [the case $x = 0$ is trivial in (7.4)]. Let $\varepsilon > 0$, and set $u = tg(x/t) + \varepsilon$. Pick $\delta > 0$ small enough and $s \in (0, t)$ close enough to t so that $sg((x+\delta)/s) < u$. Then

$$\begin{aligned} P\big\{\Gamma\big((0,0), [nx], [nt]\big) > [nu]\big\} &= P\big\{L\big((0,1),([nu],[nt])\big) < [nx]\big\} \\ &\leq P\big\{L\big((0,0),([nu],[ns])\big) < [nx]\big\}, \end{aligned}$$

where the last inequality is valid for n large enough to have $[ns] \leq [nt] - 1$. However, this last probability tends to 0 by (7.2) because $sg((x+\delta)/s) < u$ is equivalent to $x < \Psi(u, s) - \delta$. The other half of (7.4) is proved similarly.

As indicated in the second paragraph after the statement of Theorem 1, for $y > qx/p$ it is elementary to deduce the limiting value $\Psi(x, y) = x$. Construct a nondecreasing path from $(0,0)$ to $([nx],[ny])$ according to this rule: if the next horizontal edge is marked, move right, otherwise move up. Once either level $[nx]$ or $[ny]$ is reached, take the direct route to $([nx],[ny])$. A run of marked edges has on the average p/q edges, while $L((0,0),([nx],[ny]))$ is deterministically bounded by $[nx]$, so for large n, the above strategy gives a path of $[nx] + o(n)$ marked edges with high probability.

This implies that $g(x) = x$ for $x \leq p/q$. To calculate $g(x)$ for the remaining values of x we turn to study the particle system $z(\cdot)$.

LEMMA 7.1. *Let $z^n = (z_i^n)_{i\in\mathbf{Z}}$ be a sequence of random initial configurations in $\mathbf{Z}$ satisfying condition* (4.2)*, and let $z^n(t)$ be the process defined by* (3.4) *from initial configuration z^n. Fix $x \in \mathbf{R}$ and $t > 0$. For $r < x$ let*

$$\zeta^{n,r} = \min_{[nr]<i\leq[nx]} \big\{z_i^n + \Gamma((z_i^n, 0), [nx] - i, [nt])\big\}. \tag{7.5}$$

Suppose that for each $\varepsilon > 0$ there exists a constant $0 < A < \infty$ such that

$$\sup_n P\big\{z_{[nx]}^n \geq An\big\} \leq \varepsilon. \tag{7.6}$$

Then for each $\varepsilon > 0$ there exists an $r < x$ such that

$$\sup_n P\{z^n_{[nx]}([nt]) \neq \zeta^{n,r}\} \leq \varepsilon. \tag{7.7}$$

PROOF. Fix $\varepsilon > 0$. Pick $A > 0$ so that (7.6) holds and pick $B > 0$ and $\delta > 0$ so that assumption (4.2) holds. For the duration of the proof we suppose that the events

$$\{z^n_{[nx]} \leq An\} \quad \text{and} \quad \{z^n_i \geq i(1-\delta)/p \text{ for } i \leq -Bn\} \tag{7.8}$$

hold. Pick $\delta_1 > 0$ so that $(1-\delta/2)/p = (p+\delta_1)^{-1}$, and then $r < \min\{-B, x\}$ small enough so that $|r|^{-1}(A + |x|/(p+\delta)) < \delta_1/2p$. With these choices, some algebra shows that

$$[nx] - i \geq (p+\delta_1)(z^n_{[nx]} - z^n_i) \quad \text{for } i \leq [nr].$$

Fix $i \leq [nr]$ for the moment, and write $m = m^n_i = z^n_{[nx]} - z^n_i$. In the next calculation, first reason as in the proof of Lemma 5.1 and then apply Stirling's formula:

$$\begin{aligned}
&P\{L((z^n_i, 1), (z^n_{[nx]}, [nt])) \geq [nx] - i\} \\
&\quad \leq P\{L((0,0), (m, [nt])) \geq m(p+\delta_1)\} \\
&\quad \leq \binom{m + [nt]}{m} \exp[-C(\delta_1)m] \\
&\quad \leq C_1 \exp\left\{-m\left[C(\delta_1) - \log\left(\frac{m+[nt]}{m}\right) - \frac{[nt]}{m}\log\left(\frac{m+[nt]}{[nt]}\right)\right]\right\}.
\end{aligned}$$

By the exclusion rule (3.3), $m \geq [nx] - [nr]$, so the last line can be made

$$\leq C_1 \exp[-mC(\delta_1)/2]$$

uniformly over $i \leq [nr]$ and over all n by choosing $r \ll x$ small enough.

