CURVE APPROXIMATION BY MEANS OF FUNCTIONS ANALOGOUS TO THE HERMITE POLYNOMIALS.

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I. Introduction

In an article by J. P. Gram entitled "Ueber die Entwickelung reeler Functionem in Reihen mittelst der Methode der kleinsten Quadrate"[1] a unique procedure is set forth which leads to a very great simplification in the usual method of curve fitting by the method of least squares. That this method has not been given more consideration is probably due to lack of knowledge of its existence, rather than to lack of appreciation of its merit. Edward Condon,[2] Raymond T. Birge and John D. Shea[3] developed formulas by means of which curves can be fitted to certain types of data. Later, Harold T. Davis and Voris V. Latshaw[4] developed specific formulas, with tables of coefficients, by means of which curves of the second to the seventh degree can be fitted to the data with a minimum amount of computation. In a later paper, Professor Davis[5] has employed Gram's method and in this way has developed a set of functions analagous to the Legendre polynominals.

The purposes of the present paper are:

(1) To develop formulas for fitting curves of the second to the sixth degree to given data by the method of least squares where the $n+1$ frequencies of the data have the terms of the expansion of $(\frac{1}{2}+\frac{1}{2})^n$ as weighting factors.

[1]Journal fur Mathematik, Vol. 94, 1894, pp. 41-73, especially pp. 42-46.

[2]"The Rapid Fitting of a Certain Class of Empirical Formulae by the Method of Least Squares." Univ. of California Pub. in Math. Vol. 2, No. 4, pp. 55-66, March 1927.

[3]"A Rapid Method for Calculating the Least Squares Solution of a Polynomial of any Degree." University of California Publications in Mathematics. Vol. 2, No. 5, pp. 67-118, March 1927.

[4]"Formulas for the Fitting of Polynomials to Data by the method of Least Squares." Annals of Mathematics, Second Series, Vol. 31, No. 1, Jan. 1930, pp. 52-78.

(2) To develop by Gram's method a set of functions analogous to the Hermite polynomials, by means of which curves of the second to the eighth degree can be fitted to data under the same conditions as in (1).

(3) To study the properties of these functions, finding a generating function, a recurrence formula, a second order difference equation, and giving other methods for deriving them.

(4) To apply the functions in finding a curve to fit given data.

(5) To furnish tables to facilitate rapid calculation of the coefficients of the required equation.

II. Development By Ordinary Method of Least Squares

Suppose we have given data in which the variates, x , are equally spaced, the observations being weighted with the binomial coefficients, having the origin at the mean. Thus, let the given data be

x	x_{-p}	x_{-p+1}	x_{-p+2}	$\cdots$	x_{-1}	x_0	x_1	$\cdots$	x_p
y_x	y_{-p}	y_{-p+1}	y_{-p+2}	$\cdots$	y_{-1}	y_0	y_1	$\cdots$	y_p
w_x	C_{-p}	C_{-p+1}	C_{-p+2}	$\cdots$	C_{-1}	C_0	C_1	$\cdots$	C_p

where $x_{i+1} - x_i$ is constant,

and $$C_x = \frac{(2p)!}{(p+x)!\,(p-x)!}\left(\frac{1}{2}\right)^{p+x}\left(\frac{1}{2}\right)^{p-x}$$

Since the x differences are constant, it is possible, without the loss of generality, to replace the x's by their corresponding subscripts. It is evident that the problem as set forth deals with

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[5] "Polynomial Approximation by the Method of Least Squares." Annals of Mathematics.

data having an odd number of classmarks. If there should be an even number, the data can be modified by leaving out one of the end variates, or by adding an item by extrapolation. It is also possible to transform these functions by moving the origin to the extreme left, thus having x vary from o to n. If this is done, the functions in part 3 of this paper will reduce to those generated by equation (18) in Gram's article.

It is required to find the coefficients $a_{i,k}$ in the equation

$$(1) \quad y = y'C_x = \left[a_{n,o} + a_{n,1}x + a_{n,2}x^2 + \cdots\cdots a_{n,n}x^n\right]C_x$$

such that

$$\sum_{-p}^{p} C_x(y_x - y')^2 = \sum_{-p}^{p} C_x\left[y_x - a_{n,o} - a_{n,1}x - a_{n,2}x^2 \cdots - a_{n,n}x^n\right]^2$$

shall be a minimum.

In this section, the ordinary method of least squares is used, leading to the $n+1$ equations

$$(2) \quad \begin{aligned} M_o &= a_{n,o}m_o + a_{n,2}m_2 + a_{n,4}m_4 + \cdots\cdots \\ M_1 &= a_{n,1}m_2 + a_{n,3}m_4 + a_{n,5}m_6 + \cdots\cdots \\ M_2 &= a_{n,o}m_2 + a_{n,2}m_4 + a_{n,4}m_6 + \cdots\cdots \\ M_3 &= a_{n,1}m_4 + a_{n,3}m_6 + a_{n,5}m_8 + \cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

where

$$(3) \quad M_r = \sum_{x=-p}^{p} y_x x^r \quad \text{and} \quad m_r = \sum_{x=-p}^{p} C_x x^r$$

(It is evident from the symmetry of the distribution that all moments, m_r, with r odd will be identically zero.)

These $n+1$ equations can be solved more readily by dividing them into two sets, one containing the coefficients of subscripts $a_{n,2r}$, the other set containing the subscripts $a_{n,2r+1}$. Thus we get

$$M_0 = a_{n,0} m_0 + a_{n,2} m_2 + a_{n,4} m_4 + \cdots\cdots$$

$$(4) \quad M_2 = a_{n,0} m_2 + a_{n,2} m_4 + a_{n,4} m_6 + \cdots\cdots$$

$$M_4 = a_{n,0} m_4 + a_{n,2} m_6 + a_{n,4} m_8 + \cdots\cdots$$

. . . .

and

$$M_1 = a_{n,1} m_2 + a_{n,3} m_4 + a_{n,5} m_6 + \cdots\cdots$$

$$(4a) \quad M_3 = a_{n,1} m_4 + a_{n,3} m_6 + a_{n,5} m_8 + \cdots\cdots$$

$$M_5 = a_{n,1} m_6 + a_{n,3} m_8 + a_{n,5} m_{10} + \cdots\cdots$$

. . . .

The computation of the moments, m_r, may be accomplished in the following manner:

$$(5) \qquad C_x = \frac{(2p)!}{(p+x)!(p-x)!}\,\frac{1}{2^{2p}}$$ is the

general term in the expansion of $(\frac{1}{2}+\frac{1}{2})^{2p}$,

$$\frac{C_{x+1}}{C_x} = \frac{(2p)!}{(p+x+1)!(p-x-1)!}\,\frac{1}{2^{2p}} \cdot \frac{(p+x)!(p-x)!}{(2p)!}\,2^{2p} = \frac{p-x}{p+x+1}$$

$$(6) \qquad \therefore\; C_{x+1}(p+x+1) = (p-x)C_x .$$

Multiplying each side of (6) by $(x+1)^k$ and summing from

$x=-p$ to $x=p$, we get

$$\sum_{-p}^{p} C_{x+1}(p+x+1)(x+1)^{k} = \sum_{-p}^{p}(p-x)(x+1)^{k} C_x \qquad \text{or}$$

$$\sum_{-p}^{p} C_{x+1}(p)(x+1)^{k} + \sum_{-p}^{p} C_{x+1}(x+1)^{k+1} = \sum_{-p}^{p} p(x+1)^{k} C_x - \sum_{-p}^{p} x(x+1)^{k} C_x$$

$$= p\left[\sum_{-p}^{p} C_x x^{k} + \binom{k}{1}\sum_{-p}^{p} C_x x^{k-1} + \binom{k}{2}\sum_{-p}^{p} C_x x^{k-2} + \cdots\cdots\right]$$

$$-\sum_{-p}^{p}\left[C_x x^{k+1} + \binom{k}{1} C_x x^{k} + \binom{k}{2} C_x x^{k-1} + \cdots\cdots\right].$$

By virtue of (3), this becomes

$$p m_k + m_{k+1} = p\left[m_k + \binom{k}{1} m_{k-1} + \binom{k}{2} m_{k-2} + \cdots\right]$$

$$(7) \qquad - \left[m_{k+1} + \binom{k}{1} m_k \binom{k}{2} m_{k-1} + \cdots\right].$$

Combining the terms in m_i , and recalling that all odd moments are identically zero, we reduce equation (7) to

$$(8) \qquad 2m_{k+1} = \left[\binom{k}{1}p - \binom{k}{2}\right] m_{k-1} + \left[\binom{k}{3}p - \binom{k}{4}\right] m_{k-3} + \left[\binom{k}{5}p - \binom{k}{6}\right] m_{k-5} + \cdots$$

By means of this recurrence relationship, the moments are found to be

$$m_0 = 1$$

$$m_2 = \frac{p}{2}$$

$$m_4 = \frac{3p^{(2)}}{4} + \frac{p}{2}$$

$$m_6 = \frac{15p^{(3)}}{8} + \frac{15p^{(2)}}{4} + \frac{p}{2}$$

$$m_8 = \frac{105p^{(4)}}{16} + \frac{105p^{(3)}}{4} + \frac{63p^{(2)}}{4} + \frac{p}{2}$$

$$m_{10} = \frac{945p^{(5)}}{32} + \frac{1575p^{(4)}}{8} + \frac{2205p^{(3)}}{8} + \frac{255p^{(2)}}{4} + \frac{p}{2}$$

$$m_{12} = \frac{10395p^{(6)}}{64} + \frac{51975p^{(5)}}{32} + \frac{65835p^{(4)}}{16} + 2640p^{(3)}$$
$$+ \frac{1023p^{(2)}}{4} + \frac{p}{2}$$

$$m_{14} = \frac{135135p^{(7)}}{128} + \frac{945945p^{(6)}}{64} + \frac{945945p^{(5)}}{16} + 75075p^{(4)}$$
$$+ \frac{195195p^{(3)}}{8} + \frac{4095p^{(2)}}{4} + \frac{p}{2}$$

where $p^{(n)} = p(p-1)(p-2)\cdots(p-n+1)$.

When expanded, the values are

$$m_0 = 1$$

$$m_2 = \frac{p}{2}$$

$$m_4 = \frac{p(3p-1)}{4}$$

$$m_6 = \frac{p(15p^2-15p+4)}{8}$$

$$m_8 = \frac{p(105p^3-210p^2+147p-34)}{16}$$

$$m_{10} = \frac{p(945p^4-3150p^3+4095p^2-2370p+496)}{32}$$

$$m_{12} = \frac{p(10395p^5-51975p^4+107415p^3-111705p^2}{64}$$
$$+ \frac{56958p-11056)}{64}$$

$$m_{14} = \frac{p(135135p^6 - 945945p^5 + 2837835p^4 - 4579575p^3}{128}$$

$$+ \frac{4114110p^2 - 1911000p + 349504)}{128}$$

Knowing the moments, it is now possible, by explicit calculation, to find the coefficients, $a_{n,i}$, in (1) for special cases.

Case I. Linear

$$y = (a_{1,0} + a_{1,1}x)\, C_x .$$

(4) and (4a) both reduce to single equations

(9) (a) $M_0 = m_0 a_{1,0}$, $a_{0,1} = \dfrac{M_0}{m_0} = M_0$,

(b) $M_1 = m_2 a_{1,1}$, $a_{1,1} = A_1 M_1$, where $A_1 = \dfrac{2}{p}$.

