The Annals of Probability
1998, Vol. 26, No. 1, 112–131

WINDINGS OF BROWNIAN MOTION AND RANDOM WALKS IN THE PLANE

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We are interested in the almost sure asymptotic behavior of the windings of planar Brownian motion. Both the usual lim sup and Chung's lim inf versions of the law of the iterated logarithm are presented for the so-called "big" and "small" winding angles. Our method is based on some very accurate estimates of the winding clock. The corresponding problem for a spherically symmetric random walk in $\mathbb{R}^2$ is also studied. A strong approximation using the Brownian big winding process is established. Similar results are obtained.

1. Introduction. Let $\{Z(t) = X(t) + iY(t);\ t \geq 0\}$ be a planar Brownian motion (two-dimensional Wiener process), starting at $z_0 \neq 0$. Since it never hits any given point at positive time (almost surely), there exists a continuous determination of $\theta(t)$, the total angle wound by the Brownian motion around the origin up to time t [with, say, $\theta(0) = 0$]. To be precise, θ records the angle and keeps track of the number of times the Brownian path has wound around the origin, counting clockwise loops -2π and counterclockwise loops 2π. Spitzer [34] showed the weak convergence of θ:

$$\frac{2}{\log t}\theta(t) \overset{(d)}{\to} C \quad \text{as } t \to \infty, \tag{1.1}$$

with C denoting a symmetric Cauchy variable of parameter 1. Insightful explanations of convergence (1.1) and deep results have been presented on the distributional asymptotics of winding numbers of Brownian motion (or even stable processes). See, for example, [6], [12, 13], [22, 23], [25]–[28], [30] and [35]. We refer to [36] for a detailed survey. Apart from its own interest, the winding problem also appears naturally in several branches of mathematical physics (for example, in the study of polymer entanglements). See [8] for references.

There have been several recent contributions on the *almost sure* asymptotic behavior of the planar Brownian windings. Let us mention the papers by Bertoin and Werner [4, 5] and Shi [32]. See also [18] and [16] for Brownian motion valued in a compact space. Recall the upper functions of θ due to [4] and [5].

Received August 1995; revised August 1997.
AMS 1991 *subject classifications*. Primary 60J65, 60J15; second 60F15.
Key words and phrases. Winding angle, Brownian motion, random walk, strong approximation.

THEOREM 1.1 (Bertoin and Werner [4, 5]). *For any nondecreasing function* $f > 0$,

$$\limsup_{t\to\infty} \frac{\theta(t)}{f(t)\log t} = 0 \quad \text{or } \infty, \text{ a.s.},$$

accordingly as

$$\int^{\infty} \frac{dt}{tf(t)\log t}$$

converges or diverges.

In words, Bertoin and Werner's Theorem 1.1 tells us that asymptotically for big t's, $\theta(t)$ exceeds $(\log t)\log\log t$ infinitely often, but stays below $(\log t)(\log\log t)^{1+\varepsilon}$ for any $\varepsilon > 0$.

The lower functions of θ are also known.

THEOREM 1.2 [32]. *Let $g > 0$ be a nonincreasing function such that $g(t)$ $\log t$ is nondecreasing, then*

$$\mathbb{P}\Big[\sup_{0\le u\le t} |\theta(u)| < g(t)\log t \ i.o.\Big] = \begin{cases} 0, \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{dt}{tg(t)\log t}\exp\Big(-\frac{\pi}{4g(t)}\Big)\begin{cases} < \infty, \\ = \infty, \end{cases}$$

where "i.o." stands for "infinitely often as t tends to ∞." In particular,

$$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t} \sup_{0\le u\le t} |\theta(u)| = \frac{\pi}{4} \quad a.s.$$

In order to get some deeper understanding of the asymptotic behavior of θ, Messulam and Yor [26] introduced the so-called "big windings" $\theta_+(t)$ and "small windings" $\theta_-(t)$:

$$\theta_+(t) = \int_0^t \mathbb{1}_{\{|Z_u|>1\}}\, d\theta(u), \qquad \theta_-(t) = \int_0^t \mathbb{1}_{\{|Z_u|<1\}}\, d\theta(u). \tag{1.2}$$

Joint convergence (in distribution) for big and small windings was obtained by Messulam and Yor [26], and further exploited by Pitman and Yor [27, 28]. See also [8]. One of the reasons why θ_+ and θ_- are interesting is that the windings for a very large class of two-dimensional random walks behave rather more like θ_+ than θ. See the discussion in Section 5. Throughout the paper, we write

$$\eta(t) = a\theta_+(t) + b\theta_-(t), \qquad (a, b) \in \mathbb{R}^2. \tag{1.3}$$

In Section 3, we give an integral test for η.

THEOREM 1.3. *For any $a \in \mathbb{R}$ and $b \neq 0$, Theorem 1.1 remains true with $\theta(t)$ replaced by either $\pm\eta(t)$, or $\sup_{0\le u\le t}\eta(u)$, or $\sup_{0\le u\le t}\eta(u) - \inf_{0\le u\le t}\eta(u)$.*

Taking $a = b = 1$ in Theorem 1.3, Bertoin and Werner's Theorem 1.1 is obtained as a special case. The situation is, however, considerably different when $b = 0$.

THEOREM 1.4. *With probability* 1,

$$\limsup_{t\to\infty} \frac{\theta_+(t)}{\log t \log\log\log t} = \limsup_{t\to\infty} \frac{\sup_{0\le u\le t}|\theta_+(u)|}{\log t \log\log\log t} = \frac{1}{\pi}. \tag{1.4}$$

In Section 4, we are interested in the lim inf behavior of the big and small windings. Our main result for η is the following Chung-type law of the iterated logarithm (LIL).

THEOREM 1.5. *For any* $(a, b) \in \mathbb{R}^2$,

$$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t} \sup_{0\le u\le t} |\eta(u)| = \frac{\pi|a|}{4} \quad a.s., \tag{1.5}$$

$$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t} \left[\sup_{0\le u\le t} \eta(u) - \inf_{0\le u\le t} \eta(u) \right] = \frac{\pi|a|}{2} \quad a.s. \tag{1.6}$$

It is immediately noticeable that when $a = 0$, the r.h.s. terms in (1.5) and (1.6) vanish. Indeed, this special case needs to be treated separately.

THEOREM 1.6. *For any positive function* f *such that both* $f(t)$ *and* $(\log t)/f(t)$ *are nondecreasing,*

$$\liminf_{t\to\infty} \frac{f(t)}{\log t} \sup_{0\le u\le t} |\theta_-(u)| = \begin{cases} 0, \\ \infty \end{cases} a.s. \Leftrightarrow \int^{\infty} \frac{dt}{tf(t)\log t} \begin{cases} = \infty, \\ < \infty. \end{cases}$$

COROLLARY 1.7. *We have, with probability* 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\delta}}{\log t} \sup_{0\le u\le t} |\theta_-(u)| = \begin{cases} 0, & \text{if } \delta \le 1, \\ \infty, & \text{otherwise}. \end{cases}$$

In particular,

$$\liminf_{t\to\infty} \left(\frac{1}{\log t} \sup_{0\le u\le t} |\theta_-(u)| \right)^{1/\log\log\log t} = \frac{1}{e} \quad a.s.$$

REMARK. It is seen from the above theorems that, asymptotically, big windings contribute nothing to the lim sup behavior of θ, whereas in its lim inf behavior, small windings have no effect at all. This somewhat surprising fact will be explained in Section 4 when the upper and lower tails of θ_+ and θ_- are estimated.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries on the skew-product representation of planar Brownian motion, and, to illustrate our method which is based on some accurate estimates of the Brownian winding clock, we provide a simple proof of Bertoin and Werner's Theorem 1.1. In Section 3, Theorems 1.3 and 1.4 are proved in the same spirit. The lim inf behavior of the windings is studied in Section 4, where we establish Chung-type LIL's (Theorems 1.5 and 1.6). In Section 5, we are interested in the case of random walk and obtain a strong invariance principle (the forthcoming Theorem 5.1) for the winding angle of a spherically symmetric random walk in $\mathbb{R}^2$. It will be shown that the latter behaves asymptotically like a Brownian big winding process. Lim sup and lim inf versions of the LIL (Theorem 5.3) are obtained.