Now notice that if $z^n_{[nx]}([nt]) \neq \zeta^{n,r}$, then necessarily,

$$L((z^n_i, 1), (z^n_{[nx]}, [nt])) \geq [nx] - i \quad \text{for some } i \leq [nr], \tag{7.9}$$

because whenever

$$z^n_i + \Gamma((z^n_i, 0), [nx] - i, [nt]) > z^n_{[nx]},$$

the index i can have no influence on the infimum in (3.4). Put the event (7.9) together with the complements of the events in (7.8) to get the bound

$$\begin{aligned}
P\{z^n_{[nx]}([nt]) \neq \zeta^{n,r}\} &\leq \varepsilon + \varepsilon + \sum_{i \leq [nr]} C_1 \exp[-m^n_i C(\delta_1)/2] \\
&\leq 2\varepsilon + C_2 \exp[-C(\delta_1)([nx] - [nr])/2].
\end{aligned}$$

This bound is valid uniformly over n, so the lemma is proved. $\square$

PROOF OF THEOREM 2. Let the number $v(x,t)$ be defined by (4.4). Let $c > v(x,t)$ and pick $y \le x$ so that

$$v_0(y) + tg\left(\frac{x-y}{t}\right) < c.$$

Since

$$z^n_{[nx]}([nt]) \le z^n_{[ny]} + \Gamma\big((z^n_{[ny]}, 0), [nx]-[ny], [nt]\big)$$

with probability 1, it follows from assumption (4.1) and the limit in (7.4) that

$$\lim_{n\to\infty} P\{n^{-1} z^n_{[nx]}([nt]) \le c\} = 1.$$

To complete the proof, we may assume that $v(x,t) > -\infty$. Inequality (7.6) follows from assumption (4.1) so the conclusion of Lemma 7.1 holds. By (7.7) it suffices to show that, for any $r < x$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P\big\{n^{-1}\zeta^{n,r} \le v(x,t) - \varepsilon\big\} = 0. \tag{7.10}$$

By the continuity of g, pick a partition

$$r = r_0 < r_1 < r_2 < \cdots < r_s = x$$

such that

$$\left| tg\left(\frac{x-r_l}{t}\right) - tg\left(\frac{x-r_{l+1}}{t}\right)\right| \le \frac{\varepsilon}{4} \quad \text{for } l = 0, 1, \ldots, s-1. \tag{7.11}$$

Then

$$\zeta^{n,r} \ge \min_{0\le l\le s-1}\big\{z^n_{[nr_l]} + \Gamma\big((z^n_{[nr_l]}, 0), [nx]-[nr_{l+1}], [nt]\big)\big\}.$$

From this and (4.4) it follows that the probability in (7.10) is at most

$$\begin{aligned}\sum_{l=0}^{s-1}\Bigg(& P\left\{\frac{1}{n} z^n_{[nr_l]} \le v_0(r_l) - \frac{\varepsilon}{2}\right\} \\ & + P\left\{\frac{1}{n}\Gamma\big((z^n_{[nr_l]}, 0), [nx]-[nr_{l+1}], [nt]\big) \le tg\left(\frac{x-r_l}{t}\right) - \frac{\varepsilon}{2}\right\}\Bigg),\end{aligned}$$

which vanishes as $n \to \infty$ by assumption (4.1), by the limit in (7.4), and by property (7.11) of the partition. □

Next we calculate $g(x)$ for $x > p/q$. Let

$$g^+(u) = \sup_{x\ge 0}\{xu - g(x)\}, \qquad u \ge 0, \tag{7.12}$$

be the monotone conjugate of g (see [20], page 111). Fix $u \in [1, 1/p)$ for the moment. Define a random initial configuration (z_i) for the process as follows: $z_0 = 0$ with probability 1, and $(\eta_i = z_i - z_{i-1}\colon i \in \mathbf{Z})$ are i.i.d. with common distribution α from (3.10) and expectation $E[\eta_i] = u$. By Proposition 1 the interparticle distance process has i.i.d. α distribution at all successive times too. From the development in Section 6 we see that each jump x_0 of particle z_0 has distribution β from (6.5). The position $z_0(n)$ of particle z_0 at time n is a sum of n such jumps to the left. Thus

$$E[n^{-1}z_0(n)] = -\sum_x x\beta_x = -f(u) \tag{7.13}$$

with $f(u)$ given by

$$\begin{aligned} f(u) &= p(1-up)q^{-1}\sum_{x=1}^{\infty} x(u-1)^x(uq)^{-x} \\ &= (1-up)^{-1}pu(u-1). \end{aligned} \tag{7.14}$$

On the other hand, (4.1) holds now for $z^n = z$ and $v_0(y) = uy$, and (4.2) follows from the strong law applied to the i.i.d. (η_i). Thus by Theorem 2 that was proved above,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} z_0(n) &= v(0,1) \quad \text{in probability} \\ &= \inf_{y\le 0}\{uy + g(-y)\} \\ &= -g^+(u). \end{aligned} \tag{7.15}$$