Case II. Quadratic.

$$y = (a_{2,0} + a_{2,1}x + a_{2,2}x^2)\, C_x .$$

This leads to the two sets of equations,

(a)
$$M_0 = m_0 a_{2,0} + m_2 a_{2,2},$$
$$M_2 = m_2 a_{2,0} + m_4 a_{2,2},$$

and

(b) $M_1 = m_2 a_{2,1}$.

(Notice that one set of equations in each case is identical with a set in the preceding case. Therefore, only one of the two sets needs to be solved.)

(b) is identical with (b) in (9),

$$a_{2,1} = a_{1,1} = A_1 M_1 .$$

Solving (a), we have

$$(10) \qquad \begin{aligned} a_{2,0} &= A_2 M_0 + B_2 M_2 , \\ a_{2,2} &= B_2 M_0 + C_2 M_2 , \end{aligned}$$

where $A_2 = \dfrac{3p-1}{2p-1}$, $B_2 = -\dfrac{2}{2p-1}$,

$$C_2 = \frac{4}{p(2p-1)} .$$

Case III. Cubic.

$$y = (a_{3,0} + a_{3,1} x + a_{3,2} x^2 + a_{3,3} x^3) C x$$

The resulting equations are

$$(a) \qquad \begin{aligned} M_0 &= m_0 a_{3,0} + m_2 a_{3,2} , \\ M_2 &= m_2 a_{3,0} + m_4 a_{3,2} , \end{aligned}$$

$$(b) \qquad \begin{aligned} M_1 &= m_2 a_{3,1} + m_4 a_{3,3} , \\ M_3 &= m_4 a_{3,1} + m_6 a_{3,3} , \end{aligned}$$

$$
(11)\quad
\begin{aligned}
a_{3,0} &= a_{2,0} = A_2 M_0 + B_2 M_2 , \\
a_{3,2} &= a_{2,2} = B_2 M_0 + C_2 M_2 , \\
a_{3,1} &= A_3 M_1 + B_3 M_3 , \\
a_{3,3} &= B_3 M_1 + C_3 M_3 ,
\end{aligned}
$$

where
$$
A_3 = \frac{2(15p^2 - 15p + 4)}{d_3},
$$
$$
B_3 = \frac{-4(3p-1)}{d_3},
$$
$$
C_3 = \frac{8}{d_3},
$$
$$
d_3 = 3p(p-1)(2p-1).
$$

Case IV. Quartic.

$$
y = (a_{4,0} + a_{4,1}x + a_{4,2}x^2 + a_{4,3}x^3 + a_{4,4}x^4)C_x.
$$

The coefficients with the second subscript odd, are identical with those in case III. The other coefficients are found from the equations

$$
\begin{aligned}
M_0 &= m_0 a_{4,0} + m_2 a_{4,2} + m_4 a_{4,4} , \\
M_2 &= m_2 a_{4,0} + m_4 a_{4,2} + m_6 a_{4,4} , \\
M_4 &= m_4 a_{4,0} + m_6 a_{4,2} + m_8 a_{4,4} .
\end{aligned}
$$

$$
\therefore\ a_{4,1} = a_{3,1} ,
$$
$$
a_{4,3} = a_{3,3} .
$$

and

$$
a_{4,0} = A_4 M_0 + B_4 M_2 + C_4 M_4 ,
$$

$$
(12) \qquad
\begin{aligned}
a_{4,2} &= B_4 M_0 + D_4 M_2 + E_4 M_4 , \\
a_{4,4} &= C_4 M_0 + E_4 M_2 + F_4 M_4 ,
\end{aligned}
$$

where

$$A_4 = \frac{(15p^2-25p+6)}{2d'_4},$$

$$B_4 = \frac{-10(p-1)}{d'_4},$$

$$C_4 = \frac{2}{d'_4}.$$

$$d'_4 = (2p-1)(2p-3).$$

$$D_4 = \frac{4(24p^2-39p+17)}{d_4},$$

$$E_4 = -\frac{8(3p-2)}{d_4},$$

$$F_4 = \frac{8}{d_4},$$

$$d_4 = 3p(p-1)(2p-1)(2p-3).$$

Case V. Quintic.

$$y = (a_{5,0} + a_{5,1}x + a_{5,2}x^2 + a_{5,3}x^3 + a_{5,4}x^4 + a_{5,5}x^5)C_x.$$

As in the previous case, we have at once

$$a_{5,0} = a_{4,0},$$

$$a_{5,2} = a_{4,2},$$

$$a_{5,4} = a_{4,4}.$$

and

$$M_1 = m_2 a_{5,1} + m_4 a_{5,3} + m_6 a_{5,5},$$

$$M_3 = m_4 a_{5,1} + m_6 a_{5,3} + m_8 a_{5,5},$$

$$M_5 = m_6 a_{5,1} + m_8 a_{5,3} + m_{10} a_{5,5},$$

from which

$$a_{5,1} = A_5 M_1 + B_5 M_3 + C_5 M_5,$$

$$a_{5,3} = B_5 M_1 + D_5 M_3 + E_5 M_5, \tag{13}$$

$$a_{5,5} = C_5 M_1 + E_5 M_3 + F_5 M_5,$$

where

$$A_5 = \frac{(525p^4 - 2100p^3 + 2835p^2 - 1480p + 276)}{d_5},$$

$$B_5 = \frac{-(20)(21p^3 - 63p^2 + 56p - 12)}{d_5},$$

$$C_5 = \frac{4(15p^2 - 25p + 6)}{d_5},$$

$$D_5 = \frac{40(12p^2 - 27p + 16)}{d_5},$$

$$E_5 = \frac{-80(p-1)}{d_5},$$

$$F_5 = \frac{16}{d_5},$$

$$d_5 = 15p(p-1)(p-2)(2p-1)(2p-3).$$

Case VI. Sextic.

$$y = (a_{6,0} + a_{6,1}x + a_{6,2}x^2 + a_{6,3}x^3 + a_{6,4}x^4 + a_{6,5}x^5 + a_{6,6}x^6)C_x.$$

As before, we have

$$a_{6,1} = a_{5,1},$$

$$a_{6,3} = a_{5,3},$$

$$a_{6,5} = a_{5,5}.$$

From the equations

$$M_0 = m_0 a_{6,0} + m_2 a_{6,2} + m_4 a_{6,4} + m_6 a_{6,6},$$

$$M_2 = m_2 a_{6,0} + m_4 a_{6,2} + m_6 a_{6,4} + m_8 a_{6,6},$$

$$M_4 = m_4 a_{6,0} + m_6 a_{6,2} + m_8 a_{6,4} + m_{10} a_{6,6},$$

$$M_6 = m_6 a_{6,0} + m_8 a_{6,2} + m_{10} a_{6,4} + m_{12} a_{6,6},$$

we obtain

$$
(14)\quad
\begin{aligned}
a_{6,0} &= A_6 M_0 + B_6 M_2 + C_6 M_4 + D_6 M_6, \\
a_{6,2} &= B_6 M_0 + E_6 M_2 + F_6 M_4 + G_6 M_6, \\
a_{6,4} &= C_6 M_0 + F_6 M_2 + H_6 M_4 + I_6 M_6, \\
a_{6,6} &= D_6 M_0 + G_6 M_2 + I_6 M_4 + J_6 M_6,
\end{aligned}
$$

where

$$A_6 = \frac{3(35p^3 - 140p^2 + 147p - 30)}{2d_6'},$$

$$B_6 = -\frac{7(15p^2 - 48p + 28)}{d_6'},$$

$$C_6 = \frac{14(3p-5)}{d_6'},$$

$$D_6 = -\frac{4}{d_6'},$$

$$d_6' = 3(2p-1)(2p-3)(2p-5).$$

$$E_6 = \frac{2(3465p^4 - 18270p^3 + 33915p^2 - 25950p + 7216)}{d_6}$$

$$F_6 = -\frac{20(171p^3 - 681p^2 + 846p - 304)}{d_6},$$

$$G_6 = \frac{8(45p^2 - 105p + 46)}{d_6},$$

$$H_6 = \frac{40(51p^2 - 147p + 110)}{d_6},$$

$$I_6 = -\frac{80(3p-4)}{d_6},$$

$$J_6 = \frac{32}{d_6},$$

$$d_6 = 45p(p-1)(p-2)(2p-1)(2p-3)(2p-5).$$

Tables for all the coefficients, $A_1, A_2, B_2 \cdots$ to J_6, to ten significant figures for p from 1 to 20 will be found at the end of this paper.

Special attention is directed to the last coefficient in each case, $a_{r,r}$, as reference will be made to it later. It is desirable to be able to compute this coefficient without having to solve a set of equations.

Let $P_{(n)}$ represent the determinant

$$(15a) \qquad \begin{vmatrix} m_0 & m_2 & m_4 & \cdots & m_n \\ m_2 & m_4 & m_6 & \cdots & m_{n+2} \\ m_4 & m_6 & m_8 & \cdots & m_{n+4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ m_n & m_{n+2} & m_{n+4} & \cdots & m_{2n} \end{vmatrix}$$

for n an even integar, and

$$
(15b) \qquad \begin{vmatrix} m_2 & m_4 & m_6 & \cdots & m_{n+1} \\ m_4 & m_6 & m_8 & \cdots & m_{n+3} \\ m_6 & m_8 & m_{10} & \cdots & m_{n+5} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ m_{n+1} & m_{n+3} & m_{n+5} & \cdots & m_{2n} \end{vmatrix}
$$

for n odd. By definition

$$P_{(-2)} = P_{(-1)} = P_{(0)} = 1.$$

Let $P_{(n,M)}$ denote the same determinants, the last column being replaced by $M_0, M_2, M_4 \cdots\cdots M_n$, or by $M_1, M_3, M_5 \cdots M_n$ according to whether n is even or odd. Similarly, $P_{(n,x)}$ will represent the original determinants, the last column being replaced by $1, x^2, x^4, \cdots\cdots x^n$ or by $x, x^3, x^5, \cdots\cdots x^n$ for n even or odd respectively. It is clear from the normal equations in case r, that

$$
(16) \qquad a_{r,r} = \frac{P_{(r,M)}}{P_{(r)}} = \frac{\begin{vmatrix} m_0 & m_2 & \cdots & M_0 \\ m_2 & m_4 & \cdots & M_2 \\ \cdot & \cdot & \cdots & \cdot \\ m_r & m_{r+2} & \cdots & M_r \end{vmatrix}}{\begin{vmatrix} m_0 & m_2 & \cdots & m_r \\ m_2 & m_4 & \cdots & m_{r+2} \\ \cdot & \cdot & \cdots & \cdot \\ m_r & m_{r+2} & \cdots & m_{2r} \end{vmatrix}} \quad \text{for } n \text{ even.}
$$

Thus, for $r=4$

we have, $a_{4,4} = \dfrac{\begin{vmatrix} m_0 & m_2 & M_0 \\ m_2 & m_4 & M_2 \\ m_4 & m_6 & M_4 \end{vmatrix}}{\begin{vmatrix} m_0 & m_2 & m_4 \\ m_2 & m_4 & m_6 \\ m_4 & m_6 & m_8 \end{vmatrix}}$

The case with which the determinants, $P_{(n)}$, can be evaluated is not atonce evident. It will be shown later that

$$(17) \qquad P_{(n)} = \frac{n!(2p)^{(n)}}{2^{2n}} P_{(n-2)} \qquad \text{(See (47)}$$

(18) where $(2p)^{(n)} = (2p)(2p-1)(2p-2)\cdots(2p-n+1)$.