Throughout the paper we write indifferently $\xi(t)$ or ξ_t for any stochastic process ξ. The symbol N stands for a Gaussian $N(0,1)$ random variable, and $\{W(t);\ t \geq 0\}$ for a standard linear Brownian motion, which are independent of each other, and independent of all other variables and processes.

2. Preliminaries and a simple proof of Theorem 1.1. Our approach is based on the well-known skew-product decomposition of planar Brownian motion, together with some accurate estimates of the random winding clock. To give an illustration, we provide a new simple proof of Theorem 1.1 due to Bertoin and Werner. The elegant proofs of Theorem 1.1 presented in Bertoin and Werner [4, 5], relying either on the exact distribution of Brownian windings or on the stationarity of Ornstein–Uhlenbeck processes, are unfortunately limited to the winding angle θ. On the other hand, ours can be easily applied to θ_+ and θ_- as shown in Sections 3 and 4.

A basic heuristic idea in the study of Brownian windings is that, asymptotically, θ behaves like a symmetric Cauchy process. This is remarked first in [35] and further exploited in [12], [13], [25]–[28]. Although there is no weak convergence in the Skorohod topology of the Brownian winding to a symmetric Cauchy process (since the former is continuous in time), it will be seen later on that one can somehow manage to "neglect" the difference between these two processes.

Recall first of all the following well-known results on linear Brownian motion W:

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{0\leq u\leq 1} W(u) < x,\ \inf_{0\leq u\leq 1} W(u) > -y\Big]\\
&\qquad = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2}{2(x+y)^2}\right)\sin\frac{(2k+1)\pi x}{x+y} \qquad (2.1)\\
&\qquad\qquad \forall x>0,\ y>0,
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right) \leq \mathbb{P}\Big[\sup_{0\leq u\leq 1}|W(u)| < x\Big]\\
&\qquad \leq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right) \qquad \forall x>0. \qquad (2.2)
\end{aligned}$$

The joint distribution (2.1) for $[\sup_{0\leq u\leq 1} W(u),\ \inf_{0\leq u\leq 1} W(u)]$ was computed in [15], page 342. By taking $y = x$, it implies immediately (2.2), which was obtained previously by Chung [7], page 206. Note that the lower bound here for $\mathbb{P}[\sup_{0\leq u\leq 1}|W(u)| < x]$ is not optimal, but sufficient for our needs.

Let us keep the notation previously introduced. Without loss of generality, we assume $z_0 = 1$. Let R be the radial part of the Brownian motion, that is, $R^2 = X^2 + Y^2$. The well-known skew-product representation for a two-dimensional Brownian motion goes back at least to Itô and McKean [20], page 270:

$$\log R(t) = \beta(C_t), \qquad \theta(t) = \gamma(C_t), \tag{2.3}$$

where (β, γ) is a planar Brownian motion starting from 0, and

$$C_t = \int_0^t \frac{du}{R_u^2} = \inf\left\{u > 0: \int_0^u e^{2\beta(v)}\, dv > t\right\}. \tag{2.4}$$

(Recall that $R_0 = 1$ according to our assumption). Note that the above representation actually confirms that the Brownian motion γ and the log-clock C are *independent*. Now the big and small windings introduced in (1.2) may be written as stochastic integrals:

$$\theta_+(t) = \int_0^{C_t} \mathbb{1}_{\{\beta_u > 0\}}\, d\gamma(u), \qquad \theta_-(t) = \int_0^{C_t} \mathbb{1}_{\{\beta_u < 0\}}\, d\gamma(u). \tag{2.5}$$

Define the first hitting time processes of β and R respectively:

$$\sigma_t = \inf\{u > 0: \beta_u > t\}, \tag{2.6}$$

$$T_t = \inf\{u > 0: R_u > t\}. \tag{2.7}$$

The following estimates of the winding clock C play an important rôle in our proof of the theorems.

LEMMA 2.1. *For all positive numbers s and t,*

$$\mathbb{P}[C_t \le \sigma_s] \le 2\exp(-t\exp(-2s)), \tag{2.8}$$

$$\mathbb{P}[C_t \ge \sigma_s] \le 4\exp(-\exp(2s)/16t), \qquad s \ge 1. \tag{2.9}$$

PROOF. From (2.3) it is easily seen that

$$\mathbb{P}[C_t \le \sigma_s] = \mathbb{P}\left[\sup_{0 \le u \le t} R(u) \le e^s\right]. \tag{2.10}$$

Let $\hat{W}$ be an independent copy of W. Since R has the same distribution as $((1 + \hat{W})^2 + W^2)^{1/2}$, the above probability is less than or equal to

$$\mathbb{P}\left[\sup_{0 \le u \le t} |W_u| \le e^s\right] = \mathbb{P}\left[\sup_{0 \le u \le 1} |W(u)| \le \frac{e^s}{\sqrt{t}}\right],$$

which implies (2.8), using (2.2) and the inequalities $4/\pi < 2$ and $\pi^2/8 > 1$. Finally, by (2.10),

$$\mathbb{P}[C_t \ge \sigma_s] = \mathbb{P}\left[\sup_{0 \le u \le t} R(u) \ge e^s\right] = \mathbb{P}\left[\sup_{0 \le u \le t} \left(W_u^2 + \hat{W}_u^2\right)^{1/2} \ge e^s - 1\right].$$

Since $\hat{W}$ is a copy of W, this is less than or equal to

$$2\mathbb{P}\left[\sup_{0 \le u \le 1} |W(u)| \ge \frac{e^s - 1}{\sqrt{2t}}\right] \le 4\mathbb{P}\left[\sup_{0 \le u \le 1} W(u) \ge \frac{e^s - 1}{\sqrt{2t}}\right],$$

by symmetry. Noting that $\sup_{0\le u\le 1} W(u) \overset{(d)}{=} |N|$, and that $e^s - 1 \ge e^s/2$ for any $s \ge 1$, (2.9) follows from the above inequality and from the well-known estimation of the Gaussian tail $\mathbb{P}[|N| > x] \le \exp(-x^2/2)$. $\square$

REMARK. Obviously, more precision could have been given on estimates (2.8) and (2.9), but the present form of Lemma 2.1 is sufficient for our needs. It is worth noting that from Lemma 2.1 one can easily deduce characterizations of the upper and lower functions of the winding clock C previously obtained by Bertoin and Werner [4, 5] and Gruet and Shi [19], respectively.