Since $z_0(n)$ is a sum of n identically distributed steps, a uniform bound

$$\sup_n E[(n^{-1}z_0(n))^2] < \infty$$

is immediate, and then the convergence in (7.15) holds also in expectation. Comparison of (7.13) and (7.15) yields

$$f(u) = g^+(u) \quad \text{for } 1 \le u < 1/p.$$

From double duality (Theorem 12.4 of [20]),

$$g(x) = \sup_{u\ge 0}\{xu - g^+(u)\} = \sup_{1\le u<1/p}\{xu - f(u)\}. \tag{7.16}$$

The possibility of restricting u to $[1, 1/p)$ follows from (7.3) (convexity forces $g'(x) \ge 1$ from which $g^+(u) = 0$ for $u \in [0,1]$) and from $f((1/p)-) = \infty$. From (7.16) and (7.14) follows (3.14) for g. Invert $g(x)$ to get $h(x)$, and then by homogeneity, for $x \ge 0$ and $y > 0$,

$$\Psi(x,y) = yh(x/y) = \begin{cases} x, & \text{if } y \ge qx/p, \\ \left(\sqrt{px}+\sqrt{qy}\right)^2 - y, & \text{if } y < qx/p. \end{cases} \tag{7.17}$$

Notice now that $\lim_{y\searrow 0}\Psi(x, y) = px$, so in fact Ψ is continuous on the closed quadrant $[0,\infty)^2$ and (7.17) is valid for all $x, y \geq 0$. Theorem 1 follows from (7.17) and (3.15). □

Acknowledgment. I thank an anonymous referee for numerous suggestions that improved the presentation of the paper.

REFERENCES

[1] ALEXANDER, K. S. (1993). A note on some rates of convergence in first-passage percolation. *Ann. Appl. Probab.* 3 81–90.

[2] ALEXANDER, K. S. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. *Ann. Probab.* 25 30–55.

[3] BARDI, M. and EVANS, L. C. (1984). On Hopf's formulas for solutions of Hamilton–Jacobi equations. *Nonlinear Anal.* 8 1373–1381.

[4] BOIVIN, D. (1990). First-passage percolation: the stationary case. *Probab. Theory Related Fields* 86 491–499.

[5] COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* 9 583–603.

[6] DURRETT, R. and LIGGETT, T. (1981). The shape of the limit set in Richardson's growth model. *Ann. Probab.* 9 186–193.

[7] EVANS, L. C. (1984). Some Min-Max methods for the Hamilton–Jacobi equation. *Indiana Univ. Math. J.* 33 31–50.

[8] FERRARI, P. A. (1996). Limit theorems for tagged particles. *Markov Processes Related Fields* 2 17–40.

[9] FONTES, L. and NEWMAN, C. M. (1993). First-passage percolation for random colorings of $\mathbb{Z}^d$. *Ann. Appl. Probab.* 3 746–762.

[10] HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Bernoulli-Bayes-Laplace Anniversary Volume* (J. Neyman and L. Le Cam, eds.) 61–110. Springer, Berlin.

[11] HÄGGSTRÖM, O. and MEESTER, R. (1995). Asymptotic shapes for stationary first-passage percolation. *Ann. Probab.* 23 1511–1522.

[12] HOWARD, C. D. and NEWMAN, C. M. (1997). Euclidean models of first-passage percolation. *Probab. Theory Related Fields* 108 153–170.

[13] KESTEN, H. (1986). Aspects of first-passage percolation. *Lecture Notes in Math.* 1180 125–264. Springer, Berlin.

[14] KESTEN, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* 3 296–338.

[15] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Royal Statist. Soc. Ser. B* 30 499–510.

[16] KIPNIS, C. (1986). Central limit theorems for infinite series of queues and applications to simple exclusion. *Ann. Probab.* 14 397–408.

[17] LIONS, P. L. (1982). *Generalized Solutions of Hamilton–Jacobi Equations.* Pitman, London.

[18] NEWMAN, C. M. and PIZA, M. S. T. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.* 23 977–1005.

[19] REZAKHANLOU, F. (1994). Evolution of tagged particles in non-reversible particle systems. *Comm. Math. Phys.* 165 1–32.

[20] ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

[21] SAADA, E. (1987). A limit theorem for the position of a tagged particle in a simple exclusion process. *Ann. Probab.* 15 375–381.

[22] SEPPÄLÄINEN, T. (1996). A microscopic model for the Burgers equation and longest increasing subsequences. *Electronic J. Probab.* 1 1–51.

[23] SEPPÄLÄINEN, T. (1998). Hydrodynamic scaling, convex duality, and asymptotic shapes of growth models. *Markov Processes Related Fields* 4 1–26.

[24] VAHIDI-ASL, M. Q. and WIERMAN, J. (1992). A shape result for first-passage percolation on the Voronoi tessellation and Delaunay triangulation. In *Random Graphs* 2 (A. Frieze and T. Łuczak, eds.) 247–262. Wiley, New York.

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