Starting with $P_{(0)} = 1$, we have

$$P_{(2)} = \frac{2!(2p)^{(2)}}{2^{2.2}} \cdot 1 ,$$

$$P_{(4)} = \frac{4!(2p)^{(4)}}{2^{2.4}} \cdot \frac{2!(2p)^{(2)}}{2^{2.2}} = \frac{4!2!(2p)^{(4)}(2p)^{(2)}}{2^{2(2+4)}} ,$$

$$P_{(6)} = \frac{6!\,4!\,2!\,(2p)^{(6)}(2p)^{(4)}(2p)^{(2)}}{2^{2(6+4+2)}}.$$

Similarly, since $p_{(1)} = \frac{2p}{2^2}$,

$$P_{(3)} = \frac{3!\,1!\,(2p)^{(3)}(2p)}{2^{2(3+1)}}.$$

It is clear that for n even, or odd,

$$(19)\quad P_{(n)} = \frac{n!\,(n-2)!\,(n-4)!\,\cdots\cdots(2p)^{(n)}(2p)^{(n-2)}(2p)^{(n-4)}\cdots}{2^{2(n+\overline{n-2}+\overline{n-4}\,\cdots\cdots)}},$$

each series ending with $n-r=0$ or 1, for n even, or odd.

With the recurrence formula (17) or the general formula (19) it is a simple matter to evaluate $P_{(n)}$ a list of which follows:

$$P_{(0)} = 1,$$

$$P_{(1)} = \frac{p}{2},$$

$$P_{(2)} = \frac{p(2p-1)}{4},$$

$$P_{(3)} = \frac{3p^2(p-1)(2p-1)}{16},$$

$$P_{(4)} = \frac{3p^2(p-1)(2p-1)^2(2p-3)}{32},$$

$$P_{(5)} = \frac{45p^3(p-1)^2(p-2)(2p-1)^2(2p-3)}{256},$$

$$P_{(6)} = \frac{135p^3(p-1)^2(p-2)(2p-1)^3(2p-3)^2(2p-5)}{1024},$$

(20)

$$P_{(7)} = \frac{14{,}175p^4(p-1)^3(p-2)^2(p-3)(2p-1)^3(2p-3)^2(2p-5)}{16{,}384},$$

$$P_{(8)} = \frac{42{,}525p^4(p-1)^3(p-2)^2(p-3)(2p-1)^4}{32{,}768}$$

$$\frac{(2p-3)^3(2p-5)^2(2p-7)}{32{,}768}.$$

III. Development By Gram's Method.

Let the variates, x_i, the observations, y_x, and the weights, C_x, be given as before. We assume that there exists a set of functions

$$\phi_0(x), \quad \phi_1(x), \quad \phi_2(x) \cdot \cdot \cdot \cdot \phi_n(x)$$

of degrees 0, 1, 2, ... n, respectively, each of the form

(21) $$\phi_r(x) = b_0 + b_1 x + b_2 x^2 + \cdot \cdot \cdot \cdot \cdot b_r x^r,$$

and such that

(22) $$\begin{cases} (a) \sum_{-p}^{p} C_x \phi_m(x) \phi_n(x) = 0 & \text{for } m \neq n \\ (b) \sum_{-p}^{p} C_x \left[\phi_r(x)\right]^2 = S_r \neq 0, & \end{cases}$$

the value of S_r to be determined later.

We wish to approximate by means of these functions a function of x,

$$(23) \quad y = y'C_x = \left[a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x)\right]C_x,$$

which will be the equation of a curve fitting the given data.

Multiplying each side of (23) by $\phi_r(x)$ and summing from $x = -p$ to $+p$, — applying (22), — we have

$$(24) \quad \sum_{-p}^{p} y_x \phi_r(x) = a_r \sum_{-p}^{p} C_x\left[\phi_r(x)\right]^2 = a_r S_r.$$

Substitute in the left member of (24) the value of $\phi_r(x)$ in (21) and we have

$$b_0\sum_{-p}^{p} y_x + b_1\sum_{-p}^{p} y_x x + b_2\sum_{-p}^{p} y_x x^2 + \cdots + b_r\sum_{-p}^{p} y_x x^r = a_r S_r,$$

which, by (3), becomes

$$(25) \quad a_r = \frac{b_0 M_0 + b_1 M_1 + b_2 M_2 + \cdots + b_r M_r}{S_r}.$$

This value of a_r is identical with $a_{r,r}$ found by the first method as may be shown in the following manner:

Let

$$(26) \quad J = \sum_{-p}^{p} C_x\left[y'_x - a_0\phi_0(x) - a_1\phi_1(x) - \cdots - a_n\phi_n(x)\right]^2,$$

which is to be minimized, where $y'_x C_x = y_x$.

Taking the partial derivative of J with respect to a_r, we have

$$\frac{\partial J}{\partial a_r} = -2\sum_{-p}^{p} C_x \phi_r(x)\left[y'_x - a_0\phi_0(x) - a_1\phi_1(x) \cdots\cdots - a_n\phi_n(x)\right] = 0,$$

which reduces, by (22), to

$$(27) \qquad \sum_{-p}^{p} y'_x C_x \phi_r(x) - a_r \sum_{-p}^{p} C_x \phi_r^2(x) = 0,$$

$$\text{or} \qquad a_r S_r = \sum_{-p}^{p} y_x \phi_r(x) \qquad \text{as in (24)}$$

We are, therefore, able to write the values of $\frac{b_i}{S_r}$ in (25) by comparing the coefficients of the moments, M_i, in (25) with those in $a_{r,r}$ in equations (9) to (14), or in (16).

Thus, for $r = 4$, we have

$$a_{4,4} = C_4 M_0 + E_4 M_2 + F_4 M_4,$$

$$a_4 = \frac{b_0 M_0}{S_4} + \frac{b_1 M_1}{S_4} + \frac{b_2 M_2}{S_4} + \frac{b_3 M_3}{S_4} + \frac{b_4 M_4}{S_4},$$

$$(28) \qquad \therefore \frac{b_0}{S_4} = C_4, \; \frac{b_2}{S_4} = E_4, \; \frac{b_4}{S_4} = F_4.$$

$$\frac{b_1}{S_4} = \frac{b_3}{S_4} = 0.$$

$$b_0 = C_4 S_4, \qquad b_2 = E_4 S_4, \qquad b_4 = S_4 F_4,$$

$$(29) \qquad \therefore \phi_4(x) = S_4 C_4 + S_4 E_4 x^2 + S_4 F_4 x^4.$$

Now the value of $S_r = \sum_{-p}^{p} C_x \left[\phi_r(x)\right]^2$ can be found by comparing the coefficients of $C_x x^r M_r$ in the expansions given by the two different methods. In case IV, we have

$$y = (\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; + a_{4,4} x^4) C_x$$

$$= (F_4 M_4 x^4 + E_4 M_2 x^4 + C_4 M_0 x^4 + \text{terms of lower degree in } x) C_x ,$$

so the desired coefficient is F_4.

By (23), we have

$$y = a_4 \phi_4(x) C_x + a_3 \phi_3(x) C_x + \cdots\cdots$$

$$= \left(\frac{b_0}{S_4} M_0 + \cdots\cdot + \frac{b_4}{S_4} M_4\right)(S_4 F_4 x^4 + S_4 E_4 x^2 + S_4 C_4) C_x + \cdots$$

$$= \frac{b_4}{S_4} M_4 \cdot S_4 F_4 x^4 C_x + \cdots\cdots$$

But by (28) $b_4 = S_4 F_4$ and the desired coefficient is

$$\frac{S_4 F_4}{S_4} \cdot S_4 F_4 = S_4 F_4^{\,2} .$$

Equating these coefficients, we have

$$S_4 F_4^{\,2} = F_4 ,$$

$$\text{(30)} \qquad \therefore \; S_4 = \frac{1}{F_4} .$$

Therefore, S_r is equal to the reciprocal of the coefficient of M_r in $a_{r,r}$. But examination of (16) shows that this coefficient is $\dfrac{P_{(r-2)}}{P_{(r)}}$.

(31) $$\therefore S_r = \frac{P(r)}{P(r-2)}.$$

The coefficient of x^i in $\phi_r(x)$ can now be found by multiplying the coefficient of M_i in $a_{r,r}$ by $\frac{P(r)}{P(r-2)} = S_r$. Indeed, it is possible to express $\phi_r(x)$ as the quotient of two determinants. The coefficient of x^i in $P(r, x)$ will be identical with that of M_i in $P(r, M)$. We may, therefore, write

(32) $$\phi_r(x) = \frac{P(r,x)}{P(r)} \cdot \frac{P(r)}{P(r-2)} = \frac{P(r,x)}{P(r-2)}.$$

Proceeding in this manner, we obtain

$$\phi_0(x) = 1,$$

$$\phi_1(x) = x,$$

$$\phi_2(x) = x^2 - \frac{p}{2},$$

$$\phi_3(x) = x^3 - \frac{3p-1}{2}x,$$

$$\phi_4(x) = x^4 - (3p-2)x^2 + \frac{3p(p-1)}{4},$$

(33) $$\phi_5(x) = x^5 - 5(p-1)x^3 + \frac{(15p^2 - 25p + 6)x}{4},$$

$$\phi_6(x) = x^6 - 5\left(\frac{3p-4}{2}\right)x^4 + \frac{45p^2 - 105p + 46}{4}x^2 - \frac{15p(p-1)(p-2)}{8},$$

$$\phi_7(x) = x^7 - 7\frac{3p-5}{2}x^5 + \frac{105p^2 - 315p + 196}{4}x^3$$

$$-\left(\frac{105p^3 - 420p^2 + 441p - 90}{8}\right)x,$$

$$\phi_8(x) = x^8 - 14(p-2)x^6 + \frac{7(15p^2-55p+44)}{2}x^4$$

$$-\frac{105p^3-525p^2+742p-264}{2}x^2$$

$$+\frac{105}{16}p(p-1)(p-2)(p-3).$$

The coefficient, a_i, of $\phi_i(x)$ is given below. In addition to the values obtained from (9) to (14), a_7 and a_8 have been added. The values of $A_1, B_2, \cdot\cdot\cdot\cdot\cdot\cdot J_6$ are given in the tables at the end of this paper.

$$a_0 = M_0,$$

$$a_1 = A_1 M_1,$$

$$a_2 = B_2 M_0 + C_2 M_2,$$

$$a_3 = B_3 M_1 + C_3 M_3,$$

(34) $$a_4 = C_4 M_0 + E_4 M_2 + F_4 M_4,$$

$$a_5 = C_5 M_1 + E_5 M_3 + F_5 M_5,$$

$$a_6 = D_6 M_0 + G_6 M_2 + I_6 M_4 + J_6 M_6.$$

$$a_7 = D_7 M_1 + G_7 M_3 + I_7 M_5 + J_7 M_7.$$

$$a_8 = A_8 M_0 + B_8 M_2 + C_8 M_4 + D_8 M_6 + E_8 M_8,$$

where
$$D_7 = -\frac{8(105p^3-420p^2+441p-90)}{d_7},$$

$$G_7 = \frac{16(105p^2-315p+196)}{d_7},$$

$$I_7 = -\frac{224(3p-5)}{d_7},$$

$$J_7 = \frac{64}{d_7},$$

$$d_7 = 315p(p-1)(p-2)(p-3)(2p-1)(2p-3)(2p-5),$$

and
$$A_8 = \frac{2}{3(2p-1)(2p-3)(2p-5)(2p-7)},$$

$$B_8 = \frac{-16(105p^3-525p^2+742p-264)}{d_8},$$

$$C_8 = \frac{112(15p^2-55p+44)}{d_8},$$

$$D_8 = \frac{-448(p-2)}{d_8},$$

$$E_8 = \frac{32}{d_8},$$

$$d_8 = 315p(p-1)(p-2)(p-3)(2p-1)(2p-3)(2p-5)(2p-7).$$

It was shown above how the coefficients of x^i in $\phi_r(x)$ could be found from those of M_i in a_{rr}. It is evident from that, that with $\phi_r(x)$ known, the value of a_r can be immediately determined by changing x^i to M_i and multiplying the result by $F_r = \frac{P(r-2)}{P_r}$. It will be shown later that these ϕ's can be determined independent of the determinants previously used, and more easily. a_7 and a_8 were determined from $\phi_7(x)$ and $\phi_8(x)$, respectively, and then checked by means of (16).