PROOF OF THEOREM 1.1. We begin with the convergent part. Suppose that the integral $\int^{\infty} dt/tf(t)\log t$ is finite. From the skew-product representation (2.3),

$$\mathbb{P}\Big[\sup_{0\le u\le t} \theta_u > x\Big] \le 2\mathbb{P}\big[|\gamma(C_t)| > x\big] = 2\mathbb{P}\Big[\sqrt{C_t}\,|\gamma(1)| > x\Big].$$

Using (2.9), it is easily seen that this is less than or equal to

$$\begin{aligned}&2\mathbb{P}[C_t \ge \sigma_s] + 2\mathbb{P}\Big[\sqrt{\sigma_s}\,|\gamma(1)| > x\Big]\\ &\qquad \le 8\exp\left(-\frac{e^{2s}}{16t}\right) + 2\mathbb{P}\left[|C| > \frac{x}{s}\right] \quad \text{for any } s \ge 1,\end{aligned}$$

where C stands for a symmetric Cauchy variable of parameter 1. By noting that $\mathbb{P}[|C| > \lambda] \le 2/\lambda$ $(\forall \lambda > 0)$, and taking $s = \frac{1}{2}\log(16t\log x)$, we get that, when $t\log x$ is large,

$$\mathbb{P}\Big[\sup_{0\le u\le t} \theta_u > x\Big] \le \frac{8}{x} + \frac{2\log(16t\log x)}{x}.$$

Let $t_k = \exp(e^k)$ and let $x = f(t_k)\log t_k$. It follows that when k is sufficiently large,

$$\mathbb{P}\Big[\sup_{0\le u\le t_{k+1}} \theta_u > f(t_k)\log t_k\Big] \le \frac{8 + 3\log t_{k+1}}{f(t_k)\log t_k} \le \frac{4e}{f(t_k)}.$$

Since

$$\sum_k \frac{1}{f(t_k)} = \sum_k \int_{t_{k-1}}^{t_k} \frac{dt}{tf(t_k)\log t} \le \sum_k \int_{t_{k-1}}^{t_k} \frac{dt}{tf(t)\log t} = \int^{\infty} \frac{dt}{tf(t)\log t} < \infty,$$

it follows from the Borel–Cantelli lemma that

$$\limsup_{k\to\infty} \frac{1}{f(t_k)\log t_k} \sup_{0\le u\le t_{k+1}} \theta_u \le 1 \quad \text{a.s.}$$

For any $t \in [t_k, t_{k+1}]$, we have $\sup_{0\le u\le t}\theta_u \le \sup_{0\le u\le t_{k+1}}\theta_u$ and $f(t)\log t \ge f(t_k)\log t_k$. Thus

$$\limsup_{t\to\infty} \frac{1}{f(t)\log t} \sup_{0\le u\le t} \theta_u \le 1 \quad \text{a.s.}$$

Since f can be replaced by any multiple of f, the convergent part of Theorem 1.1 is proved. Now suppose $\int^\infty dt/tf(t)\log t = \infty$. Let us recall the definitions (2.6) and (2.7) of σ and T. It is easily seen from (2.3) that $C(T_t) = \sigma_{\log t}$, thus $\{\theta(T(e^t));\ t \geq 0\}$ is a symmetric Cauchy process. From the general theory of stable processes (see, e.g., [17], Theorem 11.2) we get that for any $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{\theta\big(T(e^t)\big)}{tf(e^{(2-\varepsilon)t})} = \infty \quad \text{a.s.},$$

that is,

$$\limsup_{t\to\infty} \frac{\theta(T_t)}{f(t^{2-\varepsilon})\log t} = \infty \quad \text{a.s.}$$

By the usual LIL (see, e.g., [29], page 53), almost surely for all sufficiently large t, $T_t \geq t^{2-\varepsilon}$. Therefore

$$\limsup_{t\to\infty} \frac{\theta(T_t)}{f(T_t)\log T_t} = \infty \quad \text{a.s.},$$

as desired. □

3. Upper limits. As in the last section, we focus on the lim sup behavior of the windings. Define

$$G_+(t) = \int_0^t \mathbb{1}_{\{\beta_u > 0\}}\,du, \qquad G_-(t) = \int_0^t \mathbb{1}_{\{\beta_u < 0\}}\,du, \tag{3.1}$$

the total occupation times of β in, respectively, the positive and negative parts of $\mathbb{R}$ before time t. The following is a collection of some identities and inequalities concerning the (joint) distribution of $(G_+(\sigma_1), G_-(\sigma_1))$ [σ being the first hitting time process of β as introduced in (2.6)].

LEMMA 3.1. *For all positive numbers μ and ν,*

$$\mathbb{E}\exp\big[-\mu G_+(\sigma_1) - \nu G_-(\sigma_1)\big] = \left(\cosh\sqrt{2\mu} + \frac{\sqrt{\nu}}{\sqrt{\mu}}\sinh\sqrt{2\mu}\right)^{-1}; \tag{3.2}$$

$$\begin{aligned} &\mathbb{E}\exp\big[-\mu G_+(\sigma_1)\big] = 1/\cosh\sqrt{2\mu}, \\ &\mathbb{E}\exp\big[-\nu G_-(\sigma_1)\big] = 1/\big(1+\sqrt{2\nu}\big); \end{aligned} \tag{3.3}$$

$$G_+(\sigma_1) =_{(d)} \Big(\sup_{0\leq u\leq 1} |W(u)|\Big)^{-2}. \tag{3.4}$$

PROOF. The joint Laplace transform (3.2) can be found in [27], page 744, which yields trivially (3.3), using analytic continuation. Since $1/\cosh\sqrt{2\mu}$ is also the Laplace transform of $H_1 \equiv \inf\{u\colon |W(u)| = 1\}$, the first hitting time of

1 by $|W|$, we have, for any $x > 0$,

$$\mathbb{P}[G_+(\sigma_1) > x] = \mathbb{P}[H_1 > x] = \mathbb{P}\Big[\sup_{0\le u\le x} |W(u)| < 1\Big]$$
$$= \mathbb{P}\Big[x^{1/2} \sup_{0\le u\le 1} |W(u)| < 1\Big],$$

by the scaling property. The above identity readily yields (3.4) $\square$

In order to prove Theorem 1.4, we need a preliminary result.

LEMMA 3.2. *Let Y and W be two independent linear Brownian motions starting from* 0, *and let $H_t = \inf\{u > 0\colon |W(u)| > t\}$ be the first hitting time process of $|W|$, then*

$$\limsup_{t\to\infty} \frac{Y(H_t)}{t \log\log t} \ge \frac{2}{\pi} \quad a.s.$$

REMARK. Actually an equality holds in the above relation, but the inequality will be sufficient for our needs.

PROOF OF LEMMA 3.2. For any $x > 0$ and $s < t$,

$$\begin{aligned}
&\mathbb{P}[Y(H_t) - Y(H_s) > x]\\
&\quad = \mathbb{P}\big[Y(1)(H_t - H_s)^{1/2} > x\big]\\
&\quad = \frac{1}{2}\mathbb{P}\Big[H_t - H_s > \frac{x}{N^2}\Big]\\
&\quad = \frac{1}{2}\mathbb{P}\Big[\sup_{0\le u\le x^2/N^2} W_u < t - s,\ \inf_{0\le u\le x^2/N^2} W_u > -(t+s)\Big],
\end{aligned}$$

using the Markov property. By (2.1), the above expression is

$$\begin{aligned}
&= \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\mathbb{E}\exp\Big[-\frac{(2k+1)^2\pi^2x^2}{8t^2N^2}\Big]\sin\frac{(2k+1)\pi(t-s)}{2t}\\
&= \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\Big[-\frac{(2k+1)\pi x}{2t}\Big]\sin\frac{(2k+1)\pi(t-s)}{2t},
\end{aligned}$$

as $\mathbb{E}\exp(-\lambda N^{-2}) = e^{-\sqrt{2\lambda}}$ for $\lambda > 0$. Using the trivial relation $|\sin u| \le |u|$, we get that

$$\begin{aligned}
&\mathbb{P}[Y(H_t) - Y(H_s) > x]\\
&\quad \ge \frac{2}{\pi}\exp\Big(-\frac{\pi x}{2t}\Big)\sin\frac{(t-s)\pi}{2t} - \frac{t-s}{t}\sum_{k=1}^{\infty}\exp\Big[-\frac{(2k+1)\pi x}{2t}\Big]\\
&\quad \ge \frac{2}{\pi}\exp\Big(-\frac{\pi x}{2t}\Big)\sin\frac{(t-s)\pi}{2t} - \frac{\exp(-3\pi x/2t)}{1-\exp(-\pi x/2t)}.
\end{aligned}$$

Let $t_n = n^n$ and $x = x_n \equiv (2/\pi) t_n \log\log t_n$, and we obtain

$$\mathbb{P}\left[Y\left(H_{t_n}\right) - Y\left(H_{t_{n-1}}\right) > x_n \right] \geq \frac{2}{\pi \log t_n} \sin\left[\frac{\pi}{2}\left(1 - \frac{t_{n-1}}{t_n}\right)\right] - \frac{(\log t_n)^{-3}}{1-(\log t_n)^{-2}}$$
$$\geq \frac{1}{\pi n \log n} - \frac{1}{n \log n\left[n^2 (\log n)^2 - 1\right]},$$

which is the general term of a divergent series. Since the process $\{Y(H_t)\}_{t \geq 0}$ has independent increments, an application of the Borel–Cantelli lemma yields

$$\limsup_{n\to\infty} \frac{Y\left(H_{t_n}\right) - Y\left(H_{t_{n-1}}\right)}{t_n \log\log t_n} \geq \frac{2}{\pi} \quad \text{a.s.} \tag{3.5}$$

On the other hand, it follows from Chung's LIL (see, e.g., [29], page 53) that $\limsup_{t\to\infty} H_t/t^2 \log\log t = 8/\pi^2$, almost surely. Thus, by the usual LIL ([29], page 53), we have

$$\limsup_{n\to\infty} \frac{\left|Y\left(H_{t_{n-1}}\right)\right|}{t_n \log\log t_n} \leq \frac{4}{\pi} \limsup_{n\to\infty} \frac{t_{n-1} \log\log t_{n-1}}{t_n \log\log t_n} = 0,$$

which, by means of (3.5), completes the proof of Lemma 3.2. □

Proof of Theorem 1.4. Recall the definition (2.5) of the big winding angle θ_+. It follows from Knight's theorem (see for example [31], Theorem IV.34.16) that there exists a real Brownian motion α independent of β, such that

$$\theta_+(t) = \alpha\left(G_+(C_t)\right). \tag{3.6}$$

Let $\rho_t = \inf\{s > 0: G_+(s) > t\}$. Then it is well-known that $\{\beta(\rho_t);\ t \geq 0\}$ is a reflecting Brownian motion (see [27], page 746 for an insightful explanation in terms of Brownian excursions). Therefore

$$\theta_+(T_t) = \alpha\left(G_+\left(C(T_t)\right)\right) = \alpha\left(G_+\left(\sigma_{\log t}\right)\right) = \alpha\left(\inf\left\{u > 0: \beta(\rho_u) > \log t\right\}\right).$$

By Lemma 3.2, we have

$$\limsup_{t\to\infty} \frac{\theta_+(T_t)}{\log t \log\log\log t} \geq \frac{2}{\pi} \quad \text{a.s.}$$

On the other hand, the usual and Chung's LIL's confirm that $\lim_{t\to\infty} \log T_t / 2\log t = 1$ almost surely. Thus,

$$\limsup_{t\to\infty} \frac{\theta_+(t)}{\log t \log\log\log t} \geq \frac{1}{\pi} \quad \text{a.s.}$$

It remains to prove the upper bound in Theorem 1.4. Let us note by (3.6) that for any $x > 0$ and $s \geq 1$,

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le u\le t}|\theta_+(u)| > x\Big] &\le 2\mathbb{P}\big[N^2G_+(C_t) > x\big]\\ &\le 2\mathbb{P}[C_t > \sigma_s] + 2\mathbb{P}\big[N^2G_+(\sigma_s) > x\big]\\ &\le 8\exp\Big(-\frac{e^{2s}}{16t}\Big) + \frac{8}{\pi}\mathbb{E}\exp\Big(-\frac{\pi^2x^2}{8s^2N^2}\Big),\end{aligned}$$

using (2.9), (3.4) and (2.2). Recalling that $\mathbb{E}\exp(-\lambda N^{-2}) = e^{-\sqrt{2\lambda}}$ for any $\lambda > 0$, and taking $s = \frac{1}{2}\log(xt) - \frac{1}{2}\log\log(xt) + \frac{1}{2}\log(16\pi)$, we obtain that, for $xt \geq \exp(16\pi)$,

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le u\le t}|\theta_+(u)| > x\Big] &\le 8\exp\Big(-\frac{\pi x}{\log(xt)}\Big) + \frac{8}{\pi}\exp\Big(-\frac{\pi x}{2s}\Big)\\ &\le 11\exp\Big(-\frac{\pi x}{\log(xt)}\Big),\end{aligned}$$

since $8 + 8/\pi < 11$. Now choose a rational number $p > 1$. Let $t_n = \exp(p^n)$ and $x = x_n \equiv ((1+\varepsilon)/\pi)\log t_n \log\log\log t_n$. When n is sufficiently large, we have $(1+\varepsilon)/\log(xt_n) \geq (1+\varepsilon/2)/\log t_n$, which yields

$$\mathbb{P}\Big[\sup_{0\le u\le t_n}|\theta_+(u)| > x_n\Big] \le 11\exp\Big[-\Big(1+\frac{\varepsilon}{2}\Big)\log\log\log t_n\Big] \le \frac{11}{(n\log p)^{1+\varepsilon/2}}.$$

An application of the Borel–Cantelli lemma then gives

$$\limsup_{n\to\infty}\frac{\sup_{0\le u\le t_n}|\theta_+(u)|}{\log t_n \log\log\log t_n} \le \frac{1}{\pi} \quad \text{a.s.}$$

The upper bound in Theorem 1.4 is readily obtained using a monotonicity argument. □