IV. Properties of the ϕ Function.

The similarity between the ϕ functions just derived and the Hermite polynomials, of which these may be said to be the analog, is evident, and leads one to expect that there must be a generating function analogous $e^{-x^2/2}$ by means of which all of the ϕ's can be found. Also, one would naturally expect to find a recurrence formula and a second order difference equation analogous to the relationships existing between the Hermite polynomials.

This proves to be true. We have, in fact,

$$(35) \quad C_x \phi_n(x) = \left(-\tfrac{1}{2}\right)^n \Delta^n C_x (p+x)^{(n)}, \qquad \text{where}$$

$$(p+x)^{(n)} = (p+x)(p+x-1)(p+x-2)\cdots(p+x-n+1).$$

Expand the right member of (35) and divide both sides by C_x, and we obtain

$$(36) \quad \begin{aligned} \phi_n(x) = \left(-\tfrac{1}{2}\right)^n \Big[& (p-x)^{(n)} - n(p-x)^{(n-1)}(p+x) \\ & + \binom{n}{2}(p-x)^{(n-2)}(p+x)^{(2)} \cdots + (-1)^{n-1}(p-x)(p+x)^{(n-1)} + (-1)^n (p+x)^{(n)} \Big]. \end{aligned}$$

A study of the expression in the bracket brings out the following facts:

(37) *The coefficient of* x^n *is* $(-2)^n$, for it is seen to be equal to

$$(-1)^n - \binom{n}{1}(-1)^{n-1}(1) + \binom{n}{2}(-1)^{n-2}(1)^2 - \binom{n}{3}(-1)^{n-3}(1)^3 + \cdots$$

$$(-1)^n (1)^n = (-1)^n \left[1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots \binom{n}{n}\right] = (-1)^n 2^n.$$

(38). *If* n *is even, only even powers of* x *appear in the expanded form of the bracket; for* n *odd, only odd powers of* x *will occur.*

Consider the coefficients of x^{n-1}, x^{n-3}, x^{n-5}, etc. With n even, the terms of odd degree in x in $(p-x)^{(n)}$ will be negative, and cancel the corresponding terms in the last parenthesis, $(p+x)^{(n)}$. If n is odd, the factor $(-1)^n$ causes the corresponding terms of even degree in the same two expansions to have opposite signs, and, therefore, to vanish. Similar reasoning holds for every pair of products in the bracket which have the equal coefficients, $\binom{n}{r}=\binom{n}{n-r}$. If n is odd, this will include every term. If n is even, the middle term will be

$$(p-x)^{(n/2)}(p+x)^{(n/2)}=(p-x)(p+x)(p-x-1)(p+x-1)(p-x-2)(p+x-2)\ldots,$$

the odd powers of x having zero coefficients.

Therefore, either all terms are of even, or all are of odd degree.

It is necessary first to prove that the relations (22) hold; that is,

$$\text{(a)} \quad \sum_{-p}^{p} C_x \phi_m(x)\phi_n(x)=0, \qquad \text{for } m\neq n.$$

$$\text{(b)} \quad \sum_{-p}^{p} C_x\left[\phi_m(x)\right]^2 = S_m = \frac{m!(2p)^{(m)}}{2^{2m}}.$$

To prove the first relationship, we may proceed as follows:

Let

$$\text{(39)} \qquad \begin{aligned}\sum_{-p}^{p} C_x\phi_m(x)\phi_n(x) &= \sum_{-p}^{p+1}\phi_m(x)\cdot C_x\phi_n(x)\\ &= \Delta^{-1}\left[\phi_m(x)\cdot C_x\phi_n(x)\right]_{p}^{p+1},\end{aligned}$$

where we may assume without loss of generality that $n>m$, for if it is not, $\phi_m(x)$ and $\phi_n(x)$ can be interchanged. Using the formula for finite integration,

$$\Delta^{-1} u_x v_x = u_x \Delta^{-1} v_x - \Delta u_x \Delta^{-2} v_{x+1} + \Delta^{2} u_x \Delta^{-3} v_{x+2} - \Delta^{3} u_x \Delta^{-4} v_{x+3} + \cdots\cdots,$$

we have

$$\Delta^{-1}\phi_m(x)\cdot C_x\phi_n(x) = \phi_m(x)\Delta^{-1}C_x\phi_n(x) - \Delta\phi_m(x)\Delta^{-2}C_{x+1}\phi_n(x+1)$$

$$+\Delta^2\phi_m(x)\Delta^{-3}C_{x+2}\phi_n(x+2)\,..\,+(-1)^m\Delta^m\phi_m(x)\Delta^{-m-1}C_{x+m}\phi_n(x+m)$$

(40)

$$+(-1)^{m+1}\Delta^{m+1}\phi_m(x)\Delta^{-m-2}C_{x+m+1}\phi_n(x+m+1)+\cdots\cdots$$

between the limits $-p$ and $p+1$.

Now $\Delta^{-1}C_x\phi_n(x) = \Delta^{-1}\left[(-\tfrac{1}{2})^n\Delta^n C_x(p+x)^{(n)}\right]$, (by 35)

$$=\left(-\tfrac{1}{2}\right)^n\Delta^{n-1}C_x(p+x)^{(n)}$$

$$=\left(-\tfrac{1}{2}\right)^n C_x\Big[(p-x)^{(n-1)}(p+x)-\binom{n-1}{1}(p-x)^{(n-2)}(p+x)^{(2)}$$

$$+\binom{n-1}{2}(p-x)^{(n-3)}(p+x)^{(3)}\cdots\cdot(-1)^{n-1}(p+x)^{(n)}\Big]$$

$$=\left(-\tfrac{1}{2}\right)^n C_x(p+x)\Big[f\{(p-x),(p+x-1)\}\Big]$$

$$=\left(-\tfrac{1}{2}\right)^n \frac{(2p)!}{2^{2p}(p+x)!(p-x)!}\cdot(p+x)\Big[f\{(p-x),(p+x-1)\}\Big]$$

For the lower limit, $-p$, the factor $p+x=0$, and for $x=p+1, \frac{1}{(p-x)!}=\frac{1}{(-1)!}=\frac{1}{\infty}=0.$

$$\therefore \Delta^{-1}C_x\phi_n(x)\Big|_{-p}^{p+1}=0$$

Similarly, each of the succeeding terms up to and including the term $\Delta^m\phi_m(x)\Delta^{-m-1}C_{x+m}\phi_n(x+m)$ becomes identically zero because of the factor $p+x$ or C_x for the lower and upper limits, respectively.

Since $\phi_m(x)$ is of the m-th degree in x, $\Delta^{m+1}\phi_m(x)$ and all higher differences are identically zero.

$$(41)\quad \therefore \sum_{-p}^{p} C_x\phi_m(x)\phi_n(x)=0, \qquad \text{for } m\neq n.$$

If $m=n$, the first m terms vanish as in the preceding case. The $(m+1)th$ term is the only term left in the series.

$$(42)\quad \therefore \sum_{-p}^{p} C_x\phi_m(x)\phi_m(x)=(-1)^m\Delta^m\phi_m(x).\Delta^{-m-1}C_{x+m}\phi_m(x+m)$$

$$=(-1)^m m!\,\Delta^{-m-1}C_{x+m}\phi_m(x+m).$$

But

$$\Delta^{-m-1}C_{x+m}\phi_m(x+m)=\Delta^{-m-1}\left\{\left(-\tfrac{1}{2}\right)^m\Delta^m C_{x+m}(p+x+m)^{(m)}\right\}$$

$$=\left(-\tfrac{1}{2}\right)^m\Delta^{-1}\frac{(2p)!\,(p+x+m)^{(m)}}{2^{2p}(p+x+m)!(p-x-m)!}$$

(43)

$$= \frac{(-1)^m (2p)!}{2^{2p+m}} \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}.$$

It is now necessary to find a function, u_x, such that

$$\Delta u_x = \frac{1}{(p+x)!(p-x-m)!}.$$

Since $\Delta = (E-1)$, we may write

$$\Delta u_x = (E-1) \to u_x = \frac{1}{(p+x)!(p-x-m)!},$$

$$u_x = \frac{-1}{1-E} \to \frac{1}{(p+x)!(p-x-m)!}$$

$$= -\left[1+E+E^2+E^3+\cdots\right] \to \frac{1}{(p+x)!(p-x-m)!}.$$

(44) $$\therefore \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}\Bigg|_{-p}^{p+1} = u_x \Bigg|_{-p}^{p+1} =$$

$$-\frac{1}{(p+x)!(p-x-m)!} - \frac{1}{(p+x+1)!(p-x-m-1)!} - \frac{1}{(p+x+2)!(p-x-m-2)} + \cdots\cdots,$$

between the limits $-p$ and $p+1$. Substitution of the upper limit, $p+1$, makes every term zero, because $(p-x-m-k)! = \infty$.

For $x = -p$, the right member becomes

$$\frac{1}{0!(2p-m)!} + \frac{1}{1!(2p-m-1)!} + \frac{1}{2!(2p-m-2)!} + \cdots\cdots$$

$$+ \frac{1}{(2p-m-1)![2p-m-(2p-m-1)]!} + \frac{1}{(2p-m)!0!},$$

all succeeding terms being zero because of the second factor.

$$\therefore \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}$$

$$= \frac{1}{(2p-m)!}\left[1+\binom{2p-m}{1}+\binom{2p-m}{2}+\cdots\binom{2p-m}{2p-m-1}+\binom{2p-m}{2p-m}\right]$$

or

$$(45) \quad = \frac{2^{2p-m}}{(2p-m)!}$$

Returning to (43) and then to (42), we have

$$(46) \quad \sum_{-p}^{p} C_x\left[\phi_m(x)\right]^2 = (-1)^m m!\cdot\frac{(-1)^m(2p)!}{2^{2p+m}}\cdot\frac{2^{2p-m}}{(2p-m)!}$$

$$= \frac{m!(2p)^{(m)}}{2^{2m}} = S_m, \text{ by (22),}$$

[By (31), $S_m = \dfrac{P_{(m)}}{P_{(m-2)}}$.

$$(47) \quad \therefore P_{(m)} = \frac{m!(2p)^{(m)}}{2^{2m}} P_{(m-2)}. \qquad \text{See (17).}]$$

Therefore, $C_x\phi_n(x) = (-\frac{1}{2})^n \Delta^n C_x(p+x)^{(n)}$ satisfies both conditions of (22).