PROOF OF THEOREM 1.3. Assume that

$$f(t) \geq (\log\log t)^{1/2}. \tag{3.7}$$

In this case, Theorem 1.4 tells us that $\limsup_{t\to\infty}\theta_+(t)/f(t)\log t = 0$ almost surely, and Theorem 1.3 immediately follows from Theorem 1.1 using the trivial identity $\eta(t) = b\theta(t) + (a-b)\theta_+(t)$. Let $\zeta(t)$ be either $\pm\eta(t)$, or $\sup_{0\le u\le t}\eta(u)$, or $\sup_{0\le u\le t}\eta(u) - \inf_{0\le u\le t}\eta(u)$. Let $f > 0$ be an arbitrary nondecreasing function. If $\int^\infty dt/tf(t)\log t$ converges, then by a monotonicity argument it is easily seen (see, e.g., [33], page 865) that (3.7) holds for sufficiently large t, which, by what we have just shown, implies that $\limsup_{t\to\infty}\zeta(t)/tf(t)\log t = 0$ almost surely. It remains to treat the case when $\int^\infty dt/(tf(t)\log t) = \infty$. We have to prove that

$$\limsup_{t\to\infty}\frac{\zeta(t)}{tf(t)\log t} = \infty. \tag{3.8}$$

Let $\tilde{f}(t) = \max(f(t), (\log\log t)^{1/2})$, which is nondecreasing. We distinguish two possible cases. First, if there is a sequence (t_k) increasing to infinity such that $f(t_k) < (\log\log t_k)^{1/2}$, then $\tilde{f}(t_k) = (\log\log t_k)^{1/2}$ for any $k \ge 1$. Thus

$$\int^{t_k} \frac{dt}{t\tilde{f}(t)\log t} \ge \frac{\log\log t_k}{\tilde{f}(t_k)} = (\log\log t_k)^{1/2} \qquad \forall k \ge 1,$$

which in turn yields $\int^\infty dt/(t\tilde{f}(t)\log t) = \infty$. Since $\tilde{f}$ satisfies (3.7) in the place of f, by what we have shown, we obtain that $\limsup_{t\to\infty} \zeta(t)/(t\tilde{f}(t)\log t) = \infty$, which trivially implies (3.8). The other possible case is that there is no such (t_k). In this situation, (3.7) holds for sufficiently large t, which implies again the desired conclusion (3.8). The proof of Theorem 1.3 is therefore complete. $\square$

4. Chung-type LIL's. Let us inherit the notation introduced previously.

LEMMA 4.1. *Let f be a square-integrable function with $F(t) = \int_0^t f^2(u)\,du$, and let $M_t = \int_0^t f(u)\,d\gamma_u$, where γ is a linear Brownian motion. Then for any $0 < s < t$, $0 < x < y$ and $\delta > 0$,*

$$\frac{2}{\pi}\exp\left[-\frac{\pi^2 F(t)}{8x^2}\right] \le \mathbb{P}\left[\sup_{0\le u\le t}|M_u| < x\right] \le \frac{4}{\pi}\exp\left[-\frac{\pi^2 F(t)}{8x^2}\right]; \tag{4.1}$$

$$\begin{aligned}&\mathbb{P}\left[\sup_{0\le u\le s}|M_u| < x,\ \sup_{0\le u\le t}|M_u| < y\right]\\ &\qquad \le \frac{16}{\pi^2}\exp\left[-\frac{\pi^2 F(s)}{8x^2} - \frac{\pi^2(F(t)-F(s))}{8y^2}\right];\end{aligned} \tag{4.2}$$

$$\mathbb{P}\left[\sup_{0\le u\le t} M_u - \inf_{0\le u\le t} M_u < x\right] \le K\frac{F(t)}{x^2}\exp\left[-\frac{\pi^2 F(t)}{2x^2}\right], \tag{4.3}$$

where $K > 0$ is a universal constant.

PROOF. Since M is a continuous martingale, by the Dubins–Schwarz theorem (see, e.g., [31], Theorem IV.34.1), we can write $M_t = W(F(t))$ for all $t \ge 0$, with W a linear Brownian motion. Therefore (4.1) immediately follows from (2.2) using the Brownian scaling property. A proof of inequality (4.2) can be found in [32], based on (2.2) and a general property of Gaussian measures for shifted balls. By writing $M_t = W(F(t))$ again, (4.3) is reduced to the following estimate:

$$\mathbb{P}\left[\sup_{0\le u\le 1} W(u) - \inf_{0\le u\le 1} W(u) < x\right] \le \frac{K}{x^2}\exp\left(-\frac{\pi^2}{2x^2}\right).$$

This was implicitly shown by Csáki [10], replying on the exact density function of the Brownian range previously evaluated by Feller [14]. $\square$

PROOF OF THEOREM 1.5. The case $a = 0$ being treated in Theorem 1.6, we suppose $a \neq 0$. Using the trivial relation $\sup_{0 \le u \le t} \eta(u) - \inf_{0 \le u \le t} \eta(u) \le 2 \sup_{0 \le u \le t} |\eta(u)|$, it is easily seen that only the upper bound in (1.5) and the lower bound in (1.6) need proving. We begin with the latter. Let $\varepsilon > 0$. Fix a sufficiently small (rational) number $\delta > 0$ such that

$$(1 - \delta)^{1/2}(1 + \varepsilon) > 1 + \frac{\varepsilon}{2}. \tag{4.4}$$

It follows from (4.3) that

$$\mathbb{P}\Big[\sup_{0 \le u \le t} M_u - \inf_{0 \le u \le t} M_u < x\Big] \le K_\delta \exp\left(-\frac{(1-\delta)\pi^2 F(t)}{2x^2}\right),$$

where $K_\delta > 0$ denotes a finite constant depending only on δ. Let us estimate the lower tail $q \equiv \mathbb{P}[\sup_{0 \le u \le t} \eta(u) - \inf_{0 \le u \le t} \eta(u) < x]$. By the above inequality, we have $q \le K_\delta \mathbb{E} \exp[-(1-\delta)\pi^2 a^2 G_+(C_t)/2x^2]$, dropping the negative part $-G_-(C_t)$ in the exponential. Now using (2.8) and (3.3), we have

$$\begin{aligned} q &\le K_\delta \mathbb{P}[C_t \le \sigma_s] + K_\delta \mathbb{E} \exp\big[-(1-\delta)\pi^2 a^2 G_+(\sigma_s)/2x^2\big] \\ &\le 2K_\delta \exp(-te^{-2s}) + 2K_\delta \exp\big[-(1-\delta)^{1/2}\pi|a|s/x\big], \end{aligned}$$

for any $s > 0$. Taking $s = \frac{1}{2}\log t - \log\log t$, and we obtain

$$\begin{aligned} q &\le 2K_\delta \exp\big(-(\log t)^2\big) \\ &\quad + 2K_\delta \exp\left[-\frac{(1-\delta)^{1/2}\pi|a|}{2x}\log t + \frac{\pi|a|}{x}(\log\log t)^2\right]. \end{aligned}$$

Let $t_n = \exp(r^n)$ with $r > 1$, and let $x = \pi|a|\log t_n / 2(1+\varepsilon)\log\log\log t_n$. Then for any sufficiently large n, by (4.4),

$$\begin{aligned} &\mathbb{P}\left[\sup_{0 \le u \le t_n} \eta(u) - \inf_{0 \le u \le t_n} \eta(u) < \frac{\pi|a|}{2}\frac{\log t_n}{\log\log\log t_n}\right] \\ &\quad \le 2K_\delta \exp(-r^{2n}) \\ &\qquad + 2K_\delta \exp\big[-(1-\delta)^{1/2}(1+\varepsilon)\log(n\log r) + 3(\log t_n)^{-1/2}\big] \\ &\quad \le 2K_\delta \exp(-r^{2n}) + \frac{3K_\delta}{(n\log r)^{1+\varepsilon/2}}, \end{aligned}$$

which is summable for n. Using the Borel–Cantelli lemma gives

$$\liminf_{n \to \infty} \frac{\log\log\log t_n}{\log t_n}\left[\sup_{0 \le u \le t_n} \eta(u) - \inf_{0 \le u \le t_n} \eta(u)\right] \ge \frac{\pi|a|}{2} \quad \text{a.s,}$$

which readily yields the lower bound in (1.6) using a monotonicity argument. It remains to verify the upper bound in (1.5). By (4.1), we have

$$\mathbb{P}\Big[\sup_{0 \le u \le t} |\eta(u)| < x\Big] \ge \frac{2}{\pi}\mathbb{E}\exp\left[-\frac{\pi^2 a^2 G_+(C_t)}{8x^2} - \frac{\pi^2 b^2 G_-(C_t)}{8x^2}\right].$$