Recurrence Formula.

It is necessary to note that

$$\Delta C_x(p+x)^{(n)} = \frac{(2p)!}{2^{2p}}\left[\frac{(p+x+1)^{(n)}}{(p+x+1)!(p-x-1)!} - \frac{(p+x)^{(n)}}{(p+x)!(p-x)!}\right]$$

$$= C_x(p+x)^{(n-1)}\left[\frac{(p+x+1)(p-x)}{p+x+1} - (p+x-n+1)\right]$$

(48) $$= C_x(p+x)^{(n-1)}(-2x+n-1).$$

$$\therefore \Delta^{n+1}C_x(p+x)^{(n+1)} = \Delta^n\left[\Delta C_x(p+x)^{(n+1)}\right]$$

$$= \Delta^n\left[C_x(p+x)^{(n)}(-2x+n)\right]$$

$$= (-2x+n)\Delta^n C_x(p+x)^{(n)} + n(-2)\Delta^{n-1}C_{x+1}(p+x+1)^{(n)}.$$

But $C_{x+1}(p+x+1)^{(n)} = (1+\Delta)C_x(p+x)^{(n)}$.

$$\therefore \Delta^{n+1}C_x(p+x)^{(n+1)} = (-2x+n)\Delta^n C_x(p+x)^{(n)}$$

$$-2n\Delta^n C_x(p+x)^{(n)} - 2n\Delta^{n-1}C_x(p+x)^{(n)}$$

$$= -2x\Delta^n C_x(p+x)^{(n)} - n\Delta^{n-1}\left[\Delta C_x(p+x)^{(n)}\right] - 2n\Delta^{n-1}C_x(p+x)^{(n)}$$

$$= -2x\Delta^n C_x(p+x)^{(n)} - n\Delta^{n-1}C_x(p+x)^{(n-1)}(-2x+n-1)$$

$$-n\Delta^{n-1}C_x(p+x)^{(n-1)}(2p+2x-2n+2)$$

$$= -2x\Delta^n C_x(p+x)^{(n)} - n(2p+1-n)\Delta^{n-1}C_x(p+x)^{(n-1)}.$$

Now $\Delta^n C_x(p+x)^{(n)} = (-2)^n C_x \phi_n(x)$,

$$\therefore (-2)^{n+1} C_x \phi_{n+1}(x) = -2x(-2)^n C_x \phi_n(x) - n(2p+1-n)(-2)^{n-1} C_x \phi_{n-1}(x),$$
or

$$(49) \qquad 4\phi_{n+1}(x) - 4x\phi_n(x) + n(2p+1-n)\phi_{n-1}(x) = 0.$$

Difference Equation.

To simplify the reductions later in this development, it is desirable to have the following identities:—

(50)

$$\text{(a)} \quad \Delta C_x = C_x\left[\frac{p-x}{p+x+1} - 1\right] = C_x \frac{-2x-1}{p+x+1},$$

$$\text{(b)} \quad \Delta^2 C_x = C_x \frac{4x^2+8x+2-2p}{(p+x+2)(p+x+1)},$$

$$\text{(c)} \quad C_{x+1} = C_x \frac{p-x}{p+x+1},$$

$$\text{(d)} \quad \Delta C_{x+1} = C_{x+1}\frac{-2x-3}{p+x+2} = C_x \frac{p-x}{p+x+1} \cdot \frac{-2x-3}{p+x+2},$$

$$\text{(e)} \quad C_{x+2} = C_x \frac{(p-x)(p-x-1)}{(p+x+2)(p+x+1)}.$$

Let $u_x = C_x (p+x)^{(n)}$,

$$\Delta^n u_x = \Delta^n C_x (p+x)^{(n)} = (-2)^n C_x \phi_n(x),$$

$$\Delta u_x = C_x (p+x)^{(n-1)}(-2x+n-1), \qquad \text{(by 48)}$$

and, multiplying both sides by $(p+x-n+1)$, we get

$$(p+x-n+1)\Delta u_x = (-2x+n-1)u_x.$$

Difference this equation $n+1$ times, having

$$(51) \quad \begin{aligned}&(p+x-n+1)\Delta^{n+2}u_x + (n+1)\Delta^{n+1}(u_{x+1}) \\ &= (-2x+n-1)\Delta^{n+1}u_x - 2(n+1)\Delta^n u_{x+1}\end{aligned}$$

or, since $u_{x+1} = (1+\Delta) \to u_x$, we have

(52) $(p+x-n+1)\Delta^{n+2}u_x + (n+1)\Delta^{n+2}u_x + (n+1)\Delta^{n+1}u_x$

$$= (-2x+n-1)\Delta^{n+1}u_x - 2(n+1)\Delta^{n+1}u_x - 2(n+1)\Delta^{n}u_x .$$

Combining like differences, we get

$$(p+x+2)\Delta^{n+2}u_x + (2x+2n+4)\Delta^{n+1}u_x + 2(n+1)\Delta^{n}u_x = 0.$$

(53) $(p+x+2)\Delta^2(-2)^n C_x\phi_n(x) + (2x+2n+4)\Delta(-2)^n C_x\phi_n(x)$

$$+2(n+1)(-2)^n C_x\phi_n(x) = 0.$$

This may be simplified by making the following reductions:

$$\Delta^2 C_x\phi_n(x) = \phi_n(x)\cdot\Delta^2 C_x + 2\Delta\phi_n(x)\cdot\Delta C_{x+1} + \Delta^2\phi_n(x)\cdot C_{x+2}$$

$$= \phi_n(x)\cdot C_x \frac{4x^2+8x+2-2p}{(p+x+2)(p+x+1)} + 2\Delta\phi_n(x)\cdot C_x \frac{(p-x)(-2x-3)}{(p+x+2)(p+x+1)}$$

$$+ \Delta^2\phi_n(x)\cdot C_x \frac{(p-x)(p-x-1)}{(p+x+2)(p+x+1)},$$

$$\Delta C_x\phi_n(x) = \phi_n(x)\cdot\Delta C_x + \Delta\phi_n(x)\cdot C_{x+1}$$

$$= \phi_n(x)\cdot C_x \frac{-2x-1}{p+x+1} + \Delta\phi_n(x)\cdot C_x \frac{p-x}{p+x+1}.$$

Substituting these values in (53), noticing that C_x is a common factor, and that $\frac{1}{p+x+1}$ can be made a factor by multiplying the last term by $\frac{p+x+1}{p+x+1}$, we have

(54)

$$(p-x)(p-x-1)\Delta^2\phi_n(x)+\left[2(p-x)(-2x-3)+(p-x)(2x+2n+4)\right]\Delta\phi_n(x)$$

$$+\left[4x^2+8x+2-2p+(2x+2n+4)(-2x-1)+2(n+1)(p+x+1)\right]\phi_n(x)=0.$$

The coefficient of $\phi_n(x)$ reduces to $2n(p-x)$, and that of $\Delta\phi_n(x)$, to $(p-x)(2_n-2-2x)$ Dividing by $(p-x)$, we obtain the desired second order difference equation

(55) $$(p-x-1)\Delta^2\phi_n(x)+2(n-1-x)\Delta\phi_n(x)+2n\phi_n(x)=0.$$

V. Other Methods of Deriving These Functions,— 3rd. Method.

Equations (2) were divided into two groups, (4) and (4a), from which these functions were developed. It would be expected that the functions could be derived from (2). This is easily seen to be true.

Let $Q_{(n)}$ be the determinant formed by the coefficients of $a_{n,i}$ in (2).

(56) Let $$Q_{(n)}=\begin{vmatrix} m_0 & 0 & m_2 & 0 & m_4 & \cdots \\ 0 & m_2 & 0 & m_4 & 0 & \cdots \\ m_2 & 0 & m_4 & 0 & m_6 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & m_{2n} \end{vmatrix}$$

for $n > 0$ In the special case $n = -1$, we will define $Q_{(-1)} = 1$.

$$\text{Let } Q_{(n,x)} = \begin{vmatrix} m_0 & 0 & m_2 & 0 & \cdots & 1 \\ 0 & m_2 & 0 & m_4 & \cdots & x \\ m_2 & 0 & m_4 & 0 & \cdots & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & x^n \end{vmatrix}$$

Then $\phi_n(x)$ is easily seen to be

$$\phi_n(x) = \frac{Q_{(n,x)}}{Q_{(n-1,)}}. \tag{57}$$

Explicitly, we will have

$$\phi_0(x) = \frac{Q_{(0,x)}}{Q_{(-1)}} = 1,$$

$$\phi_1(x) = \frac{Q_{(1,x)}}{Q_{(0)}} = \frac{\begin{vmatrix} m_0 & 1 \\ 0 & x \end{vmatrix}}{m_0} = x,$$

$$\phi_2(x) = \frac{\begin{vmatrix} m_0 & 0 & 1 \\ 0 & m_2 & x \\ m_2 & 0 & x^2 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 \\ 0 & m_2 \end{vmatrix}} = x^2 \frac{-m_2^2}{m_0 m_2} = x^2 - \frac{\rho}{2},$$

$$\phi_3(x) = \frac{\begin{vmatrix} m_0 & 0 & m_2 & 1 \\ 0 & m_2 & 0 & x \\ m_2 & 0 & m_4 & x^2 \\ 0 & m_4 & 0 & x^3 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 \\ 0 & m_2 & 0 \\ m_2 & 0 & m_4 \end{vmatrix}} = x^3 + x\,\frac{(m_2^2 m_4 - m_4^2)}{m_2(m_0 m_4 - m_2^2)}$$

$$= x^3 - \frac{m_4}{m_2}x = x^3 - \frac{3p-1}{2}x,$$

and so on.

The coefficients of the ϕ's can be found from the formula

$$(58) \qquad a_i = \frac{Q(i,M)}{Q_{(i)}},$$

where $Q_{(i,M)}$ is $Q_{(i)}$, the last column being replaced with M_0, $M_1, M_2, \cdots \cdots M_i$. Thus, we have

$$a_0 = \frac{M_0}{m_0} = M_0,$$

$$a_1 = \frac{\begin{vmatrix} m_0 & M_0 \\ 0 & M_1 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 \\ 0 & m_2 \end{vmatrix}} = \frac{m_0 M_1}{m_0 m_2} = \frac{2}{p} M_1 = A_1 M_1,$$

$$a_2 = \frac{\begin{vmatrix} m_0 & 0 & M_0 \\ 0 & m_2 & M_1 \\ m_2 & 0 & M_2 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 \\ 0 & m_2 & 0 \\ m_2 & 0 & m_4 \end{vmatrix}} = \frac{-m_2 M_0}{m_0 m_4 - m_2^2} + \frac{m_0 M_2}{m_0 m_4 - m_2^2}$$

$$= -\frac{2}{2p-1} M_0 + \frac{4}{p(2p-1)} M_2$$

$$= B_2 M_0 + C_2 M_2,$$

$$a_3=\frac{\begin{vmatrix} m_0 & 0 & m_2 & M_0 \\ 0 & m_2 & 0 & M_1 \\ m_2 & 0 & m_4 & M_2 \\ 0 & m_4 & 0 & M_3 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 & 0 \\ 0 & m_2 & 0 & m_4 \\ m_2 & 0 & m_4 & 0 \\ 0 & m_4 & 0 & m_6 \end{vmatrix}}=\frac{M_1(m_2^2 m_4-m_0 m_4^2)+M_3 m_2(m_0 m_4-m_2)^2}{m_6 m_2(m_0 m_4-m_2^2)+m_4^2(m_0 m_4-m_2^2)}$$

$$=\frac{-\frac{p(3p-1)}{4}M_1+\frac{p}{2}M_3}{\frac{3}{16}p^2(2p-1)(p-1)}$$

$$=B_3 M_1+C_3 M_3.$$

This method becomes more difficult than the first because of the higher order determinants, and is, therefore, of less value in deriving the functions.

A fourth method of developing these polynomials is to build up a set of orthogonal functions in the following manner:[1]

[1]This method is given in a thesis by Harry R. Mathias, entitled "Properties of Orthogonal and Biorthogonal Functions from the Standpoint of Integral Equations," written at Indiana University August, 1925. He cites as his reference E. Goursat—"Recherches sur les equations intégrales linéaires," Ann. de la Fac. De Toulouse, t. 10, 2nd series, 1908, pp. 5-98, especially page 66.

Assume a set of functions

$f_0(x)=1,\ f_1(x)=x,\ f_2(x)=x^2,\ f_3(x)=x^3,\ f_4(x)=x^4.$

It is required to find a set of functions, $\phi_i(x)$, such that

$$\sum_{-p}^{p} C_x \phi_m(x)\phi_n(x)=0, \qquad \text{for } m \neq n.$$

Let $\phi_0(x)=f_0(x)=1.$

We may then form the equations

$$\sum_{-p}^{p} C_x\left[f_i(x)-a_i f_0(x)\right]f_0(x)=0$$, obtaining

(59)
$$\sum_{-p}^{p} C_x\left[x-a_1\cdot 1\right]1=0,$$

$$\sum_{-p}^{p} C_x\left[x^2-a_2\cdot 1\right]1=0,$$

$$\sum_{-p}^{p} C_x\left[x^3-a_3\cdot 1\right]1=0,$$

$$\sum_{-p}^{p} C_x\left[x^4-a_4\cdot 1\right]1=0,$$

the solutions of which are

$$a_1=0,\ a_2=m_2=\frac{p}{2},\ a_3=0,\ a_4=m_4=\frac{p(3p-1)}{4}.$$

Let $\theta_1(x) = x - a_1 \cdot 1 = x = \phi_1(x),$

$$\theta_2(x) = x^2 - \frac{p}{2},$$

$$\theta_3(x) = x^3,$$

$$\theta_4(x) = x^4 - \frac{p(3p-1)}{4}.$$

Form a set of equations, similar to (59), using the Θ's .

$$\sum_{-p}^{p} C_x\left(x^2 - \frac{p}{2} - b_1 x\right)x = 0,$$

$$\sum_{-p}^{p} C_x\left(x^3 - b_2 x\right)x = 0,$$

$$\sum_{-p}^{p} C_x\left(x^4 - \frac{p(3p-1)}{4} - b_3 x\right)x = 0.$$

From these equations, we get

$$b_1 = 0, \qquad b_2 = \frac{3p-1}{2}, \qquad b_3 = 0.$$

Let $\psi_2(x) = \left(x^2 - \frac{p}{2}\right) = \phi_2(x),$

$$\psi_3(x) = \left(x^3 - \frac{3p-1}{2}x\right),$$

$$\psi_4(x) = \left(x^4 - \frac{p(3p-1)}{4}\right),$$

Similarly, the equations

$$\Sigma\, C_x\left[x^3 - \frac{3p-1}{2}x - c_1\left(x^2 - \frac{p}{2}\right)\right]\left(x^2 - \frac{p}{2}\right) = 0,$$

$$\Sigma\, C_x\left[x^4 - \frac{p(3p-1)}{4} - c_2\left(x^2 - \frac{p}{2}\right)\right]\left(x^2 - \frac{p}{2}\right) = 0,$$

lead to the values

$$C_1 = 0, \qquad C_2 = (3p-2).$$

C_1 is easily seen to equal 0 , as the term independent of C_1 is $\Sigma\{C_x$ multiplied by an odd function in x . Expanding the second equation, we get

$$C_2\left[\sum_{-p}^{p} C_x\left(x^2-\frac{p}{2}\right)^2\right] = \sum_{-p}^{p} C_x\left[x^4 - \frac{p(3p-1)}{4}\right]\left(x^2-\frac{p}{2}\right),$$

$$C_2 = \frac{\sum_{-p}^{p} C_x\left[x^6 - \frac{p}{2}x^4 - \frac{p(3p-1)}{4}x^2 + \frac{p^2}{8}(3p-1)\right]}{\sum_{-p}^{p} C_x\left[x^4 - px^2 + \frac{p^2}{4}\right]}$$

$$= \frac{m_6 - \frac{p}{2}m_4 - \frac{p(3p-1)}{4}m_2 + \frac{p^2}{8}(3p-1)m_0}{m_4 - pm_2 + \frac{p^2}{4}m_0},$$

and substituting the values of the moments, we reduce C_2 to the value above, $C_2 = (3p-2)$.

Let $\lambda_3(x) = x^3 - \frac{3p-1}{2}x = \phi_3(x)$,

$$\lambda_4(x) = x^4 - \frac{p(3p-1)}{4} - (3p-2)\left(x^2 - \frac{p}{2}\right)$$

$$= x^4 - (3p-2)x^2 + \frac{3p(p-1)}{4},$$

This method may be continued to obtain as many of the ϕ's as desired. It is to be noted that $\lambda_4(x)$ is $\phi_4(x)$. However, the proof of this would necessitate adding another function, $f_5(x) = x^5$, at the beginning, and carrying out the process with another equation in each set.

Method 5.

Gram's equation, number 18, is

$$\phi_m'(x) = n^{(m)} - \binom{m}{1}(n-1)^{(m-1)} \cdot 2x + \binom{m}{2}(n-2)^{(m-2)} \cdot 2^2 x^{(2)} + \cdots (1)$$

This gives ϕ's for the origin at the left end of the distribution, the variates running from O to n. Let this equation be transformed as follows:

Let $m = n$,

$x = x' + p$, the primes being dropped,

$n = 2p$

$$\phi_n'(x) = (2p)^{(n)} - \binom{n}{1}(2p-1)^{(n-1)} \cdot 2(p+x) + \binom{n}{2}(2p-2)^{(n-2)} \cdot 2^2 (p+x)^{(2)}$$

If the values of $n = 0, 1, 2, \ldots\ldots$ be substituted, we will have $\phi_n(x) = \left(-\frac{1}{2}\right)^n \phi_n'(x)$ and the required functions are found again.

VI. Example.

As an example to illustrate the use of this development, I have chosen one used by Karl Pearson,[2] the data of which he attributes to T. N. Thiele,[3] and are the frequencies in a game of "patience." On page 295, Vol. I, Pearson states, "Now either of the curves in Illustration I and II is a good example of the impossibility of using the method of least squares for systematic curve fitting." (The set of data below is his illustration II.)

In volume 2, using the Method of Moments, he fits curves of

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[1]Gram uses the notation $n^{m|-1}$ instead of $n^{(m)}$, and $(m)_r$ for $\binom{m}{r}$.

[2]On the Systematic Fitting of Curves to Observations and measurements. Biometrika, vol. 1, (1902), pp. 265-303; vol. (1903) pp/1-27.

[3]Thiele—Forelaesninger over Almindelig Iagttagelseslaere, Copenhagen, (1889) p. 12.

the second to the sixth degree to this data. Noting that his statement, page 18, "taking the sixth parabola as the best fit" is correct, it is found that the sum of the squares of the errors is more than 1400. His results for the skew frequency curve gives as the sum of the squares of the errors, 782, which indicates the second curve to be a more accurate fit.

Let the given data be

Value of Character	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	0	0	3	7	35	101	89	94	70	46	30	15	4	5	1
Class Marks (x)	—7	—6	—5	—4	—3	—2	—1	0	1	2	3	4	5	6	7

The mean of this distribution being 11.86, the origin has been chosen as at 12; p, therefore, is 7.

Suppose we wish to find the curve

$$y = \left[a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)\right.$$

$$\left. + a_4\phi_4(x) + a_5\phi_5(x) + a_6\phi_6(x)\right] C_x$$

The values of the coefficients, a_i, are

$a_0 = M_0,$

$a_1 = A_1 M_1,$

$a_2 = B_2 M_0 + C_2 M_2,$

$a_3 = B_3 M_1 + C_3 M_3,$

$a_4 = C_4 M_0 + E_4 M_2 + F_4 M_4,$

$a_5 = C_5 M_1 + E_5 M_3 + F_5 M_5,$

$a_6 = D_6 M_0 + G_6 M_2 + I_6 M_4 + J_6 M_6.$

The moments, $M_i = \sum_{-p}^{p} y_x x^i$, are computed and found to be

$$M_0 = 500, \qquad M_3 = 1466,$$

$$M_1 = -70, \qquad M_4 = 26,664,$$

$$M_2 = 2088, \qquad M_5 = 64,010,$$

$$M_6 = 607,368.$$

From Table I, for $p = 7$, we find

A_1 = 0.28571 42857, C_5 = .(2)50260 85026,

B_2 = -0.15384 61538, E_5 = -.(2)10656 01066,

C_2 = 0.043956 04396, F_5 = .(4)35520 03552,

B_3 = -0.048840 04884, D_6 = -.(2)10360 01036,

C_3 = (2)48840 04884, G_6 = .(3)99719 21083,

C_4 = .(1)13986 01399, I_6 = -.(3)11182 23340,

E_4 = -.(2)84360 08436, J_6 = .(5)26311 13742,

F_4 = .(3)44400 04440,

Substituting in the above equations, and putting $p = 7$ in $\phi_i(x)$ we have

$$a_0 = 500 \qquad \phi_0(x) = 1,$$

$$a_1 = -20 \qquad \phi_1(x) = x,$$

$$a_2 = 14.85714, \qquad \phi_2(x) = x^2 - 3.5,$$

$$a_3 = 10.57875, \qquad \phi_3(x) = x^3 - 10x,$$

$$a_4 = 1.21744, \qquad \phi_4(x) = x^4 - 19x^2 + 31.5,$$

$$a_5 = 0.39564, \qquad \phi_5(x) = x^5 - 30x^3 + 141.5x,$$

$$a_6 = 0.18056, \qquad \phi_6(x) = x^6 - 42.5x^4 + 379x^2 - 393.75.$$

Multiplying the corresponding values, and collecting like powers of x , we have

$$y = C_x y' = (415.25836 - 69.80444x + 60.15802x^2 - 1.29045x^3$$

$$-6.45636x^4 + 0.39564x^5 + 0.18056x^6)C_x.$$

[I would call attention here to the fact, that if it is desired to have an equation of the second, third, fourth, or fifth degree, the above data can be used, without change, using only as many functions as the degree required. Thus for the fourth degree equation, we have

$$y = (a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x))C_x,$$

$$y = (486.34936 - 125.78750x - 8.27422x^2 + 10.57875x^3.$$

$$+1.21744x^4)C_x.]$$

Substituting values of x from -7 to $+7$ in the above equation, we obtain the following results:

x	y'	C_x (From Table II)	$y = y'C_x$	y to nearest integer	Observed y_x	Error $y - y_x$
−7	3385.714	.(4)61035 156	.20	0	0	0
−6	258.775	.(3)85449 219	.22	0	0	0
−5	−20.817	.(2)55541 992	−.12	0	3	−3
−4	421.199	.(1)22216 797	9.36	9	7	2
−3	713.454	.(1)61096 191	43.59	44	35	9
−2	701.412	0.12219 238	85.71	86	101	−15
−1	539.835	0.18328 857	98.94	99	89	10
0	415.254	0.20947 266	86.98	87	94	−7
1	398.437	0.18328 857	73.02	73	70	3
2	426.868	0.12219 238	52.16	52	46	6
3	417.224	.(1)61096 191	25.49	*26	30	−4
4	507.857	.(1)22216 797	11.28	11	15	−4
5	1431.276	.(2)55541 992	7.95	8	4	4
6	5016.640	.(3)85449 219	4.28	4	5	−1
7	14822.246	.(4)61035 156	.90	1	1	0

The sum of the squares of the errors for this curve is 562, as compared with more than 1400 in Pearson's first method, and 782 for his skew frequency curve.