Using (3.2) and (2.9), this is greater than or equal to

$$\frac{2}{\pi}\mathbb{E}\exp\left[-\frac{\pi^2 a^2 G_+(\sigma_s)}{8x^2}-\frac{\pi^2 b^2 G_-(\sigma_s)}{8x^2}\right]-\frac{2}{\pi}\mathbb{P}[C_t>\sigma_s]$$
$$\geq \frac{2|a|}{\pi(|a|+|b|)}\exp\left(-\frac{\pi|a|s}{2x}\right)-\frac{8}{\pi}\exp\left(-\frac{e^{2s}}{16t}\right).$$

Now, choosing $x=\pi|a|\log t/4\log\log\log t$ and $s=\frac{1}{2}\log t+\log\log t$, we obtain for all large t that

$$\mathbb{P}\left[\sup_{0\le u\le t}|\eta(u)|<x\right]$$
$$\geq \frac{2|a|}{\pi(|a|+|b|)}\exp\left[-\log\log\log t-\frac{2}{(\log t)^{1/2}}\right]-\frac{8}{\pi}\exp\left(-\frac{(\log t)^2}{16}\right)$$
$$\geq \frac{|a|}{\pi(|a|+|b|)\log\log t}.$$

Let $t_n=\exp(e^n)$ and $A_n=\{\sup_{0\le u\le t_n}|\eta(u)|<x_n\}$ with $x_n=\pi|a|\log t_n/4\log\log\log t_n$, then for large n,

$$\mathbb{P}(A_n)\geq\frac{|a|}{\pi(|a|+|b|)n}. \tag{4.5}$$

On the other hand, it follows from (4.2) that for $i<j$,

$$\mathbb{P}(A_iA_j)\le\frac{16}{\pi^2}\mathbb{E}\exp\left[-\frac{\pi^2a^2G_+(C_{t_i})}{8x_i^2}-\frac{\pi^2a^2\left(G_+(C_{t_j})-G_+(C_{t_i})\right)}{8x_j^2}\right],$$

which, as $x_i<x_j$, is smaller than $16\pi^{-2}(\mathrm{I}+\mathrm{II})$, with

$$\mathrm{I}=\mathbb{P}\left[C_{t_i}\le\sigma_{s_i}\text{ or }C_{t_j}\le\sigma_{s_j}\right];$$
$$\mathrm{II}=\mathbb{E}\exp\left[-\frac{\pi^2a^2G_+(\sigma_{s_i})}{8x_i^2}-\frac{\pi^2a^2\left(G_+(\sigma_{s_j})-G_+(\sigma_{s_i})\right)}{8x_j^2}\right].$$

By (2.8), the first term "I" is bounded above by $2\exp(-t_ie^{-2s_i})+2\exp(-t_je^{-2s_j})$. Write $r_1=\pi|a|s_i/2x_i$ and $r_2=\pi|a|(s_j-s_i)/2x_j$. By conditioning on $\{\beta_u;\ u\le\sigma_{s_i}\}$ and using the strong Markov property of Brownian motion,

$$\mathrm{II}=\mathbb{E}\exp\left[-\frac{\pi^2s_i^2a^2G_+(\sigma_1)}{8x_i^2}\right]\mathbb{E}\exp\left[-\frac{\pi^2a^2}{8x_j^2}\int_0^{\sigma_{s_j-s_i}}\mathbb{1}_{\{\beta_u>-s_i\}}\,du\right]$$
$$\le\mathbb{E}\exp\left(-\frac{r_1^2}{2}G_+(\sigma_1)\right)\mathbb{E}\exp\left(-\frac{r_2^2}{2}G_+(\sigma_1)\right)$$
$$\le 4\exp(-r_1-r_2),$$

using (3.3). Pick $s_k = \frac{1}{2}\log t_k - \log\log t_k$ $(k = i, j)$ and we obtain that, for large i and j (say, for $j > i \ge n_0$),

$$\begin{aligned}\mathbb{P}(A_i A_j) &\le \frac{32}{\pi^2}\Big[\exp\big(-(\log t_i)^2\big) + \exp\big(-(\log t_j)^2\big)\Big]\\ &\quad + \frac{64}{\pi^2}\exp\Bigg[-\log\log\log t_i + \frac{2}{(\log t_i)^{1/2}}\\ &\qquad - \frac{\log(t_j/t_i)}{\log t_j}\log\log\log t_j + \frac{2}{(\log t_j)^{1/2}}\Bigg]\\ &\le \exp(-i) + \exp(-j) + \frac{65}{\pi^2 ij}\exp(\exp(i-j)\log j).\end{aligned}$$

Let $E_1 = \{n_0 \le i < j \le n\colon\ j - i < 2\log\log n\}$ and $E_2 = \{n_0 \le i < j \le n\colon\ j - i \ge 2\log\log n\}$. Then $\sum_{(i,j)\in E_1}\mathbb{P}(A_i A_j) \le 2\log\log n\sum_{i=1}^n \mathbb{P}(A_i)$. If $(i,j)\in E_2$, we have

$$\mathbb{P}(A_i A_j) \le 2\exp(-i) + \frac{65}{\pi^2 ij}\exp(1/\log n) \le 2\exp(-i) + \frac{66}{\pi^2 ij},$$

which implies

$$\sum_{(i,j)\in E_2}\mathbb{P}(A_i A_j) \le 2\sum_{i=1}^{\infty} ie^{-i} + \frac{33}{\pi^2}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2.$$

By (4.5), we have

$$\liminf_{n\to\infty}\sum_{1\le i,}\sum_{j\le n}\mathbb{P}(A_i A_j)\Bigg/\left(\sum_{i=1}^{n}\mathbb{P}(A_i)\right)^2 \le \frac{66(|a|+|b|)^2}{a^2}.$$

According to Kochen and Stone's version [21] of the Borel–Cantelli lemma, this inequality together with the divergence of $\sum\mathbb{P}(A_i)$ implies

$$\mathbb{P}\left[\liminf_{t\to\infty}\frac{\log\log\log t}{\log t}\sup_{0\le u\le t}|\eta(u)| \le \frac{\pi|a|}{4}\right] \ge \frac{a^2}{66(|a|+|b|)^2},$$

which yields the upper bound in (1.5) using an argument of the zero–one law. $\square$

Proof of Theorem 1.6. Let $t_k = \exp(e^k)$. First of all, notice that the series $\int^{\infty} dt/tf(t)\log t$ and $\sum_k 1/f(t_k)$ converge or diverge simultaneously. By using (2.8) (with $s = \frac{1}{4}\log t$) and (3.3), it is immediately seen that

$$\mathbb{P}\left[\sup_{0\le u\le t}|\theta_-(u)| < \frac{\log t}{f(t)}\right] \le \frac{8}{\pi}\exp(-\sqrt{t}) + \frac{32}{\pi^2 f(t)}.$$

This implies the second part of Theorem 1.6 by an application of the Borel–Cantelli lemma. Now, let $B_n = \{\sup_{0\le u\le t_n}|\theta_-(u)| < x_n\}$ and $x_n = \log t_n/f(t_n)$. By making use of an argument similar to that adopted to prove (4.5), we get that $\mathbb{P}(B_n) \ge 2/\pi^2 f(t_n) - 8\pi^{-1}\exp(-\exp(2n)/16)$, whereas by (2.8) and (3.3) we obtain that for any s_i and s_j,

$$\mathbb{P}(B_i B_j) \le \frac{32}{\pi^2}\left[\exp(-t_i\exp(-2s_i)) + \exp\big(-t_j\exp\big(-2s_j\big)\big)\right] + \frac{16}{\pi^2 r_3}\left(\frac{s_i}{s_j} + \frac{1}{r_4}\right),$$

with $r_3 \equiv \pi s_i/2x_i$ and $r_4 \equiv \pi s_j/2x_j$. By choosing $s_k = \frac{1}{2}\log t_k - \log\log t_k$ ($k = i$ or j), we can manage to arrive at the estimate

$$\liminf_{n\to\infty} \sum_{1\le i,\ j\le n}\mathbb{P}(B_i, B_j)\Bigg/\left(\sum_{i=1}^n \mathbb{P}(B_i)\right)^2 \le \tfrac{257}{4},$$

which yields the first part of Theorem 1.6 by means of Kochen and Stone's Borel–Cantelli lemma [21]. The details are omitted as they are quite similar to that in the proof of Theorem 1.5. □

5. Random walks. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed $\mathbb{R}^2$-valued random variables, and let $S = \{S_n,\ n\ge 1\}$ be the random walk defined by $S_n = \sum_1^n X_k$. We consider $\Theta(n)$ the winding of S at time n, that is, the total angle would by S around the origin up to time n. To be precise, $\Theta(n) = \sum_{k=1}^n \phi(k)$, where $\phi(k) \in (-\pi, \pi)$ is the unique solution to $S_{k-1}e^{i\phi(k)}/\|S_{k-1}\| = S_k/\|S_k\|$ if S_{k-1}, S_k and the origin are not colinear, and $\phi(k) = 0$ otherwise. We refer to [1]–[3] for references on the weak convergence of Θ. In particular, [1] shows (under suitable assumptions of regularity) that

$$\frac{2}{\log n}\Theta(n) \overset{(d)}{\to} W_+ \quad \text{as } n\to\infty,$$

where W_+ is a standard hyperbolic secant variable (i.e., with density function $\frac{1}{2}\operatorname{sech}(\frac{1}{2}\pi x)$ for $x\in\mathbb{R}$). Note that it was proved in [26] and [27] that $2\theta_+(t)/\log t \to^{(d)} W_+$, with θ_+ standing for the Brownian big winding process defined in (2.5). So, the above results suggest that Θ might behave somewhat like θ_+. This is confirmed by [2] who obtained a weak invariance principle using a Brownian embedding (stated below).

Our main result in this section is a strong approximation theorem. As usual, we write $\log x$ for $\log(\max(1, x))$.

THEOREM 5.1. *Assume that the distribution of X_1 is spherically symmetric such that*

$$\mathbb{E}\left[\|X_1\|^2\left(\log\|X_1\|\right)^2\right] < \infty,$$

then (after a possible redefinition on a larger space) there exists a planar Brownian motion Z starting from 0 *such that for all* $\varepsilon > 0$,

$$(5.1) \qquad \max_{1 \le k \le n} |\Theta(k) - \theta_+(k)| = o\big((\log n)^{3/4} (\log\log n)^{1+\varepsilon}\big) \quad a.s.$$

Here, θ_+ *is the big winding angle of* Z.

REMARK. The technical assumption of the spherical symmetry, which considerably simplifies the calculation, is borrowed from [2]. Without this condition, the Brownian embedding described in the proof can still be constructed, but the computation for the windings becomes simply horrible, and, as far as I can see, does not provide any useful information.

Let us recall a classical result concerning the Csörgő–Révész large increments of Brownian motion.

THEOREM 5.2 (Csörgő and Révész [11]). *Let* $\{W(t)\colon t \ge 0\}$ *be a linear Brownian motion, and let* a_t $(t \ge 0)$ *be a nondecreasing function of* t *such that* $0 < a_t \le t$ *and that* t/a_t *is nondecreasing. Then*

$$\limsup_{t\to\infty} \frac{\sup_{0 \le u \le a_t} \sup_{0 \le s \le t-a_t} |W(s+u) - W(s)|}{\big(2a_t(\log(t/a_t) + \log\log t)\big)^{1/2}} = 1 \quad a.s.$$

PROOF OF THEOREM 5.1. Let μ be the probability measure on $\mathbb{R}_+$ defined by $\mu((a,b]) = \mathbb{P}[a < \|X_1\| \le b]$, for any $0 \le a < b < \infty$. Let $\{Z(t);\ t \ge 0\}$ be a planar Brownian motion starting from the origin, and let $U_1, U_2, \dots$ be a sequence of independent nonnegative random variables, having the common distribution μ, that are independent of Z. We define $\tau_0 = 0$ and $\tau_k = \inf\{t \ge \tau_{k-1};\ \|Z(t) - Z(\tau_{k-1})\| = U_k\}$, for $k \ge 1$, and $Y_n = Z(\tau_n) - Z(\tau_{n-1})$. Obviously $\{Y_n\}_{n\ge1}$ is a copy of $\{X_n\}_{n\ge1}$ such that $\sum_1^n Y_k = Z(\tau_n)$. The Brownian embedding was used by Bélisle [2] to show that $(\Theta(n) - \theta_+(n))/\log n \to^{P\cdot} 0$ as $n \to \infty$. We will exploit the same construction to prove the stronger result stated in (5.1). Since $\mathbb{E}\|Y_1\|^2 < \infty$ according to our assumption, we can assume $\mathbb{E}\|Y_1\|^2 = 2$ without loss of generality. In this case $\mathbb{E}\tau_1 = 1$. The proof of (5.1) is then formulated in two steps, which together will imply the statement.

Step 1. For any $\varepsilon > 0$,

$$(5.2) \qquad \max_{1 \le k \le n} |\Theta(k) - \theta_+(\tau_k)| = o\big((\log n)^{1/2} (\log\log n)^{1/2+\varepsilon}\big) \quad \text{a.s.}$$

PROOF. Write $M_n = \Theta(n) - \theta_+(\tau_n)$ for notational simplification. It was shown in [2] that M is a martingale with $\mathbb{E}M_n^2 \le K \log n$ for some finite constant $K > 0$. Thus by Doob's inequality,

$$\mathbb{P}\Big[\max_{1\le k\le n} |M_k| > \lambda\Big] \le \frac{1}{\lambda^2} \max_{1\le k\le n} \mathbb{E}M_k^2 \le \frac{K\log n}{\lambda^2},$$

for any $\lambda > 0$ and $n \geq 1$. Taking $n_m = [\exp(e^m)]$ [the integer part of $\exp(e^m)$] and $\lambda = (\log n_m)^{1/2}(\log\log n_m)^{1/2+\varepsilon}$, we get

$$\mathbb{P}\left[\max_{1 \le k \le n_m} |M_k| > (\log n_m)^{1/2}(\log\log n_m)^{1/2+\varepsilon}\right] \le \frac{K}{m^{1+2\varepsilon}},$$

which implies that

$$\limsup_{m\to\infty} \frac{\max_{1\le k\le n_m}|M_k|}{(\log n_m)^{1/2}(\log\log n_m)^{1/2+\varepsilon}} = 0 \quad \text{a.s.},$$

since ε can be arbitrarily small. A monotonicity argument then completes the proof of (5.2).