The fourth degree curve, found by this method, gives 1170 for the sum of the squares of the errors.

*Taken as 26 so that $\Sigma y = \Sigma y_x = 500$.

TABLE I

(Number in parenthesis indicates the number of ciphers between the decimal point and the first significant figure.)

p	A_1	A_2	B_2	C_2	p
1	2.00000 00000	2.00000 00000	- 2.00000 00000	4.00000 00000	1
2	1.00000 00000	1.66666 66667	- 0.66666 66667	0.66666 66667	2
3	0.66666 66667	1.60000 00000	- 0.40000 00000	0.26666 66667	3
4	0.50000 00000	1.57142 85714	- 0.28571 42857	0.14285 71429	4
5	0.40000 00000	1.55555 55556	- 0.22222 22222	.(1)88888 88889	5
6	0.33333 33333	1.54545 45455	- 0.18181 81818	.(1)60606 06061	6
7	0.28571 42857	1.53846 15385	- 0.15384 61538	.(1)43956 04396	7
8	0.25000 00000	1.53333 33333	- 0.13333 33333	.(1)33333 33333	8
9	0.22222 22222	1.52941 17647	- 0.11764 70588	.(1)26143 79085	9
10	0.20000 00000	1.52631 57895	- 0.10526 31579	.(1)21052 63157	10
11	0.18181 81818	1.52380 95230	-.(1)95238 09524	.(1)17316 01732	11
12	0.16666 66667	1.52173 91304	-.(1)86956 52174	.(1)14492 75362	12
13	0.15384 61538	1.52000 00000	-.(1)80000 00000	.(1)12307 69231	13
14	0.14285 71429	1.51851 85185	-.(1)74074 07407	.(1)10582 01058	14
15	0.13333 33333	1.51724 13793	-.(1)68965 44828	.(2)91954 02298	15
16	0.12500 00000	1.51612 90322	-.(1)64516 12903	.(2)80645 16129	16
17	0.11764 70588	1.51515 15152	-.(1)60606 06061	.(2)71301 24778	17
18	0.11111 11111	1.51428 57143	-.(1)57142 85714	.(2)63492 06349	18
19	0.10526 31579	1.51351 35135	-.(1)54054 05405	.(2)56899 00427	19
20	0.10000 00000	1.51282 05128	-.(1)51282 05128	.(2)51282 05128	20

p	A_3	B_3	C_3	p
2	3.77777 77778	- 1.11111 11111	0.44444 44444	2
3	2.08888 88889	- 0.35555 55556	.(1)88888 88889	3
4	1.46031 74602	- 0.17460 31746	.(1)31746 03175	4
5	1.12592 59259	- 0.10370 37037	.(1)14814 81481	5
6	0.91717 17172	-.(1)68686 86869	.(2)80808 08081	6
7	0.77411 47741	-.(1)48840 04884	.(2)48840 04884	7
8	0.66984 12698	-.(1)36507 93651	.(2)31746 03175	8
9	0.59041 39434	-.(1)28322 44009	.(2)21786 49238	9
10	0.52787 52437	-.(1)22612 08577	.(2)15594 54190	10
11	0.47734 48773	-.(1)18470 41847	.(2)11544 01154	11
12	0.43566 09574	-.(1)15371 10232	.(3)87834 87044	12
13	0.40068 37607	-.(1)12991 45299	.(3)68376 06838	13
14	0.37091 30375	-.(1)11124 67779	.(3)54266 72093	14
15	0.34526 54625	-.(2)96332 78599	.(3)43787 62999	15
16	0.32293 90681	-.(2)84229 39068	.(3)35842 29391	16
17	0.30332 73916	-.(2)74272 13310	.(3)29708 85324	17
18	0.28596 32742	-.(2)65981 94834	.(3)24898 84843	18
19	0.27048 10073	-.(2)59006 37480	.(3)21073 70528	19
20	0.25659 01934	-.(2)53081 42149	.(3)17993 70220	20

p	A_4	B_4	C_4	p
2	2.66666 66667	− 3.33333 33333	0.66666 66667	2
3	2.20000 00000	− 1.33333 33333	0.13333 33333	3
4	2.08571 42857	− 0.85714 28571	.(1)57142 85714	4
5	2.03174 60317	− 0.63492 63492	.(1)31746 03175	5
6	2.00000 00000	− 0.50505 05051	.(1)20202 02020	6
7	1.97902 09790	− 0.41958 04196	.(1)13986 01399	7
8	1.96410 25641	− 0.35897 43590	.(1)10256 41025	8
9	1.95294 11765	− 0.31372 54902	.(2)78431 37255	9
10	1.94427 24458	− 0.27863 77709	.(2)61919 50464	10
11	1.93734 33584	− 0.25062 65664	.(2)50125 31328	11
12	1.93167 70186	− 0.22774 32712	.(2)41407 86749	12
13	1.92695 65217	− 0.20869 56522	.(2)34782 60870	13
14	1.92296 29630	− 0.19259 25926	.(2)29629 62963	14
15	1.91954 02299	− 0.17879 94891	.(2)25542 78416	15
16	1.91657 39711	− 0.16685 20578	.(2)22246 94105	16
17	1.91397 84946	− 0.15640 27370	.(2)19550 34213	17
18	1.91168 83117	− 0.14718 61472	.(2)17316 01732	18
19	1.90965 25096	− 0.13899 61390	.(2)15444 01544	19
20	1.90783 09078	− 0.13167 01317	.(2)13860 01386	20

p	D_4	E_4	F_4	p
2	7.77777 77778	− 1.77777 77778	0.44444 44444	2
3	1.71851 85185	− 0.20740 74074	.(1)29629 62963	3
4	0.77777 77778	− .(1)63492 06349	.(2)63492 06349	4
5	0.44656 08466	− .(1)27513 22751	.(2)21164 02116	5
6	0.29046 01571	− .(1)14365 88103	.(3)89786 75645	6
7	0.20424 02042	− .(2)84360 08436	.(3)44400 04440	7
8	0.15152 62515	− .(2)53724 05372	.(3)24420 02442	8
9	0.11692 08424	− .(2)36310 82062	.(3)14524 32824	9
10	.(1)92970 98957	− .(2)25685 12785	.(4)91732 59947	10
11	.(1)75704 41255	− .(2)18834 96620	.(4)60757 95549	11
12	.(1)62843 75850	− .(2)14220 88379	.(4)41826 12878	12
13	.(1)53006 31735	− .(2)10999 62840	.(4)29728 72538	13
14	.(1)45312 71198	− .(3)86826 75349	.(4)21706 68837	14
15	.(1)39181 82002	− .(3)69735 85518	.(4)16217 64074	15
16	.(1)34217 03127	− .(3)56853 29378	.(4)12359 41169	16
17	.(1)30140 11078	− .(3)46959 15512	.(5)95835 01045	17
18	.(1)26751 17185	− .(3)39234 54904	.(5)75451 05584	18
19	.(1)23903 60285	− .(3)33115 82259	.(5)60210 58653	19
20	.(1)21487 88465	− .(3)28206 34400	.(5)48631 62758	20

ρ	A_5	B_5	C_5	ρ
3	5.31555 55556	− 2.31111 11111	0.19555 55556	3
4	3.15206 34920	− 0.86984 12698	.(1)46349 20635	4
5	2.28176 36680	− 0.46490 29982	.(1)18059 96473	5
6	1.79717 17172	− 0.29090 90909	.(2)88888 88889	6
7	1.48530 58058	− 0.19962 25996	.(2)50260 85026	7
8	1.26686 60968	− 0.14562 47456	.(2)31176 23118	8
9	1.10499 84438	− 0.11098 66169	.(2)20666 04420	9
10	0.98009 86125	−.(1)87421 16730	.(2)14402 01812	10
11	0.88072 97705	−.(1)70654 75135	.(2)10436 86658	11
12	0.79975 28076	−.(1)58297 25830	.(3)78047 55631	12
13	0.73247 36327	−.(1)48925 37414	.(3)59889 86859	13
14	0.67567 56659	−.(1)41647 89943	.(3)46958 80252	14
15	0.62707 92809	−.(1)35883 40140	.(3)37500 17543	15
16	0.58502 29532	−.(1)31239 29587	.(3)30421 80906	16
17	0.54826 66002	−.(1)27442 67469	.(3)25019 32673	17
18	0.51586 56594	−.(1)24299 01253	.(3)20824 49142	18
19	0.48708 78486	−.(1)21666 60247	.(3)17517 73888	19
20	0.46135 66062	−.(1)19440 22295	.(3)14875 87453	20