Step 2. For any $\varepsilon > 0$,

$$\max_{1\le k\le n} \left|\theta_+(\tau_k) - \theta_+(k)\right| = o\big((\log n)^{3/4}(\log\log n)^{1+\varepsilon}\big) \quad \text{a.s.}$$

PROOF. By Kolmogorov's law of large numbers, $\mathbb{P}[n/2 \le \tau_n \le 2n$ eventually$] = 1$. Consequently, it suffices to prove that

$$\max_{1\le k\le n}\ \sup_{k/2\le s\le 2k} \left|\theta_+(s) - \theta_+(k)\right| = o\big((\log n)^{3/4}(\log\log n)^{1+\varepsilon}\big) \quad \text{a.s.} \tag{5.3}$$

Assume that the Brownian motion Z starts from 1 since it contributes nothing to the big windings before reaching the unit sphere. Recall from the skew-product representation (2.5) that $\theta_+(t) = \int_0^{C_t} \mathbb{1}_{\{\beta_u > 0\}}\, d\gamma(u)$, with γ independent of β (thus of C as well). Again by Knight's theorem, we can write $\alpha_+(t) = W(G_+(C_t))$, where W is a linear Brownian motion, starting from 0, independent of β and C, and $G_+(t) = \int_0^t \mathbb{1}_{\{\beta_u > 0\}}\, du$ as in (3.1). Let

$$\Omega_1 = \left\{C(2n) \le \sigma\left(\tfrac{1}{2}\log n + \log\log n\right), \text{eventually}\right\},$$

$$\Omega_2 = \left\{C(n/2) \ge \sigma\left(\tfrac{1}{2}\log n - \log\log n\right), \text{eventually}\right\},$$

$$\Omega_3 = \left\{G_+\left(\sigma\left(\tfrac{1}{2}\log n + \log\log n\right)\right) \le (\log n)^2(\log\log n)^{2+\varepsilon}, \text{eventually}\right\}.$$

Here, $\sigma(t) = \inf\{u;\ \beta(u) > t\}$ as in (2.6). We have $\mathbb{P}(\Omega_1) = 1$ since by (2.9),

$$\mathbb{P}\left[C(2n) > \sigma\left(\frac{1}{2}\log n + \log\log n\right)\right] \le 4\exp\left(-\frac{(\log n)^2}{32}\right),$$

which is summable for $n \ge 1$. Similarly, the identity $\mathbb{P}(\Omega_2) = 1$ follows from (2.9) and the Borel–Cantelli lemma. By Hirsch's theorem (see, e.g., [9]), for any $\delta > 0$, we have $\liminf_{T\to\infty} T^{-1/2}(\log T)^{1+\delta}\sup_{0\le t\le T}\beta_t = \infty$ almost surely, which readily implies $\mathbb{P}(\Omega_3) = 1$ using the trivial relation $G_+(\sigma(x)) \le \sigma(x)$. Thus, $\mathbb{P}(\bigcap_{i=1}^3 \Omega_i) = 1$. By means of Theorem 5.2, the proof of (5.3) is reduced to showing the following:

$$\begin{aligned}&\max_{1\le k\le n}\left[G_+\left(\sigma\left(\tfrac{1}{2}\log k + \log\log k\right)\right) - G_+\left(\sigma\left(\tfrac{1}{2}\log k - \log\log k\right)\right)\right]\\ &\qquad = o\big((\log n)^{3/2}(\log\log n)^{1+2\varepsilon}\big) \quad \text{a.s.}\end{aligned} \tag{5.4}$$

Let $V_n = G_+(S_n) \equiv \sum_1^n \lambda_k$, with $\lambda_k = G_+(\sigma_k) - G_+(\sigma_{k-1}) = \int_{\sigma_{k-1}}^{\sigma_k} \mathbb{1}_{\{\beta_u > 0\}}\, du$. The λ_n's are obviously independent (but not identically distributed). By (2.9) we have $\mathbb{E}\exp(-aV_n) = 1/\cosh\sqrt{2an^2}$ for any $a \geq 0$, which implies $\mathbb{E}V_n = n^2$ and $\mathbb{E}V_n^2 = 10n^4/3$. By analytic continuation, the generating function $\mathbb{E}\exp(aV_n)$ is finite for $a < \pi^2/8n^2$. By [24], Theorem 2.2.2 [taking $\phi_n = (\log n)^{1+\varepsilon}$], one can find Ω_4 with $\mathbb{P}(\Omega_4) = 1$ such that for any $\omega \in \Omega_4$, there exists a finite number $N(\omega)$ with

$$\left| \sum_{n-(\log n)^{1+\varepsilon} \leq k \leq n} (\lambda_k - \mathbb{E}\lambda_k) \right| \leq K n^{3/2} (\log n)^{1+\varepsilon},$$

for any $n > N(\omega)$, where $K > 0$ is a finite constant. Since $\mathbb{E}\lambda_k = \mathbb{E}V_k - \mathbb{E}V_{k-1} = 2k - 1$, we have

$$G_+(\sigma(n)) - G_+\Big(\sigma\Big(n - (\log n)^{1+\varepsilon}\Big)\Big) \leq K n^{3/2} (\log n)^{1+\varepsilon},$$

which yields

$$G_+(\sigma(n)) - G_+(\sigma(n - 2\log\log n)) \leq K n^{3/2} (\log n)^{1+\varepsilon}.$$

Accordingly, for any $n > \exp(2N(\omega))$,

$$\begin{aligned} &G_+\Big(\sigma\Big(\tfrac{1}{2}\log n + \log\log n\Big)\Big) - G_+\Big(\sigma\Big(\tfrac{1}{2}\log n - \log\log n\Big)\Big) \\ &\qquad \leq K(\log n)^{3/2} (\log\log n)^{1+\varepsilon}, \end{aligned}$$

which implies (5.4).

Theorem 5.1 is therefore proved. □

Now we investigate the upper and lower limits of Θ.

THEOREM 5.3. *Under the assumptions of Theorem* 5.1, *we have*

$$\limsup_{n\to\infty} \frac{\Theta(n)}{\log n \log\log\log n} = \limsup_{n\to\infty} \frac{\max_{1\leq k\leq n} |\Theta(k)|}{\log n \log\log\log n} = \frac{1}{\pi} \quad a.s.,$$

$$\liminf_{n\to\infty} \frac{\log\log\log n}{\log n} \max_{1\leq k\leq n} |\Theta(k)| = \frac{\pi}{4} \quad a.s.,$$

$$\liminf_{n\to\infty} \frac{\log\log\log n}{\log n} \Big[\max_{1\leq k\leq n} \Theta(k) - \min_{1\leq k\leq n} \Theta(k) \Big] = \frac{\pi}{2} \quad a.s.$$

PROOF. In view of the strong approximation (5.1), it is sufficient to verify

$$\lim_{n\to\infty} \frac{\log\log\log n}{\log n} \max_{0\leq k\leq n-1} \sup_{0\leq s\leq 1} |\theta_+(k+s) - \theta_+(k)| = 0 \quad \text{a.s.}$$

This, however, immediately follows from (5.3). □

Acknowledgments. I am grateful to Jean Bertoin, Jean-Claude Gruet and Marc Yor for helpful discussions, and to David Mason for a reference.

Many thanks are due to an anonymous referee for insightful comments which led to improvements in the paper.

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