ρ	D_5	E_5	F_5	ρ
3	1.27407 40741	− 0.11851 85185	.(1)11851 85185	3
4	0.31746 03175	−.(1)19047 61905	.(2)12698 41270	4
5	0.12768 95944	−.(2)56437 38977	.(3)28218 69488	5
6	.(1)64197 53086	−.(2)22446 68911	.(4)89786 75645	6
7	.(1)36852 03685	−.(2)10656 01066	.(4)35520 03552	7
8	.(1)23117 62312	−.(3)56980 05698	.(4)16280 01628	8
9	.(1)15458 03507	−.(3)33198 46457	.(5)82996 16142	9
10	.(1)10847 37988	−.(3)20639 83488	.(5)45866 29974	10
11	.(2)79052 85098	−.(3)13501 76788	.(5)27003 53577	11
12	.(2)59393 10287	−.(4)92017 48332	.(5)16730 45152	12
13	.(2)45755 21097	−.(4)64862 67356	.(5)10810 44559	13
14	.(2)35996 92489	−.(4)47031 15814	.(6)72355 62791	14
15	.(2)28829 97519	−.(4)34930 30313	.(6)49900 43304	15
16	.(2)23447 56961	−.(4)26484 45363	.(6)35312 60483	16
17	.(2)19326 72711	−.(4)20444 80223	.(6)25556 00279	17
18	.(2)16118 23181	−.(4)16033 34937	.(6)18862 76396	18
19	.(2)13582 79996	−.(4)12750 47715	.(6)14167 19683	19
20	.(2)11552 71331	−.(4)10266 67694	.(6)10807 02835	20

p	A_6	B_6	C_6	p
3	31.86666 66667	− 2.95555 55556	1.24444 44444	3
4	9.86349 20646	− 1.68888 88889	0.31111 11111	4
5	6.24338 62423	− 1.20740 74074	0.14814 81481	5
6	4.87864 28716	− 0.94276 09428	.(1)87542 08754	6
7	4.17145 81715	− 0.77415 17742	.(1)58016 05802	7
8	3.75275 83528	− 0.65703 18570	.(1)41336 44134	8
9	3.47631 97577	− 0.57083 96179	.(1)30970 33685	9
10	3.28104 57516	− 0.50471 27623	.(1)24079 80736	10
11	3.13617 37676	− 0.45235 63811	.(1)19263 84589	11
12	3.02462 67843	− 0.40986 52428	.(1)15764 04780	12
13	2.93620 42788	− 0.37468 59903	.(1)13140 09662	13
14	2.86445 51800	− 0.34507 78315	.(1)11121 84648	14
15	2.80510 85568	− 0.31981 26863	.(2)95359 72754	15
16	2.75522 87178	− 0.29799 91485	.(2)82670 73153	16
17	2.71273 35648	− 0.27897 43935	.(2)72358 73754	17
18	2.67610 66890	− 0.26223 52558	.(2)63864 45096	18
19	2.64421 82442	− 0.24739 28474	.(2)56784 05678	19
20	2.61620 96162	− 0.23414 18341	.(2)50820 05082	20

p	D_6	E_6	F_6	p
3	− 0.88888 88889	10.85234 56791	− 3.56543 20985	3
4	−.(1)12698 41270	28.89171 07594	− 5.51675 48502	4
5	−.(2)42328 04233	1.42790 35865	− 0.19461 49324	5
6	−.(2)19240 01924	0.86407 78152	−.(1)91881 78077	6
7	−.(2)10360 01036	0.58217 60133	−.(1)50816 67304	7
8	−.(3)62160 06216	0.41986 26065	−.(1)31099 76443	8
9	−.(3)40221 21669	0.31750 86255	−.(1)20431 95246	9
10	−.(3)27519 77984	0.24868 72810	−.(1)14149 24384	10
11	−.(3)19656 98560	0.20013 83997	−.(1)10205 88045	11
12	−.(3)14529 07631	0.16458 67560	−.(2)76047 24004	12
13	−.(3)11042 09800	0.13776 06312	−.(2)58190 22630	13
14	−.(4)85882 98444	0.11701 40143	−.(2)45522 17628	14
15	−.(4)68114 09110	0.10063 52146	−.(2)36284 76722	15
16	−.(4)54930 71863	.(1)87476 39340	−.(2)29389 67830	16
17	−.(4)44943 31524	.(1)76744 10943	−.(2)24138 37900	17
18	−.(4)37238 74691	.(1)67875 46964	−.(2)20068 35826	18
19	−.(4)31200 03120	.(1)60462 05713	−.(2)16865 01032	19
20	−.(4)26400 02640	.(1)54201 64924	−.(2)14309 22600	20

p	G_6	H_6	I_6	p
3	0.26864 19754	1.26419 75310	−.(1)98765 43210	3
4	0.24409 17107	1.19223 98589	−.(1)56437 38977	4
5	.(2)60764 25632	.(1)30570 25279	−.(2)10346 85479	5
6	.(2)22147 39993	.(1)11372 98915	−.(3)29928 91882	6
7	.(3)99719 21083•	.(2)51964 49641	−.(3)11182 23340	7
8	.(3)51454 71812	.(2)27108 69377	−.(4)49333 38267	8
9	.(3)29218 90503	.(2)15524 53840	−.(4)24473 22709	9
10	.(3)17816 50932	.(3)95299 97834	−.(4)13250 26436	10
11	.(3)11479 15011	.(3)61737 49551	−.(5)76774 75857	11
12	.(4)77282 94532	.(3)41752 74961	−.(5)46962 67091	12
13	.(4)53932 11190	.(3)29248 26113	−.(5)30029 01554	13
14	.(4)38778 42203	.(3)21098 48164	−.(5)19924 01348	14
15	.(4)28596 27483	.(3)15602 20206	−.(5)13639 45170	15
16	.(4)21549 40811	.(3)11786 12681	−.(6)95910 77856	16
17	.(4)16546 77743	.(4)90694 43517	−.(6)69030 58224	17
18	.(4)12915 92277	.(4)70928 04898	−.(6)50706 35472	18
19	.(4)10229 00232	.(4)56268 09892	−.(6)37922 29454	19
20	.(5)82061 36861	.(4)45209 40193	−.(6)28818 74227	20

p	J_6
3	.(2)79012 34568
4	.(2)28218 69488
5	.(4)37624 92651
6	.(5)85511 19662
7	.(5)26311 13742
8	.(6)98666 76533
9	.(6)42562 13407
10	.(6)20385 02210
11	.(6)10589 62187
12	.(7)58703 33864
13	.(7)34318 87490
14	.(7)20972 64577
15	.(7)13306 78214
16	.(8)87191 61687
17	.(8)58749 43169
18	.(8)40565 08378
19	.(8)28620 59966
20	.(8)20584 81591

TABLE II

$$C_x = C_{-x} = \frac{(2p)!}{2^{2p}(p-x)!\,(p+x)!}$$

(The number in parenthesis indicates the number of ciphers between the decimal point and the first significant figure.)

x	$p=3$	$p=4$	$p=5$	$p=6$
0	0.31250 00000	0.27343 75000	0.24609 37500	0.22558 59375
1	0.23437 50000	0.21875 00000	0.20507 81250	0.19335 93750
2	0.09375 00000	0.10937 50000	0.11718 75000	0.12084 96094
3	0.01562 50000	0.03125 00000	0.04394 53125	0.05371 09375
4		0.00390 62500	.(2)97656 25000	.(1)16113 28125
5			.(3)97656 25000	.(2)29296 87500
6				.(3)24414 06250

x	$p=7$	$p=8$	$p=9$	$p=10$
0	0.20947 26563	0.19638 06152	0.18547 05810	0.17619 70520
1	0.18328 85742	0.17456 05469	0.16692 35229	0.16017 91382
2	0.12219 23828	0.12219 23828	0.12139 89258	0.12013 43536
3	.(1)61096 19141	.(1)66650 39062	.(1)70816 04003	.(1)73928 83301
4	.(1)22216 79688	.(1)27770 99609	.(1)32684 32617	.(1)36964 41650
5	.(2)55541 99219	.(2)85449 21875	.(1)11672 97363	.(1)14785 76660
6	.(3)85449 21875	.(2)18310 54687	.(2)31127 92968	.(2)46205 52063
7	.(4)61035 15625	.(3)24414 06250	.(3)58364 86816	.(2)10871 88721
8		.(4)15258 78906	.(4)68664 55078	.(3)18119 81201
9			.(5)38146 97265	.(4)19073 48633
10				.(6)95367 43164

x	$p=11$	$p=12$	$p=13$	$p=14$
0	0.16818 80951	0.16118 02578	0.15498 10171	0.14944 59808
1	0.15417 24205	0.14878 17764	0.14391 09445	0.13948 29154
2	0.11859 41696	0.11689 99672	0.11512 87556	0.11332 98688
3	.(1)76239 10904	.(1)77933 31146	.(1)79151 01945	.(1)79997 55442
4	.(1)40660 85815	.(1)43837 48770	.(1)46559 42321	.(1)48887 39437
5	.(1)17789 12544	.(1)20629 40598	.(1)23279 71160	.(1)25730 20756
6	.(2)62785 14862	.(2)80225 46768	.(2)98019 83833	.(1)11578 59340
7	.(2)17440 31906	.(2)25334 35822	.(2)34306 94342	.(2)44108 92725
8	.(3)36716 46118	.(3)63335 89554	.(3)98019 83833	.(2)14034 65867
9	.(4)55074 69177	.(3)12063 98010	.(3)22277 23598	.(3)36612 15305
10	.(5)52452 08740	.(4)16450 88196	.(4)38743 01910	.(4)76275 31886
11	.(6)23841 85791	.(5)14305 11475	.(5)48428 77388	.(4)12204 05102
12		.(7)59604 64477	.(6)38743 01910	.(5)14081 59733
13			.(7)14901 16119	.(6)10430 81284
14				.(8)37252 90298

x	$p=15$	$p=16$	$p=17$
0	0.14446 44481	0.13994 99341	0.13583 37596
1	0.13543 54201	0.13171 75850	0.12828 74396
2	0.11153 50518	0.10976 46542	0.10803 15281
3	.(1)80553 09299	.(1)80879 21888	.(1)81023 64605
4	.(1)50875 63768	.(1)52571 49227	.(1)54015 76403
5	.(1)27981 60072	.(1)30040 85273	.(1)31919 40602
6	.(1)13324 57177	.(1)15020 42636	.(1)16653 08140
7	.(2)54509 61180	.(2)65306 20158	.(2)76326 62309
8	.(2)18959 86497	.(2)24489 82559	.(2)30530 64924
9	.(3)55299 60617	.(3)78367 44189	.(2)10568 30166
10	.(3)13271 90548	.(3)21098 92666	.(3)31313 48640
11	.(4)25522 89516	.(4)46886 50370	.(4)78283 71599
12	.(5)37811 69653	.(5)83725 89946	.(4)16196 63089
13	.(6)40512 53200	.(5)11548 39992	.(5)26994 38482
14	.(7)27939 67724	.(6)11548 39992	.(6)34831 46429
15	.(9)93132 25746	.(8)74505 80597	.(7)32654 49777
16		.(9)23283 06436	.(8)19790 60471
17			.(10)58207 66091

x	$p=18$	$p=19$	$p=20$
0	0.13206 05996	0.12858 53206	0.12537 06876
1	0.12511 00417	0.12215 60546	0.11940 06549
2	0.10634 35354	0.10470 51897	0.10311 87474
3	.(1)81023 64606	.(1)80908 55565	.(1)80701 62839
4	.(1)55243 39504	.(1)56284 21262	.(1)57163 65345
5	.(1)33626 41437	.(1)35177 63289	.(1)36584 73821
6	.(1)18214 30779	.(1)19699 47442	.(1)21106 57973
7	.(2)87428 67737	.(2)98497 37209	.(1)10944 15245
8	.(2)36989 05581	.(2)43776 60982	.(2)50812 13640
9	.(2)13699 65030	.(2)17197 95385	.(2)21025 71161
10	.(3)44034 59025	.(3)59303 28916	.(3)77094 27591
11	.(3)12147 47317	.(3)17790 98675	.(3)24869 12126
12	.(4)28344 10407	.(4)45912 22386	.(4)69944 40355
13	.(5)54859 55626	.(4)10043 29897	.(4)16956 21904
14	.(6)85718 05666	.(5)18260 54358	.(5)34909 86273
15	.(6)10390 06747	.(6)26853 74056	.(6)59845 47897
16	.(8)91677 06595	.(7)30689 98921	.(7)83118 72080
17	.(9)52386 89483	.(8)25574 99101	.(8)89858 07654
18	.(10)14551 91523	.(9)13824 31946	.(9)70940 58673
19		(11)36379 78808	.(10)36379 78807
20			.(12)90949 47018

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